A positivity bootstrap technique for validating the generating function of loop-decorated maps

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Workshop on random matrices, maps, and gauge theories
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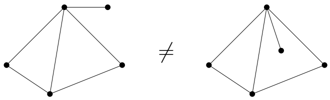
What is . . . a map?

A *planar map* is a *proper* embedding (i.e. no crossing edges) of a finite connected graph into the sphere $\mathbb{S}^2$, viewed up to the orientation-preserving homeomorphisms of $\mathbb{S}^2$.

⚠ planar map
$\neq$ planar graph



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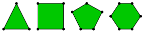
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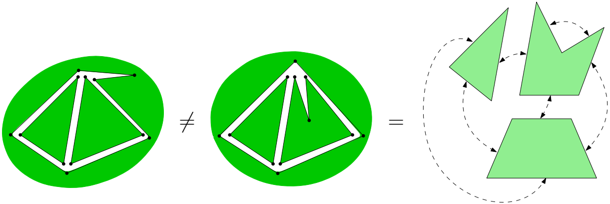

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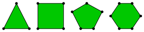
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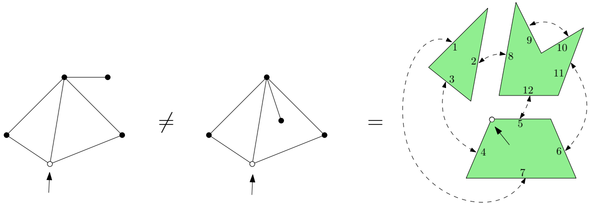

To avoid symmetry issues, we consider planar maps with a special *corner* (the *root*). Their edges/vertices/faces can be enumerated deterministically.
external face = face containing the root, *perimeter* = degree of external face

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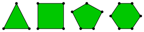

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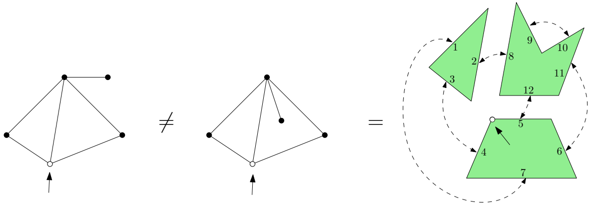

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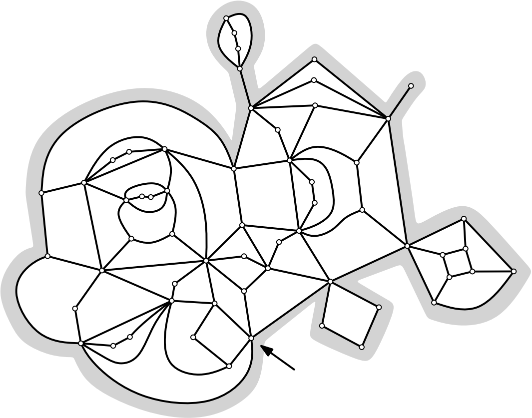
In the following, we will consider *quadrangulations with boundary*, that is, (rooted planar) maps in which all internal faces have degree *four*.

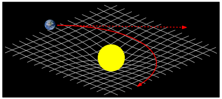
What is . . . a map?

$\mathcal{Q}_p :=$ {quadrangulations with boundary of perimeter $2p$}

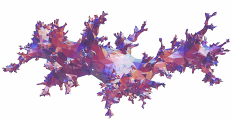


$\in \mathcal{Q}_{12}$

random maps
$\rightarrow$ (discretized) random metric of 2D space-time (Liouville quantum gravity)



$\rightsquigarrow$



Motivations

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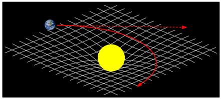
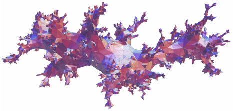
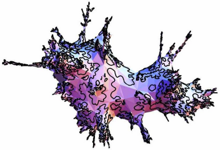
$\to$ (discretized) random metric of 2D space-time (Liouville quantum gravity)
random maps + a statistical physics model
$\to$ random metric of space-time coupled to a matter field

 $\rightsquigarrow$  $\rightsquigarrow$ 

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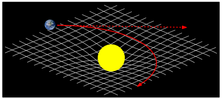
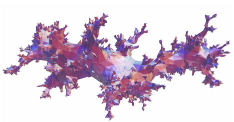
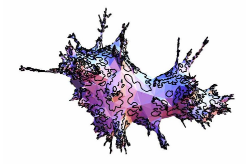
 $\rightsquigarrow$  $\rightsquigarrow$ 

Also:

- integrable models
- universality
- $\cdots$

and, nice pictures !

Definition of model

$O(n)$ model on quadrangulations

Definition (rigid loop-decorated quadrangulation)

Let $\mathfrak{q}$ be a quadrangulation with boundary. A *loop configuration* on $\mathfrak{q}$ is a set of *disjoint simple closed paths* on the dual of $\mathfrak{q}$ avoiding the external face. It is *rigid* if the loops always enter and exit from the *opposite sides* of a face.

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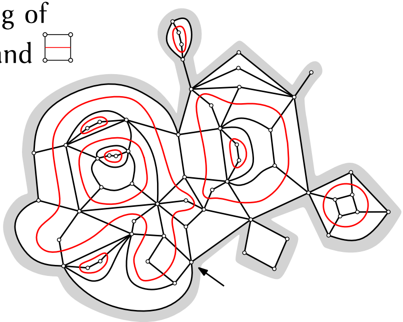

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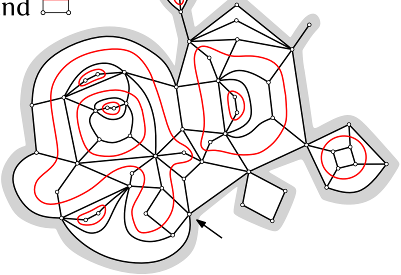

$$\rightsquigarrow \quad \mathbb{P}^{(12)}_{g,h,n}(\,\cdot\,) = \frac{g^8\, h^{38}\, n^9}{F_{12}(g,h,n)}$$

$$\mathbb{D} := [0,\infty) \times (0,\infty) \times (0,2)$$

For $(g,h,n) \in \mathbb{D}$, let

$$F_p(g,h,n) := \sum_{(\mathfrak{q},\boldsymbol{\ell}) \in \mathcal{LQ}_p} g^{\#\square} h^{\#\boxminus} n^{\#\bigcirc}$$

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A triple (g,h,n) is *admissible* if $F_p(g,h,n) < \infty$. (This is independent of p).

Definition

Fix $p \geq 1$. For each admissible triple $(g,h,n) \in \mathbb{D}$, we define a probability distribution on $\mathcal{LQ}_p$ by

$$\mathbb{P}^{(p)}_{g,h,n}(\mathfrak{q},\boldsymbol{\ell}) := \frac{g^{\#\square} h^{\#\boxminus} n^{\#\circlearrowleft}}{F_p(g,h,n)}$$

Results

Theorem (Borot–Bouttier–Guitter '12)

Assume that $(g, h, n) \in \mathbb{D}$ is admissible, then as $p \to \infty$,

$$F_p(g, h, n) \sim C \cdot \gamma^{2p} p^{-a}$$

where $C, \gamma > 0$ and $a \in \{\frac{3}{2}, \frac{5}{2}, 2-b, 2+b\}$ with $b = \frac{1}{\pi} \arccos(\frac{n}{2}) \in (0, \frac{1}{2})$.

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$$\mathcal{D} := \{(g, h, n, \gamma) \,:\, (g, h, n) \in \mathbb{D}, 0 < \gamma \le h^{-1/2}\}$$
$$\mathring{\mathcal{D}} := \{(g, h, n, \gamma) \in \mathcal{D} \,:\, \gamma < h^{-1/2}\}$$

Theorem (Borot–Bouttier–Guitter '12, C. '18)

There exist explicit functions $\mathfrak{h}$, $\mathfrak{f}$ and $\mathfrak{g}$ defined respectively on $\mathcal{D}$, $\mathring{\mathcal{D}}$ and $\mathbb{D}$ such that a triple $(g, h, n) \in \mathbb{D}$ is admissible and ...

$a = 3/2$ if and only if $\mathfrak{h}(g, h, n, \gamma) = 1$ and $\mathfrak{f}(g, h, n, \gamma) > 0$ for some γ.

$a = 5/2$ if and only if $\mathfrak{h}(g, h, n, \gamma) = 1$ and $\mathfrak{f}(g, h, n, \gamma) = 0$ for some γ.

$a = 2 - b$ if and only if $\mathfrak{h}(g, h, n, h^{-\frac{1}{2}}) = 1$ and $\mathfrak{g}(g, h, n) > 0$.

$a = 2 + b$ if and only if $\mathfrak{h}(g, h, n, h^{-\frac{1}{2}}) = 1$ and $\mathfrak{g}(g, h, n) = 0$.

Moreover, the four cases are all non-empty.

Phase diagram

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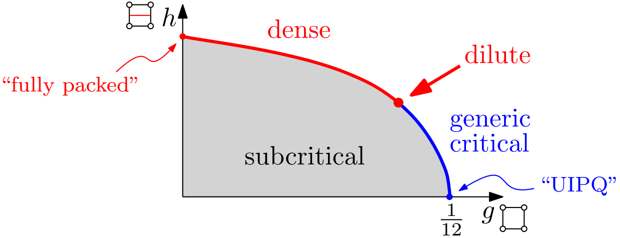

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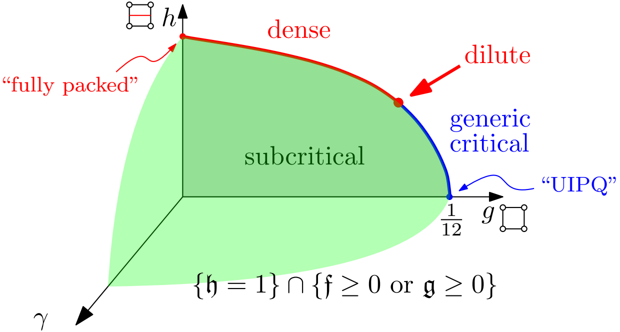
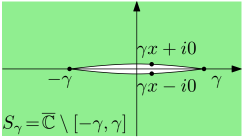


$\{\mathfrak{h} = 1\} \cap \{\mathfrak{f} \geq 0 \text{ or } \mathfrak{g} \geq 0\}$

$$\mathcal{W}_{g,h,n}(x) := 1 + \sum_{k=1}^{\infty} F_k(g,h,n)x^{-2k} \qquad (x \in \overline{S}_\gamma)$$

$$\rho_{\ldots}(x) := \frac{\mathcal{W}_{\ldots}(\gamma x - i0) - \mathcal{W}_{\ldots}(\gamma x + i0)}{2\pi i\, x} \qquad (x \in [-1,1])$$



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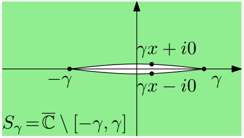

Proposition (Borot–Bouttier–Guitter '12) (Equation of resolvent)

- If $(g,h,n) \in \mathbb{D}$ is admissible, then *there exists* $\gamma \in (0, h^{-1/2}]$ such that the function $\mathcal{W}(x) \equiv \mathcal{W}_{g,h,n}(x)$ satisfies

$$\begin{cases} \mathcal{W} \text{ is an even } \textit{holomorphic function on } \overline{S}_\gamma \text{ such that for all } x \in (-\gamma,\gamma), \\ \qquad \mathcal{W}(x - i0) + \mathcal{W}(x + i0) + n\,\mathcal{W}((hx)^{-1}) \;=\; n + x^2 - gx^4 \,. \end{cases} \quad (*)$$

Moreover, $\rho_{g,h,n}$ is a *non-negative* continuous even function on $[-1,1]$.

- *For any* $(g,h,n) \in \mathbb{D}$ and $\gamma \in (0, h^{-1/2}]$, $(*)$ has a unique solution $\mathcal{W}^{(\gamma)}_{g,h,n}$.

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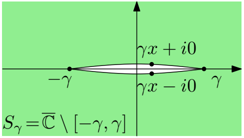

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Additional observations:
- $\mathcal{W}_{g,h,n}(\infty) = 1$.
- $\forall p,\ F_p(g,h,n) = \gamma^{2p} \int_{-1}^{1} x^{2p} \rho_{g,h,n}(x)\mathrm{d}x$, so $\rho_{g,h,n} \geq 0 \Rightarrow F_p(g,h,n) \geq 0$.

Questions:
- How to show that a tripe (g,h,n) is admissible ?
- How to characterize the γ in the combinatorial solution ?

$(g, h, n) \in \mathbb{D}$, $\gamma \in (0, h^{-1/2}]$.

Proposition (analytic condition of admissibility.)

If $\mathcal{W}^{(\gamma)}_{g,h,n}(\infty) = 1$ and $\rho^{(\gamma)}_{g,h,n} \geq 0$, then (g,h,n) is admissible and $\mathcal{W}^{(\gamma)}_{g,h,n} = \mathcal{W}_{g,h,n}$

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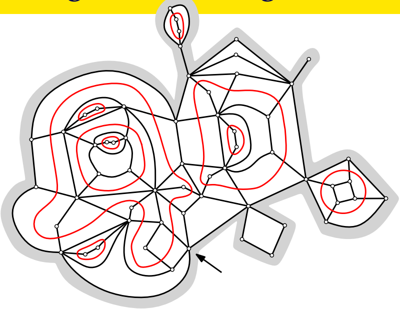
Proposition (positivity bootstrap)

$\rho^{(\gamma)}_{g,h,n}(x) \geq 0$ for all $x \in [-1, 1]$ if and only if $\rho^{(\gamma)}_{g,h,n}(x) \geq 0$ for all x close to 1. More precisely,

- When $\gamma < h^{-1/2}$, $\rho^{(\gamma)}_{g,h,n}(x) = \mathfrak{f}(g, h, n, \gamma) \cdot (1 - x^2)^{1/2} + O((1 - x)^{3/2})$, and $\rho^{(\gamma)}_{g,h,n}(x) \geq 0$ for all $x \in [-1, 1]$ if and only if $\mathfrak{f}(g, h, n, \gamma) \geq 0$.
- When $\gamma = h^{-1/2}$, $\rho^{(\gamma)}_{g,h,n}(x) = \mathfrak{g}(g, h, n) \cdot (1 - x^2)^{1-b} + O((1 - x)^{1+b})$, and $\rho^{(\gamma)}_{g,h,n}(x) \geq 0$ for all $x \in [-1, 1]$ if and only if $\mathfrak{g}(g, h, n) \geq 0$.

Idea of Proof

Background: the gasket decomposition



$(\mathfrak{q}, \boldsymbol{\ell}) \in \mathcal{LQ}_p$

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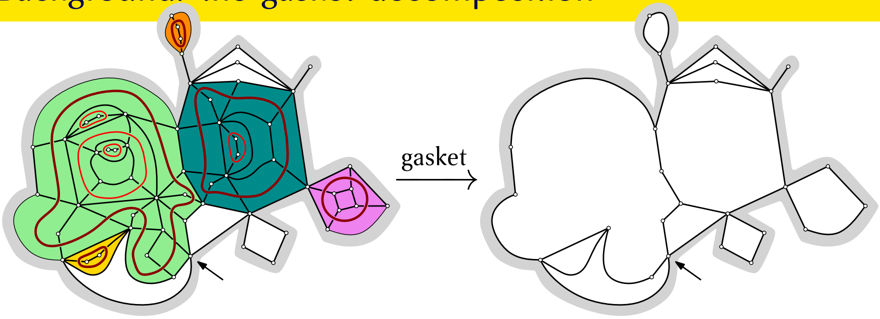

$(\mathfrak{q}, \boldsymbol{\ell}) \in \mathcal{LQ}_p$ $\qquad$ $\mathcal{G}(\mathfrak{q}, \boldsymbol{\ell}) \in \mathcal{M}_p$

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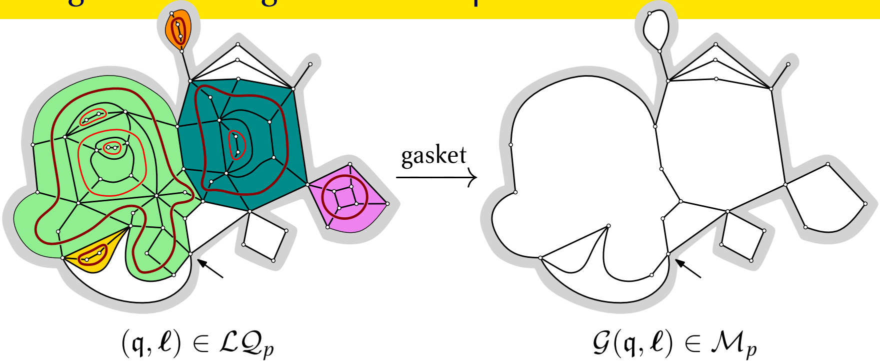

The gasket decomposition consists of:

- A mapping $\mathcal{G} : \mathcal{LQ}_p \to \mathcal{M}_p := \{\text{bipartite maps of perimeter } 2p\}$.
- For each $\mathfrak{m} \in \mathcal{M}_p$, a bijection

$$\mathcal{G}^{-1}(\mathfrak{m}) \leftrightarrow \mathcal{LQ}_{p_1} \times \mathcal{LQ}_{p_2} \times \cdots \times (\mathcal{LQ}_2 \cup \{\square\}) \times \cdots$$

where $2p_1, 2p_2, \ldots$ are the degrees of the faces of $\mathfrak{m}$.

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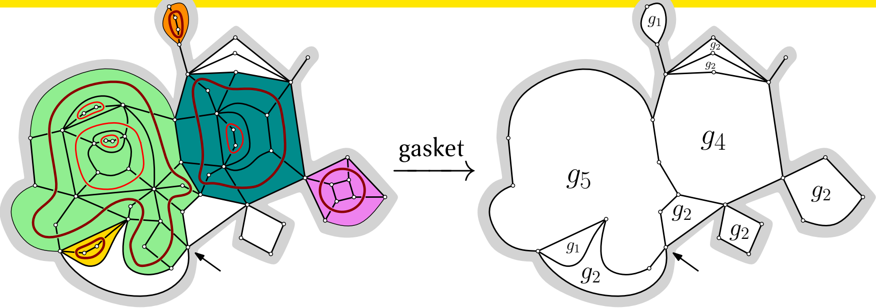

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Consequences

- Fixed point equation

$$\begin{cases} F_p(g, h, n) = B_p(g_1, g_2, \ldots) \\ g_k = g\delta_{k,2} + n\, h^{2k}\, F_k(g, h, n) \end{cases}$$

where $B_p(g_1, g_2, \ldots) = \sum_{\mathfrak{m} \in \mathcal{M}_p} \left(\prod_f g_{\frac{1}{2}\deg f} \right)$

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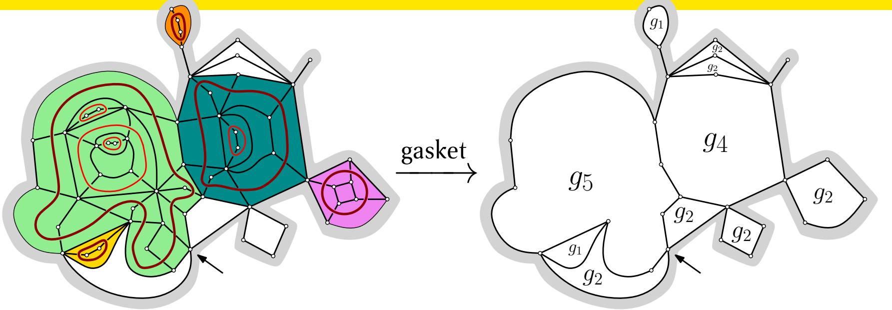

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- Recursive sampling algorithm of an $O(n)$-quadrangulation

⇝ Sample a bipartite map $\mathfrak{m}$ with Boltzmann weights $(g_1, g_2, \dots)$

⇝ Fill each face with a “necklace” + an $O(n)$-quadrangulation

⇝ Repeat

Background: enumeration of bipartite maps

Lemma (well-known results in disguise)

A sequence of weights $(g_1, g_2, \ldots)$ is admissible (i.e. $B_k(g_1, g_2, \ldots) < \infty, \forall k$) if and only if there exists $\gamma > 0$ such that the (unique) solution of the system

$$\begin{cases} \mathcal{W} \text{ is an even } \textit{holomorphic function on } \overline{\mathbb{S}}_\gamma \text{ such that for all } x \in (-\gamma, \gamma), \\ \mathcal{W}(x - i0) + \mathcal{W}(x + i0) \; = \; x^2 - \sum_{k=1}^{\infty} g_k x^{2k} \,. \end{cases}$$

satisfies $\mathcal{W}(\infty) = 1$ and $\lim_{x \to 1^-} \frac{\rho(x)}{\sqrt{1-x^2}} \geq 0$.

In this case, $\mathcal{W}(x) = 1 + \sum_{k=1}^{\infty} B_k(g_1, g_2, \ldots) x^{-2k}$, and ρ is *non-negative* and continuous on $[-1, 1]$.

Background: enumeration of bipartite maps

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Assume that (g, h, n) is admissible, then:

The fixed point equation $g_k = g\delta_{k,2} + n\, h^{2k}\, B_p(g_1, g_2, \ldots)$

$\Rightarrow \sum_{k=1}^{\infty} g_k x^{2k} = g x^4 + n(\mathcal{W}((hx)^{-1}) - 1)$ and $\gamma \leq h^{-1/2}$.

$\Rightarrow$ The equation of the resolvent for $\mathcal{W}_{g,h,n}$.

Admissibility condition

Inversely, fix some (g, h, n) and assume that the equation of resolvent has a solution such that $\mathcal{W}^{(\gamma)}_{g,h,n}(\infty) = 1$ and $\rho^{(\gamma)}_{g,h,n}(x) \geq 0$ for all $x \in [-1, 1]$.
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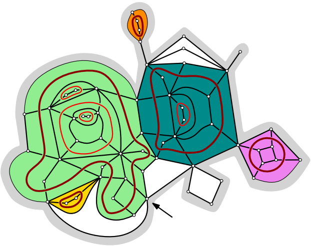
$\Rightarrow$ $(g_1, g_2, \ldots)$ is admissible.
$\rightsquigarrow$ *Is (g, h, n) admissible* ? Yes: use the recursive algorithm from the gasket decomposition to sample an $O(n)$-quadrangulation of parameters (g, h, n).



Proposition (Budd '18+)

For any admissible weight sequence $(g_k)_{k\geq 1}$ such that $g_k = g\delta_{k,2} + nh^{2k}B_p(g_1, g_2, \ldots)$ for some $g, h, n \geq 0$, the sampling algorithm almost surely stops.

(The *number of vertices* discovered by the sampling algorithm is bounded from above by some explicit super-martingale.)

Positivity bootstrap

Lemma (Integral equation for the spectral density)

Let $\tau = \gamma^2 h \in [0,1]$, then for all $x \in [-1,1]$,

$$-\frac{2\pi\,\rho^{(\gamma)}_{g,h,n}(x)}{\sqrt{1-x^2}} \;=\; -\gamma^2 + g\gamma^4(x^2+1/2) + n\int_{-1}^{1} \frac{\tau^2 y^2}{1-\tau^2 x^2 y^2}\,\frac{\rho^{(\gamma)}_{g,h,n}(y)\,\mathrm{d}y}{\sqrt{1-\tau^2 y^2}}\;,$$

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Consequences:

- $x \mapsto \rho^{(\gamma)}_{g,h,n}/\sqrt{1-x^2}$ extends to an analytic function on $[-\tau^{-2},\tau^{-2}]$. In particular, if $\tau < 1$, then $\mathfrak{f}(g,h,n,\gamma) := \lim_{x\to 1^-} \rho^{(\gamma)}_{0,h,n}(x)/\sqrt{1-x^2}$ exists.
- $(x; g,h,n,\gamma) \mapsto \rho^{(\gamma)}_{g,h,n}/\sqrt{1-x^2}$ is continuous on this extended domain.
- $\rho^{(\gamma)}_{g,h,n} \geq 0$ on $[-1,1]$, then $\rho^{(\gamma)}_{g,h,n} > 0$ on $(-1,1)$.

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Claim: For all $h > 0$, $n \in (0,2)$, $\gamma \in (0, h^{-1/2})$, we have $\mathfrak{f}(0,h,n,\gamma) > 0$.

$\Rightarrow$ The set $\left\{(g,h,n,\gamma) \,:\, \mathfrak{f}(g,h,n,\gamma) \geq 0\right\}$ is connected.

Thank you for your attention !