DETERMINANTAL PROCESSES

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A determinantal process on a nice topological measured space S is a random discrete collection of points such that the correlation functions – loosely speaking, the density of probability of seeing a finite subcollection of points at given locations – exist, and may be written in the form

$$\rho_m(x_1, \ldots, x_m) = \det[(K(x_i, x_j))_{1 \le i,j \le m}] \,,$$

where $K : S^2 \to \mathbb{C}$ is a two-point function, also called a kernel.

There are many examples of stochastic models which give rise to interesting determinantal processes, many of which find their origin in mathematical physics. The corresponding kernel can sometimes be computed rather explicitly, which enables the study of fine properties of the model. We may broadly distinguish two classes of processes according to the topology of S: discrete ones (e.g. $S = \mathbb{Z}^d$) and continuous ones (e.g. $S = \mathbb{R}^d$). Examples of discrete processes include random spanning forests on finite and infinite graphs; examples of continuous processes include eigenvalues of certain random matrices of finite or infinite size.

The goal of this course will be to present a theory of determinantal processes, namely to present what is common to these examples beyond their particularities. For that matter, we will focus on the better understood case where K is self-dual, namely when the symmetry $K(x,y) = \overline{K(y,x)}$ holds.

We will start with the case where S is finite, for which a very complete understanding is available. The kernel K is then a Hermitian matrix, and the process is completely described in terms of linear algebra in $\mathbb{C}^S$, and its Euclidean geometry. This allows to explain the link to theoretical physics in quite a transparent way. This part of the theory may easily be extended to the case where S is countable, where now $\ell^2(S)$ is the relevant geometry. To move on to the case of uncountable S requires extra caution, and the dictionary between kernels and Euclidean geometry now needs to be enhanced to the setup of bounded integral operators on the Hilbert space $L^2(S)$.

Examples we will present, at least superficially, include: uniform spanning forests of infinite lattices; zeros of the Gaussian analytic function on the unit disc; eigenvalues of a random Hermitian matrix distributed according to the Gaussian unitary ensemble.

Here is a short list of references, which will be updated in due course:

- A. Borodin. Determinantal point processes. *Oxford handbook of random matrix theory*, pp 231–249, Oxford Univ. Press, 2011.
- R. Lyons. Determinantal probability measures. *Publ. Math. Inst. Hautes Études Sci.*, (98):167–212, 2003.
- A. Soshnikov. Determinantal random point fields. *Uspekhi Mat. Nauk*, 55(5(335)):107–160, 2000.

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Date: January 20, 2020.