Integrable dynamical systems and Geometry
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ABSTRACTS

Mini-courses

Flavors of bicycle mathematics

Sergei Tabachnikov

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This minicourse concerns a naive model of bicycle motion: a bicycle is a segment of fixed length that can move so that the velocity of the rear end is always aligned with the segment. Surprisingly, this simple model is quite rich and has connections with several areas of research, including completely integrable systems. Here is a sampler of problems that I hope to touch upon:

(i) The trajectory of the front wheel and the initial position of the bicycle uniquely determine its motion and its terminal position; the monodromy map sending the initial position to the terminal one arises. This mapping is a Moebius transformation, a remarkable fact that has various geometrical and dynamical consequences.

(ii) The rear wheel track and a choice of the direction of motion uniquely determine the front wheel track; changing the direction to the opposite, yields another front track. These two front tracks are related by the bicycle (Backlund, Darboux) correspondence, which defines a discrete time dynamical system on the space of curves. This system is completely integrable and it is closely related with another, well studied, completely integrable dynamical system, the filament (a.k.a binormal, smoke ring, local induction) equation.

(iii) Given the rear and front tracks of a bicycle, can one tell which way the bicycle went? Usually, one can, but sometimes one cannot. The description of these ambiguous tire tracks is an open problem, intimately related with Ulam's problem in flotation theory (in dimension two): is the round ball the only body that floats in equilibrium in all positions? This problem is also related to the motion of a charge in a magnetic field of a special kind. It turns out that the known solutions are solitons of the planar version of the filament equation.

(iv) Bicycle geodesics are bicycle paths whose front track's length is critical among all bicycle paths connecting two given placements of the line segment. In the plane, these geodesic front tracks are elastica, and in space they are Kirchhoff rods.

(v) Is it possible to ride bicycle so that the rear wheel tracks coincides with the front wheel one (other than going straight, of course)? Such "unicycle" tracks tend to behave very chaotically, but these are mostly only experimental observations.

On the persistence of periodic tori for symplectic twist maps in all dimensions

Marie-Claude Arnaud

Université Paris-Diderot, France

In the first part of this mini-course, I shall present our results on the persistence of Lagrangian periodic tori for symplectic twist maps of the $2d$-dimensional annulus and the rigidity of completely integrable maps. In the second part, I shall give the main arguments of the proofs. Our analysis is based on some geometric and dynamical properties of Lagrangian periodic tori, which is of its owns interest. Joint work with Jessica Massetti and Alfonso Sorrentino.

Integrable systems on Lie algebras and their applications in differential geometry

Alexey Bolsinov

Loughborough University, United Kingdom

The lectures will be devoted to an interesting (and unexpected) relationship between some ideas and notions well known in the theory of integrable systems on Lie algebras and various topics in Differential Geometry including projectively equivalent Riemannian metrics, pseudo-Riemannian symmetric spaces and Yano-Obata conjecture.

Talks

The conformal and projective transformations in Newtonian dynamics from Goursat and Halphen

Alain Albouy

CNRS, Observatoire de Paris, France

Two examples of transformations concerning the Kepler problem were the foundation of "the transformation of the equations of dynamics", a prestigious theory at the beginning of the 20th century. Levi-Civita created from it his theory of geodesically equivalent metrics, his famous connection and his regularizations of the binary collisions in celestial mechanics. I will explain this old material as well as the relation between the surfaces with closed geodesics and the Kepler problem. I will briefly present some new results obtained with Lei Zhao.

Absolutely Periodic Billiard Orbits of Arbitrarily High Order

Keagan Callis

University of Maryland, USA

We show that for any natural number n, the set of domains containing absolutely periodic orbits of order n are dense in the set of bounded strictly convex domains with smooth boundary. The proof that such an orbit exists is an extension to billiard maps of the results of a paper by Gonchenko, Shilnikov, and Turaev, where it is proved that such maps are dense in Newhouse domains in regions of real-analytic area-preserving two-dimensional maps. Our result is a step toward disproving a conjecture that no absolutely periodic billiard orbits of infinite order exist in Euclidean billiards and is also an indication that Ivrii's Conjecture about the measure set of periodic orbits may not be true.

Elliptic fixed points with an invariant foliation: some facts and more questions

Alain Chenciner

Observatoire de Paris, France

We are interested in the existence of an analytic conjugacy to a normal form of an analytic local diffeomorphism of $\mathbb{R}^2$ with a non-resonant elliptic fixed point. A prerequisite is the existence of an analytic conjugacy Φ to a local diffeomorphism which preserves the foliation by circles centered on the fixed point but whose angular behavior is free, a difficult problem whose conservative analogue would be the search for invariant Lagrangian tori on which the dynamics is not constrained to be conjugated to a rotation. Assuming that such a conjugacy Φ exists we show on examples associated to Arnold's family of diffeomorphisms of the circle that obstructions to the existence of an analytic conjugacy to a normal form still exist.
This is joint work with David Sauzin, Shanzhong Sun and Qiaoling Wei.

Periodic orbits in quadratic differentials of high complexity

Vincent Delecroix

CNRS, Université de Bordeaux, France

Polygonal billiards are somehow intermediate between convex and hyperbolic billiards. A lot of recent development shed new lights on rational polygonal billiards of high complexity, for example in (convex) polygon with all its angles in the interval $(\pi - \epsilon, \pi)$. I will discuss the problem of counting periodic orbits in this kind of situation. Somehow unexpectedly, infinite dimensional integrable systems show up via the so-called topological recursion of Eynard-Orantin.

From diamonds to length spectrum rigidity

Corentin Fierobe

IST Austria

As discovered by Carminati, Marmi, Sauzin, for a small analytic perturbation of the integrable standard map the usual invariant KAM solution can be extended to the complex plane as an analytic map with respect to the rotation number, which are defined everywhere outside a set of so-called diamonds. The authors further showed that these diamonds contain singularities of the analytic extension. We will present here how their work can be adapted to a certain class of twist billiard maps, and also how it can be interesting to investigate spectral rigidity questions of billiards.

Discrete group of integrable flows

Vladimir Fock

IRMA, Strasbourg

We give a explicit construction in terms of elementary geometry of plane curves of an Abelian group acting on the phase space of Goncharov-Kenyon integrable system associated to a Newton polygon. A particular case of this system is the Poncelet celebrated porism published exactly 200 years from now.

Polynomial entropy, one-way horseshoes and convex billiards

Jean-Pierre Marco

Institut de Mathématiques de Jussieu, Paris, France

It is known that amongst the convex billiards maps, the only one which has polynomial entropy 1 is associated to the circle, and any other has polynomial entropy larger or equal to 2. We conjecture that the only case for which the polynomial entropy is 2 is that of the non-degenerate ellipse. In this talk I will recall some basic facts about the polynomial entropy, give some ideas about the conjecture and introduce the notion of one-way horseshoe due to Roth, Roth and Snoha (applied to one dimensional maps). For twist maps on the annulus, this yields interesting consequences towards the previous conjecture.

Integrability in Bianchi and Thurston's geometries

Alexander Veselov

Loughborough University, United Kingdom

I will review the known results on integrability of the geodesic flows on 3D manifolds with Thurston's geometries and on the compact quotients of 3D Lie groups classified by Bianchi, including the new analysis of the Bianchi-VII case.
The talk is based on a joint work with Yiru Ye.