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Real and Complex Dynamical Systems

dedicated to Yulij Ilyashenko's 80th Birthday

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Book of Abstracts

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Modern theory of dynamical systems is the study of iterations of mappings and of flows of vector fields (discrete- and continuous-time dynamical systems). The fundamental questions concern long-time behavior of orbits, both of individual dynamical systems and of family of systems depending on parameters. There are two subfields of dynamical systems that are closely related to each other: real dynamics (hyperbolic theory, attractors, ergodic theory, real foliations) and complex dynamics (analytic theory of differential equations in the complex domain, holomorphic dynamics, holomorphic foliations). In the study of a real-analytic dynamical system, it is often helpful to consider its complexification and to use the methods of complex analysis, algebraic geometry, and the theory of complex dynamical systems. Reciprocally, the use of the classical methods of real dynamics (hyperbolic and ergodic theories, Lyapunov exponents) led to a considerable progress in complex dynamics and the theory of holomorphic foliations. The main theme of the conference is the interface of real and complex dynamics, real and complex geometry and analysis. The talks of the conference will present recent results on this interface that belong to the area of dynamical systems and to related areas in real and complex geometry and analysis; including holomorphic foliations, finite-dimensional integrable systems, billiards and mathematical physics.

The goal of the conference is to bring together leading experts in the field, along with researchers in the early stage of career and graduate students.

Contents

Abstracts of talks and posters

Dynamical systems in asymptotic behavior of solutions to higher-order nonlinear differential equations and its asymptotic equivalence

I. V. Astashova (Lomonosov Moscow State University and Plekhanov Russian University of Economics, Russia)

We discuss the role of dynamical systems in the study of asymptotic behavior of solutions to equations

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) + p(x)\,|y(x)|^k \operatorname{sgn} y(x) = f(x) \qquad (1)$$

with $n \geq 2,\ k > 1$, and continuous functions p, f and a_j. We consider this equation as a perturbation of more simple equation with $f = 0$. This equation, in its turn, we consider as a perturbation of the equation with $p = 0$ or/and $a_j = 0$.

Some previous results are formulated in [1]–[3]. In particular, the asymptotic behavior of solutions vanishing at infinity is described.

The work is partially supported by RSF (Project 20-11-20272).

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Optimal Flows in Dynamic Transport Networks and Replicator Dynamical Systems

Armen Bagdasaryan (American University of the Middle East, Kuwait)

In 1952 John Wardrop formulated two principles of optimality of flow distribution in networks that describe user equilibrium and the system optimum. In this talk, we introduce the concept of a *Wardrop optimal network* that admits *Wardrop optimal flows* that are both the user equilibrium and the system optimum. The Wardrop optimal networks are the only networks for which the *price of anarchy* is exactly equal to its least value 1. The novelty of this work is its pioneering solution to the problem, posed by S. Dafermos in 1968, by means of a complete and comprehensive characterization of Wardrop optimal networks.

The geometric structure of the set of all Wardrop optimal networks, for which *a priori* given flow is a Wardrop optimal flow, is also studied. We propose a stochastic matrix approach to the construction of Wardrop optimal networks, and present affine transformations that transform any convex differentiable network into Wardrop optimal network. It is shown how any *a priori* given flow can be transformed into the Wardrop optimal flow. The optimality conditions of flows on parallel-series networks will also be discussed.

We propose a novel dynamical model for optimal flows in Wardrop optimal networks by means of a replicator dynamical system generated by convex differentiable functions and by Schur potential functions. The equilibrium and stability conditions of the replicator equation dynamics are described. For the proposed replicator dynamical system, the Nash equilibrium, the Wardrop equilibrium, and the system optimum represent the same flow in the network.

Disconnected large bifurcation supports and Cartesian products of bifurcations

Timur Bakiev (HSE University, Russia)

A bifurcation that occurs in a multiparameter family is a Cartesian product if it splits into two factors: one bifurcation takes place in one part of the phase portrait, another one – in another part, and they are in a sense independent, do not interact with each other. To understand how a family bifurcates, it is sufficient to study it in a neighbourhood of the so called large bifurcation support (LBS). We prove that, if the LBS is disconnected and the restriction of the original family to some neighbourhood of each connected component is structurally stable (plus some mild extra conditions), then the original family is a Cartesian product of the bifurcations that occur near the components of the LBS.

2D invariant manifolds in 3D flows: perturbed locations under general perturbations and instantaneous flux

Sanjeeva Balasuriya (University of Adelaide, Australia)

This talk introduces a geometric Melnikov method to analyze two-dimensional stable or unstable manifolds associated with a saddle point in three-dimensional non-volume preserving autonomous systems. The time-varying perturbed locations of such a manifold is obtained under very general, non-volume preserving and with arbitrary time-dependence, perturbations. The explicit computability of the leading-order spatio-temporal location of the manifold is demonstrated. In unperturbed situations with a two-dimensional heteroclinic manifold, the theory is adapted to quantify the splitting into a stable and unstable manifold, and thereby obtain an instantaneous flux quantification in terms of a Melnikov function. The time-varying instantaneous flux theory does not require any intersections between perturbed manifolds, nor rely on descriptions of lobe dynamics. The theory has specific application to transport in fluid mechanics, where the flow is in three dimensions and flow separators in forward/backward time are two-dimensional stable/unstable manifolds. Separators and transport are computed for both the classical and the swirling versions of Hill's spherical vortex. This work is in collaboration with Sula Priyankara and Erik Bollt.

Homeomorphisms on three-dimensional manifolds with pseudo-Anosov attractors and repellers

E.E. Chilina, V.Z. Grines, O.V. Pochinka

The work is devoted to the study of homeomorphisms on three-dimensional manifolds with a two-dimensional nonwandering set such that the restriction of a certain degree of the map to each of connected components is topologically conjugate to a pseudo-Anosov homeomorphism. It is proved that such maps are ambiently omega-conjugate to model homeomorphisms constructed in the work.

The work was supported by the Russian Science Foundation under grant 22-11-00027 and by the Laboratory of Dynamical Systems and Applications NRU HSE, grant of the Ministry of Science and Higher Education of the Russian Federation (agreement No. 075-15–2022-1101).

Multiple limit cycles that appear from hyperbolic polycycles

Andrei Dukov (Steklov Mathematical Institute, Russia)

Consider a smooth vector field on a 2-manifold. Let the vector field have a hyperbolic polycycle (a directed graph such that its vertexes are hyperbolic saddles and its edges are separatrix connections). Then after a small perturbation the multiplicity of any appearing limit cycle is not greater than n, where n is the number of edges. The behaviour of multiple limit cycles connects with a polynomial system. In particular, if the system has (does not have) a solution, then a limit cycle of required multiplicity appears (does not appear).

Singularities in generic two-parameter families of vector fields on 2-sphere

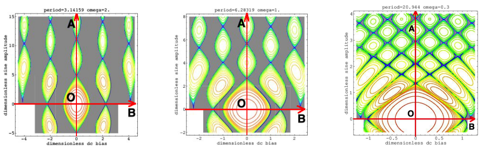
Dmitry Filimonov (HSE University, Russia)

In 1974 Jorge Sotomayor published his famous work describing all six types of degenerate planar vector fields that can occur in generic one-parameter families. Since that time the bifurcation theory advanced a lot but still there is no complete description of generic two-parameter families. A closely related question about generic families is whether all these families are structurally stable. Recent results give a positive answer for one-parameter families and a negative one for three-parameter families. The answer for two-parameter families is yet unknown. In the present work we will focus on the first step of a classification of degenerate vector fields that can occur in generic two-parameter families. Our main theorem describes all singular points and their combinations in generic two-parameter families of vector fields.

Model of Josephson junction, dynamical systems on $\mathbb{T}$, isomonodromic deformations and Painlevé 3 equations

Alexey Glutsyuk (HSE University, CNRS (UMPA, ENS de Lyon), and IITP (Moscow))

The tunneling effect predicted by B.Josephson (Nobel Prize, 1973) concerns the *Josephson junction:* two superconductors separated by a narrow dielectric. It states existence of a supercurrent through it and equations governing it. The *overdamped* Josephson junction is modeled by a family of differential equations on 2-torus depending on 3 parameters: B (abscissa), A (ordinate), ω (frequency). We study its *rotation number* $\rho(B, A; \omega)$ as a function of (B, A) with fixed ω. The *phase-lock areas* are those level sets $L_r := \{\rho(B, A) = r\} \subset \mathbb{R}^2$ that have non-empty interiors. They exist only for integer rotation number values r: this is the *rotation number quantization effect* discovered by V.M.Buchstaber, O.V.Karpov and S.I.Tertychnyi. They are analogues of the famous Arnold tongues. Each L_r is an infinite chain of domains going vertically to infinity and separated by points called *constrictions* (expect for those with $A = 0$). See the phase-lock area portraits below for $\omega = 2, 1, 0.3$.



As one can see at the above pictures and at similar pictures in physics books on Josephson effect[1], *in each phase-lock area L_r all its constrictions lie in the vertical line $\Lambda_r := \{B = \omega r\}$.* This

[1] See, e.g., Likharev, K.K. *Dynamics of Josephson junctions and circuits.* Gordon and Breach Science Publishers, 1986; p. 339, fig. 11.4.

was proved in a joint work of Yu.P.Bibilo and the author by using an equivalent description of the model by a family of linear systems of differential equations on the Riemann sphere (found by V.M.Buchstaber, O.V.Karpov and S.I.Tertychnyi), Stokes phenomena theory and isomonodromic deformations governed by Painlevé 3 equation.

In this poster we present a new, four-parameter family of dynamical systems on torus that includes the above-mentioned model of Josephson junction, for which the rotation number quantization also takes place and there is an equivalent description by a family linear systems. The parameter space of the new family is foliated by curves corresponding to isomonodromic families of linear systems, along which the corresponding flows on torus are analytically conjugated and thus, have constant rotation number. We will present results obtained by using this bigger family and a survey of open problems.

Renormalization operators and Arnold tongues

Nataliya Goncharuk (Texas A&M University, USA)

Many studies in circle dynamics are devoted to Arnold tongues: level sets of the rotation number in parametric families of circle maps.

E. Risler proved in 1999 that in analytic families of circle diffeomorphisms, Arnold tongues that correspond to Herman rotation numbers are analytic curves. In contrast to this result, Llave and Luque observed in 2011 using numerical investigations that these Arnold tongues are only finitely smooth at critical circle maps.

With M.Yampolsky, we provided explanations of these effects in terms of renormalization operators. I am going to outline main ideas of our proofs.

Speed of convergence of linear functionals in DPP with Bessel kernel

Sergei Gorbunov (MIPT, Russia)

Let $b_\tau(x) = b(x/\tau)$ be a function from Schwartz space. For some $f \in \mathbb{L}_1(\mathbb{R}_+)$ denote by $B(f)$ an operator with following kernel

$$K_B(x, y) = \int_0^\infty \sqrt{xyt} J_\nu(xt) J_\nu(yt) f(t) dt,$$

acting on $\mathbb{L}_2[0, 1]$, where $J_\nu(x)$ is a Bessel function of order $\nu > -1$. We derive constants c_1, c_2, depending on b, and an operator $A(b_\tau)$, which give following exact formula

$$\det{}_{\mathbb{L}_2[0,1]}(I + B(e^{b_\tau} - 1)) = e^{c_1\tau + c_2} \det{}_{\mathbb{L}_2[0,1]}(A(b_\tau)),$$

where the determinant on the right approaches 1 as τ approaches infinity. In addition, we estimate the convergence with seminorms, depending only on a derivative of e^b.

Fredholm determinant expressed above is connected with determinantal point process (DPP) with Bessel kernel $K(x, y) = \int_0^1 \sqrt{xyt} J_\nu(xt) J_\nu(yt) dt$. In particular, for a configuration X on $\mathbb{R}_+$ sum $S[b] = \sum_{x \in X} b(x)$ is called a linear functional. Then for the Bessel kernel DPP $\mathbb{E}e^{S[b]} = \det{}_{\mathbb{L}_2[0,1]}(I + B(e^b - 1))$. Formula above provides characteristic function of distribution of linear functional, and thus proves convergence to Gaussian distribution for $\tau \to \infty$, since c_1, c_2 depend linearly and quadratically on b respectively. And the estimate gives speed of the linear statistics convergence.

Dynamical Methods in Spectral Theory of Ergodic Schrodinger Operators

Anton Gorodetski (University of California Irvine, USA)

We will present a survey of classical and recent results on topological structure of the spectrum of ergodic Schrodinger operators, and show how to reformulate these results in the language of dynamical systems. Among others we will mention Almost Mathieu Operator, Anderson Model, and Fibonacci Hamiltonian. After that we will discuss a recent result on the topology of the spectrum of ergodic Schrodinger operators with iid random noise (joint with A.Avila and D.Damanik).

Framed link as topological invariant of polar flows on four-dimensional manifolds

Elena Gurevich (HSE University, Russia)

In [1] it is shown that knots in $S^2 \times S^1$ appear in a natural way as complete invariants of topological conjugacy for the simplest gradient-like diffeomorphisms on 3-manifolds. In particular, Masur knot in $S^2 \times S^1$ (a knot which is homotopic, but not isotopic to $\{x\} \times S^1$) represents a wild one-dimensional separatrix of a saddle fixed point of the diffeomorphism. This fact proves that there are at least two topologically non-conjugated diffeomorphisms whose non-wandering set consists of exactly one saddle and three nodes: one have wild and another have tame closures of separaticies. For similar flows such effect does not occur, and closures of all two- and one-dimensional separatricies are tame.

However, the closures of two dimensional separatricies of gradient-like flows given on four-dimensional manifold may be wild, and knot theory again plays significant role in studying embedding of separatricies. In report, we consider a class P of gradient-like flows on closed manifolds of dimension four such that a non-wandering set of any flow $f^t \in P$ consists exactly of one source, one sink, and an arbitrary number of saddle equilibria that have two-dimensional stable and unstable manifolds. We associate with each flow $f^t \in P$ a framed link L_{f^t}, formed by traces of separatricies on a secant 3-sphere near sink equilibrium, and prove that equivalence of the framed links is necessary and sufficient conditions of topological equivalence of flows in P.

Research is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, grant of the Ministry of science and higher education of the RF, ag. No. 075-15-2022-1101.

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New trends in the glocal bifurcation theory in the plane

Yulij Ilyashenko (Cornell University, USA, Independent University of Moscow and HSE University, Russia)

Eight years ago structurally unstable generic families of vector fields in the sphere were found. Since then many new examples of the families with numeric and functional invariants were found. The question “Who bifurcates?” was answered. This opened the way to the study of the structural stability and classification problems in the theory of bifurcations with a small (less than three) number of parameters. A survey of results due to Dukov, Goncharuk, Kudryashov, the speaker and others will be given.

Heun-function solutions of the Schrodinger equation

Artur Ishkhanyan (Institute for Physical Research, Armenia)

We review the cases for which the 1D stationary Schrödinger equation is solved in terms of the general and (multi-)confluent Heun functions. We present the possible choices for coordinate transformation that provide energy-independent potentials that are proportional to an energy-independent continuous parameter and have a shape independent of that parameter.

We show that there exist in total 29 independent Heun potentials. There are eleven independent potentials that admit the solution in terms of the general Heun functions, for nine independent seven-parametric potentials the solution is given in terms of the single-confluent Heun functions, there are three independent double-confluent and five independent bi-confluent Heun potentials (the six-parametric Lemieux-Bose potentials), and one tri-confluent Heun potential (the general five-parametric quartic oscillator). There are several independent potentials that present distinct generalizations of either a hypergeometric or a confluent hypergeometric classical potential, some potentials possess sub-cases of both hypergeometric types, and others possess particular conditionally integrable ordinary or confluent hypergeometric sub-potentials. We present several examples of explicit solutions for the latter potentials.

We show that there exist other exactly or conditionally integrable sub-potentials the solution for which is written in terms of simpler special functions. However, these are solutions of different structure. For instance, there are sub-potentials for which each of the two fundamental solutions of the Schrodinger equation is written in terms of irreducible combinations of hypergeometric functions. Several such potentials are derived with the use of deformed Heun equations. A complementary approach is the termination of the hypergeometric series expansions of the solutions of the Heun equations.

Typical rotation numbers for families of circle maps with singularities

Konstantin Khanin (University of Toronto, Canada)

I shall discuss how one can define in a natural way the notion of typical rotation numbers for families of circle maps with singularities. This problem is related to a well known fact that in the case of maps with singularities the set of parameters corresponding to irrational rotation numbers has zero Lebesgue measure. I shall also discuss a natural setting for the Kesten theorem in the case of maps with singularities.

Fibered toric varieties

Askold Khovanskii (University of Toronto, Canada)

A toric variety is called *fibered* if it can be represented as a total space of fibre bundle over toric base and with toric fiber. Fibered toric varieties form a special case of toric variety bundles whose theory was recently developed. Fibered toric varieties are easy to deal with because algebraic geometry and topology of toric varieties have a natural interpretation in convex geometry. Using fibered toric varieties one can illustrate some known and conjectural results on topology and intersection theory of general toric variety bundles. The equivariant cohomology rings of smooth complete toric varieties can be easily computed using the language of fibered toric varieties. The talk is based on a joint work with Leonid Monin (see [1]).

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Hölder regularity of stationary measures

Victor Kleptsyn (CNRS and University of Rennes, France)

One of the main tools of the theory of dynamical systems are the invariant measures; for random dynamical systems, their role is taken by stationary measures, that is, measures that are equal to the average of their images.

In a recent work with A. Gorodetski and G. Monakov (`https://arxiv.org/abs/2209.12342`), we show that these measures almost always (under extremely mild assumptions) satisfy the Hölder regularity property: the measure of any ball is bounded by (a constant times) some positive power of its radius.

Determinantal processes and decomposition of functions into series defined by values in points of a random configuration

A.V. Klimenko (Steklov Mathematical Institute and HSE University, Russia)

Determinantal processes is a class of random point fields, that is, probability measures on a set of discrete subsets (or *configurations*) of some phase space E, which show a mix of random and deterministic behavior.

A determinantal process is defined by a contraction operator on the space $L^2(E)$. In most known examples this operator is an orthogonal projection onto some subspace $H \subset L^2(E)$, which consists of sufficiently regular functions, so that one can define the values of a function $f \in H$ in each point of the space E. This allows us to close the loop between the measure on the space of configurations and the subspace H: is $f \in H$ uniquely defined by its values on X, for almost all configurations X?

This is known to be true for a wide class of determinantal processes, and moreover, as A. Bufetov [1] has shown, there is a constant $k \geq 0$, which is called *an excess* of the process, such that for almost any configuration X and any choice of k points $x_1, \ldots, x_k \in X$ a function $f \in H$ is uniquely defined by its values on $X \backslash \{x_1, \ldots, x_k\}$, and if we remove any $(k+1)$ points from almost any configuration, there exists a function $f \in H$ that vanishes on $X \setminus \{x_1, \ldots, x_{k+1}\}$.

We are dealing with a more delicate question: is it possible to reconstruct a function f from its values on $Y = X \setminus \{x_1, \ldots, x_k\}$? This problem is linear, so one can start with functions g_s such that $g_s(y) = \delta_{s,y}$ for $s, y \in Y$. Then one can expect that

$$f(x) = \sum_{s \in Y} f(s) g_s(x) \quad \text{for all } x \in E.$$

Both parts agree for $x \in Y$, so the identity holds, provided that the series in converging. We have shown that for some determinantal processes this series does converge in $L^2(E)$ for a functions f from a finite-codimension subspace of H.

The talk is based on a joint work in progress with Alexander Borichev, Alexander Bufetov, and Zhaofeng Lin.

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Divergence and convergence of Birkhoff Normal Forms

Raphaël Krikorian (CMLS Ecole Polytechnique, France)

Each real analytic symplectic diffeomorphism admitting a non resonant fixed point can be formally conjugated to a formal integrable symplectic diffeomorphism, its Birkhoff Normal Form (BNF). I shall discuss two questions. The first one is due to H. Eliasson: are there examples of divergent BNF? The second is the following: is it possible to perturb in the real analytic topology a given real analytic symplectic diffeomorphism so that its BNF converges?

Computer-readable proofs and dynamical systems

Yury Kudryashov (Texas A&M University, USA)

I am going to discuss computer-readable formalizations of mathematical proofs with focus on dynamical systems. What are pros and cons of formalization? How can formalization help in the traditional research? Which notable theorems were formalized? How large are the computer-readable proofs?

I will also discuss my ongoing project aiming to formalize Ilyashenko's proof of the Individual Finiteness Theorem, for now the version about vector fields with hyperbolic singular points only.

Gaussian limit of the process of moduli for the Ginibre and hyperbolic ensembles

Zhaofeng Lin (Aix-Marseille University, France)

In this poster, we give a short introduction of Ginibre and hyperbolic ensembles, which concerned about the Fock kernel and Bergman kernel. And we give the Gaussian limit of the process of moduli for the Ginibre and hyperbolic ensembles, which is a part of a joint work with Prof. Alexander I. Bufetov and Prof. David García-Zelada.

Neighborhoods of curves in complex surfaces

Frank Loray (CNRS and University of Rennes, France)

We consider smooth complex compact curves embedded in a (non necessarily compact) complex surface, in particular when the curve is irreducible and smooth. Then we are interested in the structure of the germ of neighborhood, i.e. of the arbirtrary small neighborhood. Even in the simplest case where the curve is rational, there are non equivalent neighborhoods, i.e. non biholomorphic neighborhoods. We survey on old and recent results about the classification of such neighborhoods up to biholomorphisms. In a recent work with F. Touzet and S. Voronin, we classified neighborhoods of elliptic curves with torsion normal bundle.

Structure of Feigenbaum Henon maps

M. Lyubich (Stony Brook University, USA)

Feigenabaum Henon maps are infinitely renormalizable quadratic automorphisms of the real or complex 2D spaces. Over the past 20 years a rich theory of strongly dissipative maps of this class has been developed. It includes Universality and (non-)Rigidity phenomena, description of an intricate heteroclinic web, and construction of wild attractors and Julia sets of positive measure. Some of these features are similar to their 1D counterparts but some are strikingly different. In the talk we will give an overview of this theory. If time permits, we will mention the current work in progress on the axiomatic non-perturbative theory of “unimodal Henon maps" based upon quantitative Pesin theory.

Based upon joint work with Artur Avila, Marco Martens, and many other people.

Surfaces of section for geodesic flows of closed surfaces

Marco Mazzucchelli (CNRS and ENS de Lyon, France)

A surface of section for the flow of a nowhere vanishing vector field on a closed 3-manifold N is a compact surface in N, with interior transverse to the vector field, and boundary tangent to the vector field. A surface of section is global when it intersects any orbit segment of length T, for some $T > 0$. Surfaces of section are objects of great interest in dynamics, as they allow to reduce the study of a 3-dimensional flow to the study of a surface diffeomorphism. In this talk, I will present a few results on surfaces of section for geodesic flows of closed surfaces, culminating with the existence of global surfaces of section for all those geodesic flows satisfying the C^∞ generic Kupka–Smale condition (joint work with Gonzalo Contreras, Gerhard Knieper, and Benjamin Schulz). As an application, I will present a characterization of the Anosov condition, which implies the validity of the C^2-structural stability conjecture for geodesic flows of closed surfaces (joint work with Gonzalo Contreras).

On classes of stable isotopic connectivity of gradient-like diffeomorphisms of surfaces

E. V. Nozdrinova (HSE University, Russia)

Recall that a diffeomorphism of a closed surface $f : M^2 \to M^2$ is called *gradient-like* if its non-wandering set Ω_f consists of a finite number of hyperbolic periodic points and invariant manifolds of various saddle points do not intersect. One of the problems formulated in 1975 by J. Palis and C. Pugh [1], is a description of a criterion that allows us to determine whether two such isotopic diffeomorphisms are connected by a stable arc (preserving its qualitative properties with small movements) in the space of diffeomorphisms.

Some obstacles to the existence of stable arcs between gradient-like diffeomorphisms of a surface were first discovered by P. Blanchard [2] (also see the review of [3] on the obstacles to the existence of stable arcs between diffeomorphisms of manifolds known to date) and expressed in terms of periodic data of diffeomorphisms. For orientation-preserving gradient-like diffeomorphisms of the 2-sphere, a complete classification up to stable isotopicity was obtained by E. Nozdrinova and O. Pochinka in [4]. A similar classification is obtained for orientation-changing gradient-like diffeomorphisms of the 2-sphere. In more detail.

Consider the circle $\mathbb{S}^1$ as the equator of the 2-sphere $\mathbb{S}^2$. Then the structurally stable diffeomorphism of a circle with exactly two periodic orbits of the period $m \in \mathbb{N}$ and the rotation number $\frac{k}{m}$, can be continued to the orientation-changing diffeomorphism $\psi_{k,m} : \mathbb{S}^2 \to \mathbb{S}^2$, having a periodic source of the period 2 at the north and south poles. Denote by $C_{k,m}$ the component of stable isotopic connectivity of diffeomorphism $\psi_{k,m}$ and by $C^-_{k,m}$ the component of stable isotopic connectivity of diffeomorphism $\psi^{-1}_{k,m}$. Denote by C_0 the component of the stable isotopic connectivity of the source-sink diffeomorphism —- of the diffeomorphism $\psi_0 \in G$ with a non-wandering set consisting of exactly one source and one drain.

The main result of the work is the following theorem.

Theorem. *Any orientation-reversing gradient-like diffeomorphism of a two-dimensional sphere $\mathbb{S}^2$ belongs to one of the components C_0, $C_{k,m}$, $C^-_{k,m}$, k, $m \in \mathbb{N}$, $k < m/2$, $(k, m) = 1$.*

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Shadowing in hyperbolic and nonhyperbolic dynamical systems

Sergei Yu. Pilyugin (St. Petersburg State University, Russia)

The problem on shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is well studied for systems with hyperbolic structure [1].

In this talk, we discuss several results on shadowing for systems with nonhyperbolic behavior. We describe the following approaches to obtaining conditions of shadowing.

1. Method of pairs of Lyapunov type functions [2].

2. Method of control of one-step errors in a neighborhood of a nonisolated fixed point [3].

3. Method of multiscale conditional shadowing [4].

4. Method of generalized Perron operators for systems on simple time scales [5].

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On a structure of non-wandering set of an omega-stable 3-diffeomorphism possessing a hyperbolic attractor

Olga Pochinka (HSE University, Russia)

This topic belongs to a series of items devoted to the study of the structure of the non-wandering set of an A-diffeomorphism. We study such set $NW(f)$ for an omega-stable diffeomorphism f, given on a closed connected 3-manifold M. Namely, we prove that if all basic sets in $NW(f)$ are trivial except attractors, then every non-trivial attractor is either one-dimensional non-orientable or two-dimensional expanding.

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Analytic classification of semi-hyperbolic maps germs

P.A. Shaikhullina (Chelyabinsk State University, Russia)

A germ of a biholomorphic map $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is called *semi-hyperbolic* if one of its multipliers is equal to one and the second one is hyperbolic. For example, the 1-time shift $F_{\omega_{\lambda a\beta}} = g^1_{v_{\omega_{\lambda a\beta}}}$ along the vector field

$$\omega_{\lambda a\beta}(x) = v(x)\frac{\partial}{\partial x} + y\left(\lambda + x\beta(x)\right)\frac{\partial}{\partial y}$$

were

$$\lambda \in \mathbb{C},\ |\lambda| \neq 0, 1,\ n \in \mathbb{N}$$

$$v(x) = \frac{x^{n+1}}{1 + ax^n}, a \in \mathbb{C},\ \beta \in \mathbb{C}[x] \textit{ is polinomial of degree} < n,$$

is semi-hyperbolic.

Two semi-hyperbolic germs will be called *strictly* equivalent if the local change of coordinates conjugating them has the form $(x + o(x^{n+1}), y + o(x^n)),\ x \to 0$. Let $\mathbf{F}_{\lambda a\beta}$ be the class of semi-hyperbolic germs strictly formally equivalent to the germ $F_{\lambda a\beta}$. We will show, that strict analytic classification of $\mathbf{F}_{\lambda a\beta}$ has functional modules.

Consider the class $\mathbf{M}_{\lambda a\beta}$ consisting of collections $(A^s_\pm, B^s_\pm, A^s, B^s, C^s)$, $s \in \mathbb{Z}_n$, such that:

A^s_+, B^s_+ are holomorphic in $(\mathbb{C},\infty)\times(\mathbb{C},0)$ and $A^s_+(t,\tau) = O(1)$, $B^s_+(t,\tau) = O(t^{-1})$, $t\to\infty$;

A^s_-, B^s_- are holomorphic in $(\mathbb{C}^2,0) A^s_-(t,\tau) = O(t^2)$, $B^s_-(t,\tau) = O(t)$, $t\to 0$;

A^s, B^s are holomorphic in $(\mathbb{C},0)$ and $A^s(\tau) = O(\tau)$, $B^s(\tau) = O(\tau^2)$, $\tau\to 0$;

$C^s\in\mathbb{C}$.

Theorem. *For all* $\mathsf{F}\in\mathbf{F}_{\lambda a\beta}$ *there is exist the functional module* $m_F\in\mathbf{M}_{\lambda a\beta}$ *and:*
(1) $\forall\mathsf{F},\mathsf{G}\in\mathbf{F}_{a\lambda\beta}$*:* $\mathsf{F}\sim\mathsf{G}$ *(strictly)* $\iff\ m_F = m_G$*;*
(2) $\forall m\in\mathbf{M}_{\lambda a\beta}\ \exists\mathsf{F}\in\mathbf{F}_{\lambda a\beta}:\ m = m_F$*;*
(3) If the family F_t *of germs* $\mathbf{F}_{\lambda a\beta}$ *analytically depends on* t*, then the family* m_{F_t} *of invariants also analytically depends on it.*

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Attractors with non-invariant interior

S. Minkov (Brook Institute of Electronic Control Machines, Russia), A. Okunev (Pennsylvania State University, USA), I. Shilin (HSE University, Russia)

While it is conjectured that C^1-generic diffeomorphisms of a connected manifold whose non-wandering set has a non-empty interior are transitive [2], there are open examples of endomorphisms that have non-trivial attractors with non-empty interior (see, e.g., [4], [5]).

Moreover, for such endomorphisms the interior of the attractor (or of the non-wandering set) may also be non-invariant, and in a persistent way. That is, there are maps that robustly take a point in the interior of the attractor to the boundary of it. Although not mentioned by the authors explicitly, this is the case in [3] for the attractors in a C^3-open set of maps. We present a simpler construction that yields a C^1-open set of endomorphisms (of arbitrary manifold of dimension at least 2) whose attractors and non-wandering sets have non-invariant interiors.

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Dynamical systems without fixed points on Seifert fiber spaces

Danil Shubin (HSE University, Russia)

It is well known that topology and dynamical systems are interlinked. Various topological methods are used in the theory of dynamical systems. One of the basic questions for the theory is: Can a dynamical system with certain properties be defined on a certain topological space? Our talk is dedicated to the class of dynamical systems with continuous time, and whose non-wandering set consists of three hyperbolic periodic orbits (one is attracting, one is repelling, and one is saddle). We show, that if the saddle orbit is twisted, then the ambient manifold is either lens space or Seifert fiber space with three singular fibers.

Ergodic properties of certain classes of interval translation mappings

Alexandra Skripchenko (HSE University, Russia)

Interval Translation Mappings (ITM) were introduced by M. Boshernitzan and I. Kornfeld in 1994 as a natural generalisation of interval exchange transformation (IET). It is simply a piecewise continuous map on the interval into itself, such that on each interval of continuity it is a translation. Despite of obvious similarity with IET whose ergodic properties were widely studied in the last several decades, dynamics of ITMs is not very well understood. Some observations for the certain classes of ITM were obtained by S. Troubetzkoy and H. Bruin and by H. Bruin and G. Clack around 10 years ago but many important questions remained open even for those classes.

Recently we introduced a new approach that appears to be rather efficient at least for ITM with small number of intervals. The key idea is to use a new renormalisation that can be seen as a markovian multidimensional continued fraction. I will try to explain our results and formulate the most ambitious open problems.

The talk is based on a joint paper with M. Artigiani, P. Hubert and Ch. Fougeron and current work in progress with M. Artigiani and P. Hubert.

On neighborhoods of embedded complex tori

Laurent Stolovitch (Université Côte d'Azur, France)

In this joint work with X. Gong (Wisconsin-Madison U.), we show that an n-dimensional complex torus embedded in a complex manifold of dimensional $n+d$, with a split tangent bundle, has a neighborhood biholomorphic to a neighborhood of the zero section in its normal bundle, provided the latter has (locally constant) Hermitian transition functions and satisfies a *non-resonant Diophantine* condition. This generalizes works by Arnold and Il'yashenko-Pyartli.

Bicycling geodesics and elastic curves

Sergei Tabachnikov (Penn State University, USA)

One models a bicycle as a directed segment of a fixed length that can move so that the velocity of the rear end is always aligned with the segment. A bicycle path is a motion of the segment, and the length of the path, by definition, is the length of the front track. This defines a problem of sub-Riemannian geometry, and one wants to describe the respective geodesics. In this talk I shall present three variations on this theme: the planar bicycle motion, the bicycle motion in multidimensional Euclidean space, and the planar motion of a 2-linkage (a tricycle?) Somewhat unexpectedly, in each case, the classical elastica play a prominent role.

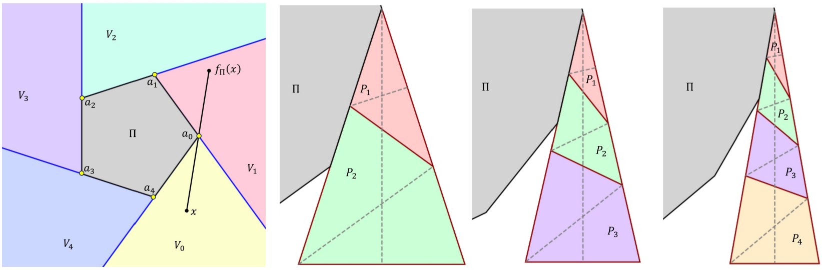
Aperiodic points for dual billiards

Vladlen Timorin (HSE University, Russia)

Joint project with A. Kanel-Belov, Ph. Rukhovich, and V. Zgurskii.

A *Euclidean dual billiard* outside a convex figure Π in the plane is the map sending a point $x \in \mathbb{R}^2 \setminus \Pi$ to a point y such that the line xy supports Π (say, on the right), and a point of contact with Π bisects the line segment $[x; y]$. Dual billiards were suggested by J. Moser as a crude model of planetary motion. In this context, the principal question was that of stability: are all orbits bounded? This is the case when the figure is strictly convex and has sufficiently smooth boundary (J. Moser, R. Douady). It is also the case (Vivaldi–Shaidenko, Kołodziej, Gutkin–Simanyi) for the so called *quasi-rational polygons*, which include all lattice polygons as well as all regular polygons. However, as was discovered by R. Schwartz, dual billiards outside convex quadrilaterals can have unbounded orbits.

Polygonal dual billiards are arguably the principal examples of *Euclidean piecewise rotations*, which serve as a natural generalization of interval exchange transformations. They also found applications in electrical engineering, in particular, in the study of digital filters.



Previously known rigorous results on dual billiards outside regular N-polygons are, apart from "trivial" cases of $N = 3, 4, 6$, based on dynamical *self-similarities*. A prototypical result in this direction is that of S. Tabachnikov, for $N = 5$. Cases $N = 8, 10, 12$ can be described similarly, whereas cases $N = 7, 9$ are more involved (have multi-fractal structure) but also include self-similarities. In his ICM 2022 address, R. Schwartz asked whether *outer billiard on the regular N-gon has an aperiodic orbit if $N \neq 3, 4, 6$*. We answer this question in affirmative for N not divisible by 4. Our methods are not based on self-similarity. Rather, *scissor congruence invariants* (including that of Sah–Arnoux–Fathi) play a key role.

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Quantum Gaudin Model and Isomonodromic Deformations

Ilya Tolstukhin (HSE University, Russia)

The Gaudin model is a quantum integrable system originally introduced to describe the interaction of multiple charged particles on a line. It consists of n commuting Hamiltonian operators, dependent on n pairwise distinct complex parameters and acting on the tensor product of n irreducible representations of the Lie algebra $\mathfrak{sl}_2$. One of the main tasks of the Gaudin model is to diagonalize these operators and understand how their joint spectrum changes as the parameters vary. In joint work with Natalia Amburg, branched coverings of the parameter space with joint spectra of Hamiltonians were studied in the case of $n = 3$. The base of such coverings is the Riemann sphere, and algebraic curves act as total spaces. The remarkable structure of these curves will be described, along with their connection to isomonodromic deformations.

Characteristic space of orbits of gradient-like diffeomorphisms of surfaces

E. V. Tsaplina (HSE University, Russia)

The classical approach to the study of dynamical systems consists in representing the dynamics of the system in the form of a "source-sink", that means a choice of a dual pair of attractor-repeller, which are attractive and repelling sets for all other trajectories of the system. If it is possible to choose a dual pair of attractor-repeller so that the space of orbits in their complement (the characteristic space of orbits) is connected, then this creates prerequisites for finding complete topological invariants of the dynamical system.

Let $f : M^n \to M^n$ is a Morse-Smale diffeomorphism given on a closed connected n-manifold. Denote by Ω_f^0, Ω_f^1, Ω_f^2 the set of sinks, saddles and sources of diffeomorphism f. For any (possibly empty) f-invariant set $\Sigma \subset \Omega_f^1$ denote by W_Σ^u the union of unstable manifolds of all points from Σ. For a set of Σ such that $cl(W_\Sigma^u) \setminus W_\Sigma^u \subset \Omega_f^0$, put

$$A_\Sigma = \Omega_f^0 \cup W_\Sigma^u,\ R_\Sigma = \Omega_f^2 \cup W^s_{\Omega_f^1 \setminus \Sigma}.$$

The sets A_Σ and R_Σ are *dual attractor and repeller*, respectively, and the set

$$V_\Sigma = M^n \setminus (A_\Sigma \cup R_\Sigma)$$

is called *characteristic space*, and the orbit space $\hat{V}_\Sigma = V_\Sigma/f$ of the action of f on V_Σ is called *characteristic orbit space.* It is known, that any Morse-Smale diffeomorphism given on a manifold of dimension $n > 3$ also has a connected characteristic space of orbits. For Morse-Smale diffeomorphisms on a surface, this is not true in the general case. So, on an orientable surface of any kind, an orientation-reversing gradient-like diffeomorphism that does not have a connected characteristic space is constructed and will be presented.

The orientation type of the saddle point σ of the period m_σ is the pair $\varsigma_\sigma = (\nu_\sigma, \lambda_\sigma)$, where $\nu_\sigma = +1\,(-1)$ if $f^{m_\sigma}|_{W^s_\sigma}$ preserving (reversing) orientation; $\lambda_\sigma = +1\,(-1)$ if $f^{m_\sigma}|_{W^u_\sigma}$ preserving (reversing) orientation. Denote by $\bar{\Omega}^1_f$ the set of saddle points with the orientation type $(-1, +1)$ such that there is no sink ω such that $cl(W^u_\sigma) \setminus W^u_\sigma \subset \mathcal{O}_\omega$. Denote by $\bar{\Omega}^0_f$ the set of ω sinks such that $f^{m_\omega}|_{W^s_\omega}$ reversing orientation and there is no saddle point $\sigma \in \bar{\Omega}^1_f$ such that $cl(W^u_\sigma) \setminus W^u_\sigma \subset \mathcal{O}_\omega$.

The subgraph $\bar{\Gamma}_f \subset \Gamma_f$ is called *a special subgraph* if all its vertices belong to the set $\bar{\Omega}^0_f \cup \bar{\Omega}^1_f$ and contain exactly one point from each orbit of the set $\bar{\Omega}^0_f$. If $\bar{\Omega}^0_f = \emptyset$, then put $\bar{\Gamma}_f = \emptyset$.

The main result of the work is the following theorem.

Theorem. *A gradient-like diffeomorphism $f : M^2 \to M^2$ has a connected characteristic space of orbits if and only if its graph Γ_f has a connected special subgraph $\bar{\Gamma}_f$. In this case, if $\bar{\Gamma}_f = \emptyset$ then the orbit space is homeomorphic to the torus, and is homeomorphic to the Klein bottle otherwise.*

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Normalization flow

Dmitry Treschev, (Steklov Mathematical Institute, Russia)

I propose a new approach to the theory of normal forms for Hamiltonian ODE systems near a non-degenerate equilibrium position. The traditional normalization procedure is performed step-by-step: non-resonant terms in the expansion of the Hamiltonian function are removed first in the lowest degree, then in the next one and so on. I consider the space of all Hamiltonian functions with equilibrium position at the origin and construct a differential equation in this space. Solutions of this equation move Hamiltonian functions towards their normal forms. Shifts along the flow of this equation correspond to canonical coordinate changes. So, we have a continuous normalization procedure. The formal aspect of the theory presents no difficulties. The analytic aspect and the problems of convergence of series, as usual, non-trivial.

Degenerated singular points of binary differential equations

S.M. Voronin, E.A. Cherepanova (Chelyabinsk State University, Russia)

Binary differential equation is an implicit differential equation of the form

$$ap^2 + 2bp + c = 0,\ p = \frac{dy}{dx} \tag{2}$$

with analytic in $(\mathbb{C}^2, 0)$ coefficients $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$. Point $(0, 0)$ is *degenerated singular point* of (1), if

$$a(0,0) = b(0,0) = c(0,0) = 0 \tag{3}$$

Bruce and Tari [1] proved that generic BDE (1)(2) (in real case) topologically equivalent to a linear (i.e. with linear coefficients) BDE. But already formal classification of such BDE has functional modulies.

We prove that for degenerated singular points of BDE *rigidity phenomena* take place: for generic BDE (1)(2)

– formal equivalence implies analytic one

– formal linearizability implies analytic one.

Also we discuss the question about analytic normal forms of BDE (1)(2).

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An introduction to the Theorem of Dulac

Melvin Yeung (Hasselt University, Belgium)

The Theorem of Dulac states that the amount of limit cycles, i.e. isolated periodic orbits, of a polynomial planar dynamical system is finite. First we will go over a brief history of the problem, reducing to the statement that any elementary polycycle on a real analytic surface has a neighborhood free of limit cycles. Then we will treat a simple case of this statement using the strategy of Yu. S. Ilyashenko.

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