Upper bounds of topology of complex polynomials in two variables[*]

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Dedicated to Mikhail Anatolievich Tsfasman
on the occassion of his 50-th birthday

Contents

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1 Introduction, main results and the plan of the paper

It is known that the roots and the critical points of a monic polynomial in one complex variable admit the following a priori bound in terms of the maximal module of the critical values.

Proposition 0. *If a monic polynomial in one complex variable has a critical point at 0 and all its critical values lie in the closed unit disk, then all its zeros and critical points lie in the closed disk of radius four centered at 0.*

The following more precise statement was communicated to the author by X.Buff.

Proposition 1. *Let the conditions of Proposition 0 hold, but 0 is not necessarily a critical point. Then all the roots and the critical points lie in the closed disk of radius two centered at the barycenter of the roots.*

The author believes that these statements are well-known, but he did not find them in the literature. Their proofs will be given below.

The present paper gives analogues of the above bounds for polynomials in two complex variables with a generic highest homogeneous part, see the following Definition.

Definition 1.1 We say that a homogeneous polynomial in two complex variables is *generic*, if it has only simple zero lines.

Let $H(x, y)$ be a complex polynomial in two variables, h be its highest homogeneous part, $H' = H - h$ be its lower terms. We assume that

- h is generic,

- 0 is a critical point of H,

$$H' - H(0) \not\equiv 0.$$

It is well-known that under these assumptions H has at least two distinct critical values. In the present paper we prove quantitative versions of this statement (Theorems 1.10 and 1.17). Assuming that H is appropriately normalized by affine variable changes in the image and the preimage (see Definition 1.4), we give explicit upper bounds of the following quantities:

- the lower terms $H' = H - h$ (Addendum to Theorem 1.10);

- the minimal size of a closed bidisk containing a deformation retract (i.e., all the nontrivial topology) of a level curve

$$S_t = \{H = t\} \text{ (Theorem 1.10);}$$

- the minimal lengths of representatives of cycles in $H_1(S_t, \mathbb{Z})$ vanishing along appropriate paths from t to the critical values (Lemma 1.45 and Corollary 1.48 in 1.5, these representatives are contained in the previous bidisk);

- the intersection indices of the latter cycles (Theorem 1.50 in 1.6).

The above-mentioned upper bounds of locations and lengths of vanishing cycles were used in [4] (a joint paper with Yu.S.Ilyashenko), where we studied zeros of Abelian integrals of polynomial 1- forms over real ovals of real polynomials. In [4] we gave an upper bound of the number of zeros for a wide class of real polynomials of arbitrary degree $n+1 \geq 3$ and any real polynomial 1- form of smaller degree. This class consists of ultra-Morse polynomials H (see Definition 1.18). The estimate of the number of zeros is done in terms of the minimal gap between two zero lines of the higher homogeneous part h and the ratio of that between two critical values of H over the diameter of the critical value set. The key object considered in the proofs in [4] is the Abelian integral matrix

$$\mathbb{I}(t) = (I_{ij}(t)), \ I_{ij}(t) = \int_{\delta_j \subset S_t} \omega_i, \text{ where} \tag{1.1}$$

ω_i, $i = 1, \dots, \mu = n^2$, are fixed monomial forms of appropriate degrees, $\delta_j \subset H_1(S_t, \mathbb{Z})$ are appropriate cycles (marked vanishing cycles) that form a basis.

Remark 1.2 The matrix elements $I_{ij}(t)$ are multivalued holomorphic functions with branchings at the critical values of H.

In the present paper we prove the following statements used in [4]:

- upper bound of the elements of the Abelian integral matrix (1.1) (see 1.7);

- lower bound of its determinant

$$\Delta(t) = \Delta_{H,\Omega}(t) = \det \mathbb{I}(t), \ \Omega = (\omega_1, \dots, \omega_{n^2}) \tag{1.2}$$

(see 1.8), which is called the *period determinant*. We prove these statements in full generality, without assumption that H is real.

The above-mentioned upper bound of $|I_{ij}(t)|$ is implied by the previously-mentioned upper bound of locations and lengths of vanishing cycles. The upper bound of the latter lengths is proved by using the upper bound of topology of level curve and Bezout's theorem.

It is well-known that if ω_i are homogeneous polynomial 1- forms of appropriate degrees, then the period determinant is a polynomial of the type

$$\Delta(t) = C(h,\Omega)\prod_{i=1}^{n^2}(t-a_i), \text{ where} \tag{1.3}$$

a_i are the critical values of H, see [2]. The value $C(h,\Omega)$ was partially calculated (up to a constant factor) in [6]. An explicit formula for $C(h,\Omega)$ (with the latter constant factor calculated) was obtained in [3]. The proof of the lower bound of $\Delta(t)$ is based on the latter formula.

The upper bound of topology and the lower bound of the period determinant are the principal results of the paper: their proofs take its most part.

Proof of Proposition 0. Let $p(z) = z^d + ...$ be the polynomial under consideration. Denote

$$D_1 = \{|z|<1\}, \ S = p^{-1}(\mathbb{C}\setminus\overline{D}_1).$$

The set S is mapped by p outside the closed unit disk, which contains all the critical values. Therefore, it contains neither roots, nor critical points. Hence, for the proof of the Proposition it suffices to show that the set S contains the complement to the closed disk of radius 4 centered at 0.

By construction,

$$p : S \to \mathbb{C}\setminus D_1 \text{ is a covering of degree } d$$

nonramified outside infinity, as is $z^d : \mathbb{C}\setminus\overline{D}_1 \to \mathbb{C}\setminus\overline{D}_1$. Therefore, there is a conformal 1-to-1 variable change

$$\phi : \mathbb{C}\setminus\overline{D}_1 \to S, \text{ such that } p(\phi(z)) = z^d, \ \phi(z) = z + O(1), \text{ as } z\to\infty.$$

By the condition of the Proposition, 0 is a critical point of p. Then the set S, which is the image of ϕ, does not contain 0. By Köbe $\frac{1}{4}$ theorem [5], the image S of ϕ contains the complement to the closed disk of radius 4 centered at 0. This proves Proposition 0. □

Proof of Proposition 1 (by X.Buff). Without loss of generality we assume that the barycenter of the roots is at 0 (one can achieve this by translation of the variable). Let S, ϕ be as above. Then

$$\phi : \mathbb{C}\setminus\overline{D}_1 \to \mathbb{C} \text{ is an univalent function}, \ \phi(z) = z + o(1), \text{ as } z\to\infty. \tag{1.4}$$

It is well-known that the image S of each univalent holomorphic function satisfying (1.4) contains the complement to the closed disk of radius 2 centered at 0. For example, this follows from results presented in [5]: theorem 6.11 and corollary 6.11 (pp. 154 and 157 of the Russian edition). This proves Proposition 1. □

1.1 The plan of the paper

The results concerning upper bounds of topology of level curve and of lower terms of H are stated in 1.2 and proved in 1.2 and Section 2.

The definition of vanishing cycles is recalled in 1.3.

Upper bounds of lengths of their appropriate (canonical) representatives are stated and proved in 1.5. These canonical representatives are defined in 1.4; their projections to the x- axis are piecewise-linear curves. The number of pieces is estimated at the same place (Proposition 1.38 and Corollary 1.39). The intersection indices of vanishing cycles are estimated in 1.6 (using Corollary 1.39). Proposition 1.38 is the key statement used in the estimates of lengths and intersection indices.

Upper bounds (used in [4]) of the integrals $I_{ij}(t)$ are stated and proved in 1.7.

The lower bound of the period determinant is stated in 1.8 and proved in Section 3.

1.2 Upper bounds of topology and lower terms

To state the results from the title of the Subsection, we have to normalize the polynomial in appropriate way by affine variable changes. To specify the normalization, let us firstly introduce the following definitions (see also [4], where some of them were also introduced).

Definition 1.3 The *norm* of a homogeneous polynomial is the maximal value of its module on the unit sphere; this norm is denoted by $\|h\|_{\max}$. The norm of a nonhomogeneous polynomial is the sum of norms of its homogeneous parts.

For any $r > 0$ denote

$$D_r = \{z \in \mathbb{C} \mid |z| < r\},\ D_r(a) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

For any $X, Y > 0$ denote

$$D_{X,Y} = \{(z, w) \in \mathbb{C}^2 \mid |z| \leq X,\ |w| \leq Y\}.$$

Definition 1.4 A polynomial H as at the beginning of the paper is said to be *weakly normalized*, if $||h||_{max} = 1$ and all its critical values are contained in the closed disk $\overline{D}_2$. It is said to be *normalized*, if the previous conditions hold and there is no smaller disk containing the critical values.

Remark 1.5 Any polynomial H as at the beginning of the paper can be transformed to a normalized one by (nonunique) affine variable changes in the image and the preimage so that the highest homogeneous part remains unchanged up to multiplication by constant. (The previous definition is a slightly extended version of an analogous one from [4].)

Remark 1.6 The max- norm of a polynomial (and hence, the notion of (weak) normalizedness are invariant under unitary coordinate transformations in the variable space.

Let us introduce the following function on the space of normalized polynomials.

Definition 1.7 For any polynomial H of degree $n+1$ with a generic highest homogeneous part h put $c_1(H)$ to be n multiplied by the smallest distance between two lines in the zero locus of h. The distance between two lines is taken in sense of Fubini-Study metric on the projective line $\mathbb{C}P^1$. Let

$$c'(H) = \min(c_1(H), 1).$$

Definition 1.8 We say that the topology of a level curve $S_t = \{H = t\}$ is contained in a bidisk $D_{X,Y}$, if the difference $S_t \setminus D_{X,Y}$ consists of $n+1 = \deg H$ punctured topological disks, and the restriction of the projection $(x, y) \mapsto x$ to any of these disks is a biholomorphic map onto $\{x \in \mathbb{C} \mid X < |x| < \infty\}$.

Remark 1.9 Let a complex polynomial $H(x, y)$ have a generic highest homogeneous part h. A bidisk containing the topology of a level curve of H exists, if and only if the y- axis is transversal to each zero line of h. (Then such a bidisk exists for each level curve.)

Theorem 1.10 *For any weakly normalized polynomial H of degree $n+1 \geq 3$ the Hermitian basis in $\mathbb{C}^2$ may be so chosen that the topology of all the level curves S_t for $|t| \leq 5$ will be located in a bidisk $D_{X,Y}$ with*

$$X \leq Y < (c'(H))^{-14n^3} n^{65n^3} = R_0. \tag{1.5}$$

The choice of basis in $\mathbb{C}^2$ is specified below.

Addendum In the conditions of the Theorem the norm of the lower terms $H' = H - h$ admits the following upper bound:

$$||H'||_{max} < (c')^{-13n^4} n^{64n^4}. \tag{1.6}$$

The Theorem and the Addendum are proved in Section 2.

Corollary 1.11 *In the conditions of the previous Theorem for any $t \in \mathbb{C}$ the topology of the level curve S_t is contained in the bidisk $D_{X,Y}$ with*

$$X \leq Y \leq R_0 \chi(|t|), \chi(t) = \max\{1, (\frac{|t|}{5})^{\frac{1}{n+1}}\}. \tag{1.7}$$

Proof If $|t| \leq 5$, then the statement of the Corollary follows immediately from the Theorem. If $|t| > 5$, consider the rescaled polynomial $\widetilde{H}(x,y) = \frac{5}{|t|} H(|\frac{t}{5}|^{\frac{1}{n+1}} x, |\frac{t}{5}|^{\frac{1}{n+1}} y)$. It is also weakly normalized. Indeed, its highest homogeneous part coincides with that of H. Its critical values are equal to $\frac{5}{|t|} < 1$ times those of H, thus, their modules are no greater than 2. This implies weak normalizedness. The level curve $S_t = \{H = t\}$ is transformed by the previous rescaling to the level curve $\{\widetilde{H} = t\frac{5}{|t|}\}$, which corresponds to a level value of $\widetilde{H}$ with module 5 by construction. Hence, by the Theorem applied to $\widetilde{H}$, the topology of the latter curve is contained in the bidisk $D_{X',Y'}$, $X' \leq Y' \leq R_0$. Applying the inverse rescaling yields that the topology of the curve S_t is contained in $D_{X,Y}$, $X \leq Y \leq R_0 |\frac{t}{5}|^{\frac{1}{n+1}}$. This proves the Corollary. □

Choice of basis in $\mathbb{C}^2$. Theorem 1.10 will be proved for orthogonal coordinates (x,y) in $\mathbb{C}^2$ satisfying the following inequality (their existence is implied by the next Proposition):

$$\text{the distance of the } y- \text{ axis to the zero lines of } h \text{ is greater than } \frac{1}{\sqrt{n}}. \tag{1.8}$$

Proposition 1.12 *For any $m \in \mathbb{N}$ and any tuple of $m+1$ complex lines in $\mathbb{C}^2$ passing through the origin there exists another complex line through the origin whose distance to each line of the tuple is greater than $\frac{1}{\sqrt{m}}$.*

Proof Take the $\frac{1}{\sqrt{m}}$ neighborhood in the sphere $\mathbb{CP}^1$ of each line of the tuple. The area of the neighborhood is less than that of an Euclidean disk of the same radius, i.e., less than $\pi\frac{1}{m}$. The union of the latter neighborhoods through all the lines from the tuple does not cover the whole sphere (and this implies the Proposition). Indeed, the area of the union is less than $\frac{m+1}{m}\pi < 2\pi$, which is less than the area 4π of the whole sphere. The Proposition is proved. □

Definition 1.13 A complex polynomial $H = h + H'$ with a generic highest homogeneous part h is said to be *unit-scaled*, if

$$||h||_{max} = 1 \geq ||H'||_{max}.$$

Let us give a brief description of proof of Theorem 1.10. This Theorem and its Addendum hold true automatically for unit-scaled polynomials H, and a much stronger bound of topology follows almost immediately by elementary inequalities. Namely, one uses the fact that outside a ball whose radius can be explicitly estimated from above the foliation $H = const$ resembles the foliation by level curves of the homogeneous polynomial h. This is done in Subsection 2.3. The results of 2.3 imply the following

Proposition 1.14 *Let H be a unit-scaled polynomial of degree $n+1 \geq 3$, h be its highest homogeneous part, (x,y) be orthogonal coordinates satisfying (1.8). Then for any $t \in \mathbb{C}$, $|t| \leq 5$, the topology of the level curve S_t is contained in the bidisk D_{X_n,Y_n},*

$$X_n = n^{7n}(c'(H))^{-2n}, \quad Y_n = n^n X_n = n^{8n}(c'(H))^{-2n}. \tag{1.9}$$

In the general case, when the norm of lower terms may be large, for the proof of Theorem 1.10, we change the normalization of H to make the norm of lower terms unit, and we prove an upper bound of the rescaling rate. This bound is based on the next Theorem, which is a quantitative analogue of the first statement mentioned at the beginning of the paper. Roughly speaking, it says that if the difference $H' - H(0)$ is not too small, then the maximal distance between critical values of H admits an explicit lower bound. To formulate it, we change normalization again in such a way that in addition $H(0) = 0$.

Definition 1.15 A polynomial H as at the beginning of the paper is said to be *centrally-rescaled* , if $H(0) = 0$ and $||h||_{max} = ||H'||_{max} = 1$.

Remark 1.16 Each polynomial H from the beginning of the paper can be transformed to a centrally-rescaled one by homothety in the preimage and an affine transformation in the image. The space of centrally-rescaled polynomials is invariant under unitary transformations of the variables, as is the max- norm.

Theorem 1.17 *Each centrally-rescaled polynomial H of degree $n+1 \geq 3$ has at least one critical value with module no less than*

$$\delta_0 = (c'(H))^{13n^4} n^{-63n^4}. \tag{1.10}$$

The proof of Theorem 1.17 takes the most part of Section 2. Theorem 1.10 will be deduced from Theorem 1.17 and Proposition 1.14 in 2.2.

For the proof of Theorem 1.17 we consider projections $\pi_t : S_t \to Ox$ of level curves to the x- axis. The proof is done by combining topological arguments and analysis of arrangement configuration of critical values of projections. The topological arguments are based on connectivity of the intersection graph of marked basic vanishing cycles and Picard-Lefschetz theorem [1]. The main technical result of the analysis of critical values of projection is a lower bound of the maximal module of a critical value of π_0 (Lemma 2.6 stated in 2.1). The topological and analytical arguments are related to each other via studying appropriate (canonical) representatives of the vanishing cycles whose projections are piecewise-linear curves with vertices at the critical values of projection.

1.3 Vanishing cycles

All the definitions and the statements of the present Subsection are contained in [1] and [4].

The basic vanishing cycles are usually defined in homology of level curves of an ultra-Morse polynomial, see the following Definition.

Definition 1.18 A complex polynomial of degree $n+1$ with a generic highest homogeneous part is said to be *ultra-Morse*, if all its critical values are simple (then their number is equal to $\mu = n^2$).

Firstly we recall the definition of a local vanishing cycle.

Lemma 1.19 (Morse lemma). *A holomorphic function having a Morse critical point may be transformed to a sum of a nondegenerate quadratic form and a constant term by an analytic change of coordinates near this point.*

Corollary 1.20 *Consider a holomorphic function in $\mathbb{C}^2$ having a Morse critical point with a critical value a. There exists a ball centered at the critical point whose intersection with each level curve corresponding to a value close to a of the function is diffeomorphic to an annulus.*

Definition 1.21 A generator of the first homology group of the latter intersection annulus (considered as a cycle in the homology of the global level curve) is called a *local vanishing cycle* corresponding to a.

A local vanishing cycle is well defined up to change of orientation.

Definition 1.22 Let H be a ultra-Morse polynomial, a_j be its critical values, $j = 1, \ldots, \mu$. A path $\alpha_j : [0, 1] \to \mathbb{C}$ is called *regular* provided that

$$\alpha_j(1) = a_j, \;\; \alpha_j[0, 1) \text{ contains no critical value of } H. \tag{1.11}$$

Definition 1.23 Let α_j be a regular path, $t_0 = \alpha_j(0)$, $s \in [0, 1]$ be close to 1, $\delta_j(t), \; t = \alpha_j(s)$, be a local vanishing cycle on S_t corresponding to α_j. Consider the extension of δ_j along the path α_j up to a continuous family of cycles $\delta_j(s)$ in complex level curves $H = \alpha_j(s)$. The homology class $\delta_j = \delta_j(0) \in H_1(S_{t_0}, \mathbb{Z})$ is called a *cycle vanishing along* α_j.

Definition 1.24 Consider a set of regular paths $\alpha_1, \ldots, \alpha_\mu$, see (1.11), with a common starting point t_0. Suppose that these paths are not pairwise and self intersected. Then the set of cycles $\delta_j \in H_1(S_{t_0}, \mathbb{Z})$ vanishing along $\alpha_j, \; j = 1 \ldots, \mu$, is called a *marked set of vanishing cycles* on the level curve $H = t_0$.

Lemma 1.25 *Any marked set of vanishing cycles is a basis in the first integer homology group of the level curve.*

The extension of a cycle in $H_1(S_t, \mathbb{Z})$ along a loop (avoiding critical values) defines an operator from the homology group to itself called *monodromy operator.*

Lemma 1.26 *The images of any vanishing cycle under the monodromy operators along all the loops generate the homology group.*

The Lemma follows from Picard-Lefschetz theorem and the connectedness of the intersection graph of marked set of vanishing cycles [1].

1.4 Canonical representatives of vanishing cycles

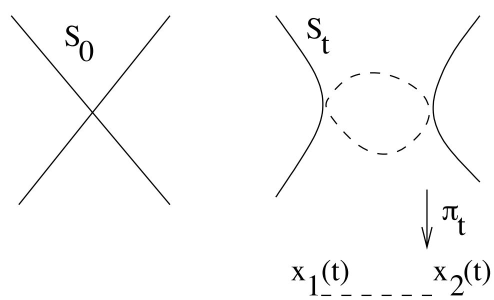
Below we define the representatives of vanishing cycles for which we prove upper bounds of lengths and locations. To do this, we use the following properties of projection of local level curve near critical point, which are well-known and follow immediately from definition.

Proposition 1.27 *Let $H(x,y)$ be an analytic function having a Morse critical point at 0, $H(0) = 0$. The singular level curve $H = 0$ is locally a union of two regularly embedded analytic curves intersecting at 0 transversally (denote L_i, $i = 1,2$, their tangent lines at 0). Let $\pi_t : S_t \to \mathbb{C}$ be a projection along a line transversal to both L_i. There exist a neighborhood U of the critical point and an $\varepsilon > 0$ such that for any $t \neq 0$ close to 0 the restriction $\pi_t|_{S_t \cap U}$ defines a degree two branched covering*

$$\pi_t : \pi_t^{-1}(D_\varepsilon) \cap U \to D_\varepsilon.$$

Addendum. The latter covering has two critical values (branching points) $x_1(t)$, $x_2(t)$ confluenting to 0, as $t \to 0$. The union of the two liftings of the segment $[x_1, x_2]$ under the covering is a closed curve representing a local vanishing cycle (the liftings are taken with inverse orientations).

Definition 1.28 Let in the previous Proposition the projection π_t be made along the y- axis. Then the corresponding closed curve from the Addendum is called the *canonical representative* of the local vanishing cycle (see Fig.1).



Canonical representative of local vanishing cycle

Figure 1

Everywhere below we assume that H is an ultra-Morse polynomial. For any non-critical value t denote

$$CS_t = \{\text{critical values of the projection } \pi_t : S_t \to Ox\}.$$

Below we define the canonical representative of a global vanishing cycle, which will be projected to a piecewise-linear curve in the x- axis with vertices in CS_t. To do this, we firstly deform the coordinate system and the path under consideration to avoid collisions of points from CS_t while t ranges along the path.

Definition 1.29 Let $H(x,y)$ be an ultra-Morse polynomial, h be its highest homogeneous part, $t \in \mathbb{C}$ be a noncritical value of H. A coordinate system (x,y) is said to be t- *regular*, if the y- axis is transversal to all the zero lines of h, all the lines tangent to the critical level curve branches at the critical points and all the lines tangent to S_t at its inflection points.

Remark 1.30 In the conditions of the previous Definition

- the canonical representative of each local vanishing cycle is well-defined;
- each critical point of the projection $\pi_t : S_t \to Ox$ is simple (i.e., of order 1);
- the latter statement holds true for projections π_τ corresponding to all but a finite number of values τ.

Remark 1.31 For any noncritical value t one can always choose a t- regular orthogonal coordinate system satisfying (1.8): the t- regularity is a generic condition; (1.8) is an open condition.

Now we construct the canonical representatives of the global vanishing cycles along appropriate piecewise-linear paths, see the next Definition. To do this, we use the following

Remark 1.32 There exists a $\mu' \in \mathbb{N}$ (depending on H) such that for all but finite number of values of t the cardinality of the set CS_t (without multiplicities) is equal to μ' (algebraicity). For a typical polynomial H all the critical values of π_t are simple and $\mu' = n(n+1) := \eta(n)$ (Bezout's theorem). In general, some critical values of the projection may be multiple for all t (i.e., $\mu' < \eta(n)$). This is the case for the homogeneous polynomial $x^{n+1} + y^{n+1}$, where $\mu' = n+1$.

Definition 1.33 A piecewise-linear regular (see Definition 1.22) path $\alpha : [0,1] \to \mathbb{C}$ with a finite number of edges is said to be *critically-regular*, if for any $t \in \alpha[0,1)$ the cardinality of the set CS_t is maximal (i.e., equal to μ') and each critical point of the projection π_t is simple.

Remark 1.34 Let $H(x,y)$ be an ultra-Morse polynomial, $t \in \mathbb{C}$ be its noncritical value, (x,y) be a t- regular coordinate system. Then any regular path starting from t is homotopic (outside the critical values of H) to some critically-regular path.

Let $a \in \mathbb{C}$ be a critical value of H, $A \in \mathbb{C}^2$ be the corresponding critical point, $x(A)$ be its x- coordinate. Let $t_0 \in \mathbb{C}$ be a noncritical value of H, $\alpha : [0,1] \to \mathbb{C}$ be a critically-regular path from t_0 to a. Let

$$CS_t = \{x_1(t), \dots, x_{\mu'}(t)\}.$$

The functions $x_i(t)$ are multivalued, but their restriction pull-backs $x_i(\alpha(\tau))$ along α have well-defined continuous branches with disjoint graphs (critical regularity). Let $x_1(\alpha(\tau))$, $x_2(\alpha(\tau))$ be those of them that confluent to $x(A)$, as $\tau \to 1$, $\widetilde{\delta}(\tau)$ be the canonical representative of the corresponding local vanishing cycle. This representative is well-defined for τ close to 1 as a double lifting to S_t of the segment $[x_1(t), x_2(t)]$, $t = \alpha(\tau)$. Let us construct its continuous extension (as a family of closed curves in $S_{\alpha(\tau)}$) for all $\tau \in [0, 1)$. Then the curve $\widetilde{\delta}(0)$ represents the global vanishing cycle along α and will be called its *canonical representative*.

Case 1: $x_i(\alpha(\tau)) \notin (x_1(\alpha(\tau)), x_2(\alpha(\tau)))$ whenever $\tau \in [0, 1)$, $i \neq 1, 2$. Then the previous double lifting $\widetilde{\delta}(\tau)$ of $[x_1, x_2]$ is well-defined and continuous for all $\tau \in [0, 1)$.

Case 2: α is a segment and there exists a parameter value τ_1 for which

$$\text{some } x_i,\ i \neq 1, 2,\ \text{meet } (x_1, x_2),\ \text{denote } N(\tau_1) \text{ the number of these } x_i. \tag{1.12}$$

(No x_i, $i \neq 1, 2$, can meet x_1, x_2, by critical regularity of α.) For $\tau = \tau_1$ the previous points x_i split $[x_1, x_2]$ into a union of $N(\tau_1) + 1$ smaller segments $[x_{i_r}, x_{i_{r+1}}]$, $r = 0, \dots, N(\tau_1)$. The new segments (whose ends are functions in τ) do not meet other x_j whenever τ is close enough to τ_1 (by construction and critical regularity). Then the family $\widetilde{\delta}(\tau)$ extends continuously to the values $\tau \leq \tau_1$ close to τ_1 as the union of appropriate double liftings of the previous segments $[x_{i_r}, x_{i_{r+1}}]$. This extension is well-defined until we cross a parameter value $\tau_2 < \tau_1$ for which some of the intervals $(x_{i_r}, x_{i_{r+1}})$ (which are the edges of the projection of $\widetilde{\delta}(\tau)$) meet some other x_j' s. Then we repeat the previous extension of $\widetilde{\delta}$ to smaller values $\tau < \tau_2$, etc. By algebraicity, after a finite number of steps we will extend $\widetilde{\delta}(\tau)$ to all the values $\tau \in [0, 1)$ (the number of steps and edges of the projection is estimated below using Bezout's theorem).

Case 3 (general): the path α has several edges. We extend $\widetilde{\delta}$ by induction in the number of edges of α. The induction base (one edge) is given by the previous construction (Case 2).

Induction step. Let $\alpha[0, \tau']$ be the first edge of α. By the induction hypothesis, the family of closed curves $\widetilde{\delta}(\tau)$ is extended to $\tau \in [\tau', 1)$. The previous construction applied to each edge of the projection of $\widetilde{\delta}(\tau')$ (instead of $[x_1, x_2]$) yields the desired extension of $\widetilde{\delta}(\tau)$ to $\tau \in [0, \tau')$.

Definition 1.35 Thus constructed closed curve $\widetilde{\delta}(0)$ is called the *canonical representative* of the cycle vanishing along α.

Proposition 1.36 *The canonical representative in a level curve S_t of a cycle vanishing along a critically-regular path is projected onto a piecewise-linear curve in the complex x- axis with vertices in CS_t (the edges are minimal: no edge interval contains a point of CS_t). More precisely, the representative is a finite union of couples of arcs; the arcs from each couple are disjoint (maybe except for their ends) and are projected bijectively onto the same edge. If the topology of S_t is contained in a bidisk $D_{X,Y}$, then the canonical representative is contained in the same bidisk. For any given critically-regular* *path* $\alpha : [0,1] \to \mathbb{C}$ *the canonical representatives of cycles vanishing along smaller paths* $\alpha|_{[\tau,1]}$, $0 \le \tau < 1$, *depend continuously on* τ.

This Proposition follows immediately from construction. Its statement concerning the location of the representative in $D_{X,Y}$ follows from the fact that all the critical values of projection should lie in the disk $\overline{D}_X$ (by definition) and from the convexity of the latter disk.

Remark 1.37 The projection of a canonical representative of vanishing cycle is a piecewise-linear curve with a given order of pieces (up to inversion). It may happen that several distinct arc couples are projected onto one and the same edge (the number of all the arc couples over a given edge is called its *multiplicity*). The number of arc couples from the previous Proposition is equal to the number of projection edges with multiplicities.

The upper bounds of lengths and intersection indices of vanishing cycles are based on the next Proposition (proved below by using Bezout's theorem).

For any piecewise-linear curve Γ denote

$$m(\Gamma) = \#(\text{edges of } \Gamma).$$

The number of arc couples of a canonical representative $\widetilde{\delta}$ of vanishing cycle (or equivalently, the number of edges (with multiplicities) of its projection) will be denoted by the same symbol $m(\widetilde{\delta})$. By definition,

$$m(\widetilde{\delta}) = 1, \text{ if } \widetilde{\delta} \text{ is a canonical representative of a local vanishing cycle.}$$

Proposition 1.38 *Let* $\alpha : [0,1] \to \mathbb{C}$ *be a critically-regular path,* $\alpha[0,\tau']$ *be its first edge. For any* $\tau \in [0,1)$ *let* $\widetilde{\delta}(\tau)$ *be the canonical representative of the cycle vanishing along the path* $\alpha|_{[\tau,1]}$. *Let* $m(\widetilde{\delta}(\tau))$ *be the previously defined number of edges (we put* $m(\widetilde{\delta}(1)) = 1$*). Then*

$$\log_2 m(\widetilde{\delta}(0)) - \log_2 m(\widetilde{\delta}(\tau')) \le 23n^{12}. \tag{1.13}$$

The Proposition is proved below.

Corollary 1.39 *In the previous Proposition*

$$m(\widetilde{\delta}(0)) \le 2^{E(\alpha)},\ E(\alpha) = 23n^{12}m(\alpha),\ m(\alpha) = \#(\textit{edges of } \alpha).$$

Proof of Proposition 1.38. The number of edges of the projection of a canonical representative increases exactly while crossing a parameter value $\tau'' \in [0,\tau')$ (moving τ from τ' towards 0) where a certain x_j meets some edge of the projection (the latter edge may be multiple, see the previous Remark). If only one x_j meets one edge, then this edge breaks in two pieces, their multiplicities are equal to that of the initial edge. Therefore, the number of edges with multiplicities is at most doubled after passing the value τ''. (In addition, some (new) edges may collide, but this does not change the sum of the multiplicities of all the edges.) Analogously, if there are $r > 1$ triples x_j, x_i, x_k such that x_j meets the edge $[x_i, x_k]$ at $\tau = \tau''$, then the total number of edges with multiplicities increases in at most 2^r times. Hence, the number of edges with multiplicities of the global vanishing cycle $\widetilde{\delta}(0)$ is no greater than $2^N m(\widetilde{\delta}(\tau'))$, where N is the total number of quadruples (x_1, x_2, x_3, τ), $\tau \in [0, \tau')$, such that $x_i = x_i(\alpha(\tau))$ form a collinear point triple, and they are not identically collinear in $\tau \in [0, \tau']$.

Without loss of generality we assume that $\alpha(\tau) = \tau$ for $\tau \in [0, \tau']$: one can achieve this by complex affine change of the coordinate t. Then the previous quadruples are isolated solutions with $\tau \in [0, \tau')$ of the following system of equations

$$\begin{cases} H(x_i, y_i) - \tau = 0 \\ \frac{\partial H}{\partial y}(x_i, y_i) = 0 \\ i = 1, 2, 3 \\ \operatorname{Im} \tau = 0 \\ (\operatorname{Re} x_1 - \operatorname{Re} x_2)(\operatorname{Im} x_1 - \operatorname{Im} x_3) - (\operatorname{Re} x_1 - \operatorname{Re} x_3)(\operatorname{Im} x_1 - \operatorname{Im} x_2) = 0 \end{cases} \tag{1.14}$$

We show that the number of these isolated solutions is less than $23n^{12}$ by using Bezout's theorem applied to the complexification of (1.14), see the following Definitions.

Definition 1.40 The *real form* of a complex polynomial is the tuple of its real and imaginary parts (as polynomials in the real and imaginary parts of the variables).

Definition 1.41 The *complexification* of a system of real polynomials is the system of their extensions to the complex variables. The complexification of the real form of a complex polynomial P will be briefly referred to, as the *complexification of P*.

The complexification of system (1.14) is a system of 14 complex polynomial equations in 14 complex variables that is obtained from (1.14) by replacing the (first 6) complex polynomials and the (two last) real ones by their complexifications. It consists of 6 equations of degree $n + 1$, 6 ones of degree n, one linear equation, one quadratic equation. Bezout's theorem applied to the complexification says that the number of its isolated solutions is no greater than the product of the latter degrees $2(n + 1)^6 n^6 < 23n^{12}$ (the latter inequality follows from elementary inequalities).

Proposition 1.42 *Each isolated solution of (1.14) with $\tau \in [0, \tau')$ is an isolated solution of its complexification.*

Remark 1.43 One can provide examples of real polynomial equations with isolated solutions in the real space that are not isolated solutions in the complex space: the real polynomial equation $x^2 + y^2 = 0$ has unique real solution 0 that is not an isolated solution of its complexification. V.Kharlamov have proposed the following example of 3 real polynomial equations with 3 variables, where the number of isolated solutions in the real space is greater than the product of the degrees (which we call the Bezout number):

$$\begin{cases} \prod_{k=1}^{d}(x-x_k)^2+\prod_{k=1}^{d}(y-y_k)^2=0 \\ z=0 \\ z=0 \end{cases}, \ d>2.$$

It has d^2 real solutions $(x_i, y_j, 0)$, which is greater than its Bezout number $2d$.

Proposition 1.42 (which is proved below) together with the previous discussion implies Proposition 1.38. □

Proof of Proposition 1.42. Fix an isolated solution X of (1.14). Let us show that it is an isolated solution of its complexification. To do this, we consider the subsystem (denoted (1.14)') of system (1.14) consisting of its (first 6) complex polynomial equations, which is obtained by dropping its nonholomorphic (two last) equations. Below we show (Proposition 1.44) that the local solutions at X of system (1.14)' form a regularly embedded holomorphic curve (two-dimensional real surface, denoted Γ) locally 1-to-1 projected to a domain in the τ- plane. We also show that system (1.14)' has the maximal rank at X. This implies the same statements in the complexification. In particular, the local solutions at X of the complexified system (1.14)' in $\mathbb{C}^{14}$ form a two-dimensional holomorphic surface (denoted $\widetilde{\Gamma}$) locally 1-to-1 projected to a domain in the complexified τ- plane $\mathbb{C}^2$. Afterwards we consider the restrictions to $\widetilde{\Gamma}$ of the complexified two last equations of (1.14): it suffices to show that their solution X is isolated in $\widetilde{\Gamma}$. The hyperplane $\operatorname{Im}\tau=0$ is transversal to the real surface Γ at X, hence, their intersection (denoted γ) is a regularly embedded real analytic curve. Hence, the two latter statements hold true in the complexification (denote $\widetilde{\gamma}$ the intersection of the complexifications of these hyperplane and surface, $\widetilde{\gamma}$ is a regularly embedded holomorphic curve). Now it suffices to show that the last equation of (1.14) does not hold identically on $\widetilde{\gamma}$. Indeed, it does not hold identically on the real curve γ by isolatedness of X as a solution of (1.14). Hence, X is an isolated solution of the complexification of (1.14).

Thus, the previous discussion proves Proposition 1.42 modulo the following

Proposition 1.44 *The solutions of (1.14)' with $\tau\in(0,\tau')$ form a disjoint union of graphs of analytic vector functions $(x_i(\tau), y_i(\tau))_{i=1,2,3}$ in $\tau\in[0,\tau')$. System (1.14)' has the maximal rank at these solutions.*

Proof The solutions of system (1.14)' with τ being a noncritical value of H are exactly the tuples $((x_i,y_i)_{i=1,2,3},\tau)$ where (x_i,y_i) are the critical points of the projection $\pi_\tau : S_\tau\to Ox$. By assumptions, the path α is critically-regular and contains the semiinterval $[0,\tau')$. The critical regularity implies that the values $\tau\in[0,\tau')$ are noncritical for H, and $(x_i(\tau), y_i(\tau))$ are simple critical points of the projection π_τ (in particular, they do not collide and depend holomorphically on $\tau\in[0,\tau')$). This proves the first statement of Proposition 1.44.

Let us prove its last statement saying that system (1.14)' has the maximal rank at the previous tuples $((x_i,y_i)_{i=1,2,3},\tau)$. This is equivalent to say that the pair of polynomials $H-\tau$, $\frac{\partial H}{\partial y}$ has the maximal rank at each their common zero, which is a critical point (x_i,y_i) of the projection π_τ. Indeed, the polynomial $H-\tau$ has nonzero gradient at its zero level curve S_τ, since τ is not a critical value of H. The restriction to S_τ of the second polynomial $\frac{\partial H}{\partial y}$ has nonzero derivative at each point (x_i,y_i). Indeed, since the latter is a critical point of π_τ, this is equivalent to say that $\frac{\partial^2 H}{\partial y^2}(x_i,y_i)\neq 0$. The latter inequality follows immediately from the simplicity of the critical points of the projection (critical regularity of the path). This proves the maximality of rank and finishes the proof of Propositions 1.44 and 1.42. □ □

1.5 Lengths of canonical representatives of vanishing cycles

Lemma 1.45 *Let H be an ultra-Morse polynomial of degree $n+1\geq 3$, a be its critical value, $t\in\mathbb{C}$ be a noncritical value, $S_t=\{H(x,y)=t\}$, $\pi_t:S_t\to Ox$ be the projection to the x- axis. Let (x,y) be a t- regular orthogonal coordinate system (see Definition 1.29). Let α be a critically-regular path from t to a with $m(\alpha)$ edges, $\widetilde{\delta}\subset S_t$ be the canonical representative of the corresponding vanishing cycle. Let $X,Y>0$ be such that the topology of the curve S_t be contained in the bidisk $D_{X,Y}$. Then*

$$|\widetilde{\delta}|\leq 2^{l(n)m(\alpha)}R,\ R=\max\{X,Y\},\ l(n)=24n^{12}. \tag{1.15}$$

Corollary 1.46 *Let all the conditions of the Lemma hold, but now the orthogonal coordinate system (x,y) be not necessarily t- regular, and the path α be piecewise-linear, but not necessarily critically-regular. Then the corresponding vanishing cycle admits a representative $\widetilde{\delta}$ satisfying (1.15) and lying in $D_{X,Y}$.*

Proof of Lemma 1.45. Each arc couple of $\widetilde{\delta}$ (see Proposition 1.36), which is projected onto a real segment in the complex Ox -line, lies in a real algebraic curve. This curve (denoted ψ) is the intersection of the complex level curve S_t of H and the real hyperplane in $\mathbb{C}^2=\mathbb{R}^4$ that contains the complex Oy- line and the latter segment. That is, ψ is the common zero set of the system of 3 polynomials in $\mathbb{C}^2=\mathbb{R}^4$: the real and the imaginary parts of H (both of degree $n+1$) and a linear function.

The number of the previous arc couples (edges) is already estimated in Corollary 1.39. Let us estimate the total length of one arc couple. It is no greater than the sum of lengths of its projections to the x- axis and to the real and imaginary y -axes (more precisely, we have to replace "length of projection" by "length of projection times the maximal number of preimages of a generic point"). Let us estimate the latter quantities.

The arc couple is projected to the x- axis onto a segment in $\overline{D}_X$ (hence, having length no greater than $2X$), and each point of the latter segment (except for its ends) has exactly two preimages (Proposition 1.36). Hence, the length contribution of the x-projection is no greater than $4X$.

The projection image of the arc couple to either real or imaginary y- axis is a segment lying in $\overline{D}_Y$ (hence, of length no greater than $2Y$). The number of preimages of a generic point is no greater than $(n+1)^2$. Indeed, it suffices to show that the ambient algebraic curve ψ has at most the same number of isolated intersection points with a generic real hyperplane in $\mathbb{C}^2 = \mathbb{R}^4$ parallel to a given one (here "generic" means "intersecting ψ transversally"). These points are common zeros of the previous system of 3 polynomials (defining ψ) and an additional linear function (defining the hyperplane). By transversality, these points are isolated common zeros of their complexifications. Hence, by Bezout's theorem, their number is no greater than $(n+1)^2$.

Therefore, the length contribution of the projections to either real or imaginary y-axis is no greater than $2Y(n+1)^2$.

The previous discussion implies that the total length of an arc couple is no greater than

$$4X + 4Y(n+1)^2 \leq (4+9n^2)Y \leq 10Rn^2.$$

Together with Corollary 1.39, this implies that the total length of the canonical representative is no greater than

$$10Rn^2 \times 2^{23n^{12}m(\alpha)} < 2^{24n^{12}m(\alpha)}R.$$

This proves Lemma 1.45. □

Proof of Corollary 1.46. One can choose a sequence (x_k, y_k) of t- regular orthogonal coordinate systems converging to (x, y) (by the genericity of the t- regularity property). Let us fix such a sequence (x_k, y_k). By assumption, the topology of the level curve S_t is contained in $D_{X,Y}$. Therefore, there exists a real pair sequence $(X_k, Y_k) \to (X, Y)$ such that for any k the topology of S_t is contained in D_{X_k,Y_k} in the coordinates (x_k, y_k). For any k one can slightly perturb the path α to make it critically-regular with respect to the coordinates (x_k, y_k). Applying Lemma 1.45 yields canonical representatives $\widetilde{\delta}_k$ of the vanishing cycle with uniformly bounded lengths and locations: they satisfy (1.15) with R replaced by $R_k \to R$, $R_k = \max\{X_k, Y_k\}$. Passing to a subsequence k_i one can achieve that the representatives $\widetilde{\delta}_{k_i}$ converge to some other representative $\widetilde{\delta}$ of the same vanishing cycle that satisfies (1.15). In more details, consider $\widetilde{\delta}_k$ as closed curves parametrized by their natural parameters. If their lengths (i.e., those of the parameter segments) are different, we rescale the parameter segments by homotheties (by enlarging the smaller ones) in order to make them equal. By construction, we get a sequence of closed curves $\widetilde{\delta}_k$ parametrized by one and the same segment; the derivatives of the parametrizations have modules no greater than 1. Now Arzela-Ascoli theorem implies the previous statement of (uniform) convergence of appropriate subsequence $\widetilde{\delta}_{k_i}$. The curves $\widetilde{\delta}_{k_i}$ are homotopic to their limit $\widetilde{\delta}$, hence, the latter is a representative of the same cycle. It satisfies (1.15) and lies in $D_{X,Y}$ by construction. The Corollary is proved. □

We use also the following Corollary of Lemma 1.45 giving an upper bound of length of vanishing cycle in terms of the length of the corresponding path. To formulate it, let us introduce the next Definition.

Definition 1.47 Let H be a ultra-Morse polynomial, $\beta > 0$. A regular path $\alpha : [0,1] \to \mathbb{C}$ (denote $a = \alpha(1)$) is said to be β- *regular*, if the curve $\alpha \cap D_\beta(a)$ is a connected arc of the path α, and α is disjoint from the β neighborhoods of the critical values distinct from a of the polynomial H.

Corollary 1.48 *Let H be an ultra-Morse polynomial of degree $n+1 \geq 3$, $0 < \beta < 1$. Let a be its critical value, $t \in \mathbb{C}$ be a noncritical value, α be a β- regular (but now, not necessarily piecewise-linear) path from t to a. Let $X, Y > 0$ be such that the topology of the curve S_t lies in the bidisk $D_{X,Y}$, $R = \max\{X, Y\}$. Then the cycle in $H_1(S_t, \mathbb{Z})$ vanishing along the path α admits a representative $\widetilde{\delta}$ such that*

$$\widetilde{\delta} \subset D_{X,Y}, \ |\widetilde{\delta}| \leq 2^{l(n)\frac{|\alpha|\beta^{-1}+5}{3}} R, \text{ where} \tag{1.16}$$

$l(n)$ is the same, as in (1.15).

Proof The vanishing cycle depends only on the homotopy class (modulo the critical values) of the path α. We construct a piecewise-linear path α' homotopic to α with at most $\frac{|\alpha|\beta^{-1}+5}{3}$ edges. Then Corollary 1.46 applied to α' implies (1.16) for the corresponding representative $\widetilde{\delta}$ of the cycle as vanishing along α'. To do this, consider a splitting of α into its arc $\alpha \cap D_\beta(a)$ (whose length is no less than β) and at most $\frac{|\alpha|\beta^{-1}+2}{3}$ other arcs of lengths at most 3β. Each splitting arc is homotopic outside the critical values of H to the straightline segment with the same ends. Indeed, by β- regularity, the β- neighborhood of the path end a does not contain other critical values of H. This implies that the arc $\alpha \cap D_\beta(a)$ is homotopic to the radius with the same ends. The similar statement for the other arcs, which are disjoint from the β- neighborhoods of all the critical values, is implied now by the following geometric fact: a curve lying outside a disk (or several disks) of a radius β and having a length less than $\pi\beta > 3\beta$ is always homotopic with fixed ends (outside the centers of the disks) to the straightline segment with the same ends. Therefore, the new path (denoted α') from t to a consisting of the previous straightline segments is homotopic to α (the homotopy does not meet the critical values of H). The path α' is piecewise-linear with at most $\frac{|\alpha|\beta^{-1}+5}{3}$ edges. Then this is a path α' we are looking for. This together with the previous discussions proves the Corollary. $\square$

We will also use the following upper bound of length of vanishing cycle along a path that is not β- regular, in particular, intersects the β- neighborhoods of the critical values. The bound is given in terms of the length of its part that lies in the disc $\overline{D}_3$ outside these neighborhoods and the variations of arguments of $t - a_i$ (respectively, t) along its arcs in $D_\beta(a_i)$ (respectively, outside $\overline{D}_3$).

Corollary 1.49 *Let H be a normalized ultra-Morse polynomial of degree $n+1 \geq 3$, a_i be its critical values. Let $0 < \beta \leq \nu = \frac{c''(H)}{4n^2}$, $\alpha : [0,1] \to \mathbb{C}$ be a regular path,*

$$t_0 = \alpha(0), \ a = \alpha(1), \ \tau' = \min\{\tau \in [0,1] \mid \alpha(\tau, 1] \subset D_\beta(a)\}, \ \hat{\alpha} = \alpha \setminus \alpha(\tau', 1],$$

$$\mathcal{D}^\beta=\overline{D}_3\setminus\cup_i D_\beta(a_i),\ \ \widetilde{\alpha}=\alpha\cap\mathcal{D}^\beta, \tag{1.17}$$

$$V=V_{\alpha,\beta}=\beta\sum_i \mathrm{Var}_{\hat{\alpha}\cap D_\beta(a_i)}\arg(t-a_i)+3\,\mathrm{Var}_{\hat{\alpha}\setminus\overline{D}_3}\arg t$$

(the Var- *terms are the complete variations of arguments along the corresponding pieces of* α*). Let* $X,Y>0$ *be such that the topology of the curve* S_{t_0} *lies in the bidisk* $D_{X,Y}$*,* $R=\max\{X,Y\}$*. Then the cycle in* $H_1(S_{t_0},\mathbb{Z})$ *vanishing along the path* α *admits a representative* $\widetilde{\delta}$ *such that*

$$\widetilde{\delta}\subset D_{X,Y},\ |\widetilde{\delta}|<2^{2l(n)\frac{(|\widetilde{\alpha}|+V)\beta^{-1}+5}{3}}R,\ l(n)=24n^{12}. \tag{1.18}$$

Proof Recall that each disc $D_\beta(a_i)$ contains no critical values of H except for a_i, and these disks are disjoint, since $\beta\leq\nu$. The complement $\mathbb{C}\setminus D_3$ also contains no critical values and is disjoint from the latter disks by normalizedness ($|a_i|\leq 2$).

If $\hat{\alpha}=\emptyset$, then $\alpha\subset D_\beta(a)$, and hence, α is homotopic to the segment $[t_0,a]$ modulo the critical values. In this case Lemma 1.45 applied to the segment path $[t_0,a]$ implies an upper bound stronger than (1.18).

If α is contained in $\overline{D}_3$ and is β- regular, then Corollary 1.48 implies an upper bound stronger than (1.18).

In what follows we assume that α is either not β- regular, or not contained in $\overline{D}_3$. We construct a β- regular path $\alpha'':[0,1]\to\overline{D}_3$ of length at most $|\widetilde{\alpha}|+V$ such that the composition $\phi=[t_0,\alpha''(0)]\circ\alpha''$ is homotopic to α modulo the critical values. Then we replace the path α'' by a homotopic piecewise-linear path α' with at most $\frac{|\alpha''|\beta^{-1}+5}{3}$ edges, as in the proof of Corollary 1.48. The path $\phi'=[t_0,\alpha''(0)]\circ\alpha'$ thus obtained, which is homotopic to α, is a piecewise-linear path with one edge more. Applying Corollary 1.46 to ϕ' yields (1.18).

The previous path ϕ is constructed by replacing connected components of the complement $\alpha\setminus\mathcal{D}^\beta$. Let $l=\alpha(\tau_1,\tau_2)$ be a maximal arc of α lying in one of the latter connected components (maximal means that the previous interval (τ_1,τ_2) is contained in no other interval with the same property): then either $l\subset D_\beta(a_i)$ for some i, or $l\subset\mathbb{C}\setminus\overline{D}_3$. If $l=\alpha(\tau',1]$, we do not replace it. If not, then l is either a starting arc $\alpha[0,\tau'')$ of the path α with one end $\alpha(\tau'')$ in $\partial\mathcal{D}^\beta$, or an arc with both ends in one and the same boundary circle. In the latter case we replace l by the arc (denoted l') of this circle with the same ends that is homotopic to l outside the critical values (it may be self-overlapped). Then it follows from definition that

$$|l'|\leq\beta\,\mathrm{Var}_l\arg(t-a_i),\ \text{if}\ l\subset D_\beta(a_i);\ |l'|\leq 3\,\mathrm{Var}_l\arg t,\ \text{if}\ l\cap D_3=\emptyset.$$

Now consider the former case, when l is a starting arc (either inside some $D_\beta(a_i)$, or outside $\overline{D}_3$). Then we replace l by the composition of

- the segment joining $\alpha(0)$ to the closest point of the corresponding circle (either $\partial D_\beta(a_i)$, or ∂D_3);

- an arc of the latter circle

so that their composition be homotopic to l in the punctured disk $\overline{D}_\beta(a_i)$ (respectively, in the complement of D_3).

Then it follows from construction that the length of the latter circle arc is no greater than $\beta \operatorname{Var}_l \arg(t-a_i)$ (respectively, $3\operatorname{Var}_l \arg t$).

The previous replacements give us a modified path α (denoted ϕ) that either starts in $\mathcal{D}^\beta$ (then we put $\alpha''=\phi$), or starts outside by a segment ending at $\partial\mathcal{D}^\beta$ (then we put α'' to be ϕ with the latter segment deleted). It follows from construction that α'' is β- regular. The previous inequalities imply that $|\alpha''|\leq|\widetilde{\alpha}|+V$. The Corollary is proved. $\square$

1.6 Intersection indices of vanishing cycles

Theorem 1.50 *Let H be an ultra-Morse polynomial, $t\in\mathbb{C}$ be a noncritical value, α_1, α_2 be two piecewise-linear paths starting at t and going to some critical values of H (may be to one and the same critical value). Then the module of the intersection index of the corresponding vanishing cycles is less than*

$$2^{24n^{12}(m(\alpha_1)+m(\alpha_2))}$$

Proof Without loss of generality we consider that the coordinate system under consideration is t- regular and the paths are critically-regular (one can achieve this by small perturbations of the coordinate system and the paths). Let $\widetilde{\delta}_1$, $\widetilde{\delta}_2$ be the canonical representatives of the vanishing cycles. Their projections to Ox are piecewise-linear curves with number of edges estimated from above by Corollary 1.39. For the proof of the Theorem we estimate the number of the intersection points of the projections and the contribution of each point to the intersection index.

Case 1. Two arc couples of $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$ respectively have transversally intersected projection edges (no common vertex). The contribution of this intersection point (per arc couple pair) to the intersection index of $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$ has module at most 2. Indeed, each arc couple consists of two arcs (disjoint maybe except for their ends), each arc is 1-to-1 projected to the corresponding edge (Proposition 1.36). Hence, we have at most two transversal intersection points of the corresponding unions of arcs over the intersection point.

There can be at most $2^{E(\alpha_1)+E(\alpha_2)}$ arc couple pairs (one arc couple from $\widetilde{\delta}_1$, the other one from $\widetilde{\delta}_2$) with transversally intersected projections, as above. This follows from Corollary 1.39. Thus, the total contribution of the transversal intersections of edges to the intersection index is no greater than $2^{E(\alpha_1)+E(\alpha_2)+1}$.

Case 2. There is a common vertex x of the projections $\pi_t\widetilde{\delta}_1$ and $\pi_t\widetilde{\delta}_2$ (with at most 2 adjacent edges in each $\pi_t\widetilde{\delta}_i$; thus, the total number of adjacent edges of both projections is at most 4). Firstly let us assume the latter edges are simple and no two of them are collinear. We claim that the module of the contribution of x to the intersection index of the canonical representatives is at most 16. Indeed, by assumption, each projection $\pi_t\widetilde{\delta}_i$ has two adjacent edges at x, and $\widetilde{\delta}_i$ has two arcs over each edge. Each arc is locally (in a neighborhood of the preimage $\pi_t^{-1}(x)$) a regularly embedded semicurve with a tangent line at its end over x (its end may be either a critical point of projection or not); in total there are 4 semicurves per each $\pi_t\widetilde{\delta}_i$. All these 8 semicurves are transversal by construction and the previous assumption on noncollinearity of edges. This implies that the index of the intersection over x of the two unions of 4 semicurves is no greater than $4 \times 4 = 16$.

Let now there be a common vertex of projections (with at most two adjacent edges in each $\pi_t\widetilde{\delta}_i$, as above) and either some of the (at most four) adjacent edges are collinear, or some edges coincide. Then one can avoid any of these situations by small deformation of the interiors of edges (keeping the vertices fixed) in class of smooth curves and lifting it up to a deformation of the closed curves $\widetilde{\delta}_i$. If the deformation of edges is C^1- small enough, then the number of transversal intersection points of projections (not including vertices) remains the same. Afterwards the contribution to the intersection index of each common vertex of the projections is estimated as above: it is no greater than 16.

In the general case, when there are several adjacent edges and (or) some of them are multiple, define the multiplicity of a vertex of the projection $\pi\widetilde{\delta}_i$ as the sum of the multiplicities of the adjacent edges of the same projection. The total vertex multiplicity of the projection is the sum of the multiplicities of its vertices. It follows from definition and the previous discussion that if A is a common vertex of $\pi\widetilde{\delta}_i$, $i = 1, 2$, with corresponding multiplicities μ_i, then its contribution to the intersection index is no greater than $16\mu_1\mu_2$. Hence, the total contribution of the common projection vertices to the intersection index is no greater than 16 times the product of the total vertex multiplicities of the projections.

Each edge has two vertices, hence, the total multiplicity of vertices in each $\pi\widetilde{\delta}_i$ is no greater than the double total multiplicity of edges, thus, no greater than $2^{E(\alpha_i)+1}$ (Corollary 1.39). Therefore, the total contribution of the common vertices to the intersection index is no greater than $2^{E(\alpha_1)+E(\alpha_2)+6}$.

The previous discussion implies that the intersection index of the vanishing cycles under consideration, which is the sum of contributions of transversal intersections of the projections and those of their common vertices, is no greater than $2^{E(\alpha_1)+E(\alpha_2)+7}$. Together with Corollary 1.39 and elementary inequalities, this implies that the module of the intersection index is less than $2^{24n^{12}(m(\alpha_1)+m(\alpha_2))}$. Theorem 1.50 is proved. □

1.7 Upper bounds of integrals

Below we recall and prove upper bounds of Abelian integrals used in [4]. To state them, let us firstly recall the definition (introduced in [4]) of the following function on the space of normalized ultra-Morse polynomials.

Definition 1.51 Let H be a normalized ultra-Morse polynomial of degree $n+1 \geq 3$. Define $c_2(H)$ to be n^2 times the minimal distance between its critical values. Put

$$c''(H) = \min(c_2(H), 1).$$

Theorem 1.52 *Let H be a normalized ultra-Morse polynomial of degree $n+1 \geq 3$, a be its critical value, t be a noncritical value, $|t| \leq 5$, $\alpha : [0,1] \to \mathbb{C}$ be a ε- regular path from t to a (see Definition 1.47),*

$$|\varepsilon| = \frac{c''(H)}{8n^2},\ |\alpha| \leq 36n^2 + 10. \tag{1.19}$$

Let $\delta \in H_1(S_t, \mathbb{Z})$ be the cycle vanishing along α. Let ω be a monomial 1- form of degree no greater than $2n-1$ with unit coefficient. Then

$$|I_\delta(t)| = |\int_\delta \omega| < 2^{\frac{2600n^{16}}{c''(H)}} (c'(H))^{-28n^4}. \tag{1.20}$$

Theorem 1.53 *Let H be a weakly normalized ultra-Morse polynomial of degree $n+1 \geq 3$, a be its critical value, t be a noncritical value, $0 < \beta < 1$. Let α be a β- regular path from t to a (see Definition 1.47), $\delta \in H_1(S_t, \mathbb{Z})$ be the corresponding vanishing cycle. Let ω be a monomial 1- form of degree at most $2n-1$ with unit coefficient. Then*

$$|I_\delta(t)| < 2^{10n^{12}\frac{|\alpha|+5}{\beta} - 2n} (c'(H))^{-28n^4}. \tag{1.21}$$

Theorem 1.54 *Let H be a normalized ultra-Morse polynomial of degree $n+1 \geq 3$, a be its critical value, t_0 be a noncritical value, $0 < \beta \leq \nu = \frac{c''(H)}{4n^2}$. Let $\alpha : [0,1] \to \mathbb{C}$ be a regular path, $t_0 = \alpha(0)$, $a = \alpha(1)$, $\hat{\alpha}$, $\widetilde{\alpha}$, $V = V_{\alpha,\beta}$ be the same, as in (1.17). Let $\delta \in H_1(S_{t_0}, \mathbb{Z})$ be the cycle vanishing along α. Let ω be a monomial 1- form of degree at most $2n-1$ with unit coefficient. Then*

$$|I_\delta(t_0)| < 2^{20n^{12}\frac{|\widetilde{\alpha}|+V+5}{\beta} - 2n} (c'(H))^{-28n^4} \max\{1, (\frac{|t_0|}{5})^2\}. \tag{1.22}$$

Proof of Theorem 1.52. Let us apply Theorem 1.10 to the polynomial H.

Case 1: the coordinates (x, y) satisfy the statements of Theorem 1.10. Then this Theorem implies that the topology of the curve S_t lies in the bidisk $D_{X,Y}$, $X \leq Y \leq R_0$, R_0 is the same, as in (1.5). By assumption, the conditions of Corollary 1.48 hold with $\beta = \varepsilon$, $R \leq R_0$. Let $\widetilde{\delta} \subset S_t \cap D_{X,Y}$ be the representative from this Corollary of the vanishing cycle. Then by the same Corollary and definition,

$$|\int_{\widetilde{\delta}} \omega| \leq R_0^{2n-1} |\widetilde{\delta}| \leq R_0^{2n} 2^{l(n)(\frac{|\alpha|+5\varepsilon}{3\varepsilon})},\ l(n) = 24n^{12}. \tag{1.23}$$

Substituting the values $\varepsilon = \frac{c''(H)}{8n^2}$, $R_0 = (c'(H))^{-14n^3} n^{65n^3}$ and inequality (1.19) to the latter right-hand side and applying elementary inequalities yields

$$|\int_{\widetilde{\delta}} \omega| \leq (c'(H))^{-28n^4} 2^{\frac{2600n^{16}}{c''(H)} - 4n^{16}}. \tag{1.24}$$

This proves Theorem 1.52 in Case 1.

Case 2: general. Let (x', y') be orthogonal coordinates on $\mathbb{C}^2$ satisfying the statements of Theorem 1.10. The form ω is monomial of degree at most $2n-1$ with unit coefficient. Therefore, in the new coordinates it becomes a product of a constant 1-form $A_1dx' + B_1dy'$ and at most $2n-1$ linear functions $A_ix' + B_iy'$, $i \geq 2$; the latter form and linear functions have unit Hermitian norm: $|A_i|^2 + |B_i|^2 = 1$.

The sum of modules of the coefficients of the form ω in the new coordinates is no greater than 2^n. Indeed, it is no greater than

$$\prod_i (|A_i| + |B_i|), \ |A_i| + |B_i| \leq 2\sqrt{\frac{|A_i|^2 + |B_i|^2}{2}} = \sqrt{2}$$

(the classical quadratic mean inequality). Hence, the previous product (and thus, the sum of modules of coefficients) is no greater than 2^n.

Let us repeat the discussion from Case 1 in the coordinates (x', y'). Now our form is not necessarily monomial, and the previous upper bounds of the integral should be multiplied by the sum of modules of its coefficients. This together with the previous inequality implies the same upper bound (1.24) but with the right-hand side multiplied by 2^n. The right-hand side thus modified is less than that of the inequality in Theorem 1.52. This proves Theorem 1.52. □

Proof of Theorem 1.53. By assumptions, all the critical values of H have modules at most 2. Hence,

$$|t| \leq |\alpha| + 2.$$

Let us assume that $|t| > 5$, thus, $|\alpha| + 2 > 5$ (the opposite case is treated simpler and we get a stronger inequality than in the Theorem; this case will be briefly discussed at the end of the proof). We consider only the case when the coordinates in $\mathbb{C}^2$ satisfy the statements of Theorem 1.10: then afterwards one has only to check that the upper bound thus obtained remains less than that in Theorem 1.53 after multiplication by 2^n. This will imply the Theorem in the general case, as in the proof of Theorem 1.52.

By the Corollary of Theorem 1.10 and the previous inequality on $|t|$, the topology of the curve S_t is contained in a bidisk $D_{X,Y}$,

$$X \leq Y \leq R = R_0\left(\frac{|\alpha| + 2}{5}\right)^{\frac{1}{n+1}}, \ R_0 \text{ is the same as in Theorem 1.10.} \tag{1.25}$$

Then as in (1.23), by definition and Corollary 1.48,

$$\left|\int_\delta \omega\right| \leq R^{2n} 2^{l(n)\frac{|\alpha|+5\beta}{3\beta}}, \ l(n) = 24n^{12}. \text{ Thus,}$$

$$\left|\int_\delta \omega\right| < R_0^{2n}\left(\frac{|\alpha| + 2}{5}\right)^2 2^{24n^{12}\frac{|\alpha|+5\beta}{3\beta}}.$$

Substituting the value of R_0 from Theorem 1.10 together with elementary inequalities yields

$$\left|\int_\delta \omega\right| < 2^{10n^{12}\frac{|\alpha|+5}{\beta} - 3n}(c'(H))^{-28n^4}\left(\frac{2}{5}\right)^2. \tag{1.26}$$

The latter right-hand side (even being multiplied by 2^n) is less than that of (1.21). This proves (1.21).

Let us now consider the case, when $|t| \le 5$. Then the previous upper bounds of the integral hold, but now we have to replace the ratio $\frac{|\alpha|+2}{5}$ in (1.25) and the next inequality by 1. This implies that (1.26) holds with its right-hand side multiplied by $(\frac{5}{|\alpha|+2})^2 < (\frac{5}{2})^2$. Thus modified right-hand side (even if multiplied by 2^n) is again no greater than that of (1.21). Theorem 1.53 is proved. $\square$

Proof of Theorem 1.54. The proof of Theorem 1.54 repeats that of Theorem 1.53 with obvious change: we have to substitute the length estimate of vanishing cycle given by Corollary 1.49 (instead of 1.48) and $R = R_0 \max\{1, (\frac{|t_0|}{5})^{\frac{1}{n+1}}\}$ (by Corollary 1.11). $\square$

1.8 Lower bound of period determinant

Definition 1.55 (see [4]) A tuple $\Omega = (\omega_1, \dots, \omega_{n^2})$ of monomial 1- forms ω_i of the type $x^l y^{m+1} dx$ is called *standard*, if

- their degrees are no greater than $2n-1$;
- all the forms $x^l y^{m+1} dx$, $0 \le l+m \le n-1$, are contained there;
- the number of forms of degree $2n-k$ equals k for any $k = 1, \dots, n$.

The following Theorem was stated and used in [4].

Theorem 1.56 *Let H be a normalized ultra-Morse polynomial, Ω be a standard monomial tuple of forms. Let $\mathbb{I}(t)$, $\Delta(t)$ be respectively the corresponding Abelian integral matrix (1.1) and its determinant (1.2). The standard monomial tuple Ω can be chosen so that*

$$|\Delta(t)| > (c'(H))^{6n^3} (c''(H))^{n^2} n^{-62n^3}, \tag{1.27}$$

whenever t lies outside the $\frac{c''(H)}{4n^2}$- neighborhoods of the critical values of H.

Example 1.57 Let $(l(i), m(i))$ be a lexicographic sequence of pairs (l, m), $0 \le l, m \le n-1$, numerated by $i = 1, \dots, n^2$. Put

$$\omega_i = x^{l(i)} y^{m(i)+1} dx.$$

This is a standard form tuple, which follows from definition. If in the previous Theorem the highest homogeneous part of H equals $h(x, y) = x^{n+1} + y^{n+1}$, then the latter forms satisfy its inequality. The proof of this statement, which is omitted to save the space, can be easily derived from the results of this Subsection and Section 3. In general, one can choose a generic h in such a way that the determinant $\Delta_{h,\Omega}(t)$ constructed with the above ω_i be identically equal to 0 (this follows from results of [3]): in this case the form tuple should be changed.

The proof of Theorem 1.56 is based on the following explicit formula for the period determinant, see [3]:

$$\Delta(t) = C(h,\Omega)\prod_{i=1}^{n^2}(t-a_i), \tag{1.28}$$

$$C(h,\Omega) = C_n(\Sigma(h))^{\frac{1}{2}-n}P(h,\Omega), \text{ where} \tag{1.29}$$

$\Sigma(h)$ is the discriminant of the homogeneous polynomial h:

$$\text{if } h(x,y) = c_{-1}\prod_{i=0}^{n}(y-c_ix), \text{ then } \Sigma(h) = c_{-1}^{2n}\prod_{0\leq j<i\leq n}(c_i-c_j)^2, \tag{1.30}$$

$$P(h,\Omega) = \prod_{d=n}^{2n-2}P_d(h,\Omega),$$

where P_d are the polynomials from [3], whose definition is recalled below,

$$C_n = (-1)^{\frac{n(3n-1)}{4}}\frac{(2\pi)^{\frac{n(n+1)}{2}}(n+1)^{\frac{n^2+n-4}{2}}((n+1)!)^n}{\prod_{m=1}^{n-1}(m+n+1)!}. \tag{1.31}$$

Definition 1.58 ([3]). For any given polynomial 1- form ω put $D\omega$ to be the polynomial defined by the equality

$$d\omega = D\omega dx\wedge dy.$$

Definition 1.59 Let $n\geq 2$, $d\in\mathbb{N}$, $n\leq d\leq 2n-2$, h be a homogeneous polynomial of degree $n+1$. Let $\Omega(d) = (\omega_1',\dots,\omega_s')$ be an ordered tuple of homogeneous 1- forms of degree $d+1$, the number s of the forms being equal to $s=d+1$ in the case, when $d\leq n-1$, and $s=2n-d-1$ otherwise. The *matrix* $A_d(h,\Omega(d))$ *associated to the form tuple* $\Omega(d)$ is the $(d+1)\times(d+1)$ matrix whose columns are numerated by all the monomials $y^d, y^{d-1}x,\dots,x^d$ of degree d and the lines consist of the corresponding coefficients of the following polynomials:

Case $d\leq n-1$. Take the $d+1$ polynomials $\frac{D\omega_r'}{d-r+2}$.

Case $d\geq n$. Take the $d-n+1$ polynomials $x^jy^{d-n-j}\frac{\partial h}{\partial y}$, $0\leq j\leq d-n$; the $2n-d-1$ polynomials $\frac{D\omega_r'}{n-r+1}$; the $d-n+1$ polynomials $x^jy^{d-n-j}\frac{\partial h}{\partial x}$, $0\leq j\leq d-n$.

Let Ω be a standard form tuple, $n\leq d\leq 2n-2$, $\Omega(d)$ be the tuple of the forms in Ω of degree $d+1$ (numerated in the same order, as in Ω). The number s of forms in $\Omega(d)$ is equal to $2n-d-1$ by definition. Put

$$A_d(h,\Omega) = A_d(h,\Omega(d)),\ P_d(h,\Omega) = P_d(h,\Omega(d)) = \det A_d(h,\Omega(d)), \tag{1.32}$$

$$P(h,\Omega) = \prod_{d=n}^{2n-2}P_d(h,\Omega).$$

Now all the entries of formula (1.29) are defined.

As it will be shown below, Theorem 1.56 is implied by the following

Theorem 1.60 *Let h be a given generic homogeneous polynomial of degree $n+1 \geq 3$ with $||h||_{max} = 1$. Then one can choose a standard form tuple Ω so that*

$$C(h, \Omega) > (c'(h))^{6n^3} n^{-60n^3}. \tag{1.33}$$

Theorem 1.60 will be proved in Section 3. The principal part of its proof is the lower bound of $P_d(h, \Omega)$ given by the next Lemma. Its proof takes the most of Section 3. The upper bound of $\Sigma(h)$ and lower bound of C_n are proved in Section 3 using elementary inequalities and straightforward a priori estimates for h.

Lemma 1.61 *For any generic homogeneous polynomial h of degree $n+1 \geq 3$ with $||h||_{max} = 1$ and any $d = n, \dots, 2n-2$ one can choose a collection $\Omega(d) = (\omega_1, \dots, \omega_s)$ of $s = 2n-d-1$ monomial forms of the type $x^l y^{m+1} dx$, $l+m = d$, so that*

$$P_d(h, \Omega(d)) > n^{-44n^2} (c'(h))^{6n^2}. \tag{1.34}$$

Proof of Theorem 1.56. Let $t \in \mathbb{C}$ lie outside $\frac{c''(H)}{4n^2}$- neighborhoods of the critical values of H. Then by (1.33) and (1.28),

$$|\Delta(t)| > (c'(h))^{6n^3} n^{-60n^3} (\frac{c''(H)}{4n^2})^{n^2}.$$

This together with elementary inequalities implies (1.27) and proves Theorem 1.56 modulo Theorem 1.60. □

2 Upper bounds of topology. Proof of Theorems 1.10 and 1.17

In this Section we give a proof of Theorem 1.17 (the most part of the Section). In 2.2 we prove Theorem 1.10 and its Addendum. In 2.3 we prove Proposition 1.14 and some more precise a priori bounds for unit-scaled polynomials. These bounds are used in the proof of Theorems 1.10 and 1.17.

2.1 The plan of the proof of Theorem 1.17

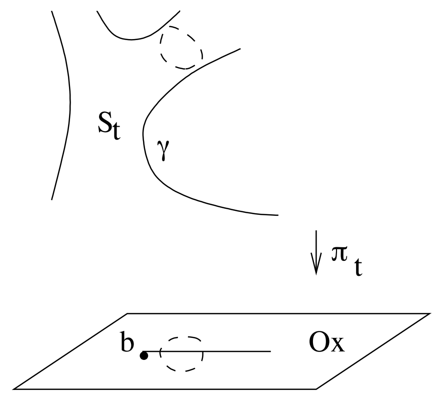
It suffices to prove Theorem 1.17 for ultra-Morse polynomials. The same statement in the general case then follows by passing to non ultra-Morse limit. Thus, everywhere below (except for Subsection 2.2 and whenever the contrary is specified) without loss of generality we consider that the polynomial H is ultra-Morse.

In the proof of Theorem 1.17 we use the following well-known and elementary topological properties of basic cycles in $H_1(S_t, \mathbb{Z})$.

Proposition 2.1 *Let H be a ultra-Morse polynomial, $t \in \mathbb{C}$ be a noncritical value, $\gamma \subset S_t$ be an embedded curve that joins two distinct points at the infinity line of the compactified curve $\overline{S}_t \subset \mathbb{CP}^2$. Then there exists a cycle in the homology group of the affine curve S_t that has a nonzero intersection index with γ (see Fig.2).*

Proof Take a cycle close to infinity and surrounding an end of γ. □

Corollary 2.2 *Given a ultra-Morse polynomial H, a noncritical value $t \in \mathbb{C}$ and a set of generators in $H_1(S_t, \mathbb{Z})$. Let $b \in Ox$ be a critical value of the projection $\pi_t : S_t \to Ox$. Then the projection of any representative of some generator intersects any ray issued from b, see Fig.2.*



A cycle intersecting a curve on S_t with infinite ends

Figure 2

Proof A generic ray issued from b has a pair of liftings to S_t under the projection, whose oriented union γ connects two distinct points at infinity of the compactified level curve (the orientations of the liftings are opposite). It follows from the previous Proposition that some generator has a nonzero intersection index with γ. Hence, the projection of any its representative intersects the ray. For nongeneric ray the same statement follows by passing to the limit. The Corollary is proved. □

We prove Theorem 1.17 by contradiction. Suppose the contrary: let there be a centrally-rescaled polynomial H whose all critical values are contained in the disc D_{δ_0}. Fix a noncritical value $t \in D_{\delta_0}$. Lemma 1.26 implies that the vanishing cycles along all the paths α from t to the critical value 0 generate the integer homology group of S_t. Then the same statement holds true for the paths α contained in D_{δ_0} (by the previous assumption on the critical values). Without loss of generality we assume that the coordinate system (x, y) under consideration is t- regular, see Definition 1.29. Then the latter paths, whose corresponding vanishing cycles generate the homology, may be chosen critically-regular by Remark 1.34. We show that in fact, the previous vanishing cycles cannot generate the homology (due to smallness of δ_0). The contradiction thus obtained will prove Theorem 1.17. To do this, we prove that there exist a critical value $b \in Ox$ of the projection π_t and a disk $D_r \subset Ox$ whose closure is disjoint from b, $|b| > r$, such that the canonical representatives of the previous vanishing cycles are all projected inside D_r. Hence, their projections do not intersect the ray issued from b away from 0. Thus, by Corollary 2.2, these cycles cannot generate the homology group.

By definition, the projection of the canonical representative of vanishing cycle is a piecewise-linear curve whose vertices are critical values of π_t. We have to show that someone of the latters lies outside some disc D_r, in particular, is distant from 0. The first step to do this is the next Lemma, which gives an a priori lower bound of the maximal distance between critical values of π_0. This is the main technical Lemma of the Section. Here the critical values are understood in the following generalized sense.

Definition 2.3 Let H be a ultra-Morse polynomial. A *generalized critical value* of the projection $\pi_t : S_t \to Ox$ is either a critical value of π_t (at a critical point where the curve S_t is regular), or the projection image of a critical point of H in S_t. For any t the set of the generalized critical values of the projection π_t will be denoted by CS_t, as in 1.4.

Remark 2.4 The projection image of a critical point of H is a double generalized critical value of the projection of the corresponding (critical) level curve, provided that the tangent lines to the local branches of this curve at the critical point are transversal to the y- axis. This follows from Proposition 1.27.

Proposition 2.5 *Let H be an ultra-Morse polynomial, and zero lines of its highest homogeneous part be transversal to the y- axis. Then for any $t \in \mathbb{C}$ the number of generalized critical values of π_t (with multiplicities) is equal to*

$$\eta(n) = n(n+1).$$

The Proposition follows from Bezout's theorem, as the similar statement of Remark 1.32.

Lemma 2.6 *Let H be a centrally-rescaled polynomial of degree $n+1 \geq 3$, (x, y) be orthogonal coordinates in $\mathbb{C}^2$ satisfying (1.8). There exists a generalized critical value $b \in CS_0$ such that*

$$|b| > r(n), \;\; r(n) = (c'(H))^{7n^2} n^{-35n^2}. \tag{2.1}$$

Lemma 2.6 is proved in 2.4.

Let ∂D_r be a circle that separates the previous value b from 0, i.e., $|b| > r$, and such that

$$dist(\partial D_r, CS_0) \geq \frac{r(n)}{2\eta(n)}, \;\; \eta(n) = n(n+1). \tag{2.2}$$

(Its existence follows from Proposition 2.5 and (2.1).) We show that the disk D_r is a one we are looking for. To do this, we prove the next Lemma.

Lemma 2.7 *Let H, (x,y), $r(n)$ be as in the previous Lemma, ∂D_r be a circle satisfying (2.2). Then the points from CS_t do not cross the circle, while t ranges in D_{δ_0}.*

Lemma 2.7 is proved in 2.5, where we also prove the following more general

Lemma 2.8 *Let H be a centrally-rescaled polynomial of degree $n+1 \geq 3$, (x,y) be orthogonal coordinates in $\mathbb{C}^2$ that satisfy (1.8). Fix arbitrary $x \in Ox \setminus CS_0$ and put*

$$\varepsilon = \min(dist(x, CS_0), 1). \text{ Then}$$

$$x \notin CS_t \text{ for any } t \in D_{\Delta(n,\varepsilon)}, \ \Delta(n,\varepsilon) = (c'(H))^{4n^3} n^{-17n^3} \varepsilon^{n(n+1)}. \tag{2.3}$$

The proofs of Lemmas 2.6 and 2.8 use a priori bounds from 2.3 for unit-scaled polynomials.

Proof of Theorem 1.17. The statement and the condition of the Theorem are invariant under orthogonal transformations in the preimage. Let us choose a noncritical value $t \in D_{\delta_0}$ and t- regular orthogonal coordinates in $\mathbb{C}^2$ (see Definition 1.29) that satisfy (1.8). Recall that the canonical representative of any cycle in S_t vanishing to 0 along a (critically-regular) path in D_{δ_0} is projected onto a piecewise-linear curve with vertices in CS_t. All the vertices lie in D_r (and hence, so does the projection itself also by convexity). For the local vanishing cycle this statement follows from definition. The same statement for the global vanishing cycle then follows from Lemma 2.7, convexity and continuity. This together with the discussion at the beginning of the Subsection proves Theorem 1.17 modulo Lemmas 2.6 and 2.7. □

2.2 From centrally-rescaled to weakly-normalized. Proof of Theorem 1.10 modulo Lemmas 2.6 and 2.7.

Here we deduce Theorem 1.10 and its Addendum from Theorem 1.17.

Let H be a weakly-normalized polynomial: then $||h||_{max} = 1$, and the critical values of H lie in the disk $\overline{D}_2$. Consider the auxiliary polynomial

$$\widetilde{H} = \lambda^{-(n+1)}(H(\lambda x, \lambda y) - H(0)), \ \lambda > 0, \text{ such that } ||\widetilde{H}'||_{max} = 1. \tag{2.4}$$

The possibility of choice of such λ follows from the condition saying that $H' - H(0) \not\equiv 0$, see the beginning of the paper. By construction, the new polynomial $\widetilde{H}$ is centrally-rescaled and has the same highest homogeneous part h. For the proof of Theorem 1.10 we prove the following upper bound of λ using Theorem 1.17.

Proposition 2.9 *Let λ be as in (2.4), δ_0 be as in Theorem 1.17. Then*

$$\lambda \leq \lambda_0 = (\frac{4}{\delta_0})^{\frac{1}{n+1}}. \tag{2.5}$$

Proof Let a_i, $\widetilde{a}_i = \lambda^{-(n+1)}(a_i - H(0))$ be the critical values of H and $\widetilde{H}$ respectively. By weak normalizedness, $|a_i| \leq 2$, and $H(0)$ is one of the a_i' s (in particular, $|H(0)| \leq 2$). This together with the previous formula implies that $|\widetilde{a}_i| \leq 4\lambda^{-(n+1)}$. On the other hand, there exists an i such that $|\widetilde{a}_i| \geq \delta_0$ (Theorem 1.17). The two latter inequalities imply (2.5). □

Denote $\widetilde{H}_{\widetilde{\lambda}}$ a polynomial given by formula (2.4) (with λ replaced by arbitrary $\widetilde{\lambda}$). Then $||\widetilde{H}'_{\lambda_0}||_{max} \leq 1$ (the previous Proposition and the monotonicity of the norm $||\widetilde{H}'_{\widetilde{\lambda}}||_{max}$ as a function in $\widetilde{\lambda}$). Thus, the polynomial $\widetilde{H}_{\lambda_0}$ is unit-scaled.

Let $t \in \mathbb{C}$, $|t| \leq 5$, $S_t = \{H = t\}$, $\widetilde{S}_\tau = \{\widetilde{H}_{\lambda_0} = \tau\}$, $\tau = \lambda_0^{-(n+1)}(t - H(0))$. By definition,

$$S_t = \lambda_0 \widetilde{S}_\tau, \ |\tau| < 5 \text{ (formula (2.4) and inequality } |H(0)| \leq 2).$$

By Proposition 1.14 applied to $\widetilde{H}_{\lambda_0}$, the topology of the curve $\widetilde{S}_\tau$ lies in the bidisk D_{X_n,Y_n}, see (1.9). This together with the previous formula implies that the topology of S_t lies in the bidisc $D_{X,Y}$, $X = \lambda_0 X_n$, $Y = \lambda_0 Y_n$. Now elementary inequalities imply that $X < Y < R_0$. This proves Theorem 1.10.

Now let us prove the Addendum to Theorem 1.10. We have proved above that $||\widetilde{H}'_{\lambda_0}||_{max} \leq 1$, $\lambda_0 = (\frac{4}{\delta_0})^{\frac{1}{n+1}}$. On the other hand, it follows from definition that $||\widetilde{H}'_{\lambda_0}||_{max} \geq \lambda_0^{-(n+1)}(||H'||_{max} - |H(0)|)$. This together with the previous inequality and the fact that $|H(0)| \leq 2$ and (1.10) yield

$$||H'||_{max} \leq \lambda_0^{n+1} + 2 = 4(c'(H))^{-13n^4} n^{63n^4} + 2 < (c'(H))^{-13n^4} n^{64n^4}.$$

This proves the Addendum.

2.3 A priori bounds of topology of unit-scaled polynomials

In the present Subsection we prove the following more precise version of Proposition 1.14.

Lemma 2.10 *Let $H(x,y) = h + H'$ be a unit-scaled polynomial (see Definition 1.13) of degree $n + 1 \geq 3$. Let the orthogonal coordinates in $\mathbb{C}^2$ satisfy (1.8). Then for any $t \in \mathbb{C}$ with $|t| \leq 5$ the topology of the level curve $S_t = \{H = t\}$ is contained in the bidisk D_{X_n,Y_n} (see Definition 1.8), where X_n, Y_n are the same, as in (1.9). Moreover, the complement $S_t \setminus D_{X_n,Y_n}$ is a union of graphs $y = y_i(x)$ of $n+1$ functions $y_0(x), \dots, y_n(x)$ holomorphic in $\mathbb{C} \setminus D_{X_n}$ such that*

$$|y_i(x) - y_j(x)| > \frac{c'(H)}{3n}|x| \text{ for any } x \in \mathbb{C} \setminus D_{X_n}, \ i \neq j. \tag{2.6}$$

Proof Consider the scalar product (,) on the two-dimensional vectors in $\mathbb{C}^2$ defined by the standard Hermitian metric. The y- axis is not a zero line of h by (1.8), so, the latter may be written as

$$h(x,y) = c_{-1}\prod_{i=0}^{n}(y - c_i x) = c_{-1}\prod_{i=0}^{n}(E, v_i),\ \ v_i = (-\bar{c}_i, 1),\ \ E = (x, y) \text{ is the Euler field.} \tag{2.7}$$

The zero lines of the homogeneous polynomial $h(x,y)$ are $y = c_i x$. As it is shown below, inequality (1.8) and unit-scaledness condition imply the following a priori estimates of the c_i' s:

$$(n+1)^{-\frac{n+1}{2}} < c_{-1} \leq 1, \tag{2.8}$$

$$|c_i| < \sqrt{n} \text{ for any } i = 0, \dots, n. \tag{2.9}$$

$$|c_i - c_j| > \frac{c'(H)}{n} \text{ for any } i, j \geq 0,\ \ i \neq j. \tag{2.10}$$

Afterwards, we prove the statement of the Lemma as follows. Fix an x, $|x| \geq X_n$, and consider the polynomial $h(x,y)$ as that with fixed x and variable y. The polynomial h has $n+1$ roots $c_i x$, $i \geq 0$. The distances between them are bounded from below by $\frac{c'(H)}{n}|x|$, see (2.10). Using this bound we show that if we add to h a polynomial H' of smaller degree and at most unit norm and then subtract a t, $|t| \leq 5$, then the new polynomial $P = h + H' - t$ has a root $y_i(x)$ in the δ- neighborhood of each $c_i x$, where

$$\delta = \frac{c'(H)}{3n}|x|. \tag{2.11}$$

(The proof of this statement is based on the next more general Proposition 2.11.) The previous neighborhoods are disjoint by (2.10), hence, the roots $y_i(x)$ are distinct and thus, depend holomorphically on $x \notin D_{X_n}$. Inequality (2.6) of the Lemma follows immediately from (2.10) and (2.11).

The Lemma says that the topology of the curve S_t, $|t| \leq 5$, is contained in D_{X_n, Y_n}. To prove this, now we have to show that $|y||_{\{(x,y)\in S_t,\ x\in D_{X_n}\}} < Y_n$. To do this, it suffices to prove the inequality

$$|y_i|(x) < Y_n, \text{ whenever } |x| = X_n,\ \ i = 0, \dots, n. \tag{2.12}$$

We will prove it at the end of the Subsection. Then the multivalued extensions of the y_i' s to the interior of D_{X_n} satisfy the same inequality by the maximum principle. This proves the Lemma.

Proof of (2.9). The distance of a zero line $l_i = \{y = c_i x\}$ of h to the y- axis is greater than $\frac{1}{\sqrt{n}}$ by (1.8). On the other hand, this distance is equal to $\arctan\frac{1}{|c_i|}$. This follows from the same statement in the case, when c_i is real (one can make an individual c_i real by applying a rotation in the coordinate x; the rotation preserves distances between lines). Therefore,

$$\arctan\frac{1}{|c_i|} > \frac{1}{\sqrt{n}}, \text{ hence } |c_i| < \frac{1}{|\tan\frac{1}{\sqrt{n}}|}. \tag{2.13}$$

This together with the classical inequality $\tan x > x$ implies (2.9). $\square$

Proof of (2.8). By unit-scaledness, $||h||_{max} = \max_{|x|^2+|y|^2=1}|h|(x,y) = 1$. The previous maximum of $|h|$ is no less than $|h|(0,1) = |c_{-1}|$. Hence, $|c_{-1}| \leq 1$, which proves the right inequality in (2.8).

Now let us prove the lower bound of c_{-1} from (2.8). Let v_i be the vectors from the expression (2.7) for h. By definition and the same formula (2.7),

$$1 = ||h||_{max} \leq |c_{-1}| \prod_{i=0}^{n} ||v_i||. \tag{2.14}$$

By definition and (2.9), $||v_i|| = \sqrt{1+|c_i|^2} < \sqrt{n+1}$. Substituting the latter inequality to (2.14) yields $1 < |c_{-1}|(n+1)^{\frac{n+1}{2}}$. This proves (2.8). □

Proof of (2.10). Fix $i \neq j \geq 0$. Consider the line $x = 1$ and its segment bounded by its intersections with the lines $y = c_i x$ and $y = c_j x$. By definition, the length of this segment is equal to $|c_i - c_j|$ and is greater than the angle between the two latter lines (since the previous length is no less than the tangent of the angle and by the inequality $\tan x > x$). The latter angle is equal to the distance between the lines, and hence, is no less than $\frac{c'(H)}{n}$ by definition. This proves (2.10). □

Proposition 2.11 *Let $p(y) = p_0(y-c_0)\dots(y-c_n)$ be a polynomial of degree $n+1$ in one variable, $q(y)$ be a polynomial of a smaller degree, $P(y) = p(y) + q(y)$. Let $\delta > 0$ be such that*

$$|c_i - c_j| > 2\delta \text{ for any } i \neq j, \tag{2.15}$$

$$\max_{|y| \leq \max_i |c_i| + \delta} |q(y)| < |p_0|\delta^{n+1}. \tag{2.16}$$

Then the polynomial $P(y) = p(y) + q(y)$ has a root in the δ- neighborhood of each root c_i of the polynomial $p(y)$.

Proof The statement of the Proposition holds true for the initial polynomial $p(y)$ with the roots c_i. To prove it for $P(y)$, we consider the auxiliary family of polynomials $P_s(y) = p(y) + sq(y)$, $s \in [0,1]$. Let us show that for any $s \in [0,1]$ the polynomial $P_s(y)$ does not vanish on the circles $|y - c_i| = \delta$ (whose closed disks are disjoint by (2.15)). This will prove that the roots of P_s (which depend continuously on s) will not leave the δ- neighborhoods of c_i, as s ranges from 0 to 1. It suffices to show that

$$|p(y)| > |q(y)|, \text{ whenever } min_i|y - c_i| = \delta. \tag{2.17}$$

The right-hand side $|q(y)|$ is less than $|p_0|\delta^{n+1}$ by (2.16). The left-hand side is greater than $|p_0|\prod|y-c_i| \geq |p_0|\delta^{n+1}$, since $|y-c_i| \geq \delta$. This proves (2.17) and the Proposition. □

Proof of (2.6). Fix an x, $|x| \geq X_n$. Let δ be the same, as in (2.11). The distance between the zeros of the polynomial $p(y) = h(x,y)$ is no less than 3δ by (2.10) and (2.11). Put $q(y) = H'(x,y) - t$. To show that the polynomial $P(y) = H(x,y) - t =$

$p(y)+q(y)$ has a root in the δ- neighborhood of each $c_i x$, let us apply Proposition 2.11 (here $p_0 = c_{-1}$). To check the conditions of the Proposition, it suffices to show that

$$|H'(x,y) - t| < |c_{-1}|\delta^{n+1} \text{ whenever } |y| \leq |x| \max_{i\geq 0} |c_i| + \delta, \ |t| \leq 5. \tag{2.18}$$

To prove the latter inequality, let us estimate its left-hand side (which is a polynomial of degree no greater than n) in terms of $||H'||_{max}$. It follows from definition that

$$|H'(x,y)| \leq ||H'||_{max}(|x|^2 + |y|^2)^{\frac{n}{2}} \leq ||H'||_{max}(|x| + |y|)^n, \text{ whenever } |x|^2 + |y|^2 \geq 1, \tag{2.19}$$

which is the case, since $|x| \geq X_n$. Substituting the inequality $||H'||_{max} \leq 1$ (unit-scaledness), the second inequality in (2.18), $\delta = \frac{c'(H)}{3n}|x|$ yields

$$|H'(x,y)| \leq (|x|(1 + \max_i |c_i| + \frac{c'(H)}{3n}))^n < (2|x|\sqrt{n})^n$$

by (2.9) and the inequality $c'(H) \leq 1$. Hence, $|H'(x,y) - t| \leq 2(2|x|\sqrt{n})^n$, since $|t| \leq 5$ and $|x| \geq X_n$.

Now for the proof of (2.18) it suffices to show that for $|x| \geq X_n$

$$2(2|x|\sqrt{n})^n < |c_{-1}|\delta^{n+1} = |c_{-1}|(\frac{c'(H)}{3n})^{n+1}|x|^{n+1}, \text{ i.e.,}$$

$$|x| > 2^{n+1}(\sqrt{n})^n(3n)^{n+1}|c_{-1}|^{-1}(c'(H))^{-(n+1)}. \tag{2.20}$$

The latter right-hand side is less than X_n by (2.8) and elementary inequalities. This proves (2.20) and (2.18). Inequality (2.6) is proved. □

Proof of inequality (2.12). For any x with $|x| = X_n$ one has $|y_i(x) - c_i x| < \delta = \frac{c'}{3n}X_n$, as it was proved before. This together with (2.9) and elementary inequalities yields (2.12). The proof of Lemma 2.10 is completed. □ □

2.4 Existence of distant branching points. Proof of Lemma 2.6

Let H be a centrally-rescaled polynomial. For simplicity denote

$$r = r(n) = (c')^{7n^2}n^{-35n^2}, \text{ see (2.1).} \tag{2.21}$$

Lemma 2.6 says that the projection $\pi : S_0 \to Ox$ has a generalized critical value with module greater than r. To prove this, we have to show that the functions $y_i(x)$, which give the roots of $H(x,y)$ (as a polynomial in y with fixed x), cannot have global holomorphic branches with disjoint graphs outside the disk $\overline{D_r}$. (A point of intersection of graphs is a critical point of H. Hence, its x- coordinate is a generalized critical value of the projection.)

Let us firstly sketch a proof of this statement in the simplest case, when the polynomial $H(x,y)$ under consideration is a product of $n+1$ linear (nonhomogeneous) functions: $H(x,y)=c_{-1}\prod_{i=0}^{n}(y-c_ix-b_i)$. We know that $H(0)=0$, so, at least one b_i is zero, say, b_0. On the other hand, H has nonconstant lower terms, moreover, their max- norm is unit. This implies that at least one b_i is not zero and the maximal module of the b_i's admits a lower bound (see (2.25) below, let b_1 have the maximal module). The line $y=c_1x+b_1$ intersects the line $y=c_0x+b_0=c_0x$ at a point with the x- coordinate $x_0=\frac{b_1}{c_0-c_1}$. An explicit calculation using (2.21), (2.10) and the lower bound for $|b_1|$ shows that $|x_0|>r$. Therefore, the graphs of the linear functions $y_i(x)=c_ix+b_i$ are not disjoint outside the disk $\overline{D_r}$.

Now let us prove the statement of the Lemma in the general case, when the functions $y_i(x)$ are not necessarily linear. If they are not holomorphic outside $\overline{D}_r$, then the Lemma follows immediately. Now suppose $y_i(x)$ are holomorphic outside $\overline{D}_r$. We show (using smallness of r) that that two of them (say, y_0 and y_1) have intersected graphs over the complement to $\overline{D}_r$. This together with the discussion at the beginning of the Subsection proves the Lemma.

By the previous holomorphicity assumption,

$$y_i(x)=c_ix+b_i+\phi_i(x),\ \phi_i(x)\to 0,\text{ as } x\to\infty,\ \phi_i(x)\text{ is holomorphic outside }\overline{D_r}. \quad (2.22)$$

Let X_n, Y_n be the constants from (1.9). Using Lemma 2.10, (2.22) and smallness of the radius r, we show that the functions $\phi_i(x)$ are small on a domain distant from D_r:

$$|\phi_i(x)|<\frac{3Y_nr}{|x|-r}\text{ outside }\overline{D_r}, \quad (2.23)$$

so, the functions $y_i(x)$ are close to the linear functions c_ix+b_i and the polynomial H is close to the product $c_{-1}\prod_{i\geq 0}(y-c_ix-b_i)$. As it will be shown below, this together with previous arguments in the case of product of linear functions imply that one of the b_i's, say b_0, is small, and the other one (say, b_1) is large:

$$\text{there exists a } b_i \text{ (say, } b_0) \text{ with } |b_0|<6Y_nr^{\frac{1}{n+1}}, \quad (2.24)$$

$$\text{there exists a } b_i \text{ (say, } b_1) \text{ with } |b_1|>\frac{1}{8n^2Y_n}, \quad (2.25)$$

We prove that the graphs of $y_0(x)$ and $y_1(x)$ are intersected over a point $x\notin\overline{D_r}$.

The graphs of the linear functions are intersected at a point with the x- coordinate

$$x_0=\frac{b_1-b_0}{c_0-c_1}.\text{ We show that } |x_0|>4r \quad (2.26)$$

(in fact, this follows from (2.10), (2.24), (2.25)). Then we show that the closeness of the $y_i(x)$' s to the linear functions implies that the difference $y_0(x)-y_1(x)$ vanishes at some point in the $\frac{|x_0|}{2}$- neighborhood of x_0. The latter neighborhood (and hence, the latter point) are disjoint from $\overline{D_r}$ by (2.26). This proves the Lemma.

In the proof of the inequalities mentioned above we use the following a priori bounds of y_i and b_i. The multivalued functions y_i satisfy the bound

$$|y_i(x)| \leq Y_n, \text{ whenever } |x| \leq X_n, \ i = 0, \dots, n \tag{2.27}$$

(Lemma 2.10, which says that the topology of S_0 is contained in D_{X_n,Y_n}). By the residue formula and (2.22),

$$b_i = \frac{1}{2\pi i}\int_{|x|=X_n} \frac{y_i(x)}{x} dx. \text{ Therefore,}$$

$$|b_i| \leq Y_n. \tag{2.28}$$

Proof of (2.23). The maximal value on $\overline{D_{X_n}}$ of the (multivalued) function ϕ_i is less than $3Y_n$:

$$|\phi_i(x)| = |y_i(x) - c_i x - b_i| \leq Y_n + |c_i|X_n + |b_i| \leq 2Y_n + \sqrt{n}X_n < 3Y_n$$

(by (2.27), (2.28), (2.9) and (1.9)). The function ϕ_i is holomorphic in $|x| > r$ and vanishes at infinity. Hence, the Cauchy formula implies that for any $x \notin \overline{D_r}$ one has

$$\phi_i(x) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\phi_i(\zeta)}{x-\zeta} d\zeta, \text{ so, } |\phi_i(x)| \leq \frac{r \max_{|\zeta|=r} |\phi_i(\zeta)|}{|x|-r}.$$

This together with the previous inequality implies (2.23). □

Proof of (2.24). By assumption, $H(x,y) = c_{-1}\prod_{i=0}^n (y - y_i(x)) = c_{-1}\prod_{i=0}^n (y - c_i x - b_i - \phi_i(x))$, $H(0,0) = 0$. Therefore, up to a polynomial vanishing at 0, H is equal to

$$(-1)^{n+1}c_{-1}(\prod_{i=0}^n b_i + \sum_i (\phi_i(x)\prod_{j\neq i} y_j(x))). \tag{2.29}$$

(In fact, the sum in (2.29) is a polynomial in x.) The polynomial (2.29) vanishes at 0, as does H. Therefore, the module of the constant term $\prod b_i$ should be no greater than the maximal module of the sum in (2.29) on any circle centered at 0, e.g., on $\partial D_1 \subset (\mathbb{C} \setminus \overline{D_r}) \cap D_{X_n}$. The latter maximal module is no greater than

$$(n+1)\max_i \max_{\partial D_1} |\phi_i| Y_n^n \leq \frac{3(n+1)Y_n^{n+1} r}{1-r} < 4r(n+1)Y_n^{n+1} \text{ (see (2.23)). Hence,}$$

$$\prod_{i=0}^n |b_i| < 4r(n+1)Y_n^{n+1}, \text{ so, } \min_i |b_i| < (4r)^{\frac{1}{n+1}}(n+1)^{\frac{1}{n+1}} Y_n < 6r^{\frac{1}{n+1}} Y_n$$

□

Proof of (2.25). By definition, $H(x,y) = c_{-1}\prod_{i=0}^n (y - c_i x - b_i - \phi_i(x))$, hence,

$$H'(x,y) = -\sum_{i=0}^n (b_i + \phi_i(x))\frac{H(x,y)}{y - y_i(x)}. \tag{2.30}$$

As it is shown below, this together with the equality $||H'||_{max} = 1$ implies that the b_i' s cannot be too small simultaneously. To do this, we use the following relation between the max- norm of a (nonhomogeneous) polynomial and the maximum of its module on the unit bidisc:

$$\text{for any polynomial } G(x,y),\ degG \leq n,\ \text{ one has } ||G||_{max} \leq 2n \max_{|x|,|y|\leq 1} |G(x,y)|. \quad (2.31)$$

Proof Let

$$G(x,y) = \sum g_{ij}x^iy^j. \text{ Then } \max_{|x|,|y|\leq 1} |G(x,y)| \geq \sqrt{\sum |g_{ij}|^2} \quad (2.32)$$

Indeed, the polynomial G can be considered as a Fourier polynomial in the harmonics of the torus $\{|x|,|y| = 1\}$, the polynomial coefficients coincide with the Fourier coefficients. The maximum of the module of the polynomial on the bidisk is equal to that (of the Fourier polynomial) on the torus. This follows from the one-dimensional maximum principle applied to G along the lines $x = const$, $y = const$. The maximal module of the Fourier polynomial is no less than its L_2- norm (divided by the square root of the area of the torus). This proves (2.32). On the other hand, it follows from definition that

$$||G||_{max} \leq \sum |g_{ij}|. \quad (2.33)$$

The total number of monomials of degree no greater than n is equal to $\frac{(n+1)(n+2)}{2}$. Now the classical mean inequality says that

$$\sum |g_{ij}| \leq \sqrt{\frac{(n+1)(n+2)}{2} \sum |g_{ij}|^2}.$$

This together with (2.32), (2.33) and elementary inequalities implies (2.31). □

Now it follows from (2.31) that

$$\max_{|x|,|y|\leq 1} |H'| \geq \frac{1}{2n}||H'||_{max} = \frac{1}{2n}. \quad (2.34)$$

On the other hand,

$$\max_{|x|,|y|\leq 1} |H'| = \max_{|x|=1,|y|\leq 1} |H'| \text{ (maximum principle).}$$

Now substituting to (2.30) the upper bounds (2.23), (2.27) of $\phi_i|_{|x|=1}$ and y_i respectively (and also the inequality $c_{-1} \leq 1$, see (2.8)), yields

$$\max_{|x|,|y|\leq 1} |H'| = \max_{|x|=1,|y|\leq 1} |H'| \leq (n+1)(\max_i |b_i| + \frac{3Y_nr}{1-r}) \max_i \max_{|x|=1,|y|\leq 1} |\frac{H(x,y)}{y-y_i(x)}|. \quad (2.35)$$

Let us firstly prove that for any i and any x, $|x| = 1$, one has

$$\max_{|y|\leq 1} |\frac{H(x,y)}{y-y_i(x)}| \leq 10^{\frac{n}{2}+1}. \quad (2.36)$$

Fix an x, $|x| = 1$, and consider the unit disc $D_1(y_0(x)) \subset Oy$ centered at $y_0(x)$. If it is disjoint from the unit disk centered at 0, then the previous fraction module maximum is no greater than the maximum of $|H|$ on the same unit disk (by definition). Otherwise, the same statement holds but for the maximums evaluated on the disk $D_3 \supset (D_1(y_0(x)) \cup D_1(0))$, since then $dist(\partial D_3, y_0(x)) > 1$. This together with the maximum principle imply that in both cases the fraction maximum from (2.36) is less than $\max_{|y|\leq 3}|H(x,y)| \leq \max_{|x|\leq 1, |y|\leq 3}|H(x,y)|$. The latter maximum is no greater than

$$||H||_{max} \max_{|x|\leq 1, |y|\leq 3}(|x|^2+|y|^2)^{\frac{n+1}{2}} = 2\times 10^{\frac{n+1}{2}} < 10^{\frac{n}{2}+1},$$

as in (2.19) (recall that $||h||_{max} = ||H||_{max} = 1$, hence $||H||_{max} = 2$). This proves (2.36)

Now substituting (2.36), (2.34) and the (elementary) inequality $\frac{3Y_nr}{1-r} < 4Y_nr$ to (2.35) yields

$$\frac{1}{2n} < 2n10^{\frac{n}{2}+1}(\max_i |b_i| + 4Y_nr), \text{ thus, } \max_i |b_i| > \frac{1-16n^2 10^{\frac{n}{2}+1}Y_nr}{4n^2 10^{\frac{n}{2}+1}}.$$

This together with elementary inequalities implies (2.25). □

Proof of (2.26). Thus, by (2.24), (2.25) we have $|b_0| < 6Y_nr^{\frac{1}{n+1}} < \frac{1}{8n^2Y_n} < |b_1|$. The intermediate inequality holds and moreover, its third term is at least twice greater than its second term (by elementary inequalities). Hence, $|b_1| > 2|b_0|$, thus, $|b_1 - b_0| > \frac{|b_1|}{2}$. Recall that $|c_1 - c_0| \leq |c_1| + |c_0| < 2\sqrt{n}$ by (2.9). This together with (2.25) and the previous inequality implies (2.26):

$$|x_0| = \frac{|b_1-b_0|}{|c_0-c_1|} > \frac{|b_1|}{2|c_0-c_1|} > \frac{1}{32n^2Y_n\sqrt{n}} = (32n^{\frac{5}{2}}Y_n)^{-1} > 4r. \tag{2.37}$$

□

Now for the proof of Lemma 2.6 it suffices to show that there exists a point x in the disk $D_{\frac{|x_0|}{2}}(x_0)$ such that $y_0(x) = y_1(x)$. To prove this, we use the argument principle: we consider the difference $y_1(x) - y_0(x)$ restricted to the circle $|x - x_0| = \frac{|x_0|}{2}$ and show that it vanishes nowhere and the increment of its argument along the circle is 2π. This is true for y_i replaced by the linear functions $c_ix + b_i = y_i(x) - \phi_i(x)$: the difference of the latters is equal to $(c_1 - c_0)(x - x_0)$. In the general case one has

$$y_1(x) - y_0(x) = (c_1 - c_0)(x - x_0) + (\phi_1(x) - \phi_0(x)).$$

The restriction to the circle of the difference of the linear functions (the first term in the right-hand side of the previous formula) has a constant module $|c_1 - c_0|\frac{|x_0|}{2}$. To show that the difference of the y_i' s has the same argument increment, as the previous first term, it suffices to prove that the second term (the difference of the ϕ_i's) has a smaller module:

$$|\phi_1(x) - \phi_0(x)| < |c_1 - c_0|\frac{|x_0|}{2} \text{ whenever } |x - x_0| = \frac{|x_0|}{2}. \tag{2.38}$$

For the proof of the latter inequality let us estimate its right-hand side. By (2.10), $|c_1 - c_0| > \frac{c'(H)}{n}$, hence,

$$|c_1 - c_0|\frac{|x_0|}{2} > \frac{c'(H)}{2n}|x_0|. \tag{2.39}$$

Now let us estimate from above the difference of the ϕ_i's on the circle $|x - x_0| = \frac{|x_0|}{2}$: by (2.23) and inequality $r < \frac{|x_0|}{4}$ (see (2.26)),

$$|\phi_1(x) - \phi_0(x)| \leq |\phi_0(x)| + |\phi_1(x)| < 2\frac{3Y_n r}{|x| - r} \leq \frac{6Y_n r}{\frac{|x_0|}{2} - r} < \frac{24Y_n r}{|x_0|}.$$

By (2.39), for the proof of (2.38) it suffices to check that $\frac{24Y_n r}{|x_0|} < \frac{c'}{n}\frac{|x_0|}{2}$, or equivalently,

$$r < c'(48nY_n)^{-1}|x_0|^2.$$

Indeed, by (2.37), the right-hand side of the latter inequality is greater than

$$c'(48nY_n)^{-1}(32n^{\frac{5}{2}}Y_n)^{-2} > c'(64nY_n)^{-1}(32n^{\frac{5}{2}}Y_n)^{-2} = c'2^{-16}n^{-6}Y_n^{-3} \geq c'n^{-22}Y_n^{-3}.$$

Recall that $Y_n = n^{8n}(c')^{-2n}$ by definition, hence, the previous right-hand side is equal to $(c')^{6n+1}n^{-24n-22} \geq (c')^{7n}n^{-35n} > r$. Inequality (2.38) is proved. The proof of Lemma 2.6 is completed.

2.5 Separation of branching points. Proof of Lemmas 2.7 and 2.8

Firstly we prove Lemma 2.8 and then deduce Lemma 2.7.

Let X_n, Y_n be the constants from (1.9). Let $y_i(x,t)$, $i = 0, \ldots, n$, be the roots of the polynomial $H(x,y) - t$ in y, which are multivalued holomorphic functions in (x,t) with branching at those points (x,t) where $x \in CS_t$. The conditions of Lemma 2.10 hold (H is unit-scaled), hence, the topology of S_t, $|t| \leq 5$, is contained in D_{X_n,Y_n}. This means that for any t, $|t| \leq 5$,

- the functions $y_i(x,t)$ are holomorphic in $x \notin \overline{D_{X_n}}$ and have disjoint graphs;
- for any branch of y_i one has $|y_i|(x) \leq Y_n$, whenever $|x| \leq X_n$.

The first statement implies that the set CS_t of generalized critical values lies in $\overline{D}_{X_n}$, whenever $|t| \leq 5$.

Now let us take any $x \in D_{X_n} \setminus CS_0$ (recall that $\varepsilon = \min(dist(x, CS_0), 1)$). Let us prove that then $x \notin CS_t$ whenever $|t| < \Delta(n, \varepsilon)$ (by definition, $\Delta(n,\varepsilon) < 5$). To do this, consider the discriminant (taken up to constant) of the polynomial H in y:

$$\Sigma_t(x) = \prod_{i<j}(y_i(x,t) - y_j(x,t))^2.$$

The zeros of Σ_t coincide with the generalized critical values of the projection of S_t. For any fixed $t \in \mathbb{C}$ $y_i(x,t) = c_i x(1+o(1))$, as $x \to \infty$. Therefore (denote $x_l(t)$ the roots of Σ_t, $l = 1, \dots, n(n+1)$),

$$\Sigma_t(x) = \prod_{0 \le i < j \le n} (c_j - c_i)^2 \prod_{l=1}^{n(n+1)} (x - x_l(t)). \tag{2.40}$$

Let $|t| < \Delta(n, \varepsilon)$ (in particular, $|t| < 4$). Let us show that x is not a zero of Σ_t. To do this, we use the following a priori lower bound of $\Sigma_0(x)$:

$$|\Sigma_0(x)| \ge (\prod_{0 \le i < j \le n} |c_j - c_i|^2) \varepsilon^{n(n+1)} > (\frac{c'(H)}{n} \varepsilon)^{n(n+1)} = (c'(H))^{n(n+1)} n^{-n(n+1)} \varepsilon^{n(n+1)}. \tag{2.41}$$

This follows from inequalities $|x - x_i| \ge \varepsilon$ (which hold by assumption), (2.40) and (2.10). To show that $\Sigma_t(x) \ne 0$ for small t, we estimate from above the derivative of $\Sigma_t(x)$ in t by using the following a priori upper bound of $\Sigma_t(x)$ valid whenever $|t| \le 5$ and $|x| \le X_n$:

$$\begin{aligned} |\Sigma_t(x)| &= \prod_{i<j} |y_i - y_j|^2 (x,t) \le (2Y_n)^{n(n+1)} \le (nY_n)^{n(n+1)} \\ &= n^{8n^2(n+1)+n(n+1)} (c')^{-2n^2(n+1)} < n^{16n^3} (c')^{-3n^3} \end{aligned} \tag{2.42}$$

(since $|y_i(x,t)| \le Y_n$ and by elementary inequalities).

The classical inequality on derivative of holomorphic function (which follows from Cauchy lemma) says that if $\psi(t)$ is a bounded function holomorphic in $t \in D_R$, $|\psi| < \mathcal{M}$, then $|\psi'(t)| \le \frac{\mathcal{M}}{R - |t|}$. Applying this inequality to the function $\Sigma_t(x)$, now with fixed x and variable $t \in D_4$, $R = 5$, $\mathcal{M} = n^{16n^3} (c')^{-3n^3}$ (see (2.42)), yields that for any t with $|t| < \Delta(n, \varepsilon) < 4$

$$|(\Sigma_t(x))'_t| \le \frac{\mathcal{M}}{5 - |t|} \le \mathcal{M} = n^{16n^3} (c')^{-3n^3}, \text{ hence,}$$

$$\begin{aligned} |\Sigma_t(x) - \Sigma_0(x)| \le \Delta(n,\varepsilon) n^{16n^3} (c')^{-3n^3} &= (c')^{4n^3} n^{-17n^3} \varepsilon^{n(n+1)} n^{16n^3} (c')^{-3n^3} \\ &= (c')^{n^3} n^{-n^3} \varepsilon^{n(n+1)} \end{aligned} \tag{2.43}$$

Now to prove that $\Sigma_t(x) \ne 0$ it suffices to show that the right-hand side of (2.43) is less than $\Sigma_0(x)$. But it is clearly less than the right-hand side in (2.41), which gives a lower bound of $|\Sigma_0(x)|$. This proves the inequality $\Sigma_t(x) \ne 0$ and Lemma 2.8.

Proof of Lemma 2.7. To show that CS_t does not meet ∂D_r whenever $|t| \le \delta_0$, we apply Lemma 2.8 to each point $x \in \partial D_r$. Then by (2.2)

$$\varepsilon = dist(x, CS_0) \ge \varepsilon' = \frac{r(n)}{2\eta(n)}, \ \eta(n) = n(n+1).$$

We show that

$$\Delta(n, \varepsilon') > \delta_0 \tag{2.44}$$

This together with the inequality $\Delta(n, \varepsilon) \geq \Delta(n, \varepsilon')$ (monotonicity) implies that $\Delta(n, \varepsilon) > \delta_0$. Together with Lemma 2.8, this yields $x \notin CS_t$ whenever $|x| = r$, $|t| \leq \delta_0$ and proves Lemma 2.7.

By (2.3),

$$\Delta(n, \varepsilon') = (c')^{4n^3} n^{-17n^3} \left(\frac{r(n)}{2n(n+1)}\right)^{n(n+1)} = (c')^{4n^3} n^{-17n^3} (r(n))^{n(n+1)} (2n(n+1))^{-n(n+1)}.$$

This together with formula (2.1) for $r(n)$ and elementary inequalities imply that the latter right-hand side (and hence, the Δ) is greater than

$$(c')^{4n^3} n^{-17n^3} (c')^{11n^4} n^{-53n^4} n^{-3n^3} \geq (c')^{13n^4} n^{-63n^4} = \delta_0.$$

This proves Lemma 2.7. □

3 Lower bounds of the formula for the main determinant. Proof of Theorem 1.60

Here we prove Theorem 1.60 (in 3.6), which gives a lower bound of $C(h, \Omega)$. To do this, we prove lower bounds of the terms of its formula (1.29). In 3.1-3.4 we prove Lemma 1.61 (lower bound of P_d). In 3.5 we prove an upper bound of the discriminant $\Sigma(h)$:

$$\Sigma(h) < n^{6n^2}. \tag{3.1}$$

In 3.7 we prove a lower bound of the constant C_n:

$$C_n > e^{-12n^2}. \tag{3.2}$$

3.1 Lower bound of P_d. The sketch of the proof of Lemma 1.61

By definition, $P_d = \det A_d(h, \Omega(d))$, where A_d is the $(d+1) \times (d+1)$- matrix defined in 1.8. Its lines are naturally identified with the vectors in the space of complex polynomials of degree $d+1$, which are split into two collections:

- the collection (denoted by Π) of $2(d-n+1)$ vectors $x^j y^{d-n-j} \frac{\partial h}{\partial y}$, $x^j y^{d-n-j} \frac{\partial h}{\partial x}$, which do not depend on the forms ω_i;

- the collection (denoted $\Pi_{\Omega(d)}$) of $2n-d-1$ vectors $\frac{D\omega_i}{n-i+1}$, $\omega_i = x^{l'(i)} y^{m'(i)+1} dx$, $l'(i) + m'(i) = d$, which depend only on $\Omega(d)$: by definition,

$$D\omega_i = (m'(i)+1) x^{l'(i)} y^{m'(i)}. \tag{3.3}$$

We have to prove a lower bound of the maximal value of P_d as a function of variable monomial form tuple $\Omega(d)$. Firstly we prove its next a priori lower bound in terms of the complex volume of the collection Π, see the following Definition.

Definition 3.1 Let $\Pi = \{v_1, \dots, v_k\} \in \mathbb{C}^m$ be a collection of vectors, $k \leq m$. Consider the real parallelogramm formed by the vectors v_j and iv_j, $j = 1, \dots, k$. The *complex volume* of Π (denoted $Vol\Pi$) is the square root of the real $2k$- dimensional standard Hermitian (Euclidean) volume of the previous parallelogramm.

Remark 3.2 The complex volume is nonzero, if and only if the vector collection is linearly independent over complex numbers. If $k = m$, then the complex volume is equal to the module of the determinant of the $m \times m$- matrix whose lines are formed by the components of the collection vectors.

Proposition 3.3 *Let h be a generic homogeneous polynomial of degree $n+1 \geq 3$, $d \in \{n, \dots, 2n-2\}$, Π be the corresponding vector collection from the beginning of the Subsection. There exists a tuple $\Omega(d)$ of $2n-d-1$ forms of the type $x^l y^{m+1} dx$, $l + m = d$, such that*

$$P_d(h, \Omega(d)) > n^{-4n} Vol\Pi. \tag{3.4}$$

The Proposition is proved at the end of the Subsection by elementary linear algebra arguments.

The principal part of the Section is the proof of lower bound of $Vol\Pi$: we show that if $||h||_{max} = 1$, then

$$Vol\Pi > n^{-42n^2} (c'(h))^{6n^2} \tag{3.5}$$

This together with Proposition 3.3 implies Lemma 1.61.

To prove (3.5), we consider the following space of two-dimensional vector polynomials:

$$V_s = \{v = (v_1(x,y), v_2(x,y)),\ degv_i = s,\ v_i \text{ are homogeneous}\},\ s \leq 2n-2, \tag{3.6}$$

equipped with the standard Hermitian scalar product. Denote $||v||_2$ the corresponding Hermitian norm. The space V_s has the standard orthonormal basis of $2(s+1)$ monomials $(x^i y^j, 0)$, $(0, x^i y^j)$, $i + j = s$.

Consider the linear operator

$$L : V_s \to V_{n+s},\ L(v) = \frac{dh}{dv} = v_1 \frac{\partial h}{\partial x} + v_2 \frac{\partial h}{\partial y} \tag{3.7}$$

Let $s \leq n-2$, $d = n+s$. By definition, the vectors of the collection Π are the images under L of the previous basic monomials. Denote

$$\nu(L) = ||L^{-1}|_{LV_s}||^{-1} = \min\{\frac{||Lv||_2}{||v||_2},\ v \in V_s \setminus 0\}.$$

It follows from definition that $Vol\Pi \geq (\nu(L))^{2(d-n+1)}$.

In Subsection 3.4 we show that $\nu(L) > n^{-21n}(c'(h))^{3n}$, or equivalently,

$$||Lv||_2 > n^{-21n}(c'(h))^{3n} \text{ for each } v \in V_s \text{ with } ||v||_2 = 1,\ s \leq n-2. \tag{3.8}$$

Proof of (3.5). The two previous inequalities imply that

$$Vol\Pi > (n^{-21n}(c'(h))^{3n})^{2(d-n+1)} \geq (n^{-21n}(c'(h))^{3n})^{2n} = n^{-42n^2}(c'(h))^{6n^2}.$$

This proves (3.5) modulo (3.8). □

To prove (3.8), we consider the following extension of the max-norm to the vector polynomials:

$$||v||_{max} = \max_{|x|^2+|y|^2=1} \sqrt{|v_1|^2+|v_2|^2}(x,y), \ v=(v_1,v_2)\in V_s. \tag{3.9}$$

Remark 3.4 The max- norm of a polynomial or of a polynomial vector field in $\mathbb{C}^2$ is invariant under orthogonal transformations, while the Hermitian norm isn't.

In 3.3 we prove an inequality similar to (3.8), but for the *max*- norm instead of the Hermitian norm:

$$||Lv||_{max} \geq (c'(h))^{3n} n^{-19n} ||v||_{max}. \tag{3.10}$$

This is the main technical inequality of the Section. Firstly we prove it for constant vector fields v. To prove it for a higher degree homogeneous polynomial vector field v, we choose appropriate orthogonal coordinates (x,y) (see Remark 3.4) and consider the space of vector polynomials in V_s proportional to the Euler vector field with polynomial coefficient:

$$V_{s,e} = \{v=(xQ(x,y), yQ(x,y))\in V_s \mid Q \text{ is a homogeneous polynomial of degree } s-1\}.$$

We decompose v as a sum

$$v = v'+v'', \ v'=(xQ,yQ)\in V_{s,e}, \ v''\perp V_{s,e}. \text{ One has } Lv = \frac{dh}{dv'}+\frac{dh}{dv''}. \tag{3.11}$$

We estimate from below the contributions of v' and v'' to Lv separately. Firstly we show that

$$||\frac{dh}{dv'}||_{max} \geq n^{-4n}||v'||_{max}. \tag{3.12}$$

Put

$$S=\{|x|^2+|y|^2=1\}.$$

On the zero lines of h one has $\frac{dh}{dv'}=0$ (by definition), hence, $\frac{dh}{dv}=\frac{dh}{dv''}$. We show that

$$\max|\frac{dh}{dv''}||_{S\cap\{h=0\}} \geq (c'(h))^{3n} n^{-12n}||v''||_{max} \text{ for any } v''\in V_s, \ v''\perp V_{s,e}. \tag{3.13}$$

This together with the previous statement implies that

$$||Lv||_{max} \geq (c')^{3n} n^{-12n} ||v''||_{max}. \tag{3.14}$$

We show that either the latter right-hand side already gives itself the desired lower bound (3.10) of $||Lv||_{max}$, or the contribution of v'' to Lv is dominated (many times) by that of v'. In the latter case lower bound (3.10) will be deduced from (3.12) and the following simple a priori bound (proved at the end of 3.3):

$$||\frac{dh}{dw}||_{max} \leq 2^{n+1}||w||_{max} \text{ for any } w \in V_s, s \in \mathbb{N} \cup 0, \text{ whenever } ||h||_{max} = 1. \quad (3.15)$$

In the proofs of the previously mentioned lower bounds of the derivatives of h along the vector fields v, v', v'' we use simple a priori bounds of coefficients of a polynomial in terms of its max- norm and the following relation between the Hermitian and the max- norms (these bounds and relation will be proved in 3.2):

$$\text{for any } s \in \mathbb{N}, Q \in V_s, \text{ one has } \frac{1}{\sqrt{s+1}}||Q||_{max} \leq ||Q||_2 \leq 2^{\frac{s}{2}}||Q||_{max}. \quad (3.16)$$

Proof of Proposition 3.3. Consider the collection $\Pi_{\Omega(d)}$ of the $2n-d-1$ lines (defined by $\Omega(d)$) of the matrix A_d. By definition, all the elements of the i-th line of the collection are zeros except for the one standing in the column numerated by the monomial $x^{l'(i)}y^{m'(i)}$. It follows from definition and (3.3) that the latter element equals $\frac{m'(i)+1}{n-i+1} \geq \frac{1}{n}$. These elements form a unique nonzero minor (denoted M) of maximal size in the collection $\Pi_{\Omega(d)}$, $|M| \geq n^{d+1-2n} > n^{-2n}$ by the previous inequality. The determinant $P_d = \det A_d$ is thus equal (up to sign) to M times the complementary minor (whose module will be denoted by M'), which is a maximal size minor in the collection Π. This together with the previous inequality implies that

$$P_d \geq n^{-2n}M'. \quad (3.17)$$

It follows from definition and Remark 3.2, that M' is equal to the complex volume of the orthogonal projection of Π along the coordinate plane generated by the monomials $x^{l'(i)}y^{m'(i)}$. We can choose the tuple $\Omega(d)$ so that the latter monomials be arbitrarily given $2n-d-1$ distinct monomials of degree d with unit coefficient, or equivalently, the complementary coordinate plane orthogonal to them be arbitrarily given coordinate plane of dimension $2(d-n+1)$. Let us choose $\Omega(d)$ so that the previous orthogonal projection of Π have the maximal possible complex volume M'.

The triangle inequality says that $Vol\Pi$ is no greater than the sum of the complex volumes of the orthogonal projections of Π to all the complex coordinate $2(d-n+1)$-planes in the space of degree d polynomials (which has dimension $d+1$). The number of all the latter planes is equal to

$$C_{d+1}^{2(d-n+1)} = \frac{(2n-d)\dots(d+1)}{(2(d-n+1))!} < (2n-d)^{2(d-n+1)} < n^{2n}.$$

Therefore, the maximal complex volume M' of projection is no less than $(C_{d+1}^{2(d-n+1)})^{-1}Vol\Pi > n^{-2n}Vol\Pi$. This together with (3.17) implies (3.4). $\square$

3.2 The max- norm and the coefficients. A priori bounds

In the proof of (3.8), (3.10) and (3.16) we use the following properties of the max-norm.

Proposition 3.5 *Let $g(x,y)$ be a complex homogeneous polynomial of degree k,*

$$g = g_0\prod_{i=1}^{k}((x,y),u_i),\ \ u_i\in\mathbb{C}^2,\ \ ||u_i||=1.\ \textit{Then}$$

$$||g||_{max}\leq|g_0|\leq||g||_{max}(2\sqrt{k})^k. \tag{3.18}$$

(Here the expression $((x,y),u_i)$ is the standard Hermitian scalar product (linear in (x,y)) of u_i and the Euler vector field (x,y).

Proof In the proof of Proposition 3.5 we use the following relation between distance of complex lines and the Hermitian scalar product.

Proposition 3.6 *Let u_1, u be vectors in $\mathbb{C}^2$, $||u_1||=||u||=1$. Let $l_1=u_1^{\perp}$ be the complex line orthogonal to u_1, λ be the complex line containing u. Then the Hermitian scalar product (u_1,u) admits the lower bound*

$$|(u_1,u)|\geq\frac{1}{2}dist(l_1,\lambda). \tag{3.19}$$

Proof It follows from definition that $|(u_1,u)|$ is equal to the length of the orthogonal projection of u along the line l_1. The latter length is equal to the sine of the minimal angle between u and a real line in l_1. The latter angle is no less than $dist(l_1,\lambda)$. The latter distance is always no greater than $\frac{\pi}{2}$ by definition. Thus,

$$|(u_1,u)|\geq\sin dist(l_1,\lambda)\geq\frac{1}{2}dist(l_1,\lambda).\text{ This proves (3.19).}$$

□

The left inequality in (3.18) follows from definition. Let us prove the right inequality. Denote $l_i=u_i^{\perp}$ the zero line of g orthogonal to u_i. By Proposition 1.12, there is a complex line λ through the origin such that $dist(\lambda,l_i)>\frac{1}{\sqrt{k}}$ for all i (let us fix such a line λ and a unit vector u on it). Then by (3.19),

$$|g(u)|=|g_0|\prod|(u,u_i)|,\ \ |(u,u_i)|\geq\frac{1}{2}dist(\lambda,l_i)>\frac{1}{2\sqrt{k}}.\text{ Hence,}$$

$$||g||_{max}\geq|g(u)|\geq|g_0|(\frac{1}{2\sqrt{k}})^k.$$

□

The max- norm of product:

$$||gQ||_{max} \geq (\frac{1}{2\sqrt{k+m}})^{k+m}||g||_{max}||Q||_{max} \text{ for any } k,m \in \mathbb{N} \tag{3.20}$$

and homogeneous polynomials $g, Q,\ k = deg(g), m = degQ$.

Proof Let us write down the decompositions preceding (3.18) of the polynomials g, Q, gQ (denote g_0, Q_0, $(gQ)_0$ the corresponding constant factors from their decompositions). By definition and the left inequality in (3.18), $|(gQ)_0| = |g_0||Q_0| \geq ||g||_{max}||Q||_{max}$. Applying the right inequality in (3.18) to gQ yields

$$||gQ||_{max} \geq |(gQ)_0|(\frac{1}{2\sqrt{k+m}})^{k+m}.$$

This together with the previous inequality proves (3.20). □

Hermitian and the max- norms: proof of (3.16). We use the inequality

$$\max_{|x|,|y|\leq 1}|Q(x,y)| \leq \max_{|x|,|y|\leq 1}(|x|^2+|y|^2)^{\frac{s}{2}}||Q||_{max} = 2^{\frac{s}{2}}||Q||_{max},\ degQ = s, \tag{3.21}$$

which follows from definition, as does (2.19). The right inequality in (3.16) follows from (3.21) and (2.32) (inequality (2.32) holds true both for scalar and vector polynomials). Let us prove the left one. It follows from definition that $||Q||_{max}$ is no greater than the sum of the modules of the vector coefficients of Q. The polynomial Q is homogeneous of degree s, hence, the number of its vector coefficients is $s+1$. By the mean inequality, the latter sum is no greater than the sum of the squared modules of the coefficients times the square root of their number $s+1$. This proves the left inequality in (3.16).

3.3 Lower bound of $||Lv||_{max}$. Proof of (3.10)

We choose the orthogonal coordinates (x,y) as in (1.8): the distance of the y- axis to each zero line of h is greater than $\frac{1}{\sqrt{n}}$.

Let us firstly prove (3.10) in the case, when v is a constant vector polynomial, i.e., $s = degv_1 = degv_2 = 0$ (we suppose that $||v||_2 = 1$, denote l the complex line containing v). Let $S = \{|x|^2+|y|^2 = 1\}$. By definition, the $\frac{c'(h)}{2n}$- neighborhoods in S of the $n+1 \geq 3$ zero lines of the polynomial h are disjoint. Therefore, the previous neighborhood of at least one zero line (denote the latter zero line by l_i) is disjoint from the line l. Let us choose a point $z \in S \cap l_i$ and show (below) that

$$|\frac{dh}{dv}|(z) > (c'(h))^{n+1}n^{-4n}. \tag{3.22}$$

The latter module of derivative is no greater than $||\frac{dh}{dv}||_{max} = ||Lv||_{max}$). This implies (3.10) in the particular case under consideration modulo (3.22). Let us prove the latter.

Let u_i be the vector from the expression for h in (3.18) that is orthogonal to the chosen zero line l_i: $(z,u_i)=0$. Then by Proposition 3.5,

$$\frac{dh}{dv}(z)=h_0(v,u_i)\prod_{j\neq i}(z,u_j),|h_0|\geq ||h||_{max}=1. \tag{3.23}$$

Let us estimate the factors in the latter right-hand side. By Proposition 3.6,

$$|(v,u_i)|\geq \frac{1}{2}dist(l_i,l),\ \ |(z,u_j)|\geq\frac{1}{2}dist(l_j,l_i).$$

The former distance is no less than $\frac{c'(h)}{2n}$, and the latter one is greater than $\frac{c'(h)}{n}$ by definition. Hence, by (3.23),

$$|\frac{dh}{dv}(z)|\geq\frac{c'}{4n}(\frac{c'}{2n})^n=(c')^{n+1}2^{-n-2}n^{-n-1}\geq(c')^{n+1}n^{-n-2-n-1}>(c')^{n+1}n^{-4n}.$$

This proves (3.22) and hence, (3.10) in the case of constant vector field v.

Now let us prove (3.10) in the case, when $1\leq s=degv_i\leq n-2$.

Proof of (3.12). By homogeneity,

$$Lv'=\frac{dh}{dv'}=Q(x,y)(x\frac{\partial h}{\partial x}+y\frac{\partial h}{\partial y})=(n+1)hQ. \tag{3.24}$$

Applying inequality (3.20) yields that $||Lv'||_{max}=(n+1)||hQ||_{max}$ is no less than $(\frac{1}{2\sqrt{n+s}})^{n+s}||h||_{max}||Q||_{max}$. By definition, $||h||_{max}=1$, $||v'||_{max}=||Q||_{max}$, hence,

$$||Lv'||_{max}\geq(\frac{1}{2\sqrt{n+s}})^{n+s}||v'||_{max}>(2\sqrt{2n})^{-2n}||v'||_{max}\geq n^{-4n}||v'||_{max}.$$

□

Proof of (3.10) modulo (3.13) and (3.15). By (3.12), (3.15) and the triangle inequality,

$$||\frac{dh}{dv}||_{max}\geq||\frac{dh}{dv'}||_{max}-||\frac{dh}{dv''}||_{max}\geq n^{-4n}||v'||_{max}-2^{n+1}||v''||_{max} \tag{3.25}$$

Let us consider the case, when $||v''||_{max}<n^{-6n}||v'||_{max}$. Then the latter inequality implies that the first term containing $||v'||_{max}$ in the right-hand side of (3.25) is at least twice greater than the second one. By the same inequality, $||v'||_{max}>\frac{1}{2}||v||_{max}$. Therefore,

$$||\frac{dh}{dv}||_{max}\geq\frac{1}{2}n^{-4n}||v'||_{max}>\frac{1}{4}n^{-4n}||v||_{max}\geq n^{-5n}||v||_{max}. \tag{3.26}$$

This proves (3.10) in the case under consideration.

Now let us consider the opposite case, when $||v''||_{max} \geq n^{-6n}||v'||_{max}$. Then

$$||v''||_{max}(1+n^{6n}) \geq ||v''||_{max} + ||v'||_{max} \geq ||v||_{max}.$$

This together with (3.14) (which follows from (3.13)) implies (3.10):

$$||\frac{dh}{dv}||_{max} \geq (c')^{3n}n^{-12n}||v''||_{max} \geq (c')^{3n}n^{-12n}(1+n^{6n})^{-1}||v||_{max} \geq (c')^{3n}n^{-19n}||v||_{max}.$$

□

In the proof of lower bound (3.13) of the contribution of v'' to Lv we use the following description of the subspace of V_s orthogonal to $V_{s,e}$:

a vector polynomial $v'' \in V_s$ is orthogonal to $V_{s,e}$, if and only if it has the form

$$v'' = r_1 y^s \frac{\partial}{\partial x} + r_2 x^s \frac{\partial}{\partial y} + R(x,y)(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}), \tag{3.27}$$

$$r_1, r_2 \in \mathbb{C},\ R \text{ is a homogeneous polynomial of degree } s-1.$$

The statement saying that each vector polynomial from (3.27) is orthogonal to $V_{s,e}$ follows from definition. The inverse statement follows from the coincidence of the dimensions of the space $V_{s,e}^{\perp}$ and the space of the vector polynomials from (3.27): the former dimension equals $dimV_s - dimV_{s,e} = 2(s+1) - s = s+2$; the latter one equals $s+2$ by definition.

Proof of (3.13). We choose appropriate zero line l_i of h (as follows) and show that

$$|\frac{dh}{dv''}||_{S\cap l_i} \geq (c')^{3n}n^{-12n}||v''||_{max}. \tag{3.28}$$

This will prove (3.13).

Consider the lines through the origin that are tangent to the vector field v''. They are defined by the equation $yv_1''(x,y) - xv_2''(x,y) = 0$ of degree $s+1$ (which does not hold identically, since $v'' \notin V_{s,e}$). Thus, the number of these lines is at most $s+1 \leq n$, which is less than the number $n+1$ of the zero lines of h. The $\frac{c'(h)}{2n}$- neighborhoods of the latters are disjoint by definition. Hence, at least one zero line of h (fix it and denote l_i) has distance no less than $\frac{c'}{2n}$ from the lines tangent to v''.

Let us prove (3.28). On the line l_i one has $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = (n+1)h = 0$, hence,

$$\frac{dh}{dv''}|_{l_i} = v_2''\frac{\partial h}{\partial y} - v_1''\frac{y}{x}\frac{\partial h}{\partial y} = (xv_2'' - yv_1'')x^{-1}\frac{\partial h}{\partial y}. \text{ Therefore,}$$

$$|\frac{dh}{dv''}||_{S\cap l_i} \geq |yv_1'' - xv_2''||\frac{\partial h}{\partial y}||_{S\cap l_i}. \tag{3.29}$$

Now we estimate from below the factors in the right-hand side of (3.29). The second factor admits the following lower bound:

$$|\frac{\partial h}{\partial y}||_{S\cap l_i} > (c')^{n+1}n^{-4n}. \tag{3.30}$$

This follows from (3.22) applied to $v = \frac{\partial}{\partial y}$. (The condition of inequality (3.22) saying that $dist(l, l_i) \geq \frac{c'}{2n}$ (in our case $l = Oy$) is satisfied by choice of the coordinates, see the beginning of the Subsection.) Now let us estimate from below the first factor in the right-hand side of (3.29): we show that

$$|yv_1'' - xv_2''|(z) \geq (c')^{2s} n^{-8s} ||v''||_{max} \text{ for any } z \in S \cap l_i. \tag{3.31}$$

This together with (3.29) and (3.30) implies (3.28), and hence, (3.13):

$$|\frac{dh}{dv''}|(z) \geq (c')^{2s} n^{-8s} ||v''||_{max} (c')^{n+1} n^{-4n} \geq (c')^{3n} n^{-12n} ||v''||_{max}, \quad z \in S \cap l_i.$$

For the proof of (3.31) let us firstly show that

$$||yv_1'' - xv_2''||_{max} \geq 2^{-2s} ||v''||_{max} \text{ for any } v'' \perp V_{s,e}. \tag{3.32}$$

Then using the assumption that the line l_i is distant from the zero lines of the polynomial $yv_1'' - xv_2''$ (which are tangent to v''), we prove (3.31).
Proof of (3.32). Firstly let us show that $||yv_1'' - xv_2''||_2 \geq ||v''||_2$. A vector polynomial $v'' \in V_{s,e}^{\perp}$ has the type (3.27), hence,

$$yv_1'' - xv_2'' = r_1 y^{s+1} - r_2 x^{s+1} + 2xyR(x, y).$$

The three terms in the latter right-hand side are orthogonal to each other, as are those in expression (3.27) for v''. Let us compare the Hermitian norms of the corresponding terms in the latter right-hand side and in (3.27). The norms of the first (second) terms are equal. The norm of the third one in the previous right-hand side is no less than the norm of that in (3.27):

$$||2xyR(x, y)||_2 = 2||R||_2 \geq \sqrt{2}||R||_2 = ||R(x, y)(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})||_2.$$

This implies that $||yv_1'' - xv_2''||_2 \geq ||v''||_2$. Now applying (3.16) twice yields

$$||yv_1'' - xv_2''||_{max} \geq 2^{-\frac{s+1}{2}} ||yv_1'' - xv_2''||_2 \geq 2^{-\frac{s+1}{2}} ||v''||_2 \geq 2^{-\frac{s+1}{2}} (s + 1)^{-\frac{1}{2}} ||v''||_{max}.$$

The total coefficient at $||v''||_{max}$ in the latter right-hand side is no less than 2^{-2s} (recall that we assume that $s \geq 1$). This proves (3.32). □

Proof of (3.31). Let $z \in S \cap l_i$. By (3.18) and (3.32), one has

$$(yv_1'' - xv_2'')(z) = q_0 \prod_{j=0}^{s} (z, u_j), \ ||u_j|| = 1, \ |q_0| \geq ||yv_1'' - xv_2''||_{max} \geq 2^{-2s} ||v''||_{max}.$$

Let $l_j' = u_j^{\perp}$ be a zero line of the polynomial $yv_1'' - xv_2''$. Recall that by the choice of l_i one has $dist(l_i, l_j') \geq \frac{c'}{2n}$ (see the beginning of proof of (3.13)). Hence, by Proposition 3.6,

$|(z,u_j)| \geq \frac{1}{2}dist(l'_j,l_i) \geq \frac{c'}{4n}$. This together with the previous formula and inequality implies that

$$|yv''_1 - xv''_2|(z) \geq 2^{-2s}||v''||_{max}(\frac{c'}{4n})^{s+1} = 2^{-4s-2}(c')^{s+1}n^{-s-1}||v''||_{max}$$

$$\geq (c')^{2s}n^{-8s}||v''||_{max}.$$

This proves (3.31). The proof of inequality (3.13) is completed. □ □

Proof of (3.15) (upper bound of derivative). Let w be a homogeneous vector polynomial. The restriction to the unit sphere of the module of the derivative $\frac{dh}{dw}$ is no greater than

$$||\nabla h||\sqrt{|w_1|^2+|w_2|^2} \leq ||\nabla h||_{max}||w||_{max} = ||(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y})||_{max}||w||_{max}. \tag{3.33}$$

Let us estimate from above the max- norm of the gradient of h, or equivalently, the maximal Hermitian norm of the gradient at points of the unit ball. To do this, we use the well-known fact that the module of the derivative of a holomorphic function (in one variable) at a given point is no greater than the maximal module of the function in the unit disk centered at the point under consideration. Therefore, the module of the derivative of h along any unit tangent vector (and hence, the Hermitian norm of the gradient) at a point of the closed unit ball is no greater than the maximal value of $|h|$ in the closed ball of radius 2. By definition, $||h||_{max} = 1$, hence,

$$|h(z)| \leq |z|^{n+1} \text{ for any } z \in \mathbb{C}^2. \text{ In particular,}$$

$$|h(z)| \leq 2^{n+1} \text{ whenever } |z| \leq 2.$$

Therefore, the max-norm of the gradient of h is no greater than 2^{n+1}. This together with (3.33) implies (3.15). □

3.4 From max- to Hermitian norm. Proof of (3.8)

Let $v \in V_s$, $s \leq n-2$, $||v||_2 = 1$. By (3.16) and (3.10),

$$||Lv||_2 \geq \frac{1}{\sqrt{n+s+1}}||Lv||_{max} > (2n)^{-\frac{1}{2}}(c')^{3n}n^{-19n}||v||_{max}$$

$$\geq n^{-2-19n}(c')^{3n}||v||_{max} \geq n^{-20n}(c')^{3n}||v||_{max}.$$

By (3.16), $||v||_{max} \geq 2^{-\frac{s}{2}}||v||_2 = 2^{-\frac{s}{2}} > 2^{-\frac{n}{2}}$. Hence, the right-hand side of the previous inequality is greater than

$$n^{-20n}2^{-\frac{n}{2}}(c')^{3n} > n^{-21n}(c')^{3n}.$$

This proves (3.8).

3.5 Upper bound of the discriminant. Proof of (3.1)

Consider a decomposition

$$h(x,y)=\prod_{i=0}^{n}((x,y),v_i) \tag{3.34}$$

of the polynomial h as a product of linear factors (not necessarily of unit norm). The vectors v_i are well-defined up to multiplications by constants (the product of the latter constants should be unit). Let $v_i=(v_{i,1},v_{i,2})$ be their components. We use the following formula for the discriminant Σ of h:

$$\Sigma=\prod_{i<j}(v_{i,1}v_{j,2}-v_{i,2}v_{j,1})^2. \tag{3.35}$$

Remark 3.7 The right-hand side of (3.35) depends only on h and does not depend on the normalization of the vectors v_i.

Consider a given pair of indices $i\neq j$ and the corresponding pair of vectors v_i, v_j. The module $|v_{i,1}v_{j,2}-v_{i,2}v_{j,1}|$ is equal to the complex volume of the vector collection (v_i,v_j) (Remark 3.2). Therefore, the previous module is no greater than $|v_i||v_j|$. Hence,

$$|\Sigma|\leq\prod_{i<j}(|v_i||v_j|)^2=\prod_{i\neq j}(|v_i||v_j|)=(\prod_i|v_i|)^{2n}. \tag{3.36}$$

Let us estimate the right-hand side of (3.36). To do this, consider the other product decomposition (preceding (3.18)) of the polynomial h. Let $h_0=g_0$ be the corresponding coefficient from (3.18). By definition and (3.18),

$$|h_0|=\prod_i|v_i|,\ \ |h_0|\leq||h||_{max}(2\sqrt{n+1})^{n+1}=(2\sqrt{n+1})^{n+1}.$$

The two previous inequalities imply that

$$|\Sigma|\leq((2\sqrt{n+1})^{n+1})^{2n}=(2\sqrt{n+1})^{2n^2+2n}<n^{2(2n^2+2n)}\leq n^{6n^2}.$$

This proves (3.1).

3.6 Lower bound of $C(h,\Omega)$. Proof of (1.33)

Recall formula (1.29) for $C(h,\Omega)$ from 1.8:

$$C(h,\Omega)=C_n(\Sigma(h))^{\frac{1}{2}-n}\prod_{d=n}^{2n-2}P_d,\ \ P_d=detA_d(h,\Omega).$$

Substituting inequalities (1.34), (3.1) and (3.2) to its right-hand side yields

$$|C(h,\Omega)|>e^{-12n^2}n^{6n^2(\frac{1}{2}-n)}(n^{-44n^2}(c')^{6n^2})^{n-1}\geq e^{-12n^2}n^{-6n^3-44n^3}(c')^{6n^3}$$
$$=n^{-12n^2(\ln n)^{-1}-50n^3}(c')^{6n^3}\geq n^{-(6(\ln 2)^{-1}+50)n^3}(c')^{6n^3}>n^{-60n^3}(c')^{6n^3}.$$

Theorem 1.60 is proved modulo (3.2).

3.7 Lower estimate of the constant C_n. Proof of (3.2)

In the proof of (3.2), we use the following inequalities:

$$\ln 2\pi > 1; \tag{3.37}$$

$$N\ln N - N + 1 < \sum_{k=1}^{N} \ln k < (N+1)\ln(N+1) - N \text{ for any } N \geq 2; \tag{3.38}$$

$$\sum_{k=a}^{b} k\ln k < (\frac{x^2\ln x}{2} - \frac{x^2}{4})|_a^{b+1} \text{ for any } a < b,\ a, b \in \mathbb{N}. \tag{3.39}$$

Inequality (3.38) follows from the inequality

$$\int_1^N \ln x dx < \sum_{k=1}^{N} \ln k < \int_1^{N+1} \ln x dx$$

and the statement that the previous integrals are equal respectively to the left- and right- hand sides of (3.38). The previous inequality follows from increasing of the function $\ln x$. Inequality (3.39) follows from the statement that its left-hand side is less than $\int_a^{b+1} x\ln x dx$ (increasing of the function $x\ln x$); the last integral is equal to the right-hand side of (3.39). By (1.31), one has

$$\ln|C_n| = \frac{n(n+1)}{2}\ln(2\pi) + \frac{n^2+n-4}{2}\ln(n+1) + n\sum_{k=1}^{n+1}\ln k - \sum_{m=1}^{n-1}\sum_{k=1}^{m+n+1}\ln k. \tag{3.40}$$

Let us estimate the first three terms in the right-hand side of (3.40). The first term is greater than $\frac{n^2}{2}$ by (3.37). The second one is greater than $\frac{n^2-2}{2}\ln n$. The third one (containing the sum till $n+1$) is greater that $n((n+1)\ln(n+1) - n) > n^2\ln n - n^2$ by (3.38). Substituting these inequalities to (3.40) and applying the right inequality in (3.38) to the inner sum of the double sum in (3.40) yields

$$\ln|C_n| > \frac{n^2}{2} + \frac{n^2-2}{2}\ln n + n^2\ln n - n^2 - \sum_{m=1}^{n-1}((m+n+2)\ln(m+n+2) - (m+n+1))$$

$$= \frac{3n^2}{2}\ln n - \frac{n^2}{2} - \ln n + \sum_{m=1}^{n-1}(m+n+1) - \sum_{k=n+3}^{2n+1} k\ln k. \tag{3.41}$$

Let us estimate the sums in the right-hand side of (3.41). The first sum is equal to

$$(n+1)(n-1) + \sum_{m=1}^{n-1} m = n^2 - 1 + \frac{n(n-1)}{2} = \frac{3n^2-n-2}{2}. \tag{3.42}$$

The second sum is estimated by inequality (3.39): it is less than

$$
\begin{aligned}
&(\frac{x^2\ln x}{2} - \frac{x^2}{4})|_{n+3}^{2(n+1)} \\
&= \frac{4(n+1)^2(\ln(n+1)+\ln 2)}{2} - \frac{(n+3)^2\ln(n+3)}{2} - (n+1)^2 + \frac{(n+3)^2}{4} \\
&< \frac{3(n+1)^2\ln(n+1)}{2} + 2(n+1)^2\ln 2 - (n+1)^2 + \frac{(n+3)^2}{4} \\
&< \frac{3(n+1)^2\ln(n+1)}{2} + \frac{(n+3)^2}{4} + (n+1)^2.
\end{aligned}
$$

Substituting the right-hand sides of (3.42) and the last inequality instead of the first (respectively, second) sum in (3.41) yields

$$
\begin{aligned}
\ln|C_n| > & \frac{3n^2\ln n}{2} - \frac{n^2}{2} - \ln n + \frac{3n^2-n-2}{2} \\
& -\frac{3(n+1)^2\ln(n+1)}{2} - (n+1)^2 - \frac{(n+3)^2}{4}.
\end{aligned} \tag{3.43}
$$

Let us simplify inequality (3.43). To do this, we use the following inequalities:

$$
\ln(n+1) < \ln n + \frac{1}{n};\ \ n+1 < 2n;\ \ n+3 \le \frac{5}{2}n. \tag{3.44}
$$

Let us estimate the fifth term in the right-hand side of (3.43). By (3.44), it is less than

$$
\frac{3(n+1)^2(\ln n + \frac{1}{n})}{2} < \frac{3(n+1)^2\ln n}{2} + 6n.
$$

The sixth term in the same place is $(n+1)^2 < 4n^2$, and the seventh one is less than $2n^2$ by (3.44). Substituting these estimates to (3.43) instead of the corresponding terms yields

$$
\begin{aligned}
\ln|C_n| > & \frac{3n^2\ln n}{2} - \frac{n^2}{2} - \ln n + \frac{3n^2-n-2}{2} - \frac{3(n+1)^2\ln n}{2} - 6n - 4n^2 - 2n^2 \\
&= \frac{3}{2}(n^2-(n+1)^2)\ln n + n^2 - \frac{n+2}{2} - \ln n - 6n - 6n^2 \\
&= -5n^2 - \frac{3}{2}(2n+1)\ln n - \ln n - \frac{13n+2}{2}.
\end{aligned}
$$

Substituting the inequality $\ln n < \frac{n}{2}$ (valid for $n \ge 2$) to the right-hand side of the previous inequality yields

$$
\ln|C_n| > -5n^2 - \frac{3}{4}(2n^2+n) - \frac{n}{2} - \frac{13n+2}{2} > -7n^2 - 9n > -12n^2.
$$

This proves (3.2).

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