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Feuilletages holomorphes, uniformisation et sous-groupes non libres dans les groupes de Lie

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Chapitre 1

Introduction

Le 16ème problème de Hilbert concerne les champs de vecteurs polynomiaux dans le plan réel. Un *cycle limite* est une orbite fermée isolée. Le problème est le suivant :

Est-il vrai que le nombre de cycles limites est toujours majoré par une constante ne dépendant que du degré maximal d'une composante du champ ?

Ce problème est ouvert et a une histoire riche de plus de 100 ans (voir [69]). Le meilleur résultat connu dit que pour tout champ polynomial, le nombre de cycles limites est fini. Cela fut démontré simultanément et indépendamment par J.Écalle [28] et Yu.S.Ilyashenko [66].

Dans les années 1950, I.G.Petrovskii et E.M.Landis [84] ont essayé de résoudre le 16ème problème de Hilbert. Leur démonstration s'est avérée fausse [62]. En même temps, ils ont suggéré une méthode intéressante : étudier un champ de vecteurs polynomial complexe sur $\mathbb{C}^2$ et ses orbites complexes, qui sont des surfaces de Riemann. Ces dernières orbites forment un feuilletage holomorphe singulier de $\mathbb{C}^2$. Les cycles limites du champ réel sont des cycles limites complexes de son complexifié : lacets non contractiles sur des orbites complexes dont l'holonomie (l'application de premier retour) est non triviale.

Il est bien connu, que les racines complexes d'une famille de polynômes de même degré sont continues en le paramètre, et leur nombre (avec multiplicités) reste constant (il est toujours égal au degré). Petrovskii et Landis ont essayé de démontrer que, grosso modo, les cycles complexes d'une famille de champs polynomiaux ont une propriété similaire.

L'étude du 16ème problème de Hilbert et les idées de Petrovskii et Landis ont motivé le développement de beaucoup de domaines dans la dynamique, l'analyse et la géométrie, en particulier,

- les feuilletages par des surfaces de Riemann et l'uniformisation de feuilles ;

- les intégrales abéliennes ;

- les invariants de classification analytique de germes d'applications conformes et de champs de vecteurs holomorphes, le phénomène de Stokes.

Yu.S.Ilyashenko a commencé l'étude des feuilletages holomorphes singuliers par des surfaces de Riemann à la fin des années 1960. Il a démontré qu'un champ de vecteurs polynomial non linéaire générique sur $\mathbb{C}^2$ d'un degré donné a toutes les orbites complexes denses, et un nombre dénombrable de cycles limites complexes [65, 60].

Dans les années 1960 D.V.Anosov a conjecturé que pour un champ polynomial générique toutes les orbites complexes sont simplement connexes (sauf pour un nombre dénombrable d'orbites). Cette conjecture est ouverte.

La plupart de mes travaux concernent les trois thèmes mentionnés ci-dessus. Mes travaux plus récents concernent

- les laminations horosphériques en dynamique holomorphe ;

- les sous-groupes non libres dans les groupes de Lie.

1.1 Résumé des travaux présentés dans ce mémoire

Ici toutes les citations sont données selon les Références à la fin du mémoire.

1.1.1 Uniformisation de feuilletages par des surfaces de Riemann (chapitre 2)

Pour étudier les cycles complexes, Ilyashenko a commencé (à la fin des années 1960) l'étude de l'uniformisation des feuilles (orbites) complexes. Le résultat d'uniformisation d'une feuille fixée est donné par le théorème classique de Poincaré et Köbe :

Théorème d'Uniformisation. *Toute surface de Riemann simplement connexe et non compacte est conformément équivalente ou bien à $\mathbb{C}$, ou bien au disque.*

Définition 0. Une surface de Riemann est *parabolique (hyperbolique)*, si son revêtement universel est conformément équivalent à $\mathbb{C}$ (resp. au disque).

À toute section transverse D simplement connexe, Ilyashenko a associé la réunion des revêtements universels des feuilles intersectant D : toute revêtement universel est celui d'une feuille avec un point marqué dans D. Cette réunion s'appelle la *variété de revêtement universels*. Pour les feuilletages holomorphes singuliers de dimension un sur une variété de Stein (par exemple $\mathbb{C}^n$), il a démontré que toute variété de revêtements universels munie de la structure complexe naturelle est une variété de Stein [63, 68]. Cette variété est un *cylindre tordu :* variété fibrée holomorphiquement au-dessus de D, dont les fibres sont des courbes holomorphes simplement connexes (revêtements universels), et qui admet une section holomorphe (donnée par D elle-même).

J'avais démontré dans ma thèse [35, 36, 37], que *toutes les feuilles d'un champ de vecteurs polynomial générique sur $\mathbb{C}^n$ sont hyperboliques.* J'avais aussi démontré un énoncé analogue pour les feuilletages sur une variété projective lisse arbitraire. Indépendamment et presqu'en même temps, des cas particuliers ont été démontrés dans le travail commun de A. Candel et X. Gómez-Mont [19] (plus tôt) et par A.Lins Neto [87]. Cela a donné une réponse à une question posée par Ilyashenko (fin des années 60).

Il est important de connaître la dépendance de l'uniformisation d'une feuille en le paramètre transverse. Le théorème classique de L. Bers [13] sur l'uniformisation simultanée concerne un feilletage holomorphe par des surfaces de Riemann compactes. Il dit que la variété fibrée de leurs revêtements universels est toujours *simultanément uniformisable :* biholomorphiquement équivalente à un ouvert de $\overline{\mathbb{C}} \times D$ fibré au-dessus de D par des domaines simplement connexes.

Ilyashenko a conjecturé (fin des années 1960), que toute variété de revêtements universels (et plus généralement, tout cylindre tordu Stein) est simultanément uniformisable. Il l'avait démontré dans un cas particulier, pour un feuilletage par des courbes algébriques compactes au voisinage d'une courbe invariante à singularités de Morse [64].

J'ai construit des contre-exemples [43, 44] : *des variétés de revêtements universels non simultanément uniformisables.* Celle de [43] est associée au feuilletage d'une surface (affine ou projective) par des courbes algébriques, pour une section transverse appropriée. Dans [44] j'ai montré *qu'il existe des surfaces complexes (tant affines que projectives) qui admettent un feuilletage holomorphe de dimension un à singularités isolées, dont aucune variété de revêtements universels n'est simultanément uniformisable. En plus, un tel feuilletage peut être choisi à feuilles denses et avec une structure affine transverse.*

Les résultats des articles [43, 44] sont présentés dans la section 2.3.

Presqu'en même temps j'avais étudié un autre problème sur une autre notion d'uniformisabilité simultanée, concernant les feuilletages (pas forcément holomorphes) par des surfaces de Riemann, où la structure complexe des feuilles est lisse en le paramètre transverse. Un exemple de base, introduit et partiellement étudié par É. Ghys [34], est un tore de dimension quelconque, feuilleté par des plans parallèles et muni d'une métrique riemannienne lisse g arbitraire. La métrique induit sur chaque feuille une structure complexe. Toute feuille est conformément équivalente à $\mathbb{C}$, et admet donc une métrique conforme plate et complète. Plus précisément, il existe sur chaque feuille une fonction positive lisse ϕ (unique à multiplication par une constante près), telle que la métrique ϕg de la feuille soit plate et complète.

É. Ghys a demandé si la fonction ϕ peut être choisie sur chaque feuille de sorte qu'elle soit lisse en le paramètre transverse. Il a démontré une réponse positive en dimension 3 dans des cas particuliers, quand ou bien les feuilles sont des cylindres, ou bien la pente du feuilletage vérifie une condition diophantienne [34].

Je l'ai démontré dans le cas général :

Théorème [45]. *Pour tout feuilletage d'un tore (de dimension quelconque) par des plans parallèles, et pour toute métrique g riemannienne lisse C^∞ sur le tore, il existe une fonction ϕ positive et lisse C^∞ sur le tore, telle que la restriction à toute feuille de la métrique ϕg soit plate.*

Dans le même article [45] j'ai obtenu d'autres résultats (positifs et négatifs) concernant d'autres feuilletages. Les résultats principaux de cet article sont présentés dans Sections 2.1 et 2.2.

La preuve de ce dernier théorème m'a permis d'obtenir une nouvelle démonstration du théorème de redressement d'une structure presque complexe lisse sur le tore de dimension deux [45, 51]. Avec des arguments classiques, cela donne une nouvelle démonstration [51] du théorème de C.B. Morrey, Jr. sur l'existence d'une application quasiconforme qui redresse une structure presque complexe bornée mesurable sur la sphère de dimension 2 [4, 94].

1.1.2 Laminations horosphériques en dynamique holomorphe (chapitre 3)

Les laminations (feuilletages topologiques) par des surfaces de Riemann et par des variétés hyperboliques apparaissent dans différents domaines des mathématiques, dont la dynamique d'itérations d'une fonction rationnelle :

$$f = \frac{P}{Q} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}.$$

En 1985 D. Sullivan [110] a introduit un dictionnaire entre deux domaines de la dynamique complexe : les itérations de fonctions rationnelles $f(z) = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ sur la sphère de Riemann et la théorie des groupes kleiniens. Ces derniers sont les sous-groupes discrets du groupe d'automorphismes conformes $PSL_2(\mathbb{C})$ de la sphère de Riemann. Ce dictionnaire a motivé beaucoup de résultats remarquables dans les deux domaines, en commençant par le célèbre théorème de Sullivan sur l'absence de composantes errantes dans la théorie des itérations de fonctions rationnelles.

L'un des objets principaux dans l'étude des groupes kleiniens est la variété hyperbolique de dimension trois associée à un groupe kleinien. C'est le quotient de l'espace hyperbolique $\mathbb{H}^3$ par l'action du groupe agissant par isométries. M. Lyubich and Y. Minsky ont suggéré d'étendre le dictionnaire de Sullivan par une construction analogue pour les itérations de fonctions rationnelles. À toute fonction rationnelle f, ils ont associé une *lamination hyperbolique* $\mathcal{H}_f$ (voir [89] et le Chapitre 3 de ce mémoire). C'est un espace topologique feuilleté par des orbifolds hyperboliques de dimension trois (qui peuvent avoir des singularités coniques), vérifiant les propriétés suivantes :

- tout point non singulier possède un voisinage homéomorphe au produit d'une partie d'un ensemble de Cantor par la boule de dimension trois ;
- la métrique hyperbolique des feuilles est continue en le paramètre transverse ;
- il existe une projection naturelle $\mathcal{H}_f \to \overline{\mathbb{C}}$, qui relève l'action $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (non inversible) à l'action par un homéomorphisme $\hat{f} : \mathcal{H}_f \to \mathcal{H}_f$, qui est une isométrie sur les feuilles ;
- l'action de $\hat{f}$ est proprement discontinue.

Le quotient $\mathcal{H}_f/\hat{f}$ est donc un "joli" espace topologique laminé par des orbifolds hyperboliques de dimension trois ; l'espace $\mathcal{H}_f/\hat{f}$ s'appelle *la lamination hyperbolique quotient*.

La lamination hyperbolique $\mathcal{H}_f$ est construite de la manière suivante. Prenons l'extension naturelle $\hat{f}$ de la dynamique de f à l'espace $\mathcal{N}_f$ de toutes les demi-orbites négatives :

$$\mathcal{N}_f = \{\hat{z} = (z_0, z_{-1}, z_{-2}, \dots) \mid z_{-j} \in \overline{\mathbb{C}},\ f(z_{-j-1}) = z_{-j}\};$$

$$\hat{f} : \mathcal{N}_f \to \mathcal{N}_f,\ (z_0, z_{-1}, \dots) \mapsto (f(z_0), z_0, z_{-1}, \dots).$$

Ce dernier espace contient toujours beaucoup de surfaces de Riemann conformément équivalentes à $\mathbb{C}$. La réunion de toutes cettes surfaces (notée $\mathcal{A}_f^n$) est invariante par la dynamique relevée $\hat{f}$. La lamination hyperbolique est obtenue par le recollement d'une copie de l'espace hyperbolique $\mathbb{H}^3$ et de chaque surface précedante (eventuellement effacée), suivi d'un raffinement de la topologie et d'une completion appropriées.

Des travaux récents sur les variétés hyperboliques associées à des groupes kleiniens ont abouti à la solution de tous les grands problèmes de la théorie, y compris une solution positive à la célèbre conjecture d'Ahlfors sur la mesure de l'ensemble limite. Ce résultat est le fruit des efforts de nombreux mathématiciens, voir les articles [3, 17] et leurs reférences. D'un autre côté, tout récemment, une conjecture analogue pour la théorie des itérations rationnelles s'est révelée fausse. X. Buff et A. Chéritat [16] ont construit des exemples de polynômes quadratiques avec ensembles de Julia de mesure positive, en utilisant une méthode complétement différente, proposée par A. Douady.

Il est espéré, que l'étude des laminations hyperboliques associées à des fonctions rationnelles éclairera la dynamique sous-jacente d'une nouvelle manière.

J'ai étudié l'arrangement d'*horosphères* dans l'espace quotient $\mathcal{H}_f/\hat{f}$ (voir les articles [48, 49]). Rappelons leur définition. L'espace hyperbolique $\mathbb{H}^3$ avec un point marqué "∞" sur sa frontière (qui est la sphère de Riemann) admet pour modèle standard le demi-espace dans l'espace euclidien de dimension trois. Ses isométries fixant l'infini sont exactément les extensions des transformations affines complexes de la frontière. Un plan horizontal dans le demi-espace s'appelle un *horosphère*. Les isométries hyperboliques de $\mathbb{H}^3$ fixant l'infini transforment des horosphères en des horosphères. Les *horosphères du quotient de* $\mathbb{H}^3$ par l'action d'un groupe discret de telles isométries sont les images des horosphères de $\mathbb{H}^3$ par la projection naturelle (toutes ces horosphères portent des structures euclidiennes induites par la restriction de la métrique hyperbolique). Ces structures euclidiennes peuvent avoir des singularités coniques. Les horosphères feuillettent l'orbifold hyperbolique ambiant.

Les feuilles des laminations $\mathcal{H}_f$ et $\mathcal{H}_f/\hat{f}$ sont aussi des quotients de $\mathbb{H}^3$ par un groupe d'isométries fixant l'infini. Toutes leurs feuilles sont donc feuilletées par des horosphères bien définies.

L'arrangement des horosphères dans $\mathcal{H}_f$ et dans son quotient $\mathcal{H}_f/\hat{f}$ est lié au comportement du cocycle des dérivées $|Df^n(z)|$ des itérations de la fonction rationnelle f.

La lamination de $\mathcal{H}_f$ par variétés hyperboliques est toujours minimale (toute feuille hyperbolique est dense), sauf pour le cas d'une fonction rationnelle ayant une orbite periodique répulsive"à ramification exceptionnelle" (pour exemple, Chebyshev ou Lattès, voir la Définition 3.1.10 dans le Chapitre 3). Dans ce dernier cas il y a des feuilles isolées, dont le nombre est toujours fini. Notons

$$\mathcal{H}'_f = \mathcal{H}_f \setminus (\text{feuilles hyperboliques isolées}).$$

La lamination hyperbolique de $\mathcal{H}'_f$ est toujours minimale.

M. Lyubich et V. Kaimanovich ont démontré, que si f appartient à la liste suivante :

$$\text{Lattés, Chebyshev, } z^{\pm d},$$

alors aucune horosphére dans $\mathcal{H}'_f/\hat{f}$ n'est dense.

J'ai démontré (en [48, 49]) une sorte de réciproque :

- si f n'appartient pas à la liste ci-dessus, alors il y a une infinité d'horosphères (explicitement présentées) denses dans le quotient $\mathcal{H}'_f/\hat{f}$ ou autrement dit, la lamination horosphérique de $\mathcal{H}'_f/\hat{f}$ est topologiquement transitive.

- si f n'appartient pas à la liste ci-dessus, et de plus est critiquement non récurrent et sans orbite périodique parabolique, alors la lamination horosphérique de $\mathcal{H}'_f/\hat{f}$ est minimale : toutes les horosphères sont denses.

Les énoncés analogues sout faux pour la lamination horosphérique de l'espace non factorisé $\mathcal{H}'_f$, déjà pour des polynômes quadratiques réels [49].

1.1.3 Sous-groupes non libres dans les groupes de Lie (chapitre 4)

Il est connu que dans un groupe de Lie dont la composante neutre est non résoluble, un couple générique (au sens de la mesure de Haar) d'éléments engendre un sous-groupe libre [29]. J'ai démontré que si, en plus, ce sous-groupe libre n'est pas discret, alors il est instable : il existe des paires arbitrairement proches qui engendrent des sous-groupes non libres.

Théorème [50]. *Soit G un groupe de Lie non résoluble, $(A, B) \in G \times G$ un couple d'élements engendrant un sous-groupe libre non discret. Alors il existe une suite $(A_k, B_k) \to (A, B)$ et une suite de mots $w_k(a, b)$ non triviaux en deux symboles abstraits (et leurs inverses) tels que $w_k(A_k, B_k) = 1$ pour tout k.*

Ce théorème répond à une question d'É. Ghys, qui a proposé d'étudier le taux d'approximation d'une paire (A, B) comme ci-dessus par des générateurs de sous-groupes non libres, dont la longueur minimale d'une relation soit donnée. Il y a une conjecture qui dit, que pour une paire (A, B) "générique", le taux optimal d'approximation est exponentiel en cette dernière longueur.

Dans le même article [50] j'ai obtenu une majoration du taux optimal, qui est exponentielle en une puissance de la longueur minimale d'une relation.

1.1.4 Intégrales abéliennes et géométrie algébrique quantitative (chapitre 5)

Tout champ de vecteurs polynomial sur le plan réel peut s'écrire comme un champ de droites de zéros d'une 1- forme à coefficients polynomiaux. Un champ de droîtes tangentes à un champ vectoriel hamiltonien polynomial s'écrit comme

$$dH = 0, \text{ ou } H \text{ est le hamiltonien.}$$

Les orbites fermées d'un champ hamiltonien forment une famille continue d'ovales : courbes fermées (non singulières) dans les courbes de niveau de l'hamiltonien.

Aucune borne uniforme pour le nombre de cycles limites n'est connue pour les champs polynomiaux proches des champs hamiltoniens (sauf pour les champs quadratiques ; un survol des résultats partiels avec références est présenté dans [69]), par exemple, dans une famille en un paramètre ε du type

$$dH + \varepsilon\omega = 0, \ \omega = A(x, y)dx + B(x, y)dy, \ degA, degB < degH.$$

Un ovale $\gamma \subset \{H = t\}$ du champ hamiltonien ($\varepsilon = 0$) peut engendrer un cycle limite du champ perturbé ($\varepsilon \neq 0$), seulement dans le cas où le niveau correspondant t est un zéro d'une fonction $I(t)$ spéciale : l'intégrale abélienne

$$I(t) = \int_\gamma \omega.$$

Cette dernière se prolonge à une fonction holomorphe sur le revêtement universel au-dessus du complémentaire de l'ensemble des valeurs critiques complexes de H.

Dans mon travail commun avec Yu.S.Ilyashenko ([52, 53]), nous avons obtenu une majoration explicite du nombre de zéros d'une intégrale abélienne pour un hamiltonien polynomial d'un degré arbitraire, de telle sorte, que :

- les droites complexes de zéros de la partie homogène supérieure sont distinctes (i.e. la partie supérieure est non dégénérée) et ne sont pas trop proches l'une de l'autre ;

- les valeurs critiques complexes sont distinctes, et la distance minimale entre deux n'est pas trop petite par rapport à la distance maximale.

Cette majoration est exponentielle en $(degH)^4$. C'est la meilleure majoration connue jusqu'à présent.

La preuve de cette majoration est basée sur une idée de Ilyashenko, la théorie de Picard-Lefschetz et mes résultats [46, 47] obtenus au cours de notre travail. Ces résultats concernent les courbes de niveau d'un polynôme complexe en deux variables, dont la partie homogène supérieure est non dégénérée.

Le résultat principal de [47] donne une formule explicite pour le déterminant d'une matrice d'intégrales abéliennes des 1- formes monomiales formant une base, le long d'une base des cycles engendrant l'homologie d'une courbe de niveau.

Il est connu que les racines et les points critiques d'un polynôme complexe unitaire normalisé admettent une borne supérieure explicite. "Normalisé" signifie, que zéro est un point critique, et toutes les valeurs critiques sont dans le disque unité.

Les résultats de [46], qui appartiennent à la "géométrie algébrique quantitative", étendent cet énoncé aux polynômes en deux variables (convenablement normalisés d'une manière analogue). Le théorème principal donne une majoration du rayon minimal d'un bidisque centré en zéro, qui contient toute la topologie non triviale d'une courbe de niveau donnée.

1.1.5 Confluence de points singuliers et phénomène de Stokes (chapitre 6)

L'holonomie (l'application de premier retour) d'un cycle limite d'un feuilletage holomorphe de codimension un est un germe d'application conforme $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ à point fixe 0. Un germe est *parabolique* s'il est tangeant à l'identité en 0 et différent de l'identité. La classification analytique (i.e. modulo conjugaison conforme) de germes paraboliques a été obtenue simultanément et indépendamment par J. Écalle [27] et S.M. Voronin [117]. Leurs invariants analytiques sont la forme normale formelle et une collection finie de germes conformes $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Cette dernière collection s'appelle le *module d'Écalle-Voronin*.

La théorie des invariants d'Écalle-Voronin est un analogue non linéaire de la théorie classique (développée dans les années 1970) des équations différentielles ordinaires linéaires en temps complexe à points singuliers irréguliers. Considérons, par exemple, une équation différentielle

$$\dot{z} = A(t)z, \ z \in \mathbb{C}^n,$$

où $A(t)$ est une fonction matricielle méromorphe. Un *point singulier* d'une telle équation est un pôle de $A(t)$. Il est *de type Fuchs* si c'est un pôle simple. Il est *irrégulier* si une certaine solution croît exponentiellement le long d'un secteur à sommet en le point singulier. La classification analytique de germes d'équations linéaires à points singuliers irréguliers a été obtenue par W. Balser, W. Jurkat, D. Lutz, A. Peyerimhoff, Y. Sibuya [10, 75, 107]. Les invariants analytiques sont la forme normale formelle et une collection d'opérateurs linéaires unipotents agissant dans les espaces de solutions au-dessus de secteurs appropriés. Ces derniers opérateurs s'appellent *les opérateurs de Stokes*.

Dans les années 1980 V.I. Arnold a proposé d'étudier une équation à point singulier irrégulier comme une limite d'équations à points singuliers Fuchsiens, qui confluent. Il avait conjecturé que certains opérateurs de monodromie de l'équation perturbée (Fuchsienne) convergent vers des opérateurs de Stokes. Une question proche a été posée et partiellement étudiée par J.-P. Ramis [104] (voir l'article [41] pour un survol de résultats partiels avec références).

Dans les articles [38, 40, 41, 42], j'ai obtenu des résultats qui relient la monodromie limite avec les opérateurs de Stokes dans le cas non résonnant général et dans certains cas résonnants. Dans [39], j'ai obtenu des analogues non linéaires de ces résultats, en particulier pour les germes paraboliques et leurs invariants d'Écalle-Voronin. Ces résultats, avec une esquisse de démonstration et un survol historique, sont présentés dans chapitre 6.

1.2 Résumé des travaux non présentés dans ce mémoire

Ici toutes les citations sont données selon la liste de publications personnelles dans la section 1.3.

Dans [1] j'ai obtenu la description combinatoire (analogue à celle de Lyashko et Loojienga) du revêtement de l'espace des polynômes complexes "équivariants" en une variable au-dessus des collections de leurs valeurs critiques.

Les travaux [2], [5], [7] et [22] font partie de ma thèse de doctorat et concernent les feuilletages holomorphes singuliers de dimension un sur $\mathbb{C}^n$ ou sur une variété projective lisse. J'ai démontré

[2,5,22] que pour un feuilletage générique, toutes les feuilles sont hyperboliques : leurs revêtements universels sont conformément équivalents au disque. Dans [7], j'ai calculé la codimension de l'ensemble des feuilletages sur $\mathbb{C}^n$ et sur $\mathbb{CP}^n$ qui ne satisfont pas les conditions suffisantes de [2,5,22] pour l'hyperbolicité de feuilles.

Dans les travaux [3] et [4] j'ai démontré que si un champ de vecteurs lisse sur $\mathbb{R}^2$ a au moins un point singulier, et en tout point du plan, toute valeur propre de sa matrice de Jacobi a une partie réelle négative, alors le point singulier est unique et globalement attractif. Cela a donné une réponse positive à la conjecture planaire de Markus et Yamabe. Presqu'en même temps (mais un peu plus tôt), deux autres solutions ont été obtenues par C. Gutierrez et R. Fessler par méthodes complétement différentes. J'ai construit un contre-exemple en dimension 3 dans [23]. Quand j'étais en train de le preparer pour publication, un contre-exemple polynomial simple en dimension 3 a été construit dans un travail commun par A. Cima, A. van den Essen, A. Gasull, E. Hubbers et F. Mañosas.

Dans [6], j'ai étudié les courbes sur le tore $\mathbb{T}^2$ sans intersections, dont les relevées sur le revêtement universel $\mathbb{R}^2$ ne sont pas bornées. On colle un cercle à l'infini du plan : des rayons différents partant de l'origine aboutissent à deux points différents du cercle. J'ai donné une description complète des ensembles de directions (comme points du cercle à l'infini), le long desquelles une telle relevée peut s'accumuler vers l'infini. Ces ensembles sont : 1) un point ; 2) deux points opposes ; 3) un segment fermé contenu dans un demi-cercle ; 4) le cercle tout entier.

L'article [12] concerne les equations différentielles linéaires à point singulier irrégulier résonnant du type générique. Les résultats obtenus sont analogues à ceux présentés dans la sous-section précedente et dans le chapitre 6.

Dans [15] j'ai construit des fractions continues"exotiques" à coefficients réels, qui donnent un contre-exemple à une affirmation trouvée dans des notes de Ramanujan.

L'article [17] avec son résultat (une formule explicite pour le déterminant d'une matrice d'intégrales abéliennes"de base") a déjà été mentionné dans la sous-section 1.1.4. Ce résultat, qui était utilisé dans la majoration du nombre de zéros d'une intégrale abélienne [18, 19], ne sera pas présenté ici.

Les résultats de l'article [9] sont brièvement mentionnés dans la sous-section précédente et dans le chapitre 6. Ceux qui ne sont pas présentés ici concernent

- les déformations d'un point fixe parabolique et les invariants d'Écalle-Voronin ;
- les déformations d'un point singulier nœud-col d'un champ vectoriel holomorphe en dimension strictement supérieure à 2, et ses variétés centrales sectorielles.

L'article [27] concerne une équation différentielle linéaire satisfaite par les intégrales hypergéométriques associées à un arrangement générique d'hyperplans réels. Cette équation a un pôle d'ordre deux à l'infini, qui est un point singulier irrégulier non résonnant. Nous avons calculé ses opérateurs de Stokes.

1.3 Liste de publications personnelles

1.3.1 Articles parus

[1] The analogue of Cayley's theorem for the cyclically-symmetric connected graphs with a single cycle that are related to the généralized Lyashko - Looijenga coverings - Uspehi Mat.Nauk, 2(1993) 233-234 (version anglaise en Russian Mathematicals Surveys 2(1993), 182-183).

[2] The hyperbolicity of phase curves of a generic polynomial vector field in $\mathbb{C}^n$ - Functsionalnyi Analiz i iego Prilozheniia, 2(1994), 1-11 (version anglaise en Functional analysis and its Applications, 2(1994), 77-84).

[3] The complete solution of the Jacobian problem for planar vector fields - Uspehi Mat.Nauk, 3(1994), 173-174.

[4] Asymptotic stability of linearizations of planar vector field with a singular point implies global stability - Functsionalnyi Analiz i iego Prilozheniia, 4(1995), 17-30 (version anglaise en Functional Analysis and its Applications 4(1995), 238-247).

[5] Hyperbolicity of leaves of a generic one-dimensional holomorphic foliation on a nonsingular projective algebraic manifold - Trudy Matematicheskogo Instituta im. V.A.Steklova, v.213 (1996), 90-111 (version anglaise en Proceedings of Steklov Mathematical Institute, v.213 (1996), 83-103).

[6] Limit sets at infinity of liftings to the plane of nonself-intersected curves in the torus - Matematicheskiie zamietki journal, v.64 (1998), No 5, 667-679 (version anglaise in Mathematical Notes, v.64 (1998), No 5, 579-589).

[7] On the codimension of the set of one-dimensional polynomial foliations on $\mathbb{C}^n$ and $\mathbb{CP}^n$ that do not satisfy the sufficient conditions for hyperbolicity of leaves (en russe) - Algebra i Analiz, v. 11 (1999), No 4, 35-63 (la traduction anglaise de ce journal sera Saint-Petersburg Mathematical Journal, v. 11 (2000), No 4).

[8] Stokes operators via limit monodromy of a generic deformation. - Journal of Dynamical and Control Systems, v.5 (1999) No 1, 101-135.

[9] Confluence of singular points and the nonlinear Stokes Phenomena - Trudy Moskovskogo Matematicheskogo Obshchestva, v.62 (2000), p.54-104 (en russe, la version anglaise de ce journal est “Proceedings of Moscow Mathematical Society”).

[10] Nonuniformizable skew cylinders : a counterexample to the simultaneous uniformization problem. - C.R.Acad.Sci.Paris, Série 1 Math., t.332 (2001), p.209-214.

[11] On simultaneous uniformization and local nonuniformizability. - C.R.Acad.Sci.Paris, Série 1 Math., t.334 (2002), p.489-494.

[12] Resonant confluence of singular points and Stokes phenomena. - Journal of Dynamical and Control Systems, vol. 10 (2004), No. 2 (April), pp. 253–302.

[13] Simultaneous metric uniformization of foliations by Riemann surfaces” Commentarii Mathematici Helvetici, vol. 79, Issue 4 (2004), pp.704-752.

[14] On the monodromy group of confluenting linear equations. - Moscow Math. J., 5 (2005), no. 1, 67-90.

[15] On convergence of generalized continued fractions and Ramanujan conjecture. - C. R. Math. Acad. Sci. Paris 341 (2005), no. 7, 427–432.

[16] Upper bounds of topology of complex polynomials in two variables. - Mosc. Math. J. 5 (2005), no. 4, 781–828.

[17] An explicit formula for period determinant. - Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 887–917.

[18] (Avec Yu.S.Ilyashenko). Restricted infinitesimal Hilbert sixteenths problem. - Doklady Academii Nauk, 2006, v. 407, No 2, 154-159 (in Russian). English translation in Doklady Mathematics, 2006, vol. 73, No 2, 185-189.

[19] (Avec Yu.S.Ilyashenko). Restricted version of the insinitesimal Hilbert 16-th problem. - Moscow Math. J. 7 (2007), no. 2, 281-325.

1.3.2 Actes de colloques

[20] Confluence of singular points and Stokes phenomena. - Proceedings of NATO Advanced Study Institute ”Normal Forms, Bifurcations and Finiteness Problems in Differential Equations”, Montreal, July 6-19 2002 (C. Rousseau and Yu. Ilyashenko, eds.), NATO Science Series II Math. Phys. Chem., (2004), vol.137, pp. 267-294. Kluwer Academic Publishers, Dordrecht.

[21] A survey on minimality of horospheric laminations associated to rational functions. - Dans Fields Institute Communications 2007, Vol : 51, pp. 269-287. Poceedings of the Partially hyperbolic dynamics, laminations, and Teichmuller flow Workshop, January 5-9, 2006.

1.3.3 Preprints et la thèse

[22] Uniformization of leaves of one-dimensional holomorphic foliations - Thèse Ph.D., Département de Mathématiques, Université d'État de Moscou, 1996 (en russe).

[23] Asymptotic stability of linearizations of a vector field in $\mathbb{R}^3$ with a singular point does not imply global stability - preprint in Communicaciones del CIMAT, Guanajuato, Mexico, 1996.

[24] Simple proofs of uniformization theorems. - À paraître dans Fields Institute Communications. Disponible sur l'archive :
http ://xxx.lanl.gov/abs/math/0510071

[25] Instability of free nondiscrete subgroups in Lie groups. -
http ://arxiv.org/abs/math/0409556

[26] On density of horospheres in dynamical laminations. -
http ://xxx.lanl.gov/abs/math.DS/0605644

[27] (Avec C.Sabot) Stokes matrices of hypergeometric integrals. -
http ://arxiv.org/abs/0712.1707

Chapitre 2

Uniformization of Riemann surface foliations

The present Chapter deals with foliations by Riemann surfaces. The main question studied here is the dependence of the uniformization of a leaf on the transversal parameter. We study this question with respect to two different notions of simultaneous uniformizability : the metric uniformizability in the sense of É Ghys (Sections 2.1, 2.2) and the holomorphic simultaneous uniformizability of holomorphic foliations in the sense of Yu.S.Ilyashenko (Section 2.3). We present positive and negative results. The main positive result is Theorem 2.1.3 stated in 2.1.2 and proved in Section 2.2. Its proof yields a new proof of the integrability of smooth almost complex structure on two-torus (Theorem 2.1.2 stated in 2.1.1 and proved in Section 2.2).

2.1 Metric uniformizability

2.1.1 Introduction : flat metrics and uniformization

The *(almost) complex structure* on a two-dimensional real surface is a family of complex structures on the tangent planes at the points of the surface. A Riemann surface with its standard complex structure carries a lot of nonstandard almost complex structures. We say that a (nonstandard) complex structure on a Riemann surface is *bounded* if it has uniformly bounded dilatation with respect to the standard complex structure (see 2.1.5).

It is well-known that each measurable bounded almost complex structure is locally integrable. This was proved in [94] and earlier under additional regularity conditions (Hölder or continuous) in [83, 86, 85]. Each measurable bounded almost complex structure on $\mathbb{C}$ is globally integrable, see the next theorem proved by M.A.Lavrentiev [85] for continuous almost complex structures and by C.Morrey Jr. [94] in the general case.

Theorem 2.1.1 *([5, 94]). For any measurable (C^∞) bounded almost complex structure σ on $\mathbb{C}$ there exists a quasiconformal homeomorphism (C^∞ diffeomorphism) $\mathbb{C} \to \mathbb{C}$ that transforms σ to the standard complex structure.*

The definition of a quasiconformal homeomorphism may be found in [4]. Theorem 2.1.1 implies that *for any C^∞ metric g on $\mathbb{R}^2$ with bounded dilatation there exists a C^∞ positive function $\phi : \mathbb{R}^2 \to \mathbb{R}_+$ such that the metric ϕg is flat and complete (the function ϕ is unique up to multiplication by constant).* This statement remains valid with $\mathbb{R}^2$ replaced by an arbitrary parabolic Riemann surface (see Definition 0).

In this section we present foliated versions of Theorem 2.1.1. Namely, we consider a real two-dimensional foliation on a compact Riemann manifold (M, g). The metric g induces an almost complex structure on each leaf. We suppose that the latter complex structure is parabolic. (This property is independent on the choice of the metric, by compactness and Theorem 2.1.1.) By the same theorem, on each individual leaf L there exists a C^∞ function $\phi : L \to \mathbb{R}_+$ such that the metric ϕg of the leaf L is flat and complete. We study the following questions.

Question 1. Is it possible to find a C^∞ function $\phi : M \to \mathbb{R}_+$ such that the restriction to each leaf of the metric ϕg be flat and complete? In other words, is it true that the previous functions ϕ may be chosen to depend smoothly on the transversal parameter?

Question 2. If yes, is it possible to find a Euclidean metric g' on the ambient manifold M that coincides with ϕg along the leaves, and for which each leaf be totally geodesic?

Positive and negative results were obtained in [45]. We present some of them here (Subsections 2.1.2-2.1.4). The main positive results (Theorems 2.1.3, 2.1.11 and 2.1.12) concern linear planar foliations on torus of arbitrary dimension equipped with a nonstandard Riemann metric (Subsections 2.1.2, 2.1.3). Counterexamples to Question 1 are discussed in 2.1.4.

The proof of Theorem 2.1.3 is based on a new proof (presented in [45, 51]) of the following classical Theorem. Both proofs are given in Section 2.2.

Theorem 2.1.2 *([Ab]) For any C^∞ almost complex structure σ on $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ there exists a C^∞ diffeomorphism of $\mathbb{T}^2$ onto appropriate complex torus (the latter torus depends on σ) that transforms σ to the standard complex structure.*

Theorem 2.1.2 is proved by showing the existence of a global nowhere vanishing σ- holomorphic differential. To do this, we use the homotopy method for the Beltrami equation with parameter. This method reduces the proof to solving a linear ordinary differential equation in $L_2(\mathbb{T}^2)$. We prove regularity of its solution by showing that the equation is bounded in any Sobolev space $H^s(\mathbb{T}^2)$.

As is shown in [51] (by classical arguments), Theorem 2.1.2 implies the Poincaré-Köbe Uniformization Theorem (modulo the contractibility of a simply connected surface) and Theorem 2.1.1. Another short proof of Theorem 2.1.1 using a different method (Fourier transformation) was earlier obtained by A.Douady and X.Buff [23].

Analogues of Question 1 were studied by A.Verjovsky [115], A.Candel and X.Gómez-Mont [19], A.Lins Neto [87] for some holomorphic foliations with singularities by hyperbolic Riemann surfaces. A.Candel [18] completely answered the analogue of Question 1 for laminations by hyperbolic Riemann surfaces, with flat metric replaced by Poincaré metric. In 1995 É.Ghys [34] proposed and partially studied Question 1. He proved the positive answer for linear foliations on $\mathbb{T}^3$ under certain Diophantine condition on the slope of the leaves. He noticed [34] that Reeb foliation of the three-sphere provides a counterexample to Question 1. Moreover, *the foliated manifold (sphere) admits no bounded Riemann metric whose restriction to each leaf be Euclidean.* Theorems 2.1.14 and 2.1.15 stated in 2.1.4 provide counterexamples to Question 1 in the class of C^∞ foliations on compact manifolds for which at least one latter Riemann metric exists and is analytic. In these examples we construct some other Riemann metric g on the foliated manifold for which there is no positive smooth function ϕ such that the metric ϕg be flat along the leaves.

2.1.2 Uniformizability of linear folations

Denote $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Consider a two-dimensional parallel plane foliation on $\mathbb{R}^n$. The standard projection $\mathbb{R}^n \to \mathbb{T}^n$ induces a foliation on the torus $\mathbb{T}^n$. This foliation is called *linear*. Take a (nonstandard) metric g on $\mathbb{T}^n$ and consider the corresponding complex structures on the leaves. Then each leaf is parabolic, by Theorem 2.1.1 and since the metric g has a bounded dilatation with respect to the standard Euclidean metric (by compactness argument).

Theorem 2.1.3 *[45]. Let F be an arbitrary linear foliation on $\mathbb{T}^n$, g be a Riemann metric on $\mathbb{T}^n$ that is analytic (respectively, C^∞ /measurable and uniformly bounded from below on $\mathbb{T}^n$ with uniformly bounded dilatation along the leaves of F). There exists an analytic (respectively, C^∞ /L_1) positive* *function $\phi : \mathbb{T}^n \to \mathbb{R}_+$ such that the restriction of the metric ϕg to each leaf (almost each in the measurable case) of the foliation F is flat (in the sense of distributions in the third case) and complete.*

Remark 2.1.4 In the previous theorem in the smooth and analytic cases the completeness of the metric ϕg follows from the nonvanishing of the function ϕ and compactness argument.

Remark 2.1.5 For any linear foliation on $\mathbb{T}^n$ either all the leaves are tori, or each leaf is dense. In the simplest case, when all the leaves are tori, Theorem 2.1.3 follows from Theorem 2.1.2 with smooth (analytic) dependence of the uniformization of the almost complex torus on the parameter of the almost complex structure, see [2]. The proof of Theorem 2.1.2 given in 2.2 also works to prove the regular dependence on the parameter. Thus, the interesting case of Theorem 2.1.3 is when the leaves are dense : then all they are either planes, or cylinders.

2.1.3 Existence of conformal Euclidean metric for which leaves are totally geodesic

Here we present positive answers to Question 2 for linear foliations on $\mathbb{T}^n$ satisfying some (sharp) Diophantine conditions on the slope. These are two different Diophantine conditions (see Definition 2.1.7) corresponding to the cases, when the metric of the torus is smooth (respectively, analytic).

Definition 2.1.6 We say that a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is *Diophantine*, if there exist constants $C > 0$, $s \geq 1$ such that for any pair $m, k \in \mathbb{Z}$, $k \neq 0$, the following inequality holds :

$$|\alpha - \frac{m}{k}| > \frac{C}{|k|^{s+1}}.$$

Definition 2.1.7 Consider a foliation on $\mathbb{R}^n$ by parallel planes : level planes of a linear vector function of rank $n-2$. Let F be the corresponding factorized linear foliation on $\mathbb{T}^n$. Let $W \subset \mathbb{R}^n$ be the $n-2$-space passing through the origin and orthogonal to the planes. Say that F is *Diophantine*, if there exist constants $C > 0$, $s \geq 1$ such that for any $N = (N_1, \dots, N_n) \in \mathbb{Z}^n \setminus 0$

$$dist(N, W) > \frac{C}{|N|^s}, \quad |N| = \sum_i |N_i|.$$

Say that F is *weakly Diophantine*, if

$$\underline{\lim}_{N \in \mathbb{Z}^n,\ |N| \to \infty} (dist(N, W))^{\frac{1}{|N|}} = 1. \tag{2.1.1}$$

Remark 2.1.8 Let $n = 3$, $x = (x_1, x_2, x_3)$ be coordinates in the space $\mathbb{R}^3$. Consider the foliation on $\mathbb{R}^3$ by level planes of the linear function $l(x) = a_1x_1 + a_2x_2 - x_3$. Then the corresponding linear foliation F on $\mathbb{T}^3$ is Diophantine, if and only if there exist constants $C > 0$, $s \geq 1$ such that for any $N = (N_1, N_2, N_3) \in \mathbb{Z}^3 \setminus 0$ the following inequality holds :

$$|N_1 + a_1N_3| + |N_2 + a_2N_3| > \frac{C}{|N|^s}, \quad |N| = |N_1| + |N_2| + |N_3|. \tag{2.1.2}$$

It is weakly Diophantine, if and only if

$$\underline{\lim}_{N \in \mathbb{Z}^3,\ |N| \to \infty} (|N_1 + a_1N_3| + |N_2 + a_2N_3|)^{\frac{1}{|N|}} = 1. \tag{2.1.3}$$

Example 2.1.9 In the notations of the previous remark let the additive subgroup in $\mathbb{R}$ generated by a_1 and a_2 contain a Diophantine number. Then the foliation F is Diophantine. It is not known to the author, whether the converse is true.

Remark 2.1.10 The limit (2.1.1) is always less than or equal to 1. A Diophantine foliation is always weakly Diophantine.

Theorem 2.1.11 *[45]. Let F be a Diophantine foliation on $\mathbb{T}^n$ (see Definition 2.1.7), g be a C^∞ Riemann metric on $\mathbb{T}^n$. There exists a C^∞ Euclidean metric $\widetilde{g}$ on $\mathbb{T}^n$ and a C^∞ function $\phi : \mathbb{T}^n \to \mathbb{R}_+$ such that*

$$\textit{each leaf } L \textit{ of the foliation } F \textit{ is totally geodesic and } \widetilde{g}|_L = \phi g|_L. \tag{2.1.4}$$

Or equivalently, let σ be the family of almost complex structures induced by the metric g on the leaves of F. There exist a discrete rank n additive subgroup $G \subset \mathbb{R}^n$ and a C^∞ diffeomorphism $\mathbb{T}^n \to \mathbb{T}_G = \mathbb{R}^n/G$ that transforms F to a linear foliation and sends σ to the standard complex structure induced by the standard Euclidean metric. Conversely, if a linear foliation on $\mathbb{T}^n$ is not Diophantine, then there exists a C^∞ metric g on the torus such that there is no C^2 Euclidean metric $\widetilde{g}$ on $\mathbb{T}^n$ satisfying (2.1.4).

Theorem 2.1.12 *[45]. Let F be a weakly Diophantine foliation on $\mathbb{T}^n$ (see Definition 2.1.7). Then for any analytic metric g on $\mathbb{T}^n$ there exists an analytic Euclidean metric $\widetilde{g}$ on $\mathbb{T}^n$ that satisfies (2.1.4). Conversely, if F is not weakly Diophantine, then there exists an analytic metric g on $\mathbb{T}^n$ such that there is no C^2 Euclidean metric $\widetilde{g}$ on $\mathbb{T}^n$ that satisfies (2.1.4).*

Let us justify the equivalence of the two statements of Theorem 2.1.11. Clearly, the second one implies the first one : the Euclidean metric from the first statement is the pull-back of the standard one under the diffeomorphism from the second statement. Let us prove the converse. Any Euclidean metric on a torus is transformed under appropriate diffeomorphism into the standard Euclidean metric on another torus (that is a quotient of the space by another lattice in general). Consider the images of leaves of the foliation. Their liftings to the space are planes, since the leaves are totally geodesic. They are parallel. Indeed, the liftings to the space of any two leaves of the initial foliation remain on a bounded distance from each other. Therefore, the same is true for the liftings of their images (by the compactness of $\mathbb{T}^n$). Hence, they are parallel. Thus, the leaves of F are transformed to the leaves of a linear foliation. This shows that statement (2.1.4) of Theorem 2.1.11 implies its second statement.

Remark 2.1.13 Earlier A.Haefliger [58] have obtained a result implying that under an a priori stronger Diophantine condition the metric ϕg on the leaves extends up to a global metric on the torus for which all the leaves are minimal surfaces.

2.1.4 Nonuniformizability. Counterexamples to Question 1

Theorem 2.1.14 *[45]. There exists a two-dimensional analytic foliation F on $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$ with the following properties.*
1) F is invariant under the translations of $\mathbb{T}^2$.
2) Any leaf is locally 1-to-1 projected to $\mathbb{T}^2$.
3) There are exactly two leaves that are horizontal tori ; any other leaf is homeomorphic to the cylinder $S^1 \times \mathbb{R}$.
4) There exists an analytic family of almost complex structures on the leaves satisfying the two following statements :

a) there is a unique continuous family of conformal flat metrics on the leaves up to multiplication by constant ; it is analytic outside the previous toric leaves ;

b) the latter family of flat metrics is not differentiable in the transversal parameter at one of the toric leaves.

Theorem 2.1.15 *[45]. There exists a two-dimensional C^∞ foliation F on $\mathbb{T}^2 \times S^2$ with the following properties.*

1) F is invariant under the translations of $\mathbb{T}^2$.
2) Any leaf is locally 1-to-1 projected to $\mathbb{T}^2$.
3) There is a big circle $S^1 \subset S^2$ such that the product $\mathbb{T}^2 \times S^1$ is a union of leaves of F; each of these leaves is a horizontal torus $\mathbb{T}^2 \times a$, $a \in S^1$.
4) Any other leaf is diffeomorphic to $\mathbb{R}^2$, and its accumulation set is the previous product $\mathbb{T}^2 \times S^1$.
5) There exists a C^∞ metric g on $\mathbb{T}^2 \times S^2$ such that on each non-toric leaf L there exists a unique function $\phi : L \to \mathbb{R}_+$ (up to multiplication by constant) such that the metric $\phi g|_L$ is flat and complete. The function $\phi(x)$ tends to infinity, as $x \to \infty$.

Let us describe briefly the construction of the foliation and the metric from Theorem 2.1.15. The foliation F is the suspension over the torus $\mathbb{T}^2$ under appropriate action of its fundamental group $\mathbb{Z}^2$ by sphere diffeomorphisms $S^2 \to S^2$. Any of these diffeomorphisms fixes only the points of the equator $S^1 \subset S^2$ and is flatly tangent to the identity at these points. Thus, the product $\mathbb{T}^2 \times S^1$ is an invariant set foliated by horizontal tori. Any other leaf L is canonically identified with $\mathbb{R}^2$ and embedded to $\mathbb{T}^2 \times S^2$ by the pair of projections $(\pi_1, \pi_2) : L \to \mathbb{T}^2 \times S^2$. The mapping $\pi_2 : L = \mathbb{R}^2 \to S^2$ is a diffeomorphism onto a hemisphere bounded by the equator. It commutes with the rotations of $\mathbb{R}^2$ around the origin and those of the hemisphere around its center. The projection $\pi_1 : L \to \mathbb{T}^2$ is a universal covering : the composition of the group quotient mapping $\mathbb{R}^2 \to \mathbb{T}^2$ with a translation of the torus. To define the metric g, we construct its restriction to the leaves and then extend it to the transversal direction in an arbitrary way. The metric on the horizontal toric leaves is the lifting of the standard Euclidean metric on $\mathbb{T}^2$. Any other leaf $L = \mathbb{R}^2$ is equipped with an appropriate rotation-invariant metric that tends to the standard Euclidean metric, as the point where it is taken tends to infinity.

For any rotation-invariant metric g on $\mathbb{R}^2$ with uniformly bounded dilatation the corresponding function $\phi : \mathbb{R}^2 \to \mathbb{R}_+$, for which the metric ϕg is flat and complete, is also rotation-invariant. The latter function ϕ can be find by an explicit formula. It appears that one can achieve appropriate asymptotic behaviors at infinity of the mapping π_2 and the metric g so that the function $\phi(x)$ tend to infinity, as $x \to \infty$, and g extends up to a C^∞ family of metrics on all the leaves of the foliation F.

2.1.5 Complex structures and Beltrami equations. Basic notations

To a (nonstandard) almost complex structure (denoted by σ) on a subset $D \subset \mathbb{C}$ we put into correspondence a $\mathbb{C}$- valued 1- form that is $\mathbb{C}$- linear with respect to σ. The latter form can be normalized to have the type

$$\omega_\mu = dz + \mu(z)d\bar{z}, \ |\mu| < 1. \tag{2.1.5}$$

The function $\mu : D \to \mathbb{C}$ is uniquely defined by σ. Vice versa, for an arbitrary complex-valued function μ with $|\mu| < 1$, the 1- form (2.1.5) defines the unique complex structure for which the form is $\mathbb{C}$- linear. We denote by σ_μ the almost complex structure thus defined (whenever the contrary is not specified). Then σ_μ is bounded, if and only if the essential supremum of the function $|\mu|$ is less than 1.

Definition 2.1.16 The *ellipse* associated to σ_μ on the tangent plane at a point z is given by the equation $|dz + \mu(z)d\bar{z}| = 1$. The *dilatation* of σ_μ is the aspect ratio of the ellpise : it is equal to $\frac{1+|\mu(z)|}{1-|\mu(z)|}$.

We will be looking for a differentiable homeomorphism $\Phi(z)$ that is holomorphic, i.e., that transforms σ_μ to the standard complex structure. This is equivalent to say that the differential of Φ (which is a closed 1- form) is a $\mathbb{C}$- linear form, i.e., has the type $f(z)(dz + \mu d\bar{z})$:

$$\frac{\partial \Phi}{\partial \bar{z}} = \mu \frac{\partial \Phi}{\partial z} \text{ (Beltrami equation).}$$

Remark 2.1.17 Conversely, let μ be C^∞ with $|\mu| < 1$. Then any C^∞ closed 1- form $f(z)(dz + \mu d\bar{z})$ is σ_μ- holomorphic, i.e., is a differential of a complex-valued C^∞ function Φ transforming σ_μ to the standard complex structure. A form $f(dz + \mu(z)d\bar{z})$ is closed if and only if

$$\partial_{\bar{z}} f = \partial_z(\mu f). \tag{2.1.6}$$

2.2 Uniformization of almost complex torus. Proof of Theorems 2.1.2 and 2.1.3

First we prove Theorem 2.1.2. At the end of the section we discuss the proof of Theorem 2.1.3 obtained by modifying the proof of Theorem 2.1.2.

2.2.1 Homotopy method. The sketch of the proof of Theorem 2.1.2

Let $\mu : \mathbb{T}^2 \to \mathbb{C}$ be a C^∞ complex-valued function with $|\mu| < 1$. Let σ_μ be the corresponding almost complex structure, $\omega_\mu = dz + \mu d\bar{z}$ be the corresponding $\mathbb{C}$- linear 1- form, see (2.1.5). Theorem 2.1.2 says that there exists a diffeomorphism transforming $(\mathbb{T}^2, \sigma_\mu)$ into appropriate complex torus equipped with the standard complex structure. We construct a C^∞ nowhere vanishing function $f : \mathbb{T}^2 \to \mathbb{C}$ such that the 1- form $f\omega_\mu$ be closed or equivalently, f satisfy partial differential Equation (2.1.6). Then the lifting to the universal cover $\mathbb{R}^2 \to \mathbb{T}^2$ of the form $f\omega_\mu$ is the differential of the mapping $\Psi : \mathbb{R}^2 = \mathbb{C} \to \mathbb{C}$, $\Psi(z) = \int_0^z f\omega_\mu$. The mapping Ψ is a diffeomorphism and transforms the integer lattice $\mathbb{Z}^2$ and its translation images to some lattice $G \subset \mathbb{C}$ and its appropriate translation images. This follows from the definition and the local diffeomorphicity of Ψ ($f \neq 0$). The factorized mapping $\mathbb{T}^2 \to \mathbb{T}^2_G = \mathbb{C}/G$ is a diffeomorphism that sends σ_μ to the standard complex structure. This implies Theorem 2.1.2.

To solve (2.1.6), we use the homotopy method. Namely, we include σ_μ into the one-parametric family of complex structures (denoted by σ_ν) defined by their $\mathbb{C}$- linear 1- forms

$$\omega_\nu = dz + \nu(z,t)d\bar{z}, \ \nu(z,t) = t\mu(z), \ t \in [0,1].$$

The complex structure corresponding to the parameter value $t = 0$ is the standard one, the given structure σ_μ corresponds to $t = 1$. We will find a C^∞ family $f(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C}$ of complex-valued nowhere vanishing C^∞ functions on $\mathbb{T}^2$ depending on the same parameter t, such that the differential forms $f(z,t)\omega_\nu$ be closed, i.e.,

$$\partial_{\bar{z}} f = \partial_z(f\nu), \text{ and } f(z,0) \equiv 1. \tag{2.2.1}$$

Then the function $f = f(z,1)$ is the one we are looking for.

To construct the above-mentioned family of functions, first we will find a family $f(z,t)$ of functions that satisfy (2.2.1) and do not vanish identically on $\mathbb{T}^2$ for any fixed parameter value t.

Lemma 2.2.1 *Let $\nu(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C}$ be a C^∞ family of C^∞ functions on $\mathbb{T}^2$ with $|\nu| < 1$, $\nu(z,0) \equiv 0$, z be the complex coordinate on $\mathbb{T}^2$. There exists a C^∞ family $f(z,t) : \mathbb{T}^2 \times [0,1] \to \mathbb{C}$ of C^∞ functions on $\mathbb{T}^2$ that are solutions of differential Equation (2.2.1) (with the boundary condition) such that for any fixed $t \in [0,1]$ one has $f(z,t) \not\equiv 0$.*

The Lemma will be proved in the next subsection.

We show that, in fact, the functions $f(z,t)$ from the lemma vanish nowhere. To do this (and only in this place) we use the local integrability of a C^∞ complex structure :

Proposition 2.2.2 *[20, 83, 85, 86]. Let $D \subset \mathbb{C}$ be a disk centered at 0, $\mu : D \to \mathbb{C}$ be a C^∞ function with $|\mu| < 1$. Let σ_μ be the corresponding almost complex structure, see (2.1.5). There exists a local C^∞ σ_μ- holomorphic univalent complex coordinate near 0.*

The proposition will be proved in Subsection 2.2.3.

Proof of Theorem 2.1.2 modulo Lemma 2.2.1 and Proposition 2.2.2. Let $f(z,t)$ be a family of functions from Lemma 2.2.1. By the previous discussion, it suffices to show that $f(z,t) \neq 0$. This inequality holds for $t = 0$, where $f(x,0) \equiv 1$.

Let us prove that $f(z,t) \neq 0$ by contradiction. Suppose the contrary. Then the set of the parameter values t corresponding to the functions $f(z,t)$ having zeroes is nonempty. Denote this set by M. Its complement $[0,1] \setminus M$ is open by definition. Let us show that the set M is open as well. This will imply that the parameter segment is a union of two disjoint open sets, which will bring us to contradiction. It is sufficient to show that the local presense of a zero of a function f persists under perturbation.

Suppose $f(z_0,t) = 0$ for some z_0 and t (let us fix them). It suffices to show that for any t' close to t the function $f(z,t')$ has a zero near z_0. Let w be the local σ_ν- holomorphic coordinate on $\mathbb{T}^2$ near z_0 from Proposition 2.2.2 with $\mu(z)$ replaced by $\nu(z,t)$ and $w(z_0) = 0$. We consider that the function $f(z,t)$ does not vanish identically on $\mathbb{T}^2$ locally near z_0. One can achieve this by changing z_0, since $f(z,t)$ does not vanish identically on $\mathbb{T}^2$. Recall that the 1- form $f(z,t)\omega_{\nu(z,t)}$ is a closed $\mathbb{C}$- linear form on $\mathbb{T}^2$ with respect to the complex structure $\sigma_{\nu(z,t)}$. Hence, it is holomorphic in the coordinate w. Therefore, $f(z,t)\omega_{\nu(z,t)} = (w^k + \text{higher terms})dw$, $k \geq 1$. Now by an index argument, the local presense of zero of f on $\mathbb{T}^2$ persists under perturbation. This together with the previous discussion proves the inequality $f(z,t) \neq 0$ and Theorem 2.1.2. □

2.2.2 Variable holomorphic differential : proof of Lemma 2.2.1

We denote by $\dot{f}$ the partial derivative in t of a function f. Differentiating (2.2.1) in t yields

$$\partial_{\bar{z}}\dot{f} - (\partial_z \circ \nu)\dot{f} = (\partial_z \circ \dot{\nu})f. \tag{2.2.2}$$

where $\partial_z \circ \nu$ ($\partial_z \circ \dot{\nu}$) is the composition of the operator of the multiplication by the function ν (respectively, $\dot{\nu}$) and the operator ∂_z. Any C^∞ solution f of equation (2.2.2) with the initial condition $f(z,0) \equiv 1$ that does not vanish identically on the torus for any value of t is a one we are looking for. Let us show that (2.2.2) is implied by a bounded linear differential equation in $L_2(\mathbb{T}^2)$ and in any Hilbert Sobolev space. To do this, we use the following properties of the operators ∂_z and $\partial_{\bar{z}}$.

Remark 2.2.3 Denote $z = x_1 + ix_2$, $x = (x_1, x_2) \in \mathbb{R}^2$. The operators ∂_z, $\partial_{\bar{z}}$ on $\mathbb{T}^2$ have common eigenfunctions $e_n(x) = e^{i(n,x)}$, $n = (n_1, n_2) \in \mathbb{Z}^2$. The corresponding eigenvalues (denote them by λ_n and λ'_n respectively) have equal moduli, more precisely,

$$\lambda'_n = -\overline{\lambda_n}. \tag{2.2.3}$$

This is implied by the fact that the operator $\partial_{\bar{z}}$ is conjugated to $-\partial_z$ in the L_2 scalar product, which follows from definition. In fact,

$$\lambda_n = \frac{i}{2}(n_1 - in_2) \text{ and } \lambda'_n = \frac{i}{2}(n_1 + in_2).$$

Corollary 2.2.4 *There exists a unique unitary operator $U : L_2(\mathbb{T}^2) \to L_2(\mathbb{T}^2)$ preserving averages and such that "$U = \partial_{\bar{z}}^{-1} \circ \partial_z$" (more precisely, $U \circ \partial_{\bar{z}} = \partial_{\bar{z}} \circ U = \partial_z$ in the sense of distributions). The operator U commutes with partial differentiations and extends up to a unitary operator to any Hilbert Sobolev space of functions on $\mathbb{T}^2$. In particular, it preserves the space of C^∞ functions.*

Proof The operator U from the corollary is defined to have the eigenfunctions e_n with the eigenvalues $\frac{\lambda_n}{\lambda'_n} = \frac{n_1 - in_2}{n_1 + in_2}$ if $n \neq 0$, and 1 if $n = 0$. Its uniqueness follows immediately from the previous operator equation on U applied to the functions e_n. The rest of the statements of the corollary follow immediately from definition and Sobolev embedding theorem (see [21], p.411). □

Let us write down equation (2.2.2) in terms of the new operator U. Applying the "operator" $\partial_{\bar{z}}^{-1}$ to (2.2.2) and substituting $U = \partial_{\bar{z}}^{-1} \circ \partial_z$ yields

$$(Id - U \circ \nu)\dot{f} = (U \circ \dot{\nu})f.$$

This equation implies (2.2.2). For any $t \in [0,1]$ the operator $Id - U \circ \nu$ in the left-hand side is invertible in $L_2(\mathbb{T}^2)$ and the norm of the inverse operator is bounded uniformly in t, since U is unitary and the modulus $|\nu|$ is less than 1 and bounded away from 1 by compactness. Thus, the last equation can be rewritten as

$$\dot{f} = (Id - U \circ \nu)^{-1}(U \circ \dot{\nu})f, \tag{2.2.4}$$

which is a linear ordinary differential equation in $f \in L_2(\mathbb{T}^2)$. The operator in its right-hand side is uniformly bounded in the operator L_2- norm. Let us show that the same operator is uniformly bounded in the Sobolev $H^s(\mathbb{T}^2)$- norms. More precisely, for any $s \in \mathbb{N}$ there exists a $c_s > 0$ such that

$$||(Id - U \circ \nu)^{-1}||_{H^s(\mathbb{T}^2)} < c_s(1 + \sum_{k \leq s;\ i_r = 1,2} \max_{\mathbb{T}^2 \times [0,1]} |\frac{\partial^k \nu}{\partial x_{i_1} \dots \partial x_{i_k}}|^s). \tag{2.2.5}$$

Proof Let us prove (2.2.5) for $s = 1$. For higher s the proof is analogous. Let

$$\delta = \max|\mu|. \text{ Then } \max_{\mathbb{T}^2 \times [0,1]} |\nu| \leq \delta < 1,\ ||U \circ \nu||_{L_2} \leq \delta < 1.$$

Hence, the operator $Id - U \circ \nu$ is invertible in $L_2 = H^0$ and

$$(Id - U \circ \nu)^{-1} = Id + \sum_{k=1}^{\infty} (U \circ \nu)^k; \tag{2.2.6}$$

$$||(U \circ \nu)^k||_{L_2} \leq \delta^k,\ ||(Id - U \circ \nu)^{-1}||_{L_2} \leq \frac{1}{1-\delta}. \tag{2.2.7}$$

To prove (2.2.5), we use (2.2.6) and estimate the H^1- norms of the terms in its sum.

Let $f \in H^1(\mathbb{T}^2)$. Let us estimate $||(U \circ \nu)^k f||_{H^1}$. We show that for any $k \in \mathbb{N}$

$$||\frac{\partial}{\partial x_r}((U \circ \nu)^k f)||_{L_2} \leq ck\delta^{k-1}||f||_{H^1}, \quad c = \delta + \max|\frac{\partial \nu}{\partial x_r}|, \quad r = 1, 2. \tag{2.2.8}$$

This together with (2.2.6) and the first inequality in (2.2.7) implies (2.2.5) ; here $c_s = c_1 = 4\sum_{k \in \mathbb{N}} k\delta^{k-1} = \frac{4}{(1-\delta)^2}$.

Let us prove (2.2.8), e.g., for $r = 1$. The derivative in the left-hand side of (2.2.8) equals

$$(U \circ \nu)^k \frac{\partial f}{\partial x_1} + \sum_{i=1}^{k} (U \circ \nu)^{k-i} \circ (U \circ \frac{\partial \nu}{\partial x_1}) \circ (U \circ \nu)^{i-1} f,$$

since U commutes with the partial differentiations. The L_2- norm of the first term in the previous formula is no greater than $\delta^k||f||_{H^1}$ by (2.2.7). Each term in the latter sum has L_2- norm no greater than $\delta^{k-1}\max|\frac{\partial \nu}{\partial x_1}|||f||_{L_2}$ by (2.2.7). This proves (2.2.8). Inequality (2.2.5) is proved. □

Ordinary differential Equation (2.2.4) is bounded in any Sobolev space $H^s(\mathbb{T}^2)$, by (2.2.5). Therefore, it has a unique solution $f(x,t)$ with the initial condition $f(x,0) \equiv 1$ that belongs to all the Sobolev spaces. This follows from the existence and uniqueness theorem for solution of ordinary differential equation in Banach space with right-hand side having bounded derivative, see [21]. This solution is C^∞ by the Sobolev embedding theorem (see [21], p.411). For any fixed value of t it does not vanish identically on $\mathbb{T}^2$ (the uniqueness of local solution). Lemma 2.2.1 is proved.

Remark 2.2.5 The solution of Equation (2.2.4) with the initial condition $f|_{t=0} \equiv 1$ admits the following formula :

$$f(x,t) = (Id - U \circ \nu)^{-1}(1) = 1 + U(\nu) + (U \circ \nu \circ U)(\nu) + \dots \qquad (2.2.9)$$

Indeed, its right-hand side is a well defined C^∞ family of C^∞ functions on $\mathbb{T}^2$, which follows from the uniform boundedness of the operators $(Id - U\circ\nu)^{-1}$ in any given Hilbert Sobolev space. By definition, f satisfies the initial condition $f(x,0) \equiv 1$. Differentiating (2.2.9) in t yields

$$\dot{f} = (Id - U \circ \nu)^{-1} \circ (U \circ \dot{\nu}) \circ (Id - U \circ \nu)^{-1}(1) = (Id - U \circ \nu)^{-1} \circ (U \circ \dot{\nu}) f.$$

Hence, the function (2.2.9) satisfies (2.2.4).

2.2.3 Zero of holomorphic differential. Proof of Proposition 2.2.2

Let us prove the existence of local holomorphic coordinate. Without loss of generality we assume that $\mu(0) = 0$. One can achieve this by applying a real-linear transformation of the plane $\mathbb{R}^2 = \mathbb{C} \supset D$ that brings the ellipse at 0 associated to σ_μ to a circle. One can achieve also that μ be arbitrarily small with derivatives of orders up to 3 by applying a homothety and taking the restriction to a smaller disk centered at 0. We consider that the disk where μ is defined is embedded into $\mathbb{T}^2$ and extend the function μ smoothly to $\mathbb{T}^2$. We assume that the extended function satisfies the inequality $||\mu||_{C^3(\mathbb{T}^2)} < \delta$; one can make δ arbitrarily small.

Let $\nu(x,t) = t\mu$, $f(x,t)$ be the corresponding function family from Lemma 2.2.1 constructed as the solution of differential equation (2.2.4) with unit initial condition. Put $f(x) = f(x,1)$. We show in the next paragraph that $f(0) \neq 0$, if the previous constant δ is small enough. Then the local coordinate we are looking for is the function

$$w(z) = \int_0^z f(dz + \mu d\bar{z}).$$

Indeed, it is well-defined and holomorphic, since the 1- form $f(dz + \mu d\bar{z})$ is closed by construction. Its local univalence follows from the nondegeneracy of its differential $f(0)(dz + \mu(0)d\bar{z})$ at 0 (the inequalities $|\mu| < 1$, $f(0) \neq 0$).

Recall that by (2.2.9),

$$f(x,t) = (Id - tU \circ \mu)^{-1}(1), \text{ where } U = (\partial_{\bar{z}})^{-1}\partial_z.$$

The functions $f(x,t)$ are equal to 1, if $\mu = 0$. Let us show that they are C^0- close to 1 (and hence, $f(0) = f(0,1) \neq 0$), whenever μ is small enough with derivatives up to order 3. For any $t \in [0,1]$ consider the operator-valued functional $\mathcal{A}(\mu) = (Id - tU \circ \mu)^{-1}$ defined for $||\mu||_{C^3} < \delta$: its value being an operator acting in $H^3(\mathbb{T}^2)$. (It is well-defined, see Inequality (2.2.5).) The derivative $\mathcal{A}'(\mu)$ exists and is uniformly bounded. Indeed, the operators $\mathcal{A}(\mu)$ are uniformly bounded by some constant $c' = c'(\delta)$ (Inequality (2.2.5)). Therefore, we can apply the usual formula for the derivative of the inverse operator : the derivative of $\mathcal{A}(\mu)$ along a vector $h \in C^3(\mathbb{T}^2)$ is equal to

$$\nabla_h \mathcal{A}(\mu) = t\mathcal{A}(\mu) \circ U \circ h \circ \mathcal{A}(\mu). \text{ Hence, } ||\nabla_h \mathcal{A}(\mu)||_{H^3} \leq ||\mathcal{A}(\mu)||^2_{H^3} ||h||_{H^3} \leq c'(\delta)||h||_{C^3}.$$

Thus, the operator-valued functional $\mathcal{A}(\mu)$ is Lipschitz (and hence, continuous) in μ. Therefore, if $||\mu||_{C^3}$ is small enough, then each function $f(x,t)$ is close to 1 in H^3 (thus, in C^0, by the Sobolev embedding theorem, and hence, $f \neq 0$). This proves Proposition 2.2.2. The proof of Theorem 2.1.2 is complete.

2.2.4 Foliated version : proof of Theorem 2.1.3

Here we present only a proof of the C^∞ version of Theorem 2.1.3. The proof of its other (analytic and measurable) versions is analogous.

Fix a projection $\mathbb{T}^n \to \mathbb{T}^2$ to appropriate coordinate two-torus whose restriction to each leaf of F be a local diffeomorphism. The universal covering $\mathbb{R}^2 \to \mathbb{T}^2$ lifts under the projection up to a universal covering of any leaf. Let us introduce an affine complex coordinate z on $\mathbb{R}^2$. Its differential dz yields well-defined complex-valued 1- forms (also denoted by dz) on $\mathbb{T}^2$ and on any leaf. Consider the complex structures on the leaves defined by the metric g. In the local coordinate z they are defined by a 1-form

$$\omega_\mu = dz + \mu d\bar{z},\ \mu : \mathbb{T}^n \to \mathbb{C} \text{ is a } C^\infty \text{ function with } |\mu| < 1,$$

as in (2.1.5). Vice versa, each function μ as above yields a C^∞ family of almost complex structures on the leaves that is defined by some C^∞ Riemann metric on $\mathbb{T}^n$. Namely let $H : \mathbb{R}^n \to \mathbb{R}^{n-2}$ be a linear vector function whose level planes are the liftings to $\mathbb{R}^n$ of the leaves of the foliation F. Then $g = |\omega_\mu|^2 + |dH|^2$ is a C^∞ Riemann metric on $\mathbb{T}^n$ that is conformal with respect to the given complex structures along the leaves.

We prove the following more precise version of Theorem 2.1.3.

Theorem 2.2.6 *Let F be a linear foliation on $\mathbb{T}^n$. Let $\mu : \mathbb{T}^n \to \mathbb{C}$ be an arbitrary C^∞ function with $|\mu| < 1$. Let z, ω_μ be as above. There exists a C^∞ nowhere vanishing function $f : \mathbb{T}^n \to \mathbb{C}$ such that the restriction to each leaf of the 1- form $f\omega_\mu$ be closed.*

The restriction to each leaf of the 1- form $f\omega_\mu$ from Theorem 2.2.6 is a nowhere vanishing holomorphic differential. Therefore, its squared modulus $|f\omega_\mu|^2$ is a flat metric on each leaf. This yields a C^∞ family of flat metrics on the leaves. These metrics are proportional to the restrictions of the C^∞ metric g to the leaves with a positive functional coefficient (which is then also C^∞). This implies Theorem 2.1.3.

Remark 2.2.7 If in the conditions of the previous theorem the leaves of the foliation are dense, then the corresponding function f is unique up to multiplication by constant.

Proof of Theorem 2.2.6. Without loss of generality we consider that each leaf is dense. In the opposite case, all the leaves are tori and Theorem 2.2.6 follows from Theorem 2.1.2 with smooth dependence of the uniformizing diffeomorphism of the almost complex torus on the parameter of the almost complex structure (see Remark 2.1.5).

The closeness of a 1- form $f\omega_\mu$ is equivalent to the partial differential Equation (2.1.6) along the leaves :

$$D_{\bar{z}} f = D_z(\mu f),\ D_z = \frac{\partial}{\partial z},\ D_{\bar{z}} = \frac{\partial}{\partial \bar{z}} : \text{ both differentiations are done along the leaves.}$$

The function f is constructed by homotopy method, as before. We include μ into the family of functions

$$\nu(x,t) = t\mu(x),\ t \in [0,1],$$

and find a solution $f(x,t)$ of the previous differential equation with μ replaced by ν :

$$D_{\bar{z}} f = D_z(\nu f) \text{ with the initial condition } f(x,0) \equiv 1. \tag{2.2.10}$$

Differentiating in t (we denote $\dot{f} = \frac{\partial}{\partial t}$) yields

$$D_{\bar{z}} \dot{f} = (D_z \circ \nu)\dot{f} + (D_z \circ \mu) f.$$

The operators D_z and $D_{\bar{z}}$ are differential operators with constant coefficients, for which the Fourier harmonics $e_N = e^{i(N,x)}$, $N \in \mathbb{Z}^n$, are thus eigenfunctions. The corresponding eigenvalues λ_N and λ'_N have equal moduli, moreover,

$$\lambda'_N = -\overline{\lambda_N},$$

since the operators D_z and $-D_{\bar{z}}$ are conjugated. One has $\lambda_N = 0$, if and only if $N = 0$. Indeed, a smooth function on $\mathbb{T}^n$ (anti)holomorphic on the leaves (in the standard complex structure given by the coordinate z) is always constant along the leaves (Liouville's theorem), and hence, is constant globally (the density of the leaves). Consider the operator $U: L_2(\mathbb{T}^n) \to L_2(\mathbb{T}^n)$ defined to have the eigenbase $\{e_N\}|_{N\in\mathbb{Z}^n}$ with the eigenvalues $\frac{\lambda_N}{-\bar{\lambda}_N}$ if $N \neq 0$ and 1 if $N = 0$. The operator U extends up to a unitary operator to each Hilbert Sobolev space of functions on $\mathbb{T}^n$ such that the equality

$$U \circ D_{\bar{z}} = D_{\bar{z}} \circ U = D_z$$

holds true on smooth functions. The equation

$$(Id - U \circ \nu)\dot{f} = (U \circ \mu)f$$

has a unique smooth solution $f(x,t)$ with unit initial condition, which satisfies (2.2.10) and vanishes nowhere, as in Subsections 2.2.1 and 2.2.2. The function $f = f(x,1)$ is a one we are looking for. This proves Theorems 2.2.6 and 2.1.3. □

2.3 Holomorphic nonuniformizability

2.3.1 Main result : nonuniformizable universal covering manifolds

Let S be an affine (or projective) smooth algebraic surface of dimension 2, F be a one-dimensional holomorphic foliation on S (with isolated irremovable singularities) tangent to a rational vector field. In this case we say briefly that the foliation F is *algebraic affine (projective).*

Remark 2.3.1 Let S, F be as above, S be affine and its projective closure $\overline{S}$ be smooth. Then F extends up to an algebraic foliation on $\overline{S}$ (called *the projective extension*, denoted $\overline{F}$).

Roughly speaking, the principal result of the section is the existence of S, F as above such that the family of leaves intersecting an arbitrary given cross-section does not admit a uniformization holomorphic in the parameter by a family of simply connected domains in the Riemann sphere. To state this result precisely, let us introduce the following

Definition 2.3.2 Let S, F be as above, $D \subset S$ be a simply connected (may be not global) transversal cross-section to F *containing no singularities*. For any $z \in D$ denote $L_z \subset S$ the leaf of F passing through z. The *universal covering manifold* (briefly *u.c.m.*) associated to D is

$$M_D = \bigcup_{z\in D}(\text{the universal covering of } L_z \text{ with the base point } z).$$

Theorem 2.3.3 *[63, 68] Let S, F, D, M_D be as above, S be affine. Then the space M_D admits a natural structure of complex manifold and it is Stein.*

Remark 2.3.4 In general, the space M_D is a complex manifold, if and only if it is Hausdorff. If S is projective, then in general M_D may be non-Hausdorff. (Such an example was proposed by the referee of the paper [44]; the foliation from this example is obtained from another one by blowing up at a nonsingular point.) But if S is projective and no leaf of F intersecting D is a once punctured sphere, then M_D is a manifold. This follows from a remark of E.Chirka and a version of Gromov compactness theorem [57]. It is not known in the latter case, whether M_D is always Stein whenever it is a manifold.

The manifold M_D admits a natural holomorphic projection $p: M_D \to D$ and a section $D \to M_D$ inverse to p defined by taking the base points of the universal coverings.

Definition 2.3.5 A u.c.m. M_D is said to be *uniformizable*, if it admits a biholomorphism (called *uniformization*) onto a domain in $\overline{\mathbb{C}} \times D$ that forms a commutative diagram with the projections. It is said to be *locally uniformizable* at a given point $z \in D$, if its restriction $p^{-1}(U) = M_U$ to a neighborhood U of z is uniformizable.

Theorem 2.3.6 *[44] There exists an affine algebraic foliation with no uniformizable u.c.m.*

Corollary 2.3.7 *For a foliation from Theorem 2.3.6 each u.c.m. is nowhere locally uniformizable.*

Addendum to Theorem 2.3.6 [44]. *In Theorem 2.3.6 the affine foliation (denoted by F) can be chosen to have the following additional properties :*

1) *F is transversally affine and admits a Liouvillian first integral (cf. 5) below);*

2) *each leaf is dense and hyperbolic : its universal covering is conformally equivalent to disc;*

3) *some leaf contains an attracting cycle (a closed curve with an attracting return mapping);*

4) *the projective extension $\overline{F}$ is well-defined, each its u.c.m is a manifold and nonuniformizable.*

5) *F is a rational pullback of the foliation on $(\mathbb{C} \setminus \pm 1) \times \mathbb{C}$ with a first integral $I(w, z) = z(1 - w)^{\alpha} + \beta \int_0^w \frac{(1-\tau)^{\alpha}}{\tau+1} d\tau$.*

A brief proof of Theorem 2.3.6 and its Addendum is given in the next two Subsections.

In late 1960-s Yu.S.Ilyashenko proposed the conjecture saying that each u.c.m. of any algebraic foliation is uniformizable. He proved uniformizability of certain u.c.m's [64]. In 1969 T.Nishino [96] independently proved the positive answer with u.c.m replaced by abstract holomorphic Stein surface fibered by complex lines (Stein skew cylinder with fiber $\mathbb{C}$, see Definition 2.3.8 below). His and Ilyashenko's results [96, 63, 68] together imply the positive answer to Ilyashenko's conjecture for the u.c.m's with fibers $\mathbb{C}$. At the end of 1999 a negative answer in the general case was proved by the author in [43]. The counterexample constructed there was locally uniformizable at a generic point. In 2001 A.A.Shcherbakov asked the following question : is it true that each u.c.m. of any algebraic foliation with hyperbolic leaves is locally uniformizable? Theorem 2.3.6, its Corollary and the Addendum give a negative answer.

The proofs of Theorem 2.3.6 and the results of [43] are based on a key result in several complex variables due to B.Berndtsson and J.Ransford ([12], see Theorem 2.3.13 below). Their result provides a very exotic subset K in the product of $\mathbb{C}$ and unit disk D with a Stein complement $V = (\mathbb{C} \times D) \setminus K$ and infinitely many $\mathbb{C}$- slices of the set K being single points with two distinct $\mathbb{C}$- coordinates. The universal covering M over V is fibered over D by simply connected Riemann surfaces. It appears that the fibered manifold M is not uniformizable in the sense of Definition 2.3.5. This was proved in [43]; the proof is presented in the next subsection. Afterwards in 2.3.3 we construct a foliation satisfying the statements of Theorem 2.3.6 and its Addendum by using the nonuniformizability of M, the Stein nature of V, and approximations of holomorphic functions on Stein manifolds embedded in $\mathbb{C}^N$ by polynomials.

2.3.2 Skew annuli and nonuniformizable Stein skew cylinders

Universal covering manifolds are particular cases of skew cylinders, see the following Definition.

Definition 2.3.8 [68] Let D be a simply-connected domain in $\mathbb{C}$, M be a two-dimensional complex manifold, $p : M \to D$ be a holomorphic surjection having nonzero derivative. We say that the triple (M, p, D) is a *skew cylinder* with the base D and the total space M, if

1) the level sets of the mapping p are connected and simply connected holomorphic curves;

2) M has a holomorphic section : a holomorphic mapping $i : D \to M$, $p \circ i = Id$.

The definition of a (locally) *uniformizable* skew cylinder coincides with that of a uniformizable u.c.m. (Definition 2.3.5). A skew cylinder is said to be *Stein*, if its total space is Stein. A u.c.m. corresponding to an algebraic foliation is a skew cylinder, whenever it is a manifold. It is Stein, if the foliation is affine (Theorem 2.3.3). Denote

$$\pi : \mathbb{C} \times D \to D \text{ the product projection.}$$

Definition 2.3.9 A domain $V \subset \mathbb{C} \times D$ is said to be a *uniformizable skew annulus* (or briefly, u.s.a.), if it satisfies the following conditions : 1) for any $z \in D$ the fiber $\pi^{-1}(z) \cap V$ is either a once punctured complex line, or a complement of $\mathbb{C}$ to a disk; 2) $V \supset c \times D$ for any $c \in \mathbb{C}$ large enough.

Remark 2.3.10 The universal covering over a uniformizable skew annulus has a natural structure of skew cylinder.

Theorem 2.3.11 *[43] There exists a pseudoconvex u.s.a. whose universal covering is a nonuniformizable Stein skew cylinder.*

Remark 2.3.12 It is easy to construct a u.s.a. with nonuniformizable (non-Stein) universal covering manifold, e.g.,

$$V = (\mathbb{C} \times D) \setminus \{w = \bar{z}\}, \ D \text{ being unit disk,}$$

w, z are the coordinates on $\mathbb{C}$ and D respectively. Other examples of nonstein nonuniformizable skew cylinders with fibers $\mathbb{C}$ may be found in [73].

We prove the statement of Theorem 2.3.11 for an exotic u.s.a. V given by the following

Theorem 2.3.13 *[12] Let D be unit disk in complex line with the coordinate z. Let $E_+ = \{\frac{1}{2}\} \cup \{\frac{n}{2n+1}\}_{n\in\mathbb{N}}$, $E_- = -E_+ \subset D$. There exists a closed subset $K \subset \mathbb{C} \times \overline{D}$ such that*
1) the complement $V = (\mathbb{C} \times D) \setminus K$ is pseudoconvex;
2) for any $z \notin E_+ \cup E_-$ the fiber $K \cap \pi^{-1}(z)$ is a disc;
3) for any $z \in E_+$ $K \cap \pi^{-1}(z) = 0 \times z$;
4) for any $z \in E_-$ $K \cap \pi^{-1}(z) = 1 \times z$.

For the completeness of presentation, we recall the construction of the set K from [12]. Let w be the coordinate in the fiber $\mathbb{C}$ on the direct product $\mathbb{C} \times D$. Let $u(z) = \ln|z - \frac{1}{2}| + \ln|z + \frac{1}{2}| + \sum_{n=1}^{+\infty} 2^{-n}(\ln|z - \frac{n}{2n+1}| + \ln|z + \frac{n}{2n+1}|)$, $A \in \mathbb{R}^+$. The function u is harmonic, $u(E_\pm) = -\infty$. Let $\psi : \overline{D} \to \mathbb{C}$ be a C^∞ function with bounded derivatives (up to the second order) that is constant in a neighborhood of each set $E_\pm$ so that $\psi|_{E_+} = 0$, $\psi|_{E_-} = 1$. Define

$$K = \{|w - \psi(z)| \leq e^{u(z)+|z|^2+A}\}. \tag{1}$$

The fibers of K over E_+ (E_-) are single points where the coordinate w is equal to 0 and 1 respectively. Its other fibers are disks. Thus, $V = (\mathbb{C} \times D) \setminus K$ is a uniformizable skew annulus. If A is large enough, then V is pseudoconvex [12].

Proof of Theorem 2.3.11. Let $V = (\mathbb{C} \times D) \setminus K$ be a skew annulus given by Theorem 2.3.13, $p_V : M \to V$ be its universal covering. The manifold M is Stein, as is V (the pseudoconvexity statement in Theorem 2.3.13), since a covering over a Stein manifold is Stein [109]. Let us prove that the skew cylinder M is nonuniformizable (by contradiction). Suppose the contrary : there exists a uniformization $g : M \to \overline{\mathbb{C}} \times D$. Let $f : M \to \overline{\mathbb{C}}$ be the $\overline{\mathbb{C}}$- component of g. The fibers of the cylinder M over $E_\pm$ are conformally equivalent to complex line. Let w be the $\mathbb{C}$- coordinate on $\mathbb{C} \times D \supset V$. Consider the multivalued holomorphic function $\ln w \circ p_V$ on M. It provides a well-defined 1-to-1 parametrization by $\mathbb{C}$ of the fibers of M over E_+. This is not the case for the fibers over E_-, where this function is multivalued and has branch points where $w \circ p_V = 0$. The function f is univalent on each fiber of M by definition. Therefore, for any $z \in E_+$ the restriction to the fiber of M over z of the function f is Möbius in the chart $\ln w \circ p_V$, and this is not the case for $z \in E_-$. Let Sf be the Schwartzian derivative of f along the fibers of M in the (multivalued) coordinate $\ln w \circ p_V$. It is a well-defined holomorphic function on $M \setminus \{w = 0\}$, since any two distinct branches of $\ln w$ differ from each other by constant. It vanishes identically on all the fibers over the set E_+, which contains a limit point $\frac{1}{2}$. Therefore, $Sf \equiv 0$ on M. On the other hand, Sf does not vanish identically on the fibers over the set E_-, since f is not Möbius in the previous coordinate on these fibers. The contradiction thus obtained proves that M is nonuniformizable. $\square$

2.3.3 Nonuniformizable universal covering manifolds. Proof of Theorem 2.3.6

The proof of Theorem 2.3.6 and its Addendum is based on Propositions 2.3.17, 2.3.20 and Theorem 2.3.18 stated and proved below. Theorem 2.3.18 follows from Proposition 2.3.20 and Lemma 2.3.22, which is the main technical statement of the subsection. Theorem 2.3.6 will be deduced from them at the end of the section.

Definition 2.3.14 An affine algebraic foliation is *geometrically nice*, if it satisfies the statements 1)-3), 5) of the Addendum to Theorem 2.3.6 (in particular, it has a dense leaf with an attracting cycle).

Definition 2.3.15 Let F be an algebraic foliation, D be a simply connected cross-section such that some leaf contains an attracting cycle starting at a point $0 \in D$ with a well-defined Poincaré return mapping $h : D \to D$ (then $h(0) = 0$). Let $hD \Subset D$. Then we say that D is (h-) *contracting*. In this case the corresponding u.c.m. M_D is also said to be *contracting*.

Definition 2.3.16 Two skew cylinders are said to be *equivalent*, if there exist biholomorphisms of their total spaces and bases that form a commutative diagram with the projections.

Proposition 2.3.17 *Let an algebraic foliation have a nonuniformizable contracting u.c.m. M_D, $0 \in D$ be the starting point of the corresponding attracting cycle. Then M_D is locally nonuniformizable at 0.*

Proof The iterations h^n converge to 0 uniformly on D, as $n \to +\infty$ (since $hD \Subset D$). For any $n \in \mathbb{N}$ the u.c.m. $M_{h^n D}$ corresponding to the smaller cross-section $h^n D$ is equivalent to M_D. Since M_D is nonuniformizable by assumption, so is $M_{h^n D}$. This together with the uniform convergence $h^n \to 0$ implies Proposition 2.3.17. $\square$

Theorem 2.3.18 *There exists a geometrically nice foliation F having at least one nonuniformizable contracting u.c.m. M_D. The foliation F and the cross-section D may be chosen so that in addition, all the u.c.m.'s associated to the projective extension $\overline{F}$ be manifolds, and the one corresponding to D be nonuniformizable.*

For the proof of Theorem 2.3.18 let us introduce the following definition.

Definition 2.3.19 Let (M, p, D) be a skew cylinder, $B \subset M$ ($B \Subset M$) be its subdomain. Then B is called a *(compact) subcylinder*, if the triple $(B, p, p(B))$ is a skew cylinder.

Proposition 2.3.20 *(by Ilyashenko, see [106]). Let a Stein skew cylinder be exhausted by an increasing sequence of uniformizable subcylinders. Then it is uniformizable.*

Remark 2.3.21 A.A.Shcherbakov [106] proved that any Stein skew cylinder can be exhausted by a growing sequence of compact subcylinders with smooth strictly pseudoconvex boundaries. His result together with Theorem 2.3.11 and Proposition 2.3.20 imply the existence of a nonuniformizable compact skew cylinder with a strictly pseudoconvex boundary.

Lemma 2.3.22 *For any Stein u.s.a. any compact subcylinder of its universal covering is equivalent to a subcylinder of a contracting u.c.m. corresponding to a geometrically nice foliation.*

Remark 2.3.23 Yu.S.Ilyashenko had shown (late 1960-ths, unpublished) that *any compact subcylinder of a Stein skew cylinder is equivalent to a subcylinder of a u.c.m. corresponding to an affine (projective) algebraic foliation.* He proved this by considering the Stein cylinder as embedded to $\mathbb{C}^N$ so that its cylinder projection be the restriction of an orthogonal projection $p : \mathbb{C}^N \to \mathbb{C}$, and then approximating its compact subcylinder by a piece of an algebraic surface S. The foliaton on S we are looking for is the fibration defined by the same orthogonal projection. The method of the proof of Lemma 2.3.22 given below was motivated by this Ilyashenko's method.

Proof of Lemma 2.3.22 (sketch). We consider the auxiliary foliation on $(\mathbb{C} \setminus \pm 1) \times \mathbb{C}$ (denoted by $F_{\alpha,\beta}$) with the first integral $I(w, z) = z(1 - w)^{\alpha} + \beta \int_0^w \frac{(1-\tau)^{\alpha}}{\tau+1} d\tau$. (The foliation $F_{\alpha,\beta}$ tends to the parallel line fibration $z = const$, as $\alpha, \beta \to 0$.)

Proposition 2.3.24 *The foliation $F_{\alpha,\beta}$ is algebraic and transversally affine. If $\alpha \notin \mathbb{R} \cup i\mathbb{R}$, $\beta \neq 0$, then all its leaves are dense. Let $h_+ : 0 \times \mathbb{C} \to 0 \times \mathbb{C}$ be the first return mapping corresponding to $F_{\alpha,\beta}$ and the circuit in $\mathbb{C} \times 0$ starting at 0×0 and going around 1×0 counterclockwise. The mapping h_+ is affine (i.e., linear nonhomogeneous) with the derivative $e^{-2\pi i\alpha}$. If $\operatorname{Im} \alpha < 0$, then h_+ is a contraction and its fixed point is $0 \times O(\beta)$, as $\alpha, \beta \to 0$.*

Proposition 2.3.24 easily follows from the definition of the foliation $F_{\alpha,\beta}$. Its statements imply that the foliation $F_{\alpha,\beta}$ becomes geometrically nice after realizing its phase space $(\mathbb{C} \setminus \pm 1) \times \mathbb{C}$ as an affine algebraic surface.

Let $V \subset \mathbb{C} \times D$ be a given Stein u.s.a., M be its universal covering, $B \subset M$ be a compact subcylinder. Denote $p_V : M \to V$ the covering projection. Recall that w and z are the coordinates on $\mathbb{C}$ and D respectively. Fix a $R > 4$ such that

$$p_V(B) \subset \{|w| < R - 4\},\ \{|w| \geq R - 4\} \times D \subset V. \text{ Put} \tag{2.3.1}$$

$$V_R = (V + (iR, 0)) \setminus (\pm 1 \times D) \subset \mathbb{C} \times D,\ p_{V,R}(B) = p_V(B) + (iR, 0) \subset V_R. \tag{2.3.2}$$

Fix a disk $D' \Subset D$ centered at 0 such that $\pi(p_V(B)) = \pi(p_{V,R}(B)) \Subset D'$. Replace the parallel line fibration $z = const$ of V_R by the restriction to V_R of the foliation $F_{\alpha,\beta}$. Consider auxiliary domains $\Sigma_1, \dots, \Sigma_4 \Subset (\mathbb{C} \setminus \pm 1) \times D$ with the following properties :

$$\Sigma_1 = \{|w - iR| < R\} \times D',\ \Sigma_1 \Subset \Sigma_2,\ \Sigma_3 \Subset \Sigma_4 \Subset (V_R \cap \Sigma_2),\quad p_{V,R}(B) \Subset \Sigma_3,\ 0 \times D' \Subset \Sigma_3,$$

the domain Σ_2 being a bidisk (whose closure is disjoint from $\pm 1 \times D$ by definition), the $\mathbb{C}$- fibers of the domain $\Sigma_{3,4}$ being diffeomorphic to an annulus. The existence of the domains $\Sigma_{2,3,4}$ follows from definition : the fibers of the skew annulus V are topological annuli. For any α, β small enough there exists a biholomorphism

$$\chi : \overline{\Sigma_2} \to \chi(\overline{\Sigma_2}) \Subset \mathbb{C} \times D,\ \chi|_{0 \times D'} = 0 \times Id_{D'},\ \chi(\Sigma_3) \Subset \Sigma_4, \tag{2.3.3}$$

that transforms the foliation $z = const$ to the foliation $F_{\alpha,\beta}$ and preserves the w- coordinate : the leaves of the foliation $F_{\alpha,\beta}$ are uniformly close to the product $\mathbb{C}$- fibers in any closed bidisk disjoint from $\pm 1 \times D$, whenever α and β are small enough.

We fix α and β such that $\alpha \notin \mathbb{R} \cup i\mathbb{R}$, $\operatorname{Im} \alpha < 0$, $\beta \neq 0$. We show that if they are small enough, then there exist a smooth affine surface S and a rational mapping $P : S \to (\mathbb{C} \setminus \pm 1) \times \mathbb{C}$ with nowhere degenerate Jacobian matrix such that the subcylinder B and the foliation $F = P_*^{-1} F_{\alpha,\beta}$ on S satisfy the statements of Lemma 2.3.22.

To construct S, P and F, we consider the Stein manifold V_R as a submanifold in some space $\mathbb{C}^N$ so that the natural inclusion $V_R \to \mathbb{C}^2$ is the restriction to V_R of an orthogonal projection $P : \mathbb{C}^N \to \mathbb{C}^2$. Let V^r be the intersection of V_R with a ball centered at 0 of a large radius r such that

$$P(V^r) \Supset \Sigma_4. \tag{2.3.4}$$

We approximate $\overline{V^r}$ by a compact piece of a smooth affine algebraic surface $S' \subset \mathbb{C}^N$ using results of [11] (cf. [43]) and approximation and extension theorems for functions on Stein manifolds. We do the approximation so that $P|_{S'}$ has a holomorphic inverse (denoted $(P|_{S'})^{-1}$) on $\overline{\Sigma}_4$. In what follows, we identify Σ_4 (and hence, Σ_3) with its image in S' under the latter inverse : thus, $\Sigma_3 \Subset \Sigma_4 \Subset S'$. Let $S = S' \setminus (Crit(P|_{S'}) \cup \{w \circ P = \pm 1\})$, $\tilde{D} = (P|_{S'})^{-1}(0 \times D')$. The foliation $F = (P|_S)_*^{-1} F_{\alpha,\beta}$ is the one we are looking for, if r is large enough and α, β are small enough : $\tilde{D}$ is a contracting cross-section to F and B is embedded to $M_{\tilde{D}}$ as a subcylinder. The latter embedding is constructed as follows. Recall that

$$\Sigma_4 \subset S', \text{ and } F|_{\Sigma_4} = F_{\alpha,\beta}, \ p_{V,R}(B) \Subset \Sigma_3 \Subset \Sigma_4$$

by construction. Let $\chi : \Sigma_3 \to \Sigma_4$ be the mapping (2.3.3). The mapping $\phi = \chi \circ p_{V,R} : B \to \Sigma_4$ sends the fibers of the skew cylinder B to leaves of the foliation $F_{\alpha,\beta}$. Two points in B are mapped by ϕ to one and the same point in Σ_4, if and only if they lie in one and the same fiber of B and the path connecting them is transformed by ϕ to a contractible closed loop in a leaf of the foliation $F_{\alpha,\beta}$. This follows from construction. Consider the projection $\psi : M_{\widetilde{D}} \to S$, which sends the universal cover of each leaf of F to the leaf itself. Consider the germ of the inverse ψ^{-1} sending $0 \times D'$ to the canonical section (a copy of $\widetilde{D}$) of the cylinder $M_{\widetilde{D}}$. This is a multivalued analytic mapping $\Sigma_4 \to M_{\widetilde{D}}$ that extends analytically along each path in any leaf of $F_{\alpha,\beta}|_{\Sigma_4}$. This implies that the corresponding composition $Q = (\psi)^{-1} \circ \phi$ yields a holomorphic mapping of B onto a subset in $M_{\widetilde{D}}$. The latter mapping is a biholomorphism : its injectivity follows from construction and the fact that no contractible loop in a leaf of F can be transformed by P to a noncontractible loop in a leaf of $F_{\alpha,\beta}$ (the maximum principle for holomorphic functions). The foliation F is geometrically nice. This follows from its construction, Proposition 2.3.24 and the discreteness of the preimage of each point in $\mathbb{C}^2$ under the mapping $P|_S$, which is a local biholomorphism. (The latter discreteness statements yields in particular that the density of leaves of the foliation $F_{\alpha,\beta}$ implies the density of leaves of its pullback F.) This proves Lemma 2.3.22. □

Proof of Theorem 2.3.18. Let V be a Stein u.s.a. with a nonuniformizable universal covering M. By Proposition 2.3.20, M contains a nonuniformizable compact subcylinder B. By Lemma 2.3.22, B is equivalent to a subcylinder of a contracting u.c.m. of a geometrically nice foliation. The latter u.c.m. is nonuniformizable as well. The proof of the second statement of Theorem 2.3.18 (on projective extension) is relatively easy and is omitted to save the space. □

Proof of Theorem 2.3.6. Let F, M_D be as in Theorem 2.3.18. By assumption, each leaf of F is dense and the cross-section D is contracting (and hence, intersects an attracting cycle in some leaf). Let $0 \in D$ be the starting point of this attracting cycle, L be the leaf of F through 0. By Proposition 2.3.17, M_D is locally nonuniformizable at 0. For any cross-section D' intersecting L the u.c.m. $M_{D'}$ is locally nonuniformizable at the points of the intersection $D' \cap L$. Now density of L implies Theorem 2.3.6. (Recall that the foliation F is geometrically nice, hence, each its leaf is dense.) Statement 4) of the Addendum follows analogously from the second statement of Theorem 2.3.18 and Proposition 2.3.17. □

Chapitre 3

On minimality of horospheric laminations associated to rational functions

This chapter deals with iterations of rational functions $f(z) = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree at least two. In 3.1.2 we recall Lyubich-Minsky construction (briefly mentioned in the Introduction), which associates to each f the following objects : affine Riemann surface lamination $\mathcal{A}_f$; lamination $\mathcal{H}_f$ by hyperbolic three-dimensional varietes (that may have singularities), the lifted dynamics $\hat{f} : \mathcal{H}_f \to \mathcal{H}_f$ and the quotient hyperbolic lamination $\mathcal{H}_f/\hat{f}$. Each leaf of $\mathcal{H}_f$ and $\mathcal{H}_f/\hat{f}$ is foliated itself by horospheres, which form the horospheric laminations of $\mathcal{H}_f$ and $\mathcal{H}_f/\hat{f}$.

In Section 3.2 we present the main results of the papers [48, 49], which concern topological transitivity and minimality of the horospheric lamination of the quotient $\mathcal{H}_f/\hat{f}$. The principal Theorem 3.2.3 says that the quotient horospherical lamination (with isolated hyperbolic leaves deleted) is topologically transitive (i.e., at least one horosphere is dense), provided that the map f does not belong to the following list of exceptions :

$$z^{\pm d}, \text{ Chebyshev polynomials, Lattès examples.} \tag{3.0.1}$$

In this case, all the horospheres "over the repelling periodic orbits" are dense.

Remark 3.0.25 For any exceptional f on the list (3.0.1), each horosphere in a nonisolated leaf of $\mathcal{H}_f/\hat{f}$ is nowhere dense in $\mathcal{H}_f/\hat{f}$ (see [78] and Corollary 3.2.2).

Theorem 3.2.4 asserts that all the horospheres are dense (outside possible isolated hyperbolic leaves) for any non-exceptional f which is critically non-recurrent without parabolic periodic points.

In the case when parabolic points are allowed, a more general Theorem 3.2.5 says that all the horospheres are dense in $\mathcal{H}_f/\hat{f}$ (outside possible isolated hyperbolic leaves), except for the horospheres "related" to the parabolic points. To prove it, we show (Theorem 3.2.6) that any horosphere in question accumulates onto some horosphere over an appropriate repelling periodic point (which is dense by Theorem 3.2.3). Theorem 3.2.7 deals with an arbitrary rational function having a parabolic periodic point. It says that each horosphere in a leaf associated to this point is closed in $\mathcal{H}_f/\hat{f}$ and does not accumulate onto itself.

Remark 3.0.26 There exist non-exceptional rational functions (even hyperbolic) such that the corresponding hyperbolic lamination $\mathcal{H}_f$ has a leaf whose horospheres are nowhere dense in $\mathcal{H}_f$. This is true, e.g., for real quadratic polynomials $f_\varepsilon(z) = z^2 + \varepsilon$ with $\varepsilon < \frac{1}{4}$, $\varepsilon \neq 0, -2$ (which are hyperbolic, e.g., whenever ε is small enough). Moreover, this is true for an open set of complex values of the parameter ε containing the above real values. The leaf with nowhere dense horospheres is associated to a repelling fixed point (which is real, if so is ε). These statements are proved in [49].

On the other hand, under some arithmetic assumptions on the multipliers of repelling periodic points, the horospherical lamination of $\mathcal{H}_f$ is topologically transitive (private communication by M. Lyubich and D. Saric).

Example 3.0.27 Let us consider once again the quadratic family $f_\varepsilon(z) = z^2+\varepsilon$. It is well-known that the quotient hyperbolic laminations $\mathcal{H}_{f_0}/\hat{f}_0$ and $\mathcal{H}_{f_\varepsilon}/\hat{f}_\varepsilon$ are homeomorphic for all $\varepsilon \neq 0$ small enough. (The homeomorphism sends leaves to leaves but not isometrically.) On the other hand, Theorem 3.2.4 implies that if $\varepsilon \neq 0$ is small enough, then each horosphere in the latter lamination is dense, while no horosphere in the former lamination (with $\varepsilon = 0$) is dense (see Corollary 3.2.2).

The necessary background material is recalled in Section 3.1 : iterates of rational functions, see 3.1.1 ; affine and hyperbolic laminations, see 3.1.2 ; horospheres and their metric properties, see 3.1.3.

Brief proofs of Theorems 3.2.3, 3.2.6 and 3.2.7 are given in Section 3.3. To prove Theorem 3.2.3, we fix a horosphere in $\mathcal{H}_f$ "over" a repelling periodic orbit and show that the orbit of this horosphere under the forward and the backward iterates of $\hat{f}$ is dense. To this end, we study the holonomies of the horosphere along loops based at a repelling periodic point. We show that the images of a point of the horosphere under consecutively applied dynamics and holonomies are dense in the fiber over the base point. To do this, we use the description of the holonomy in terms of the basic cocycle introduced in [78] (its definition and some basic properties are recalled in 3.1.3).

Recall that everywhere below we assume that the rational function $f(z) = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ under consideration has degree at least 2.

3.1 Background material : rational dynamics, laminations and horospheres

3.1.1 Rational iterations

The basic notions and facts of holomorphic dynamics recalled here are contained, e.g., in [88] and [89]. Let

$$f = \frac{P(z)}{Q(z)} : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \text{ be a rational function. Recall that}$$

- its *Julia set* $J = J(f)$ is the closure of the union of the repelling periodic points, see the next Definition. An equivalent definition of the Julia set says that its complement $\overline{\mathbb{C}} \setminus J$ (called the *Fatou set*) is the maximal open subset where the iterations f^n form a normal family (i.e., are equicontinuous on compact subsets). One has

$$f^{-1}(J) = J = f(J).$$

Definition 3.1.1 A germ of nonconstant holomorphic mapping $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ at a fixed point 0 is called *attracting (repelling / parabolic / superattracting)*, if its derivative at the fixed point respectively has nonzero modulus less than 1 (has modulus greater than 1 / is equal to a root of unity and no iteration of the mapping f is identity / is equal to zero). An attracting (repelling, parabolic or superattracting) periodic point of a rational mapping is a fixed point (of the corresponding type) of its iteration.

Definition 3.1.2 A rational function is said to be *hyperbolic*, if the forward orbit of each its critical point either is periodic itself (and hence, superattracting), or tends to an attracting (or a superattracting) periodic orbit.

Definition 3.1.3 Given a rational function. A point of the Riemann sphere is called *postcritical*, if it belongs to the forward orbit of a critical point. A rational function is called *critically-finite*, if the number of its postcritical points is finite.

Definition 3.1.4 The ω- limit set $\omega(c)$ of a point $c \in \overline{\mathbb{C}}$ is the set of limits of converging subsequences of its forward orbit $\{f^n(c) | n \geq 0\}$ (the ω- limit set of a periodic orbit is the orbit itself). A point c is called *recurrent*, if $c \in \omega(c)$.

Definition 3.1.5 A rational mapping is called *critically-nonrecurrent*, if each its critical point is either nonrecurrent, or periodic (or equivalently, each critical point in the Julia set is nonrecurrent).

Example 3.1.6 The following mappings are critically-nonrecurrent : any hyperbolic mapping ; any critically-finite mapping ; any quadratic polynomial with a parabolic periodic orbit. A hyperbolic mapping has no parabolic periodic points.

Theorem 3.1.7 *A germ of conformal mapping at an attracting (repelling) fixed point is always conformally linearizable : there exists a local conformal coordinate in which the germ is equal to its linear part (the multiplication by its derivative at the fixed point).*

Remark 3.1.8 Let $f(z) = z + z^{k+1} + \dots$ be a parabolic germ tangent to the identity. The set $\{z^k \in \mathbb{R}_+\}$ consists of k rays going out of 0 (called *repelling rays*) such that

- each repelling ray is contained in an appropriate sector S (called *repelling sector*) for which there exists an arbitrarily small neighborhood $U = U(0) \subset \mathbb{C}$ where f is univalent and such that $f(S \cap U) \supset S \cap U$ and each backward orbit of the restriction $f|_{S \cap U}$ enters the fixed point 0 asymptotically along the corresponding repelling ray ;

- there is a canonical 1-to-1 conformal coordinate t on $S \cap U$ in which f acts by translation : $t \mapsto t + 1$; if the previous sector S is chosen large enough, then this coordinate parametrizes $S \cap U$ by a domain in $\mathbb{C}$ containing a left half-plane ; the previous coordinate is well-defined up to translation and is called *Fatou coordinate* (see [27], [117]).

For any parabolic germ (not necessarily tangent to the identity) its appropriate iteration is tangent to the identity. By definition, the repelling rays and sectors of the former are those (defined above) of the latter.

Let us recall what are Chebyshev polynomials and Lattès examples.

Chebyshev polynomials. For any $n \in \mathbb{N}$ there exists a unique (real) polynomial p_n of degree n that satisfies the trigonometric identity $\cos n\theta = p_n(\cos\theta)$. It is called *Chebyshev polynomial.*

Lattès examples. Consider a one-dimensional complex torus, which is the quotient of $\mathbb{C}$ by a lattice. Consider arbitrary multiplication by a constant $\lambda \in \mathbb{C}$, $|\lambda| > 1$, that maps the lattice to itself. It induces an endomorphism of the torus of degree greater than 1. The quotient of the torus by the central symmetry $z \mapsto -z$ is a Riemann sphere. The previous endomorphism together with the quotient projection induce a rational transformation of the Riemann sphere called *Lattès example.*

Remark 3.1.9 Let f be either Chebyshev, or Lattès. Then it is critically finite. More precisely, the forward critical orbits eventually finish at repelling fixed points. The Julia set of a Chebyshev polynomial is the segment $[-1, 1]$ of the real line, while that of a Lattès example is the whole Riemann sphere. Chebyshev and Lattès functions have branch-exceptional repelling fixed points, see the following definition.

Definition 3.1.10 [77] A repelling periodic point of a rational function is called *branch-exceptional*, if any its nonperiodic backward orbit contains a critical point. In this case its periodic orbit is also called branch-exceptional.

Remark 3.1.11 (Lasse Rempe [77]). There exist rational functions with branch-exceptional repelling fixed points that are neither Chebyshev, nor Lattès.

3.1.2 Affine and hyperbolic dynamical laminations

The constructions presented here were introduced in [89]. We recall them briefly and send the reader to [89] for more details.

Recall that a *lamination* is a "topological" foliation by manifolds, i.e., a topological space that is split as a disjoint union of manifolds (called *leaves*) of one and the same dimension so that each point of the ambient space admits a neighborhood (called "flow-box") such that each connected component (local leaf) of its intersection with each leaf is homeomorphic to a ball; the neighborhood itself is homeomorphic to the product of the ball and some (transversal) topological space under a homeomorphism transforming the local leaves to the fibers of the product.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. Denote

$$\mathcal{N}_f = \{\hat{z} = (z_0, z_{-1}, \dots) \mid z_{-j} \in \overline{\mathbb{C}},\ f(z_{-j-1}) = z_{-j}\}.$$

This is a topological space equipped with the natural product topology and the projections

$$\pi_{-j} : \mathcal{N}_f \to \overline{\mathbb{C}},\ \hat{z} \mapsto z_{-j}.$$

The action of f on the Riemann sphere lifts naturally up to a homeomorphism

$$\hat{f} : \mathcal{N}_f \to \mathcal{N}_f,\ (z_0, z_{-1}, \dots) \mapsto (f(z_0), z_0, z_{-1}, \dots),\ f \circ \pi_{-j} = \pi_{-j} \circ \hat{f}.$$

First of all we recall the construction of the "regular leaf subspace" $\mathcal{R}_f \subset \mathcal{N}_f$, which is a union of Riemann surfaces that foliate $\mathcal{R}_f$ in a very turbulent way. Afterwards we take the subset $\mathcal{A}_f^n \subset \mathcal{R}_f$ of the leaves conformally-equivalent to $\mathbb{C}$. Then we refine the induced topology on $\mathcal{A}_f^n$ to make it a lamination (denoted $\mathcal{A}_f^l$) by complex lines with a continuous family of affine structures on them. Afterwards we take a completion $\mathcal{A}_f = \overline{\mathcal{A}_f^l}$ in the new topology. The space $\mathcal{A}_f$ is a lamination by affine Riemann surfaces (the new leaves added by the completion may have conical singularities). Then we discuss the three-dimensional extension of $\mathcal{A}_f$ up to a lamination $\mathcal{H}_f$ by hyperbolic manifolds (with singularities).

Let $\hat{z} \in \mathcal{N}_f$, $V = V(z_0) \subset \overline{\mathbb{C}}$ be a neighborhood of z_0. For any $j \geq 0$ denote

$$V_{-j} = \text{the connected component of the preimage } f^{-j}(V) \text{ that contains } z_{-j}.$$

$$\text{Then } V_0 = V, \text{ and } f^j : V_{-j} \to V \text{ are ramified coverings.}$$

Definition 3.1.12 We say that a point $\hat{z} \in \mathcal{N}_f$ is *regular*, if there exists a disk V containing the initial point z_0 such that the above coverings $f^j : V_{-j} \to V$ have uniformly bounded degrees. Denote

$$\mathcal{R}_f \subset \mathcal{N}_f \quad \text{the set of the regular points in } \mathcal{N}_f.$$

Example 3.1.13 Let $\hat{z} \in \mathcal{N}_f$ be a backward orbit such that there exists a $j \in \mathbb{N} \cup 0$ for which the point z_{-j} is disjoint from the ω- limit sets of the critical points. Then $\hat{z} \in \mathcal{R}_f$. If the mapping f is hyperbolic, then this is the case, if and only if $\hat{z}$ is not a (super) attracting periodic orbit. A mapping f is critically-nonrecurrent, if and only if

$$\mathcal{R}_f = \mathcal{N}_f \setminus \{\text{attracting and parabolic periodic orbits}\}, \text{ see [89]}.$$

Definition 3.1.14 Let $\hat{z} \in \mathcal{R}_f$, V, V_{-j} be as in Definition 3.1.12. The *local leaf* $L(\hat{z}, V) \subset \mathcal{R}_f$ is the set of the points $\hat{z}' \in \mathcal{R}_f$ such that $z'_{-j} \in V_{-j}$ for all j (the local leaf is path-connected by definition). We say that the previous local leaf is *univalent over* V, if the projection π_0 maps it bijectively onto V. The *global leaf* containing $\hat{z}$ (denoted $L(\hat{z})$) is the maximal path-connected subset in $\mathcal{R}_f$ containing $\hat{z}$.

Remark 3.1.15 Each leaf $L(\hat{z}) \subset \mathcal{R}_f$ carries a natural structure of Riemann surface so that the restrictions to the leaves of the above projections π_{-j} are meromorphic functions. A local leaf $L(\hat{z}, V) \subset L(\hat{z})$ (when well-defined) is the connected component containing $\hat{z}$ of the preimage $(\pi_0|_{L(\hat{z})})^{-1}(V) \subset L(\hat{z})$.

Remark 3.1.16 The above-defined objects $\mathcal{R}_f$, $\mathcal{R}_{f^n}$ corresponding to both f and any its forward iteration f^n, are naturally homeomorphic under the mapping that sends a backward orbit $\hat{z} \in \mathcal{N}_f$ to the backward orbit $(z_0, z_{-n}, z_{-2n}, \dots) \in \mathcal{N}_{f^n}$. The latter homeomorphism maps the leaves conformally onto the leaves.

We use the following

Lemma 3.1.17 (Shrinking Lemma) *[89] Let f be a rational mapping, $V \subset \overline{\mathbb{C}}$ be a domain, $V' \Subset V$ be a compact subset. Then for any sequence of single-valued branches $f^{-n} : V \to \overline{\mathbb{C}}$ the diameters of the images $f^{-n}(V')$ tend to 0, as $n \to +\infty$ (except for the cases, when f has either a Siegel disk or a Herman ring that contains an infinite number of the previous images).*

Remark 3.1.18 Parabolic leaves in $\mathcal{R}_f$ always exist (see the next two Examples) and are simply connected; hence they are conformally equivalent to $\mathbb{C}$ [89]. If f is critically-nonrecurrent, then each leaf is parabolic [89]. On the other hand, there are rational mappings such that some leaves of $\mathcal{R}_f$ are hyperbolic (e.g., if there is either a Siegel disk or a Herman ring, see [89]). J.Kahn proved [77] that if the postcritical points are dense in the Julia set, then there are always some hyperbolic leaves in $\mathcal{R}_f$.

Example 3.1.19 Let $a \in \overline{\mathbb{C}}$ be a repelling fixed point of f, $\hat{a} = (a, a, \dots) \in \mathcal{N}_f$ be its fixed orbit. Then $\hat{a} \in \mathcal{R}_f$ and the leaf $L(\hat{a})$ is parabolic (it is $\hat{f}$- invariant and the quotient of $L(\hat{a}) \setminus \hat{a}$ by $\hat{f}$ is a torus). The linearizing coordinate w of f in a neighborhood of a lifts up to a conformal isomorphism $w \circ \pi_0 : L(\hat{a}) \to \mathbb{C}$. Analogously, the periodic orbit of a repelling periodic point is contained in a parabolic leaf (see Remark 3.1.16).

Example 3.1.20 Let f have a parabolic fixed point $a \in \overline{\mathbb{C}}$, $f'(a) = 1$, $\hat{a} = (a, a, \dots) \in \mathcal{N}_f$ be its fixed orbit. Then $\hat{a} \notin \mathcal{R}_f$. On the other hand, for each repelling ray (see Remark 3.1.8) there is a unique leaf in $\mathcal{R}_f$ (denoted L_a) consisting of the backward orbits that converge to a asymptotically along the chosen ray. This leaf is parabolic : the Fatou coordinate w on the corresponding repelling sector lifts up to a conformal isomorphism $w \circ \pi_0 : L_a \to \mathbb{C}$. An analogous statement holds true in the case, when a is a parabolic periodic point (and not necessarily tangent to the identity).

Definition 3.1.21 The leaves from the two previous examples are called respectively a *leaf associated to a repelling (respectively, parabolic) periodic point.*

Proposition 3.1.22 *A point $\hat{z} \in \mathcal{N}_f$ belongs to a leaf associated to a repelling (or parabolic) fixed point a, if and only if it is represented by a backward orbit converging to a (and distinct from its fixed orbit, if the latter is parabolic).*

The Proposition follows from the Shrinking Lemma.
Denote

$$\mathcal{A}_f^n = \text{ the union of the parabolic leaves in } \mathcal{R}_f.$$

If f is hyperbolic, then $\mathcal{A}_f^n$ is a lamination with a global Cantor transversal section. In general, $\mathcal{A}_f^n$ is not a lamination in a good sense, since some ramified local leaves can accumulate to a univalent one in the product topology. The refined topology (defined in [89]) that makes it a "lamination with singularities" is recalled below. To do this, we use the following

Remark 3.1.23 Let $\hat{z} \in \mathcal{A}_f^n$. Fix a conformal isomorphism $\mathbb{C} \to L(\hat{z})$ that sends 0 to $\hat{z}$ (it is unique up to multiplication by nonzero complex constant in the source). The natural projections $\pi_{-j} : \mathcal{N}_f \to \overline{\mathbb{C}}$ induce a meromorphic function sequence $\phi_{-j,\hat{z}} = \pi_{-j}|_{L(\hat{z})}$ on the leaf $L(\hat{z}) = \mathbb{C}$:

$$\phi_{-j,\hat{z}} : \mathbb{C} \to \overline{\mathbb{C}},\ \phi_{-j+1,\hat{z}} = f \circ \phi_{-j,\hat{z}} \text{ for any } j;\ \phi_{-j,\hat{z}}(0) = z_{-j}. \tag{3.1.1}$$

The latter function sequence is uniquely defined up to the $\mathbb{C}^*$- action on the source space $\mathbb{C}$ (by multiplication by complex constants). Two points of $\mathcal{A}_f^n$ lie in one and the same leaf, if and only if the corresponding function sequences are obtained from each other by affine transformation of the variable.

Denote $\hat{\mathcal{K}}_f$ the space of the meromorphic function sequences

$$\{\phi_{-j}(t)\}_{j \in \mathbb{N} \cup 0},\ \phi_{-j} : \mathbb{C} \to \overline{\mathbb{C}},\ \phi_{-j+1} = f \circ \phi_{-j} \text{ for all } j. \tag{3.1.2}$$

This is a subset of the infinite product of copies of the meromorphic function space; the latter space is equipped with the topology of uniform convergence on compact sets. The product topology induces a topology on the space $\hat{\mathcal{K}}_f$. The groups $Aff(\mathbb{C})$ (complex affine transformations of $\mathbb{C}$), $\mathbb{C}^* \subset Aff(\mathbb{C})$ and $S^1 = \{|z| = 1\} \subset \mathbb{C}^*$ act on the space $\hat{\mathcal{K}}_f$ by variable changes in the source. Denote

$$\hat{\mathcal{K}}_f^a = \hat{\mathcal{K}}_f / \mathbb{C}^*,\ \hat{\mathcal{K}}_f^h = \hat{\mathcal{K}}_f / S^1. \tag{3.1.3}$$

(The latters are equipped with the corresponding quotients of the topology of $\hat{\mathcal{K}}_f$.) A *leaf* in $\hat{\mathcal{K}}_f^a$ (respectively $\hat{\mathcal{K}}_f^h$) is the quotient projection of an orbit of the previous action $Aff(\mathbb{C}) : \hat{\mathcal{K}}_f \to \hat{\mathcal{K}}_f$. Each leaf is naturally identified with a quotient $\Gamma \backslash Aff(\mathbb{C}) / \mathbb{C}^*$ (respectively, $\Gamma \backslash Aff(\mathbb{C}) / S^1$), where Γ is a discrete group of Euclidean isometries of $\mathbb{C}$. This equips the leaves with affine (respectively, hyperbolic) structures that vary continuously on $\hat{\mathcal{K}}_f^a$ ($\hat{\mathcal{K}}_f^h$). There is a natural (not necessarily continuous) inclusion

$$\mathcal{A}_f^n \to \hat{\mathcal{K}}_f^a.$$

Definition 3.1.24 The topological subspace $\mathcal{A}_f^l \subset \hat{\mathcal{K}}_f^a$ is the image of the space $\mathcal{A}_f^n$ under the previous inclusion (or equivalently, the space $\mathcal{A}_f^n$ equipped with the topology induced from $\hat{\mathcal{K}}_f^a$). The space $\mathcal{A}_f$ (which is called the *affine orbifold lamination associated to a rational function* f) is the closure of $\mathcal{A}_f^l$ in the space $\hat{\mathcal{K}}_f^a$. The subspace $\mathcal{H}_f^l \subset \hat{\mathcal{K}}_f^h$ is the union of the leaves in $\hat{\mathcal{K}}_f^h$ containing the S^1- orbits in $\hat{\mathcal{K}}_f$ of the function sequences (3.1.1) (which define the points of $\mathcal{A}_f^n$). Its closure (denoted $\mathcal{H}_f = \overline{\mathcal{H}_f^l}$) in $\hat{\mathcal{K}}_f^h$ is called the *hyperbolic orbifold lamination associated to* f.

Remark 3.1.25 In general, the topology of the space $\mathcal{A}_f^l$ is stronger than that of $\mathcal{A}_f^n$. The spaces $\mathcal{A}_f^l$, $\mathcal{A}_f$, $\mathcal{H}_f^l$, $\mathcal{H}_f$ consist of entire leaves. Each leaf of $\mathcal{A}_f$ is affine-equivalent either to $\mathbb{C}$ (as are the leaves from $\mathcal{A}_f^l$), or to a quotient of $\mathbb{C}$ by a discrete group of affine transformations (in this case the latters are Euclidean isometries of $\mathbb{C}$). Each leaf of $\mathcal{H}_f$ is isometric either to $\mathbb{H}^3$ (as are those of $\mathcal{H}_f^l$), or to its quotient by a discrete group of isometries of $\mathbb{H}^3$ fixing the infinity and an affine Euclidean metric on $\mathbb{C} = \partial \mathbb{H}^3 \setminus \infty$. The latter affine (hyperbolic) quotients, if nontrivial, may have singularities. The affine (hyperbolic) structures on the leaves of $\mathcal{A}_f$ (respectively, $\mathcal{H}_f$) depend continuously on the transversal parameter.

There is a natural projection

$$p : \hat{\mathcal{K}}_f^a \to \mathcal{A}_f^n$$

induced by the mapping $\hat{\mathcal{K}}_f \to \mathcal{A}_f^n$ that sends each sequence (3.1.2) of functions to the sequence of their values at 0. The latter sequence is always a regular backward orbit of f and it lies in a parabolic leaf of $\mathcal{R}_f$. The regularity follows from definition. The parabolicity follows from Picard's theorem. The composition of p with the natural inclusion $\mathcal{A}_f^n \to \hat{\mathcal{K}}_f^a$ is the identical mapping $\mathcal{A}_f^n \to \mathcal{A}_f^n$. The projection

$$\mathcal{A}_f \to \overline{\mathbb{C}} \text{ induced by } \pi_0, \ (\phi_{-j})_{j\in\mathbb{N}\cup 0} \mapsto \phi_0(0), \text{ will be also denoted by } \pi_0. \tag{3.1.4}$$

The quotient projection $\hat{\mathcal{K}}_f^h = \hat{\mathcal{K}}_f/S^1 \to \hat{\mathcal{K}}_f^a = \hat{\mathcal{K}}_f/\mathbb{C}^*$ induces a natural leafwise projection

$$\pi_h : \mathcal{H}_f \to \mathcal{A}_f, \text{ which maps } \mathcal{H}_f^l \text{ onto } \mathcal{A}_f^l, \text{ such that} \tag{3.1.5}$$

the projection of each leaf in $\mathcal{H}_f$ is a leaf in $\mathcal{A}_f$ that is canonically identified with its boundary.

The rational mapping $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ lifts up to the leafwise homeomorphism

$$\hat{f} : \hat{\mathcal{K}}_f \to \hat{\mathcal{K}}_f, \ \hat{f} : (\phi_0, \phi_{-1}, \dots) \mapsto (f \circ \phi_0, \phi_0, \phi_{-1}, \dots), \text{ which induces homeomorphisms}$$

$$\hat{f} : \mathcal{A}_f \to \mathcal{A}_f \text{ affine along the leaves and } \hat{f} : \mathcal{H}_f \to \mathcal{H}_f \text{ isometric along the leaves.}$$

The previous homeomorphisms form a commutative diagram with the projection π_h. The action $\hat{f} : \mathcal{H}_f \to \mathcal{H}_f$ is proper discontinuous, and its quotient

$$\mathcal{H}_f/\hat{f} \text{ is called the } \textit{quotient hyperbolic lamination associated to } f.$$

Proposition 3.1.26 *[89] A sequence of points $\hat{z}^m \in \mathcal{A}_f^l$ converges to a point $\hat{z} \in \mathcal{A}_f^l$, as $m \to \infty$, if and only if*

- $\pi_{-j}(\hat{z}^m) \to \pi_{-j}(\hat{z})$ for any j,

- for any $N \in \mathbb{N}$, any connected domain $V \subset \overline{\mathbb{C}}$ and any its subdomain U such that $\overline{U} \subset V$ and $\pi_{-N}(\hat{z}) \in U$, if the local leaf $L(\hat{f}^{-N}(\hat{z}), V)$ is univalent over V, then the local leaf $L(\hat{f}^{-N}(\hat{z}^m), U)$ is univalent over U, whenever m is large enough.

Remark 3.1.27 The analogous criterion holds true for convergence of a sequence of points in $\mathcal{A}_f$ to a point in $\mathcal{A}_f^l$ with the following Definition of local leaf in $\mathcal{A}_f$.

Definition 3.1.28 Let f be a rational mapping, $\mathcal{A}_f$ be the corresponding affine lamination, $L \subset \mathcal{A}_f$ be a leaf, $\hat{z} \in L$, $V \subset \overline{\mathbb{C}}$ be a domain containing its projection $\pi_0(\hat{z})$. The *local leaf* $L(\hat{z}, V)$ is the connected component containing $\hat{z}$ of the projection preimage $\pi_0^{-1}(V) \cap L$. A local leaf is called *univalent over* V, if it contains no singular points and is bijectively projected onto V.

Everywhere below for any $\hat{z} \in \mathcal{A}_f$ we denote

$$L(\hat{z}) \subset \mathcal{A}_f \text{ the leaf containing } \hat{z}, \quad H(\hat{z}) \subset \mathcal{H}_f \text{ the leaf projected to } L(\hat{z}) \text{ by (3.1.5).}$$

Corollary 3.1.29 *Let $a \in \mathbb{C}$ be a repelling fixed point of f, $\hat{a} \in \mathcal{A}_f^l$ be its fixed orbit. Let $V \subset \overline{\mathbb{C}}$ be a neighborhood of a, $\{\hat{b}^m\}_{m\in\mathbb{N}}$ be a sequence of points in $\mathcal{A}_f$ such that $\pi_0(\hat{b}^m) = a$ and the local leaves $L(\hat{b}^m, V)$ are univalent over V (see the previous Definition). Then $\hat{f}^m(\hat{b}^m) \to \hat{a}$, as $m \to +\infty$.*

Definition 3.1.30 A leaf of $\mathcal{A}_f$ is *associated to a repelling (or parabolic) periodic point* if it is contained in $\mathcal{A}_f^l$ and coincides with a leaf of $\mathcal{A}_f^n$ that is associated to the previous point (see Definition 3.1.21). In this case we also say that the corresponding leaves of $\mathcal{H}_f$ and $\mathcal{H}_f/\hat{f}$ are associated to this point.

Proposition 3.1.31 *[89] The laminations $\mathcal{A}_f$ and $\mathcal{H}_f$ are minimal (i.e., each leaf is dense), if and only if the function f does not have branch-exceptional repelling periodic orbits (see Definition 3.1.10). If f has branch-exceptional repelling periodic orbits, then each of the previous laminations has a finite number of isolated leaves (all of them are associated to the latter periodic orbits) and becomes minimal after removing the isolated leaves.*

Denote

$$\mathcal{H}'_f = \mathcal{H}_f \setminus (\text{isolated hyperbolic leaves}). \tag{3.1.6}$$

One has $\mathcal{H}'_f = \mathcal{H}_f$, if and only if f does not have branch-exceptional repelling periodic orbits.

3.1.3 Horospheres : metric properties and basic cocycle

The horospheres in the hyperbolic 3- space with a marked point "infinity" at the boundary (and in the leaves of the hyperbolic laminations) were defined in Subsection 1.1. We use the following their well-known equivalent definition. Consider the projection $\pi : \mathbb{H}^3 \to L = \partial\mathbb{H}^3 \setminus \infty$ to the boundary plane along the geodesics issued from the infinity. In the model of half-space this is the Euclidean orthogonal projection to the boundary plane. It coincides with the natural projection $\mathbb{H}^3 = Aff(\mathbb{C})/S^1 \to \mathbb{C} = Aff(\mathbb{C})/\mathbb{C}^*$, and its latter description equips the boundary with a natural complex affine structure : $L = \mathbb{C}$. The boundary admits a Euclidean affine metric (uniquely defined up to multiplication by constant).

Everywhere below whenever we consider a Riemann metric on a surface, we treat it as a length element, not as a quadratic form. If we say "two metrics are proportional", then by definition, the proportionality coefficient is the ratio of the corresponding length elements.

Consider a global section of the previous projection $\pi : \mathbb{H}^3 \to L$: a surface in $\mathbb{H}^3$ that is 1-to-1 projected to L. It carries two metrics : the restriction to it of the hyperbolic metric of the ambient space $\mathbb{H}^3$; the pullback of the Euclidean metric of L under the projection.

Definition 3.1.32 A previous section is a *horosphere*, if its latter (Euclidean) metric is obtained from the former one (the restricted hyperbolic metric) by multiplication by a constant factor. The *height* of a horosphere (with respect to the chosen Euclidean metric on L) is the logarithm of the latter constant factor. The *height of a given point* in the hyperbolic space is the height of the horosphere that contains this point.

Remark 3.1.33 The height is a real-valued analytic function $\mathbb{H}^3 \to \mathbb{R}$. In the upper half-space model the horospheres are horizontal planes, and their previously defined heights are equal to the logarithms of their Euclidean heights in the ambient Euclidean 3- space. The isometric liftings to $\mathbb{H}^3$ of the affine mappings $z \mapsto \lambda z + b$ of the boundary $\mathbb{C} = \partial\mathbb{H}^3 \setminus \infty$ transform the horospheres to the horospheres so that the height of the image equals $\ln|\lambda|$ plus the height of the preimage.

Now we discuss metric properties of the horospheres in the hyperbolic laminations. Let $\mathcal{A}_f$, $\mathcal{H}_f$ be respectively the affine and the hyperbolic laminations associated to a rational function f. Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{z} \in L$ be a nonsingular point such that the restricted projection $\pi_0|_L$ has nonzero derivative at $\hat{z}$. Fix a Hermitian metric on the tangent line to $\overline{\mathbb{C}}$ at $\pi_0(\hat{z})$. Its projection pullback to the tangent line $T_{\hat{z}}L$ extends (in unique way) up to a Euclidean affine metric on the whole leaf L. Let H be the corresponding leaf in $\mathcal{H}_f$. We denote

$$\beta_{\hat{z}} : H \to \mathbb{R} \text{ the height with respect to the latter metric, see Definition 3.1.32,} \tag{3.1.7}$$

$$\alpha = (\hat{z}, h) \in H \text{ the point such that } \pi_h(\alpha) = \hat{z} \text{ and } \beta_{\hat{z}}(\alpha) = h$$

(then we say that the point α is *situated over* $\hat{z}$ at height h),

$$S_{\hat{z},h} \subset H \text{ the horosphere containing } \alpha, \text{ i.e., such that } \beta_{\hat{z}}|_{S_{\hat{z},h}} \equiv h. \tag{3.1.8}$$

Proposition 3.1.34 *A sequence of points $(\hat{z}^k, h_k) \in \mathcal{H}_f$ converges to a point $(\hat{z}, h) \in \mathcal{H}_f$, if and only if $\hat{z}^k \to \hat{z}$ in $\mathcal{A}_f$ and $h_k \to h$.*

The Proposition follows from definition and the continuity of the family of hyperbolic structures on the leaves of $\mathcal{H}_f$.

When we extend the horospheres along loops in $\overline{\mathbb{C}}$, their heights may change. The monodromy of the heights is described by basic cocycle. Let us recall its definition.

Definition 3.1.35 Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{z}, \hat{z}' \in L$ be a pair of nonsingular points projected to one and the same $z = \pi_0(\hat{z}) = \pi_0(\hat{z}') \in \overline{\mathbb{C}}$ so that the restricted projection $\pi_0|_L$ has nonzero derivative at both points $\hat{z}$ and $\hat{z}'$. Let $H = H(\hat{z}) \subset \mathcal{H}_f$ be the corresponding hyperbolic leaf. Fix a Hermitian metric on $T_z\overline{\mathbb{C}}$, let $\beta_{\hat{z}}, \beta_{\hat{z}'} : H \to \mathbb{R}$ be the corresponding heights defined in (3.1.7). The *basic cocycle* is the difference

$$\beta(\hat{z}, \hat{z}') = \beta_{\hat{z}'} - \beta_{\hat{z}}.$$

Remark 3.1.36 In the conditions of the previous Definition the basic cocycle is a well-defined constant and depends only on $\hat{z}$ and $\hat{z}'$ (it is independent on the choice of metric). One has

$$\beta(\hat{z}, \hat{z}) = 0, \ \beta(\hat{z}, \hat{z}') = -\beta(\hat{z}', \hat{z}).$$

Each horosphere $S_{\hat{z},h} \subset H(\hat{z})$ coincides with the horosphere $S_{\hat{z}',h+\beta(\hat{z},\hat{z}')}$. The basic cocycle is $\hat{f}$-invariant :

$$\beta(\hat{z}, \hat{z}') = \beta(\hat{f}^n(\hat{z}), \hat{f}^n(\hat{z}')) \text{ for any } n \in \mathbb{N}. \tag{3.1.9}$$

For any triple of nonsingular points $\hat{z}, \hat{z}', \hat{z}'' \in \mathcal{A}_f$ lying in one and the same leaf L and projected by $\pi_0|_L$ to one and the same point $z \in \overline{\mathbb{C}}$ with nonzero derivatives one has

$$\beta(\hat{z}', \hat{z}'') = \beta(\hat{z}, \hat{z}'') - \beta(\hat{z}, \hat{z}') \text{ (the cocycle property)}. \tag{3.1.10}$$

The next proposition is well-known and follows immediately from definition.

Proposition 3.1.37 *Let $L \subset \mathcal{A}_f$ be a leaf, $\hat{c}, \hat{c}' \in L$, $\pi_0(\hat{c}) = \pi_0(\hat{c}') = c$. Let $V \subset \overline{\mathbb{C}}$ be a neighborhood of c such that the local leaves $L(\hat{c}, V), L(\hat{c}', V) \subset L$ are univalent over V (see Definition 3.1.28). Define*

$$\psi_{\hat{c},\hat{c}'} = (\pi_0|_{L(\hat{c}',V)})^{-1} \circ \pi_0|_{L(\hat{c},V)} : L(\hat{c}, V) \to L(\hat{c}', V). \tag{3.1.11}$$

Let us fix a Euclidean affine metric on the leaf L, which contains the previous local leaves. Consider the derivative modulus $|\psi'_{\hat{c},\hat{c}'}|$ in the chosen Euclidean metric. Then for any $\hat{z} \in L(\hat{c}, V)$, $\hat{z}' = \psi_{\hat{c},\hat{c}'}(\hat{z})$, one has

$$\beta(\hat{z}, \hat{z}') = -\ln|\psi'_{\hat{c},\hat{c}'}(\hat{z})|. \tag{3.1.12}$$

Corollary 3.1.38 *Let L, $\hat{c}$, $\hat{c}'$, V be as in the previous proposition. For any $z \in V$ put*

$$\hat{z} = \pi_0^{-1}(z) \cap L(\hat{c}, V), \ \hat{z}' = \pi_0^{-1}(z) \cap L(\hat{c}', V). \textit{ The function}$$

$$\beta_{\hat{c},\hat{c}'}(z) = \beta(\hat{z}, \hat{z}') \tag{3.1.13}$$

is harmonic on V (and hence, real-analytic).

3.2 Main results : density of horospheres

First let us recall the following

Theorem 3.2.1 *[78] The affine lamination $\mathcal{A}_f$ associated to a rational function f (with isolated leaves deleted) admits a continuous family of Euclidean affine metrics on the leaves, if and only if f is conformally-conjugated to a function from the list (3.0.1). In the latter case there exists a unique (up to multiplication by constant) conformal Euclidean metric on $\overline{\mathbb{C}}$ (with isolated singularities) whose pullback under the projection $\pi_0 : \mathcal{A}_f \to \overline{\mathbb{C}}$ yields the previous Euclidean metric family on the nonisolated leaves.*

Corollary 3.2.2 *Let f be a rational function from (3.0.1). Then each horosphere in its quotient hyperbolic lamination $\mathcal{H}_f/\hat{f}$ (with isolated leaves deleted) is nowhere dense.*

Proof (sketch). Let S be an arbitrary horosphere in a nonisolated leaf of $\mathcal{H}_f$. For the proof of the corollary it suffices to show that the union of the images of S under forward and backward iterations of $\hat{f}$ is nowhere dense. Denote g the singular Euclidean metric on $\overline{\mathbb{C}}$ from the previous theorem. We measure the heights of the horospheres with respect to this metric. The heights of S over all the points are all the same (by definition and Theorem 3.2.1). The mapping f has a constant modulus of derivative in the metric g, since $\hat{f}$ is leafwise affine. Hence, the heights of the iterated images of S form an arithmetic progression, thus, a discrete set of real numbers. This proves the corollary. □

Theorem 3.2.3 *[48, 49]. Let f be a rational function that does not belong to the list (3.0.1). Let $\mathcal{H}_f/\hat{f}$ ($\mathcal{H}'_f/\hat{f}$) be the corresponding quotient hyperbolic lamination (with deleted isolated leaves, if f has branch-exceptional repelling periodic orbits, see (3.1.6)); $H \subset \mathcal{H}'_f/\hat{f}$ be a leaf associated to a repelling periodic point of f (see Definition 3.1.30). Then each horosphere in H is dense in $\mathcal{H}'_f/\hat{f}$.*

Theorem 3.2.3 is the main result of the papers [48, 49]. Its proof is sketched in the next section. As it is shown below, it implies density of all the horospheres in the critically-nonrecurrent nonparabolic case and density of "almost" all the horospheres in the general critically-nonrecurrent case, with parabolics allowed, provided that $f \notin$ (3.0.1).

Theorem 3.2.4 *[48, 49]. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a critically-nonrecurrent rational function without parabolic periodic points (e.g., a hyperbolic one) that does not belong to the list (3.0.1). Then each horosphere in $\mathcal{H}_f/\hat{f}$ accumulates to $\mathcal{H}'_f/\hat{f}$.*

Theorem 3.2.5 *[48, 49]. Let f be a critically-nonrecurrent rational function that does not belong to the list (3.0.1). Let $H \subset \mathcal{H}_f$ be a leaf, $L = \pi_h(H) \subset \mathcal{A}_f$ be its boundary. Let the projection $p(L) \subset \mathcal{A}^n_f$ do not lie in a leaf associated to a parabolic periodic point of f. Let $H/\hat{f} \subset \mathcal{H}_f/\hat{f}$ be the corresponding leaf of the quotient lamination. Then each horosphere in $H/\hat{f}$ accumulates to $\mathcal{H}'_f/\hat{f}$.*

Theorem 3.2.4 follows immediately from Theorem 3.2.5. Below we deduce Theorem 3.2.5 from Theorem 3.2.3 and the following theorem.

Theorem 3.2.6 *[48, 49]. Let the conditions of Theorem 3.2.5 hold (but now f is not necessarily excluded from the list (3.0.1)). Then each horosphere in $H/\hat{f}$ accumulates to some horosphere in a leaf in $\mathcal{H}'_f/\hat{f}$ associated to appropriate repelling periodic point.*

Proof of Theorem 3.2.5. Each horosphere in $H/\hat{f}$ accumulates to some horosphere in a leaf in $\mathcal{H}'_f/\hat{f}$ corresponding to a repelling periodic point (Theorem 3.2.6). The latter horosphere is dense in $\mathcal{H}'_f/\hat{f}$ (Theorem 3.2.3). Hence, the former horosphere accumulates to $\mathcal{H}'_f/\hat{f}$. This proves Theorems 3.2.5 and 3.2.4. □

The following theorem shows the closeness of the horospheres in the leaves associated to parabolic periodic points, without the critical nonrecurrence assumption.

Theorem 3.2.7 *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an arbitrary rational function with a parabolic periodic point a. Let $H_a \subset \mathcal{H}_f$ be a leaf associated to it, $H_a/\hat{f} \subset \mathcal{H}_f/\hat{f}$ be the corresponding leaf of the quotient hyperbolic lamination. Each horosphere in H_a ($H_a/\hat{f}$) is closed in $\mathcal{H}_f$ (respectively, $\mathcal{H}_f/\hat{f}$) and does not accumulate to itself.*

3.3 Brief proofs of main results

In the next subsection we prove Theorem 3.2.3. In Subsection 3.3.2 we prove Theorem 3.2.6 (which, together with Theorem 3.2.3, implies Theorem 3.2.5 on the density of all the horospheres). In Subsection 3.3.3 we prove Theorem 3.2.7.

For simplicity, everywhere below (including the statements of lemmas and propositions) we assume that the rational function f under consideration does not have branch-exceptional repelling periodic orbits, and thus, $\mathcal{H}_f = \mathcal{H}'_f$: the proofs of Theorems 3.2.3 and 3.2.5 (given below) remain valid in the opposite case with obvious changes. Thus, the laminations $\mathcal{A}_f$ and $\mathcal{H}_f$ are minimal (Proposition 3.1.31).

3.3.1 Dense horospheres over repellers. Proof of Theorem 3.2.3

Let $a \in \overline{\mathbb{C}}$ be a repelling periodic point of f, $\hat{a} \subset \mathcal{A}_f$ be its periodic backward orbit, $L(\hat{a})$, $H(\hat{a})$ be the respectively the corresponding leaves of the laminations $\mathcal{A}_f$ and $\mathcal{H}_f$. We fix a horosphere $S \subset H(\hat{a})$, denote

$$\mathcal{S} = \cup_{m\in\mathbb{Z}} \hat{f}^m(S), \text{ and show that the closure of } \mathcal{S} \text{ in } \mathcal{H}_f \text{ contains } H(\hat{a}). \tag{3.3.1}$$

The leaf $H(\hat{a})$ is dense (minimality). This together with the previous statement implies Theorem 3.2.3.

It suffices to prove (3.3.1) with $S = S_{\hat{a},0}$. Without loss of generality everywhere below we assume that the point a is fixed : $f(a) = a$. One can achieve this by replacing f by its iteration. Then both leaves $L(\hat{a})$ and $H(\hat{a})$ are fixed by $\hat{f}$, which acts on $L(\hat{a})$ by (complex) homothety centered at $\hat{a}$ with coefficient $f'(a)$. Denote

$$\Pi_a = \{\hat{b} \in L(\hat{a}) \setminus \hat{a} \mid \pi_0(\hat{b}) = a,\ (\pi_0|_{L(\hat{a})})'(\hat{b}) \neq 0\}. \tag{3.3.2}$$

The set Π_a is nonempty and infinite. This follows from the assumption that a is not a branch-exceptional fixed point and Picard's theorem.

Each horosphere $S \subset H(\hat{a})$ is mapped by $\hat{f}$ to a horosphere in the same leaf $H(\hat{a})$ so that

$$\hat{f}^m(S_{\hat{a},0}) = S_{\hat{a},m\ln|f'(a)|},\ \hat{f}(S_{\hat{b},h}) = S_{\hat{f}(\hat{b}),h+\ln|f'(a)|} \text{ for any } m \in \mathbb{Z},\ h \in \mathbb{R} \text{ and } \hat{b} \in \Pi_a. \tag{3.3.3}$$

The monodromies of the horospheres (when defined) along loops based at a add appropriate basic cocycles to the heights (see Definition 3.1.35) so that for any $\hat{b} \in \Pi_a$, $h \in \mathbb{R}$, $m \in \mathbb{Z}$

$$S_{\hat{a},h} = S_{\hat{b},h+\beta(\hat{a},\hat{b})}, \text{ thus, } \hat{f}^m(S_{\hat{a},0}) = S_{\hat{b},h_{\hat{b},m}},\ h_{\hat{b},m} = \beta(\hat{a},\hat{b}) + m\ln|f'(a)|.$$

The main part of the proof of Theorem 3.2.3 is the next lemma, which implies that the previous height values $h_{\hat{b},m}$ are dense in $\mathbb{R}$. Theorem 3.2.3 is then deduced from it by elementary topological arguments (using Corollary 3.1.29), which are omitted to save the space.

Lemma 3.3.1 *Let f be a rational function that does not belong to the list (3.0.1), a be its repelling fixed point, Π_a be as in (3.3.2). The set*

$$\mathcal{B}f = \{\beta(\hat{a},\hat{b}) + m\ln|f'(a)| \mid \hat{b} \in \Pi_a,\ m \in \mathbb{Z}\} \tag{3.3.4}$$

is dense in $\mathbb{R}$.

Everywhere below for any $z \in \overline{\mathbb{C}}$ (with a chosen local chart in its neighborhood, the latter being equipped with the standard Euclidean metric) and $\delta > 0$ we denote

$$D_\delta(z) = \{|w - z| < \delta\} \subset \mathbb{C},\ D_\delta = D_\delta(0).$$

Lemma 3.3.1 is proved below. In its proof we use the following properties of the points from Π_a and basic cocycles.

Proposition 3.3.2 *Let f be a rational function, $a \in \overline{\mathbb{C}}$ be its repelling fixed point, Π_a be as in (3.3.2), $\hat{b}, \hat{c} \in \Pi_a$. Let $\delta > 0$ be such that the local leaves $L(\hat{a}, D_\delta(a))$, $L(\hat{b}, D_\delta(a))$, $L(\hat{c}, D_\delta(a))$ are univalent over $D_\delta(a)$, and moreover, the inverse branch f^{-1} that fixes a extends up to a univalent holomorphic function $D_\delta(a) \to D_\delta(a)$ (whose orbits in $D_\delta(a)$ thus converge to a). Let $j \in \mathbb{N}$ be such that $b_{-k} \in D_\delta(a)$ for any $k \geq j$ (see Proposition 3.1.22). Let*

$$y \in L(\hat{c}, D_\delta(a)),\ \pi_0(y) = b_{-j},\ \hat{d} = \hat{f}^j(y),\ \beta_{\hat{a},\hat{c}} \text{ be the function from (3.1.13). Then}$$

$$\beta(\hat{a}, \hat{d}) = \beta(\hat{a}, \hat{b}) + \beta_{\hat{a},\hat{c}}(b_{-j}). \tag{3.3.5}$$

Proof (sketch). One has $\hat{d} \in \Pi_a$, which easily follows from definition,

$$\beta(\hat{a}, \hat{d}) = \beta(\hat{a}, \hat{b}) + \beta(\hat{b}, \hat{d}) \text{ by (3.1.10)},\ \beta(\hat{b}, \hat{d}) = \beta_{\hat{a},\hat{c}}(\hat{b}_{-j})$$

(the two latter equalities imply (3.3.5)). Let us prove the second equality. The points $\hat{f}^{-j}(\hat{b})$ and y are projected to one and the same point b_{-j} and lie in the local leaves $L(\hat{a}, D_\delta(a))$ and $L(\hat{c}, D_\delta(a))$ respectively by construction. One has $\beta(\hat{b}, \hat{d}) = \beta(\hat{f}^{-j}(\hat{b}), y)$ (the invariance of basic cocycle, see (3.1.9), and the projection coincidence). Now $\beta(\hat{f}^{-j}(\hat{b}), y) = \beta_{\hat{a},\hat{c}}(b_{-j})$ by definition and the previous inclusion. □

Corollary 3.3.3 *Let f, a, Π_a be as in Proposition 3.3.2. The closure*

$$\mathcal{B} = \overline{\{\beta(\hat{a}, \hat{b}) \mid \hat{b} \in \Pi_a\}} \tag{3.3.6}$$

is an additive semigroup in $\mathbb{R}$.

Proof Fix arbitrary $\hat{b}, \hat{c} \in \Pi_a$. We have to show that $B' = \beta(\hat{a}, \hat{b}) + \beta(\hat{a}, \hat{c}) \in \mathcal{B}$, i.e., B' is approximated arbitrarily well by values $\beta(\hat{a}, \hat{d})$, $\hat{d} \in \Pi_a$. Let j, $\hat{d}$ be as in (3.3.5). Then

$$\beta(\hat{a}, \hat{d}) - B' = \beta_{\hat{a},\hat{c}}(b_{-j}) - \beta(\hat{a}, \hat{c}) \text{ by (3.3.5)}. \tag{3.3.7}$$

The latter difference tends to 0, as $j \to \infty$, since $\beta_{\hat{a},\hat{c}}(a) = \beta(\hat{a}, \hat{c})$ and $b_{-j} \to a$. This proves the Corollary. □

We use the following elementary property of additive semigroups.

Proposition 3.3.4 *Let $\mathcal{B} \subset \mathbb{R}$ be an additive semigroup such that for any $\varepsilon > 0$ it contains a pair of at most ε- close distinct elements. Then for any $M \in \mathbb{R} \setminus 0$ the semigroup $\mathcal{B}_M = \mathcal{B} + \mathbb{Z}M$ is dense in $\mathbb{R}$.*

By definition, one has

$$\mathcal{B} \subset \mathcal{B} + \mathbb{Z} \ln |f'(a)| \subset \overline{\mathcal{B}f}. \tag{3.3.8}$$

We show that the semigroup $\mathcal{B}$ contains distinct elements arbitrarily close to each other. Then applying Proposition 3.3.4 to $M = \ln |f'(a)|$ together with the previous inclusion implies Lemma 3.3.1.

As it is shown below, the previous statement on $\mathcal{B}$ is implied by (3.3.5) and the following

Lemma 3.3.5 (Main Technical Lemma) *Let f be a rational function that does not belong to the list (3.0.1), $a \in \overline{\mathbb{C}}$ be its repelling fixed point, $\hat{a} \in \mathcal{A}_f$ be its fixed orbit, Π_a be the set from (3.3.2). There exists a pair of points $\hat{b}, \hat{c} \in \Pi_a$ such that for any $N \in \mathbb{N}$*

$$\beta_{\hat{a},\hat{c}}|_{\{b_{-j} \mid j \geq N\}} \not\equiv const. \tag{3.3.9}$$

The proof of Lemma 3.3.5 (sketched below) uses essentially the analyticity of basic cocycle.

Proof of Lemma 3.3.1. It suffices to show that the semigroup $\mathcal{B}$ contains pairs of arbitrarily close distinct elements (see the previous discussion). Let $\hat{b}, \hat{c} \in \Pi_a$ be as in Lemma 3.3.5, j, $\hat{d}$ be as in (3.3.5). The values $B' = \beta(\hat{a}, \hat{b}) + \beta(\hat{a}, \hat{c})$ and $\beta(\hat{a}, \hat{d})$ are both contained in $\mathcal{B}$ (Corollary 3.3.3). Their difference (3.3.7) is arbitrarily small, whenever j is large enough, see the proof of the corollary. It is nonzero for an infinite number of indices j by (3.3.7) and (3.3.9). This proves Lemma 3.3.1 modulo Lemma 3.3.5. □

Proof of Lemma 3.3.5 (sketch). Fix a small neighborhood U of a where f is univalent and such that $f(U) \supset U$. Then the linearizing chart of f at a extends up to a holomorphic univalent chart on U. We take U to be convex in the linearizing chart. For any $\hat{z} \in L(\hat{a})$ there exists a $N > 0$ such that $z_{-j} \in U$ for any $j \geq N$. Then the backward orbit $z_{-N}, z_{-N-1}, \dots$ is called a *tail of* $\hat{z}$ (the previous number N is not necessarily chosen to be the minimal one satisfying the previous statement). If N is minimal, then the tail is called *complete*. The local leaf $L(\hat{a}, U)$ is well-defined and univalent over U by definition. It consists precisely of the points of $\mathcal{A}_f^l$ represented by tails.

We have to show that there exists a basic cocycle $\beta_{\hat{a},\hat{c}}$ that is nonconstant along an arbitrary tail of appropriate point $\hat{b} \in \Pi_a$. First let us show that if f does not belong to the list (3.0.1), then there exists a $\hat{c} \in \Pi_a$ such that

$$\beta_{\hat{a},\hat{c}} \not\equiv \textit{const} \text{ in a neighborhood of } a. \tag{3.3.10}$$

This is proved by showing that the contrary would imply that f belongs to (3.0.1). For any $\hat{c} \in \Pi_a$ such that $\beta_{\hat{a},\hat{c}} \equiv \textit{const}$ one has $\beta_{\hat{a},\hat{c}} \equiv 0$. Indeed, the constance of $\beta_{\hat{a},\hat{c}}$ implies that the mapping germ $\psi : (L(\hat{a}), \hat{a}) \to (L(\hat{a}), \hat{c})$ preserving the projection extends up to an affine automorphism ψ of $L(\hat{a}) = \mathbb{C}$ such that $\beta_{\hat{a},\hat{c}} \equiv -\ln|\psi'|$ and $\pi_0 \circ \psi \equiv \pi_0$. The latter identity implies that ψ cannot have attracting (repelling) fixed points; hence, $|\psi'| \equiv 1$ and $\beta_{\hat{a},\hat{c}} \equiv 0$. Now let $\beta_{\hat{a},\hat{c}} \equiv 0$ for all $\hat{c} \in \Pi_a$. Recall that the lamination $\mathcal{A}_f$ is minimal by assumption. Fix an affine Euclidean metric on the leaf $L(\hat{a})$. It extends up to a continuous family of affine Euclidean metrics on all the leaves of $\mathcal{A}_f$ that are projected to one and the same (singular) metric on $\overline{\mathbb{C}}$ (by density of $L(\hat{a})$, the vanishing and the invariance of basic cocycle). Hence, f belongs to (3.0.1) by Theorem 3.2.1. This proves the existence of a nonconstant $\beta_{\hat{a},\hat{c}}$.

Fix a $\hat{c} \in \Pi_a$ satisfying (3.3.10). Without loss of generality we consider that the local leaf $L(\hat{c}, U)$ is univalent over U (then the function $\beta_{\hat{a},\hat{c}}$ is real-analytic on U, see Corollary 3.1.38). We prove the existence of $\hat{b}$ satisfying (3.3.9) by contradiction. Suppose the contrary : $\beta_{\hat{a},\hat{c}} \equiv \textit{const}$ on some tail of each $\hat{b} \in \Pi_a$ (and hence, equals $\beta_{\hat{a},\hat{c}}(a)$ there). We show that $\beta_{\hat{a},\hat{c}} \equiv \textit{const}$ on U, - a contradiction to (3.3.10).

The level set $\beta_{\hat{a},\hat{c}} = \beta_{\hat{a},\hat{c}}(a)$ is a nontrivial real-analytic subset in U by the analyticity of $\beta_{\hat{a},\hat{c}}$ and (3.3.10). Let $A \subset U$ be the minimal analytic subset that contains a tail of each $\hat{b} \in \Pi_a$. Then A lies in the previous level set. We show that either $A = U$ (then $\beta_{\hat{a},\hat{c}} \equiv \textit{const}$), or A is a line interval in the linearizing chart. In the latter case we also show that $\beta_{\hat{a},\hat{c}} \equiv \textit{const}$.

The existence of a backward orbit converging to a along a nontrivial analytic set A implies immediately that $\arg f'(a) \in \pi\mathbb{Q}$. We then deduce that A is a finite union of line intervals passing through a with ends on ∂U. Let us show that then A is a single line interval. To do this, we use the fact, that A is f^{-1}- invariant and contains the complete tail of each $\hat{b} \in \Pi_a$. The latter statement is deduced from the former one and the convexity of U.

Suppose the contrary : the set A contains at least two distinct line intervals (let us fix them and denote l_1 and l_2). Fix a N such that $\hat{c}_{-j} \in U$ for any $j \geq N$. Consider the inverse branch $f^{-j}|_U : a \mapsto c_{-j}$ (which is single-valued by the univalence of $L(\hat{c}, U)$). We show that the germs of the analytic curves $f^{-N}(l_r)$, $r = 1, 2$, at their transversal intersection point $c_{-N} \neq a$ are contained in A. This implies that A cannot be a finite union of line intervals containing a, - a contradiction. Each l_r contains a subsequence $x_1, x_2, \dots$ (let us fix it) of a tail of some $\hat{b} \in \Pi_a$. The previous germ inclusion follows from analyticity and the fact that for any s large enough the sequence $f^{-N}(x_s), f^{-N-1}(x_s), \dots$ is a tail of some $\hat{a}^s \in \Pi_a$.

Thus, the previous analytic set A is a single line interval. By definition, $\beta_{\hat{a},\hat{c}}|_A \equiv const$. The function $\beta_{\hat{a},\hat{c}}$, whose constance we have to prove, is equal to minus the logarithm of the modulus of the derivative of a holomorphic univalent function $\psi : U \to \mathbb{C}$. The latter function is defined by the lifting

$$\psi : U \to L(\hat{c}, U) \subset L(\hat{a}),\ \pi_0 \circ \psi = Id,\ \text{and the affine identification } L(\hat{a}) = \mathbb{C}.$$

The latter identification is given by the linearizing coordinate of f at a (see Example 3.1.19). The previous derivative of ψ is taken in the linearizing chart of f on U. The modulus $|\psi'|$ is constant along A, since $\beta_{\hat{a},\hat{c}}|_A \equiv const$. The image $\psi(A)$ lies in a line (the latter line passes through $\hat{a}$ and near $\hat{a}$ it is locally projected to A). This easily follows from the $\hat{f}$- invariance of this line by an argument analogous to the proof of the previous germ inclusion. This together with the following proposition shows that the derivative (and hence, $\beta_{\hat{a},\hat{c}}$) is constant globally, - a contradiction to (3.3.10). This proves Lemma 3.3.5.

Proposition 3.3.6 *Let ψ be a conformal mapping of one domain of $\mathbb{C}$ onto another one. Let ψ map a line interval A to a line and the modulus of its derivative be constant along A. Then ψ is an affine mapping.*

□

3.3.2 Minimality. Proof of Theorem 3.2.6

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a critically-nonrecurrent rational mapping. Let $L \subset \mathcal{A}_f$ be a leaf of the corresponding affine lamination whose projection $p(L) \subset \mathcal{A}_f^n$ does not lie in a leaf associated to a parabolic periodic point. Let $H \subset \mathcal{H}_f$ be the corresponding hyperbolic leaf. Let us show that there exists a repelling periodic point $a \in \overline{\mathbb{C}}$ of f (denote $\hat{a} \in \mathcal{A}_f$ its periodic orbit) such that for each horosphere $S \subset H$ the union of its images under forward and backward iterations of $\hat{f}$ accumulates to some point of $H(\hat{a})$ (and hence, to the horosphere passing through this point). This will prove Theorem 3.2.6.

Here we prove the previous accumulation statement only in the case, when $L \subset \mathcal{A}_f^l$. The proof in the general case is similar but becomes slightly more technical.

Lemma 3.3.7 *Let f and $L \subset \mathcal{A}_f^l$ be as above, $\hat{x} \in L$ be such that $x_0 \in J = J(f)$. There exist a sequence $n_k \to +\infty$, a point $b \in J$ (that is not a parabolic periodic point) and a neighborhood $V = V(b) \subset \overline{\mathbb{C}}$ such that $x_{-n_k} \to b$ and for any $k \in \mathbb{N}$ the local leaf $L(\hat{f}^{-n_k}(\hat{x}), V)$ is well-defined and univalent over V.*

The Lemma is proved by using Mañe's theorem [89].

Let $\hat{x}$, n_k, b and V be as in the previous lemma. Without loss of generality we consider that $V = D_1$, $b = 0$. The disk V intersects the Julia set of f and hence, contains a repelling periodic point (let us fix it and denote by a). We show that a is a repelling point we are looking for.

For each local leaf $L(\hat{f}^{-n_k}(\hat{x}), V)$ and any $w \in V$ denote its lifting to this leaf by

$$\hat{w}^k \in L(\hat{f}^{-n_k}(\hat{x}), V),\ \pi_0(\hat{w}^k) = w;\ \hat{a}^k \in L(\hat{f}^{-n_k}(\hat{x}), V),\ \pi_0(\hat{a}^k) = a.$$

Fix a horosphere $S \subset H$ and denote

$$S^k = \hat{f}^{-n_k}(S) \subset H(\hat{a}^k),\quad \alpha^k \in S^k \text{ its point over } \hat{a}^k :\ \pi_h(\alpha^k) = \hat{a}^k.$$

Let s be the period of a, $\lambda = (f^s)'(a)$. We show that there exists a sequence $l_k \to +\infty$ such that the sequence $\hat{f}^{sl_k}(\alpha^k)$ contains a subsequence converging to a point $\alpha = (\hat{a}, h) \in H(\hat{a})$. The height of the point $\hat{f}^{sl_k}(\alpha^k)$ equals $l_k \ln|\lambda|$ plus the height of α^k (all the heights are measured in the standard metric on V). For the proof of the existence of the previous sequence l_k we show that the heights of

α^k tend to $-\infty$, and moreover, the heights of S^k over the local leaves $L(\hat{f}^{-n_k}(\hat{x}), V)$ tend to $-\infty$ (as functions on V, uniformly on compact sets), as $k \to \infty$. Indeed, the height of S^k over $\hat{f}^{-n_k}(\hat{x})$ (which is projected to $x_{-n_k} \to b = 0$) tends to $-\infty$: it equals the height of S over $\hat{x}$ plus $\ln |(f^{-n_k})'(x_0)|$, which tends to $-\infty$ (by the Shrinking Lemma). The previous uniform convergence to $-\infty$ then follows from the equicontinuity of the heights on compact subsets in V. The equicontinuity follows from the fact that the heights are equal (up to additive constants) to logarithms of moduli of derivatives of appropriate univalent functions $\psi : V \to \mathbb{C}$ (that can be normalized by affine transformations in the image so that $\psi(0) = 0$, $\psi'(0) = 1$) and the compactness of the space of all thus normalized univalent functions on a disk.

3.3.3 Closeness of the horospheres associated to the parabolic periodic points

Let us prove Theorem 3.2.7. Let f be a rational function with a parabolic periodic point $a \in \overline{\mathbb{C}}$. Without loss of generality we consider that a is fixed. Let $L_a \subset \mathcal{A}_f$ be a leaf associated to a of the affine lamination, $H_a \subset \mathcal{H}_f$ be the corresponding hyperbolic leaf. In the proof of Theorem 3.2.7 we use the following

Proposition 3.3.8 *Let f, a, L_a, H_a be as above. Then each horosphere in H_a is invariant under the mapping $\hat{f}$.*

Proof The Fatou coordinate is affine on the leaf L_a, and $\hat{f}$ acts by unit translation there. Hence, it preserves an Euclidean metric on L_a. This implies the proposition. □

Fix a horosphere $S \subset H_a$. We show that S is closed in $\mathcal{H}_f$ and does not accumulate to itself. This together with its invariance (Proposition 3.3.8) implies Theorem 3.2.7.

Suppose the contrary : S accumulates to some horosphere S'. Let $H \subset \mathcal{H}_f$ be the leaf containing S', $L \subset \mathcal{A}_f$ be the corresponding affine leaf. Take an arbitrary nonsingular point $\hat{b} \in L$ such that $b = \pi_0(\hat{b}) \neq a$, $(\pi_0|_L)'(\hat{b}) \neq 0$ and a neighborhood $V = V(b) \subset \overline{\mathbb{C}}$ such that the local leaf $L(\hat{b}, V)$ is univalent over V. There exists a sequence $\hat{b}^k \in L_a$ converging to $\hat{b}$ in $\mathcal{A}_f$ so that the points of S over $\hat{b}^k$ converge to that of S' over $\hat{b}$ (by definition), and in addition, $\hat{b}^k \notin L(\hat{b}, V)$. For any neighborhood $U = U(b)$, $\overline{U} \subset V$ (let us fix it) the local leaves

$$\Lambda_k = L(\hat{b}^k, U) \subset L_a \text{ are univalent over } U \text{ for all } k \text{ large enough.}$$

This follows from the convergence $\hat{b}^k \to \hat{b}$ and the definition of topology in $\mathcal{A}_f$. Without loss of generality we consider that this is true for all k,

$$U = D_1, \ b = 0 = \pi_0(\hat{b}^k), \text{ and the leaves } \Lambda_k \text{ are distinct.}$$

We equip U with the standard Euclidean metric and measure the heights of the horospheres over the local leaves with respect to this metric. We show that the heights of S over $\hat{b}^k$ tend to $+\infty$, - a contradiction to the convergence of the points of S over $\hat{b}^k$.

For the proof of the previous height asymptotics, we fix a disk $\overline{D_r(a)}$ where f is univalent, the branch of f^{-1} fixing a is single-valued and such that each backward orbit contained there in fact converges to a (this is true, whenever the disk is small enough). For any fixed k one has $b^k_{-j} \to a$, as $j \to +\infty$; denote $n_k \in \mathbb{N}$ the minimal number such $b^k_{-j} \in \overline{D_r(a)}$ for any $j \geq n_k$. Passing to a subsequence of the indices k one can achieve that $b^k_{-n_k}$ converge; then $\hat{f}^{-n_k}(\hat{b}^k)$ converge to some $\hat{x} \in \mathcal{N}_f$ in $\mathcal{N}_f$ represented by a backward orbit in $\overline{D_r(a)}$. One has $\hat{x} \in L_a$, by construction and since it is distinct from the fixed orbit of a (by definition and the inequality $0 = b^k_0 \neq a$). Moreover, $\hat{f}^{-n_k}(\hat{b}^k) \to \hat{x}$ along a local leaf around $\hat{x}$. The sequence n_k tends to infinity. The height of $S = \hat{f}^{-n_k}(S)$ over $\hat{f}^{-n_k}(\hat{b}^k)$ (measured in a metric near x_0) tends to a finite value, namely, to its height over $\hat{x}$. On the other hand, its difference with the height of S over $\hat{b}^k$ is equal to $\ln|(f^{-n_k})'(0)|$, which tends to $-\infty$ (the Shrinking Lemma). This implies that the height of S over $\hat{b}^k$ tends to $+\infty$. Together with the previous discussion, this proves Theorem 3.2.7.

Chapitre 4

Instability of nondiscrete Lie subgroups in Lie groups

4.1 Introduction : main results, open problems and history

4.1.1 Main result : instability of liberty. Plan of the chapter

Let G be a nonsolvable Lie group. It is well-known (see [29]) that almost each (in the sense of the Haar measure) pair of elements $(A, B) \in G \times G$ generates a free subgroup in G. At the same time in the case, when G is connected and semisimple, there is a neighborhood $U \subset G \times G$ of unity in $G \times G$ where a topologically-generic pair $(A, B) \in U$ generates a dense subgroup : the latter pairs form an open dense subset in U. This was proved in [14].

The pairs generating groups with relations form a countable union of surfaces (relation surfaces) in $G \times G$. We show that the relation surfaces are dense in U.

The main result of the chapter is the following

Theorem 4.1.1 *[50] Any nondiscrete free subgroup with two generators in a nonsolvable Lie group G is unstable. More precisely, consider two elements $A, B \in G$ generating a free subgroup $\Gamma = < A, B >$. Let Γ be not discrete. Then there exists a sequence $(A_k, B_k) \to (A, B)$ of pairs converging to (A, B) such that the corresponding groups $< A_k, B_k >$ have relations : there exists a sequence $w_k = w_k(a, b)$ of nontrivial abstract words in symbols a, b (and their inverses a^{-1}, b^{-1}) [1] such that $w_k(A_k, B_k) = 1$ for all k.*

Remark 4.1.2 The condition that the subgroup under consideration be nondiscrete is natural : one can provide examples of discrete free subgroups of $PSL_2(\mathbb{C})$ (e.g., the Schottky group, see [6]) that are stably free, i.e., remain free under any small perturbation of the generators.

Remark 4.1.3 The closure of a nondiscrete subgroup in a Lie group is a Lie subgroup of positive dimension (see [116], p.42). Therefore, in Theorem 4.1.1 without loss of generality we assume that the subgroup $< A, B > \subset G$ under consideration is dense in G.

The question of instability of nondiscrete free subgroups was stated by É Ghys, who also suggested to study the best rate of approximation of the pair (A, B) by pairs having a relation of a length no greater than a given l (in analogy with the approximations of irrational number by rationals, where the best approximation rate is well-known ; it is achieved by continued fractions. In our situation the pair (A, B) plays the role of an irrational number, the pairs with relations play the role of rationals.)

[1] Everywhere in the chapter, by a word in given symbols we mean a word in the same symbols and their inverses

We prove an upper bound of the best approximation rate (Theorem 4.1.29 and Corollaries 4.1.30, 4.1.31 stated in 4.1.3 and briefly proved in 4.1.3 and 4.6).

The proof of Corollary 4.1.30 uses Theorem 4.1.16 (stated in 4.1.2), which deals with a semisimple Lie group and a pair (A, B) of its elements generating a dense subgroup (briefly called an irrational pair). It provides an upper bound for the rate of approximations of the elements of the unit ball in the Lie group by words in (A, B) satisfying a bound of derivatives. These and related results and open problems are discussed in Subsections 4.1.2-4.1.4.

Theorem 4.1.16 follows (see 4.1.2) from Lemma 4.1.25 and Theorem 4.1.26, both stated in 4.1.2; their proofs are omitted here and may be found in [50]. Theorem 4.1.26 proves the statement of Theorem 4.1.16 for a Lie group whose Lie algebra satisfies the so-called weak Solovay-Kitaev inequality (see Definition 4.1.23). This inequality means a decomposition (with estimate) of each element of a Lie algebra as a sum of two Lie brackets. Lemma 4.1.25 shows that the latter inequality holds true for any semisimple Lie algebra.

Theorem 4.1.21 (recalled in 4.1.2 and proved by R.Solovay and A.Kitaev, see [22, 80, 95]) concerns the Lie groups whose Lie algebras satisfy the (strong) Solovay-Kitaev inequality (see Definition 4.1.17). This inequality says that each element of a Lie algebra is a Lie bracket (with estimate). For these Lie groups Theorem 4.1.21 provides an upper bound for the rate of approximations of its elements in the unit ball by words in a given irrational pair of elements. The bound given by Theorem 4.1.21 is stronger than that in Theorem 4.1.16. Corollary 4.1.31 follows (see 4.1.2) from Theorems 4.1.21, 4.1.29 and Remark 4.1.22.

Remark 4.1.4 In the case, when the Lie group under consideration is $PSL_2(\mathbb{R})$, Theorem 4.1.1 easily follows from the density of the elliptic elements of finite orders in an open domain of $PSL_2(\mathbb{R})$: the proof is given in Subsection 4.1.5. The case of $PSL_2(\mathbb{C})$ is already nontrivial (in some sense, this is a first nontrivial case). In this case the previous argument cannot be applied, since the elliptic elements in $PSL_2(\mathbb{C})$ are nowhere dense. At the same time, there is a short proof of Theorem 4.1.1 for dense subgroups in $PSL_2(\mathbb{C})$ that uses holomorphic motions and quasiconformal mappings. We present it in Section 4.5.

In this chapter we prove Theorem 4.1.1 only for semisimple Lie groups with irreducible adjoint. Its statement in the general case then follows (relatively easily, see [50]) by arguments using the classical radical and decomposition theorems for Lie algebras (see [116], pp. 60, 61, 151; they are briefly recalled in Subsection 4.2.1). We treate separately the cases of a Lie group with proximal elements (Section 4.3, whose arguments work, e.g., for $G = SL_n(\mathbb{R})$) and without proximal elements (Section 4.4). A reader can read the proofs in Section 4.3 assuming everywhere that $G = SL_n(\mathbb{R})$.

In 4.1.7 we formulate a more general Theorem 4.1.33 in the case of a semisimple Lie group with irreducible adjoint representation. We deduce Theorem 4.1.1 from it at the same place. We prove Theorem 4.1.33 (modulo technical details) in Sections 4.3 and 4.4.

The definition of proximal element and basic properties of groups with proximal elements will be recalled in 4.2.3.

In 4.1.4 we present a brief historical overview and some open problems.

In 4.1.6 we give a proof of a simplified analogue (Proposition 4.1.32) of Theorem 4.1.1 for the simplest solvable noncommutative Lie group $Aff_+(\mathbb{R})$, which is the group of orientation-preserving affine transformations of the real line. (The author is sure that Proposition 4.1.32 is well known to the specialists.) The proof gives a simple illustration of the basic ideas used in the proof of Theorem 4.1.1.

The basic definitions concerning Lie groups (adjoint representation, (semi) simple groups, etc.), which will be used through the chapter (mostly in proofs), are recalled in 4.2.1 and 4.2.2.

4.1.2 Approximations by values of words.

Definition 4.1.5 Let G be a Lie group. We say that a pair $(A, B) \in G \times G$ is *irrational*, if it generates a dense subgroup in G.

Proposition 4.1.6 *Let G be a semisimple Lie group. The set of irrational pairs in $G \times G$ is open. More generally, the set of M- ples of elements of G generating dense subgroups is open in the product of M copies of G.*

Proof We prove the statement of the proposition for pairs : for M- ples the proof is analogous. Let $(A, B) \in G \times G$ be an irrational pair. We have to show that there exists its neighborhood $V \subset G \times G$ such that each pair $(A', B') \in V$ is irrational. Let $G_0 \subset G$ be the unity component of G. Recall that there exists a neighborhood $U \subset G_0 \times G_0$ of unity where an open and dense set of pairs generate dense subgroups in G_0 (see the beginning of the chapter and [14]). Thus, there exists an open subset $U' = U_1 \times U_2 \subset U$ such that each pair in U' generates a dense subgroup in G_0. There exist words w_1 and w_2 such that $w_j(A, B) \in U_j$, $j = 1, 2$. By continuity, there exists a neighborhood V of (A, B) such that for any $(A', B') \in V$ one has $w_j(A', B') \in U_j$, and thus, the subgroup generated by $w_j(A', B')$ is dense in G_0 by definition. The ambient subgroup generated by (A', B') is dense in G, since its closure contains G_0 (the previous statement) and each connected component of G contains an element of $< A', B' >$. (The latter fact holds true for the subgroup $< A, B >$ (which is dense) and remains valid for $< A', B' >$ by continuity.) Thus, each pair $(A', B') \in V$ is irrational. The proposition is proved. □

Let us recall the following well-known

Definition 4.1.7 Given a metric space E, a subset $K \subset E$ and a $\delta > 0$. We say that a subset in E is a δ- net on K, if the union of the δ- neighborhoods of its elements covers K, and all these neighborhoods do intersect K.

Remark 4.1.8 A δ- net on K is always contained in the δ- neighborhood of K.

Everywhere below (whenever the contrary is not specified) for any given point a of the space $\mathbb{R}^n$ (or of a Lie group G equipped with a Riemann metric) we denote

$$D_r(a) \text{ the ball centered at } a \text{ of radius } r,\ D_r = D_r(0) \subset \mathbb{R}^n \text{ (respectively, } D_r = D_r(1) \subset G_0\text{)},$$

where G_0 is the unity component of G. Everywhere below whenever we say about a distance on a connected component of a Lie group, we measure it with respect to a given left-invariant Riemann metric on the group (if the contrary is not specified). We use the following property of left-invariant distance.

Proposition 4.1.9 *Let $\delta_1, \delta_2 > 0$, G be a connected Lie group equipped with a left-invariant metric, $K \subset G$ be an arbitrary subset. Let $\Omega, \Omega' \subset G$ be two subsets such that Ω contains a δ_1- net on K, Ω' contains a δ_2- net on the δ_1- ball $D_{\delta_1} \subset G$. Then the product $\Omega\Omega' \subset G$ contains a δ_2- net on K.*

Proof Take an arbitrary $x \in K$ and some its δ_1- approximant $\omega \in \Omega$. Then $x' = \omega^{-1}x \in D_{\delta_1}$ (the left-invariance of the metric). Take a δ_2- approximant $\omega' \in \Omega'$ of x'. Then $\omega\omega'$ is a δ_2- approximant of x :

$$dist(\omega\omega', x) = dist(\omega', x') < \delta_2.$$

This proves the proposition. □

Let $X > 0$,

$$\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+ \text{ be a decreasing function such that } \varepsilon(cx) < c^{-1}\varepsilon(x) \text{ for any } c > 1,\ x \geq X. \quad (4.1.1)$$

Example 4.1.10 For any $\kappa > 0$ the function $\varepsilon(x) = e^{-x^{\kappa}}$ satisfies (4.1.1) with appropriate X (depending on κ).

Definition 4.1.11 Let G be a Lie group (equipped with a Riemann metric). Let $(A,B) \in G \times G$ be an irrational pair, $K \subset G$ be a bounded set in the unity component G_0 of G, $\varepsilon(x)$ be a function as in (4.1.1). We say that G is $\varepsilon(x)$- *approximable on* K *by words in* (A,B), if there exist a $c = c(A,B,K) > 0$, a sequence of numbers $l_m = l_m(A,B,K) \in \mathbb{N}$ (called *length majorants*), $l_m \to \infty$, as $m \to \infty$, and a sequence $\Omega_{m,K} = \Omega_{m,K,A,B}$ of word collections such that

$$|w| \leq l_m \text{ for any } w \in \Omega_{m,K} \text{ and} \tag{4.1.2}$$

$$\text{the subset } \Omega_{m,K}(A,B) \text{ is contained in } G_0 \text{ and contains a } \varepsilon(cl_m)- \text{ net on } K. \tag{4.1.3}$$

We say that G is $\varepsilon(x)$- approximable on K by words in (A,B) *with bounded derivatives*, if $\Omega_{m,K}$ satisfying (4.1.2) and (4.1.3) may be chosen so that the union $\cup_m \Omega_{m,K}(A,B)$ is a bounded subset in G_0 and there exist a $\Delta = \Delta(A,B,K) > 0$ and a neighborhood $V \subset G \times G$ of the pair (A,B) such that for any $m \in \mathbb{N}$ and any $w \in \Omega_{m,K}$

$$\text{the mapping } G \times G \to G, \ (a,b) \mapsto w(a,b), \text{ has derivative of norm less than } \Delta \text{ on } V. \tag{4.1.4}$$

Definition 4.1.12 We say that a Lie group G is $\varepsilon(x)$- *approximable (with bounded derivatives) by words in* $(A,B) \in G \times G$, if so it is on any bounded subset of its unity component. We say briefly that G is $\varepsilon(x)$- *approximable (with bounded derivatives)*, if so it is by words in an arbitrary irrational pair and on any bounded subset of its unity component.

The following proposition shows that the $\varepsilon(x)$- approximability is equivalent to the $\varepsilon(x)$- approximability on the unit ball centered at 1.

Proposition 4.1.13 *Let $\varepsilon(x)$ be as in (4.1.1), G, G_0, (A,B) be as in Definition 4.1.11, and let the metric on G be left-invariant. Let G be $\varepsilon(x)$- approximable by words in (A,B) (with bounded derivatives) on the unit ball $D_1 \subset G_0$, $c(A,B,D_1)$, $l_m(D_1) = l_m(A,B,D_1)$, Ω_{m,D_1} be the corresponding constant and sequences of length majorants and word collections, see (4.1.2) and (4.1.3). Let $R > 1$, Ω_R be a finite collection of words whose values at (A,B) form a 1- net on $D_R \subset G_0$,*

$$l(R) = \max_{w \in \Omega_R} |w|.$$

Then G is $\varepsilon(x)$- approximable on D_R by words in (A,B) (with bounded derivatives), where

$$\Omega_{m,D_R} = \Omega_R \Omega_{m,D_1}, \ l_m(D_R) = l_m(A,B,D_R) = l(R) + l_m(D_1), \ c(A,B,D_R) = \frac{c(A,B,D_1)}{l(R)}. \tag{4.1.5}$$

Proof Let Ω_{m,D_R}, $l_m(D_R)$ be the word collections and numbers given by (4.1.5). For any $m \in \mathbb{N}$ the set $\Omega_{m,D_R}(A,B)$ contains a δ- net on D_R,

$$\delta = \varepsilon(c_1 l_m(D_1)), \ c_1 = c(A,B,D_1),$$

by Proposition 4.1.9 applied to $K = D_R$, $\Omega = \Omega_R(A,B)$, $\delta_1 = 1$, $\Omega' = \Omega_{m,D_1}(A,B)$, $\delta_2 = \delta$. (The latter satisfy the conditions of the proposition by definition and the $\varepsilon(x)$- approximability.) One has

$$|w| \leq l_m(D_R) \text{ for any } w \in \Omega_{m,D_R},$$

$$\delta \leq \varepsilon(c_1(\inf_m \frac{l_m(D_1)}{l_m(D_R)})l_m(D_R)) \leq \varepsilon(c(A,B,D_R)l_m(D_R)).$$

This follows by definition, (4.1.5), the inequality $\frac{l_m(D_1)}{l_m(D_R)} \geq \frac{1}{l(R)}$ and the decreasing of the function $\varepsilon(x)$. If in addition, the set $\cup_m \Omega_{m,D_1}(A,B)$ is bounded and the derivatives of the mappings $(a,b) \mapsto w(a,b)$, $w \in \cup_m \Omega_{m,D_1}$, are uniformly bounded on a neighborhood of (A,B) in $G \times G$, then the same holds true with Ω_{m,D_1} replaced by Ω_{m,D_R} and the same neighborhood. This follows by definition and the finiteness of the collection Ω_R. This proves the $\varepsilon(x)$- approximability on D_R (with bounded derivatives) and Proposition 4.1.13. $\square$

Corollary 4.1.14 *Any Lie group $\varepsilon(x)$- approximable by words in a given irrational pair (with bounded derivatives) on unit ball, is $\varepsilon(x)$- approximable by words in the same pair (with bounded derivatives) on any bounded subset.*

The next proposition shows that the notion of $\varepsilon(x)$- approximability is independent on the choice of the metric on G.

Proposition 4.1.15 *Let $\varepsilon(x)$ be as in (4.1.1), G, A, B, K be as in Definition 4.1.11. Let g_1, g_2 be two (complete) Riemann metrics on G. Let the group G equipped with the metric g_1 be $\varepsilon(x)$- approximable on K by words in (A,B) (with bounded derivatives), $\Omega_{m,K}$, $l_m = l_m(A,B,K)$, $c_1 = c(A,B,K)$ be respectively the corresponding word collections, majorants and constant from (4.1.2) and (4.1.3). Let*

$$p = \max_m \varepsilon(c_1 l_m),\ \ K_p \text{ be the closed } p- \text{ neighborhood of } K \text{ in the metric } g_1.$$

Then the group G equipped with the metric g_2 is also $\varepsilon(x)$- approximable on K by words in (A,B) (with bounded derivatives), with respect to the same sequences $\Omega_{m,K}$, l_m and the new constant

$$c_2 = c_2(A,B,K) = \rho^{-1} c_1,\ \ \rho = \max\{\sup_{x,y\in K_p} \frac{d_{g_2}(x,y)}{d_{g_1}(x,y)}, 1\}.$$

Proof Each set $\Omega_{m,K}(A,B)$ contains a $\varepsilon(c_1 l_m)$- net on K in the metric g_1. The latter net is contained in K_p by definition, and is a $\rho\varepsilon(c_1 l_m)$- net on K in the metric g_2 (by the definition of ρ). One has

$$\rho\varepsilon(c_1 l_m) \leq \varepsilon(\rho^{-1} c_1 l_m) = \varepsilon(c_2 l_m),\ \text{whenever } m \text{ is large enough},$$

by definition and (4.1.1). This proves the $\varepsilon(x)$- approximability in the metric g_2. Let in addition, (G, g_1) (the group G equipped with the metric g_1) be $\varepsilon(x)$- approximable with bounded derivatives, i.e., the set $\cup_m \Omega_{m,K}(A,B)$ be bounded and the derivatives of the mappings $(a,b) \mapsto w(a,b)$, $w \in \cup_m \Omega_{m,K}$, be uniformly bounded on a (bounded) neighborhood $V \subset G\times G$ of (A,B) (in the metric g_1). Then the set $\widetilde{V} = \cup_m \Omega_{m,K}(V)$ is bounded and hence, $\sup_{x,y\in\widetilde{V}} \frac{d_{g_2}(x,y)}{d_{g_1}(x,y)} < +\infty$. The latter inequality together with the previous uniform boundedness of the derivatives on V (in the metric g_1) implies their uniform boundedness on V in the metric g_2. This proves the proposition. □

The following well-known Question is open. It was stated in [95], p.624 (without bounds of derivatives) for the groups $SU(n)$.

Question 4.1. Is it true that each semisimple Lie group (having at least one irrational pair of elements) is always $\varepsilon(x)$- approximable with $\varepsilon(x) = e^{-x}$? If yes, does the same hold true with bounded derivatives ?

Theorem 4.1.16 *Let G be an arbitrary semisimple Lie group (such that there exists at least one irrational pair $(A,B) \in G\times G$). Then the group G is $\varepsilon(x)$- approximable with bounded derivatives, where*

$$\varepsilon(x) = e^{-x^{\kappa}},\ \ \kappa = \frac{\ln 1.5}{\ln 9}. \tag{4.1.6}$$

In addition, for any irrational pair $(A,B) \in G\times G$ the corresponding length majorants $l_m = l_m(A,B,D_1)$ may be chosen so that

$$l_{m+1} = 9 l_m. \tag{4.1.7}$$

Theorem 4.1.16 follows from Lemma 4.1.25 and Theorem 4.1.26 (both formulated below).

It appears that for many Lie groups the previous approximation rate can be slightly improved. To state the corresponding result, let us introduce the following

Definition 4.1.17 Let $\mathfrak{g}$ be a Lie algebra with a fixed a positive definite scalar product on it. We say that $\mathfrak{g}$ *has surjective commutator*, if for any $z \in \mathfrak{g} \setminus 0$ there exist $x, y \in \mathfrak{g}$ such that

$$[x, y] = z. \tag{4.1.8}$$

We say that $\mathfrak{g}$ satisfies *the Solovay-Kitaev inequality*, if there exists a $c > 0$ such that for any $z \in \mathfrak{g} \setminus 0$ there exist $x, y \in \mathfrak{g}$ satisfying (4.1.8) and such that

$$|x| = |y| < c\sqrt{|z|} \tag{4.1.9}$$

Theorem 4.1.18 *(G.Brown, [15]). Each complex semisimple Lie algebra and each real semisimple split Lie algebra (see [116], p.288) have surjective commutator.*

Remark 4.1.19 In fact, the latter Lie algebras satisfy the Solovay-Kitaev inequality. The author did not find a proof of this statement in the literature, but it can be obtained by minor refinement of Brown's arguments [15]. The question of the surjectivity of commutator in Lie groups has a long history, see [15], [59] and the references therein. We would like to mention one of the first results due to M.Goto [54], who have proved that in any compact semisimple Lie group each element is a commutator of appropriate two other elements.

Example 4.1.20 The Lie algebras $\mathfrak{su}_n$ satisfy the Solovay-Kitaev inequality [22, 80, 95].

Question 4.2. Is it true that each real semisimple Lie algebra has surjective commutator? If yes, is it true that it satisfies the Solovay-Kitaev inequality?

Theorem 4.1.21 *(R.Solovay, A.Kitaev, [22, 80, 95]) Let a Lie group G have a Lie algebra satisfying the Solovay-Kitaev inequality, and there exist at least one irrational pair $(A, B) \in G \times G$. Then the group G is $\varepsilon'(x)$- approximable with*

$$\varepsilon'(x) = e^{-x^{\kappa'}}, \ \kappa' = \frac{\ln 1.5}{\ln 5}. \tag{4.1.10}$$

In addition, for any irrational pair $(A, B) \in G \times G$ the corresponding length majorants $l_m = l_m(A, B, D_1)$ can be chosen so that

$$l_{m+1} = 5 l_m. \tag{4.1.11}$$

Remark 4.1.22 In fact, in Theorem 4.1.21 the Lie group is $\varepsilon'(x)$- approximable with bounded derivatives (with length majorants $l_m(A, B, D_1)$ satisfying (4.1.11)). This can be easily derived from Kitaev's proof [22, 80, 95]. See [50] for more detail.

Definition 4.1.23 Let $\mathfrak{g}$ be a Lie algebra with a fixed positive definite scalar product on it. We say that $\mathfrak{g}$ satisfies *the weak Solovay-Kitaev inequality*, if there exists a consant $c > 0$ such that for any $z \in \mathfrak{g} \setminus 0$ there exist $x_j, y_j \in \mathfrak{g}$, $j = 1, 2$, such that

$$z = [x_1, y_1] + [x_2, y_2], \ |x_j| = |y_j| < c\sqrt{|z|}. \tag{4.1.12}$$

Remark 4.1.24 The condition that a Lie algebra satisfies a (weak) Solovay-Kitaev inequality is independent on the choice of the scalar product. A Lie algebra satisfying the strong Solovay-Kitaev inequality obviously satisfies the weak one.

Lemma 4.1.25 *Each semisimple Lie algebra satisfies the weak Solovay-Kitaev inequality.*

Lemma 4.1.25 is easily deduced from basic properties of complex roots of a semisimple Lie algebra. Some of these properties are recalled in 4.2.2.

Theorem 4.1.26 *Let a Lie group G have a Lie algebra satisfying the weak Solovay-Kitaev inequality. Let $(A, B) \in G \times G$ be an irrational pair. Then the group G is $\varepsilon(x)$- approximable with bounded derivatives, where $\varepsilon(x)$, $l_m = l_m(A, B, D_1)$ are the same, as in (4.1.6) and (4.1.7) respectively.*

Theorem 4.1.26 is proved analogously to the proof of Theorem 4.1.21 given in [22, 80, 95], see its proof in [50] for more detail. Together, Lemma 4.1.25 and Theorem 4.1.26 imply Theorem 4.1.16.

4.1.3 Approximations by groups with relations

Fix a Riemann metric on a Lie group G.

Definition 4.1.27 Let G be a Lie group, $(A, B) \in G \times G$. Let $\varepsilon(x)$ be a function as in (4.1.1). We say that the pair (A, B) is $\varepsilon(x)$- *approximable by pairs with relations*, if there exist a $c = c(A, B) > 0$ and sequences of numbers $l_k \in \mathbb{N}$ (called *the length majorants*), $l_k \to \infty$, as $k \to \infty$, nontrivial words $w_k(a, b)$ of lengths at most l_k and pairs $(A_k, B_k) \to (A, B)$ such that for any $k \in \mathbb{N}$ one has

$$w_k(A_k, B_k) = 1 \text{ and } dist((A_k, B_k), (A, B)) < \varepsilon(cl_k) \text{ for any } k \in \mathbb{N}. \tag{4.1.13}$$

Remark 4.1.28 The previous Definition and the corresponding word sequence w_k are independent on the choice of the metric on G (while the constant c depends on the metric). The proof of this statement is analogous to the proof of Proposition 4.1.15.

Theorem 4.1.29 *Let G be a nonsolvable Lie group, G_{ss} be its semisimple part (see Definition 4.2.5). Let $\varepsilon(x)$ be a function as in (4.1.1). Let $A, B \in G$ and $A', B' \in G_{ss}$ be their projections. Let the pair $(A', B') \in G_{ss} \times G_{ss}$ be irrational, and the group G_{ss} be $\varepsilon(x)$- approximable with bounded derivatives by words in (A', B') (see Definition 4.1.12). Then the pair (A, B) is $\varepsilon(x)$- approximable by pairs with relations.*

Addendum to Theorem 4.1.29. *In the conditions of Theorem 4.1.29 the group G_{ss} is $\varepsilon(x)$- approximable by words in (A', B') with bounded derivatives. Let $l_m = l_m(A', B', D_1)$ be the corresponding word length majorants from (4.1.2). There exist constants $q \in \mathbb{N}$ and $c'' > 0$ depending only on (A, B) such that the pair $(A, B) \in G \times G$ is $\varepsilon(x)$- approximable by pairs with relations having length majorants*

$$l'_m = c'' l_m, \ m \geq q. \tag{4.1.14}$$

Corollary 4.1.30 *Each irrational pair of elements in a nonsolvable Lie group is $\varepsilon(x) = e^{-x^{\kappa}}$- approximable by pairs with relations, where $\kappa = \frac{\ln 1.5}{\ln 9}$, see (4.1.6). The corresponding length majorant sequence l_k can be chosen so that $l_{k+1} = 9l_k$.*

Proof Let G be a nonsolvable Lie group, $(A, B) \in G \times G$ be an irrational pair. Then its projection $(A', B') \in G_{ss} \times G_{ss}$ is also irrational. The function $\varepsilon(x) = e^{-x^{\kappa}}$ satisfies the conditions of Theorem 4.1.29 and its Addendum with a majorant sequence l_k such that $l_{k+1} = 9l_k$ (Theorem 4.1.16 applied to the semisimple part of G). This together with Theorem 4.1.29 and its Addendum, see (4.1.14), implies the corollary. $\square$

Corollary 4.1.31 *Let G be a nonsolvable Lie group such that the semisimple part of $\mathfrak{g}$ satisfies the Solovay-Kitaev inequality. Then each pair $(A, B) \in G \times G$ with irrational projection to $G_{ss} \times G_{ss}$ is $\varepsilon'(x) = e^{-x^{\kappa'}}$- approximable by pairs with relations, where $\kappa' = \frac{\ln 1.5}{\ln 5}$, see (4.1.10). The corresponding length majorant sequence l_k can be chosen so that $l_{k+1} = 5l_k$.*

Corollary 4.1.31 follows from Theorem 4.1.29 (with the Addendum), Theorem 4.1.21 and Remark 4.1.22, analogously to the above proof of Corollary 4.1.30. A proof of Theorem 4.1.29 together with its Addendum is sketched in Section 4.6.

Question 4.3. Is it true that in any nonsolvable Lie group each irrational pair of elements is e^{-x}-approximable by pairs with relations ?

By Theorem 4.1.29, a positive solution of Question 4.1 with bounded derivatives (see 4.1.2) would imply a positive answer to Question 4.3.

4.1.4 Historical remarks and further open questions

The famous Tits' alternative [112] says that any subgroup of linear group satisfies one of the two following incompatible statements :

- either it is solvable up-to-finite, i.e., contains a solvable subgroup of a finite index ;
- or it contains a free subgroup with two generators.

Any dense subgroup of a connected semisimple real Lie group satisfies the second statement : it contains a free subgroup with two generators.

The question of possibility to choose the latter free subgroup to be dense was stated in [33] and studied in [14] and [33]. É.Ghys and Y.Carrière [33] have proved the positive answer in a particular case. E.Breuillard and T.Gelander [14] have done it in the general case.

T.Gelander [32] have shown that in any compact nonabelian Lie group any finite tuple of elements can be approximated arbitrarily well by another tuple (of the same number of elements) that generates a nonvirtually free group.

A question (close to Question 4.1) concerning Diophantine properties of an individual pair $A, B \in SO(3)$ was studied in [79]. We say that a pair $(A, B) \in SO(3) \times SO(3)$ is *Diophantine* (see [79]), if there exists a constant $D > 1$ depending on A and B such that for any word $w_k = w_k(a, b)$ of length k

$$|w_k(A, B) - 1| > D^{-k}.$$

A.Gamburd, D.Jakobson and P.Sarnak have stated the following

Question 4.4 [30]. *Is it true that almost each pair $(A, B) \in SO(3) \times SO(3)$ is Diophantine ?*

V.Kaloshin and I.Rodnianski [79] proved that almost each pair (A, B) satisfies a weaker inequality with the latter right-hand side replaced by D^{-k^2}.

Question 4.5. *Is there an analogue of Theorem 4.1.1 for the group of*
- germs of one-dimensional real diffeomorphisms (at their common fixed point) ?
- germs of one-dimensional conformal diffeomorphisms ?
- diffeomorphisms of compact manifold ?

The latter question concerning conformal germs is related to study of one-dimensional holomorphic foliations. A related result was obtained in the joint paper [72] by Yu.S.Ilyashenko and A.S.Pyartli, which deals with one-dimensional holomorphic foliations on $\mathbb{CP}^2$ with isolated singularities and invariant infinity line. They have shown that for a typical foliation the holonomy group at infinity is free. Here "typical" means "lying outside a set of zero Lebesgue measure". It is not known whether this is true for an open set of foliations.

4.1.5 A simple proof of Theorem 4.1.1 for $G = PSL_2(\mathbb{R})$

Without loss of generality we assume that $\overline{< A, B >} = G$. Otherwise, $< A, B >$ would be dense in a Lie subgroup of dimension at most two, which is solvable, hence, A and B cannot generate a free subgroup.

The group $G = PSL_2(\mathbb{R})$ acts by conformal transformations of unit disk D_1. There is an open subset $U \subset G$ formed by nontrivial elliptic transformations, which are conformally conjugated to nontrivial rotations. The rotation number (which is the rotation angle divided by 2π) is a local (nowhere zero) analytic function in the parameters of U. An elliptic transformation f has finite order if and only if its rotation number $\rho(f)$ is rational.

Let $w = w(a, b)$ be a word such that $w(A, B) \in U$ (it exists by density). It suffices to show that the function $(a, b) \mapsto \rho(w(a, b))$ is not constant near (A, B) : then it follows that there exists a sequence $(a_n, b_n) \to (A, B)$ such that $\rho(w(a_n, b_n)) \in \mathbb{Q}$. Hence, $w(a_n, b_n)$ are finite order elements, thus, one has relations of the type $w^{k_n}(a_n, b_n) = 1$.

The previous function is locally analytic. Suppose the contrary : it is constant. Then by analyticity, it is constant globally and $w(a, b)$ is elliptic with one and the same nonzero rotation number for all the pairs (a, b). On the other hand, it vanishes at $(a, b) = (1, 1)$, since $w(1, 1) = 1$ - a contradiction. This proves Theorem 4.1.1 for $G = PSL_2(\mathbb{R})$.

4.1.6 Case of group $Aff_+(\mathbb{R})$.

For any $s > 0$, $u \in \mathbb{R}$ denote

$$g_s : x \mapsto sx, \ t_u : x \mapsto x + u, \ \Gamma(s) = < g_s, t_1 > \subset Aff_+(\mathbb{R}).$$

Proposition 4.1.32 *For any $s_0 > 0$ there exists a sequence $s_k \to s_0$ such that the corresponding subgroups $\Gamma(s_k)$ have relations that do not hold identically in s.*

Proof It suffices to prove the statement of the proposition for open and dense subset of the values $s_0 > 0$ (afterwards we pass to the closure and diagonal sequences). Thus, without loss of generality we assume that $s_0 \neq 1$. We also assume that $0 < s_0 < 1$, since the groups $\Gamma(s)$ and $\Gamma(s^{-1})$ coincide.

For any s the group $\Gamma(s)$ contains the elements

$$t_{s^k} = g_s^k \circ t_1 \circ g_s^{-k} \text{ and } t_{ms^k}, \ m \in \mathbb{Z}, \ k \in \mathbb{N} \cup 0.$$

We construct sequences of numbers $s_k \to s_0$ and $m_k \in \mathbb{N}$ in such a way that each group $\Gamma(s)$, $s = s_k$, has an extra relation $t_{m_k s^k} = t_1$. For obvious reasons this is not a relation that holds identically. This will prove the Proposition.

For any k take $m_k = [s_0^{-k}]$, thus, m_k is the integer number such that $m_k s_0^k$ gives a best approximation of 1, with rate less than s_0^k ; $m_k s_0^k \to 1$, as $k \to \infty$. The values s_k we are looking for are the positive solutions to the equations $m_k s^k = 1$ (they correspond to the previous relations by definition). Indeed, it suffices to show that $s_k \to s_0$, or equivalently, that the solutions u_k of the equations $\psi_k(u) = m_k(s_0 + u)^k = 1$ converge to 0. The mapping ψ_k is the composition of the homothety $u \mapsto \widetilde{u} = ku$ and the mapping $\widetilde{\psi}_k : \widetilde{u} \mapsto m_k(s_0 + k^{-1}\widetilde{u})^k$. One has

$$\widetilde{\psi}_k(\widetilde{u}) = m_k s_0^k (1 + k^{-1}\frac{\widetilde{u}}{s_0})^k \to \psi(\widetilde{u}) = e^{\frac{\widetilde{u}}{s_0}}, \text{ as } k \to \infty. \tag{4.1.15}$$

The convergence is uniform with derivatives on compact sets. The limit $\psi(\widetilde{u})$ is a diffeomorphism $\mathbb{R} \to \mathbb{R}_+$ with unit value at 0. Hence, the solutions $\widetilde{u}_k$ of the equations $\widetilde{\psi}_k(\widetilde{u}) = 1$ converge to 0. Therefore, so do $u_k = k^{-1}\widetilde{u}_k$ and $s_k = s_0 + u_k \to s_0$. The proposition is proved. □

4.1.7 Generalization in the case of semisimple Lie group with irreducible adjoint

Theorem 4.1.33 *Let G be a semisimple Lie group with irreducible Ad_G (not necessarily connected). Consider a family $\alpha(u) = (a_1(u), \dots, a_M(u))$, $M \in \mathbb{N}$, of M- ples of its elements that depend on a parameter u from some manifold (say, $\mathbb{R}^l$). Let the family $\alpha(u)$ be conj- nondegenerate at 0 (see Definition 4.2.12 in 4.2.1). Then there exist arbitrarily small values u such that the mappings $a_i(0) \mapsto a_i(u)$ do not induce group isomorphisms $< \alpha(0) > \to < \alpha(u) >$.*

Theorem 4.1.33 and Corollary 4.2.14 (stated below, in 4.2.1) imply immediately Theorem 4.1.1 in the case, when G is semisimple, Ad_G is irreducible and A, B generate a dense subgroup. Indeed, suppose the contrary : each pair (a, b) close to (A, B) generates a free subgroup, hence, the mapping $(A, B) \mapsto (a, b)$ induces an isomorphism of the corresponding subgroups. Consider the family of all the pairs (a, b) depending on the parameters in G of the elements a and b. By the previous assumption and Theorem 4.1.33 (applied to the same family), this family is *conj*- degenerate at (A, B). On the other hand, it is a priori *conj*- nondegenerate at (A, B) (Corollary 4.2.14), - a contradiction.

4.2 Background material on Lie groups

4.2.1 Lie groups, basic definitions and properties

Everywhere below the Lie algebra of a Lie group G will be denoted

$$\mathfrak{g} = T_1 G.$$

Let us firstly recall what is the adjoint action (see [116], p.32). The group G acts on itself by conjugations (the unity is fixed). The derivative of this action along the vectors of the tangent Lie algebra $\mathfrak{g}$ defines a linear representation of G in $\mathfrak{g}$ called the *adjoint representation.* The adjoint representation of an element $g \in G$ is denoted Ad_g. (If G is a matrix group, then the adjoint action is given by matrix conjugation : $Ad_g(h) = ghg^{-1}$.) The adjoint action of a Lie algebra on itself is defined by the Lie bracket, $\mathrm{ad}_x : y \mapsto [x, y]$. Let G be a Lie group with a given algebra $\mathfrak{g}$. One has

$$Ad_{\exp x} = \exp(\mathrm{ad}_x) \text{ for any } x \in \mathfrak{g}.$$

Definition 4.2.1 A Lie group is said to be *simple*, if it has dimension greater than one and the adjoint representation of its unity component is irreducible. A Lie group is said to be *semisimple*, if its unity component has no normal solvable Lie subgroup of positive dimension.

Remark 4.2.2 A Lie group is (semi)simple, if and only if so is its algebra in the following sense.

Definition 4.2.3 An *ideal* in a (real or complex) Lie algebra $\mathfrak{g}$ is a Lie subalgebra $I \subset \mathfrak{g}$ (over the corresponding field) such that $[\mathfrak{g}, I] \subset I$. A Lie algebra $\mathfrak{g}$ is said to be *simple*, if it has no nonzero ideal different from itself. A Lie algebra $\mathfrak{g}$ is said to be *semisimple*, if it has no nonzero *solvable* ideal.

Remark 4.2.4 A complex Lie algebra is semisimple, if and only if so is it as a real algebra.

It is well-known (see [116], pp. 60, 61) that each Lie algebra $\mathfrak{g}$ has a unique maximal solvable ideal (called *radical*; it may be trivial). The factor of $\mathfrak{g}$ by the radical is a semisimple Lie algebra. Analogously, each nonsolvable Lie group has a unique maximal solvable normal connected Lie subgroup and its tangent algebra coincides with the radical of the Lie algebra of the ambient group; the corresponding Lie group quotient is a semisimple Lie group.

Definition 4.2.5 The factor of a nonsolvable Lie algebra (group) by its radical (respectively, the maximal solvable normal connected Lie subgroup) is called its *semisimple part.*

Remark 4.2.6 The Lie algebra of the semisimple part of a nonsolvable Lie group G is the semisimple part of $\mathfrak{g}$.

Remark 4.2.7 Each semisimple Lie algebra is a finite direct product of simple Lie algebras (the latter product decomposition is unique, see [116], p.151).

Example 4.2.8 Let $G = SL_n(\mathbb{R})$. The adjoint action of a diagonal matrix

$$A = diag(a_1, \dots, a_n) \in G$$

is diagonalizable and has the eigenvalues 1, $\lambda_{ij} = \frac{a_i}{a_j}$, $i \neq j$. The eigenvector corresponding to the eigenvalue λ_{ij} is represented by the matrix with zeros everywhere except for the (i, j)- th element. The other (unit) eigenvalues correspond to the diagonal matrices. It is well-known that the group $SL_n(\mathbb{R})$ is simple (see, [116], pp. 150, 177).

Proposition 4.2.9 *For any semisimple (not necessary (simply) connected) Lie group G there exists a collection of semisimple Lie groups $H_1, \dots, H_s$, each one with irreducible adjoint Ad_{H_j}, and a homomorphism*

$$\pi : G \to H_1 \times \cdots \times H_s$$

that is a local diffeomorphism (in particular, $\mathfrak{g} = \prod_{j=1}^s \mathfrak{h}_j$). Moreover, the image $\pi(G)$ is projected surjectively onto each group H_j. The kernel of π is contained in the center of the unity component of G.

Proof If the adjoint Ad_G is irreducible, we put $s = 1$, $G = H_1$, and we are done. In general, $\mathfrak{g}$ is a product of simple Lie algebras. If the group G is simply connected, then it is the product of the corresponding simply connected Lie groups (which are simple, and hence, have irreducible adjoints).

Case when G is an arbitrary connected semisimple Lie group. Denote $\widetilde{G}$ its universal covering, $C(\widetilde{G})$ the center of $\widetilde{G}$ (which is a discrete subgroup in $\widetilde{G}$). Then

$$G = \widetilde{G}/\Gamma,\ \Gamma \subset C(\widetilde{G}),\ \widetilde{G} = \widetilde{H}_1 \times \cdots \times \widetilde{H}_s,\ \widetilde{H}_j \text{ are simply connected simple groups.}$$

One has $C(\widetilde{G}) = \prod_{j=1}^s C(\widetilde{H}_j)$. Therefore, there is a natural projection homomorphism

$$\pi : G = \widetilde{G}/\Gamma \to \widetilde{G}/C(\widetilde{G}) = H_1 \times \cdots \times H_s,\ H_j = \widetilde{H}_j/C(\widetilde{H}_j). \tag{4.2.1}$$

This is a homomorphism we are looking for.

Case, when G is an arbitrary semisimple Lie group. Denote $G_0 \subset G$ its unity component. We assume that Ad_G is not irreducible (the opposite case was already discussed). Let $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ be the decomposition of $\mathfrak{g}$ as a product of simple Lie algebras. The adjoint of each $g \in G$ sends any subalgebra $\mathfrak{g}_i$ to an isomorphic subalgebra $\mathfrak{g}_j$; then we say that $\mathfrak{g}_i$ is equivalent to $\mathfrak{g}_j$. To each equivalence class of the $\mathfrak{g}_j$' s we associate the product of the algebras from this class. Denote all the latter products $\mathfrak{h}_1, \dots, \mathfrak{h}_s$: by definition, $\mathfrak{g} = \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_s$. The subalgebras $\mathfrak{h}_j$ are Ad_G- invariant by construction, and $Ad_G|_{\mathfrak{h}_j}$ is irreducible for each j. Indeed, the only Ad_{G_0}- invariant subspaces in $\mathfrak{h}_j$ are the subalgebras $\mathfrak{g}_i$ from the corresponding equivalence class and their products. No one of these subspaces is Ad_G invariant, since Ad_G acts transitively on the subalgebras $\mathfrak{g}_i$ in $\mathfrak{h}_j$ by definition.

Let $\widetilde{H}_j$ be the simply connected Lie groups with algebras $\mathfrak{h}_j$, $\hat{H}_j = \widetilde{H}_j/C(\widetilde{H}_j)$. Let

$$\hat{\pi} : G_0 \to \hat{H}_1 \times \cdots \times \hat{H}_s$$

be the homomorphism (4.2.1), which is a local diffeomorphism. Consider the subset $H_j' \subset G_0$ of the elements in G_0 whose images under $\hat{\pi}$ have unit $\hat{H}_j$- component : it is the kernel of the composition of $\hat{\pi}$ with the projection to $\hat{H}_j$. This is a normal Lie subgroup in G_0. Denote $H_j^0 \subset H_j'$ its unity component. Its Lie algebra is the product of the $\mathfrak{h}_i$' s with $i \neq j$, which is Ad_G- invariant. Thus, the subgroup $H_j^0 \subset G$ is normal in G. Denote

$$H_j = G/H_j^0;\ \pi : G \to H_1 \times \cdots \times H_s$$

the homomorphism whose components are the natural projections. By construction, this is a local diffeomorphism and the projection of $\pi(G)$ to each H_j is surjective. Denote $\Gamma \subset G$ the kernel of π, which is the intersection of the subgroups $H_j^0 \subset G_0$. It is contained in G_0 and is a discrete normal subgroup there. Hence, it is contained in the center of G_0. The image of the adjoint representation $Ad_{H_j} : \mathfrak{h}_j \to \mathfrak{h}_j$ coincides with that of the previous representation $Ad_G|_{\mathfrak{h}_j}$, which is irreducible. Therefore, Ad_{H_j} is also irreducible. Proposition 4.2.9 is proved. $\Box$

Definition 4.2.10 Let G be a Lie group, $\alpha = (a_1, \dots, a_M) \in G^M$. Consider the G- action on G^M by simultaneous conjugations, $g : \alpha \mapsto g\alpha g^{-1}$, and denote $Conj(a_1, \dots, a_M) \subset G^M$ the orbit of $(a_1, \dots, a_M)$ (i.e., the joint conjugacy class).

Proposition 4.2.11 *Let G be a semisimple Lie group, $n = dimG$. Let a pair (or M- ple) of its elements be irrational, i.e., generate a dense subgroup in G. Then their joint conjugacy class is bijectively analytically parametrized (as a G- action orbit) by the quotient of the group G by its center. The space of the conjugacy classes corresponding to all the irrational pairs (M- ples) is an analytic manifold of dimension n (respectively, $(M-1)n$). The mapping $(a_1, \dots, a_M) \mapsto Conj(a_1, \dots, a_M)$ is a local submersion at the irrational M- ples $(a_1, \dots, a_M) \in G^M$.*

Proof Let $A = (A_1, \dots, A_M) \in G^M$ be an irrational M- ple : the subgroup $< A >$ generated by A is dense in G. The parametrization $g \mapsto gAg^{-1}$ of the conjugacy class of A by $g \in G$ induces its 1-to-1 parametrization by the quotient of G by its center. Equivalently, for any two distinct elements $g, h \in G$ the elements gAg^{-1}, hAh^{-1} of the conjugacy class of A coincide if and only if $g' = g^{-1}h$ lies in the center of G. Indeed, $gAg^{-1} = hAh^{-1}$, if and only if g' commutes with each A_i, or equivalently, with $< A >$. The latter commutation is equivalent to the commutation with $G = \overline{< A >}$. This proves the previous statement. The irrational M- ples form an open subset in the product of M copies of G (Proposition 4.1.6). This together with the previous parametrization statement implies the statements of Proposition 4.2.11. $\Box$

Definition 4.2.12 Let G be a semisimple Lie group, $\alpha(u) = (a_1(u), \dots, a_M(u))$ be a C^1- family of M- ples of its elements depending on a parameter u from some manifiold (say, $\mathbb{R}^l$). We say that α is *conj- nondegenerate at* $u = u_0$ if the subgroup $< \alpha(u_0) > \subset G$ is dense in G and the mapping $u \mapsto Conj(\alpha(u))$ has a rank no less than $n = dimG$ at $u = u_0$. Otherwise we say that the family $\alpha(u)$ is *conj*- degenerate at u_0. If $\alpha(u)$ is *conj*- nondegenerate at all u, then we say that $\alpha(u)$ is *conj- nondegenerate.*

Remark 4.2.13 Let G be a semisimple Lie group, $\alpha(u)$ be an arbitrary family of M- ples of its elements. Then the set of the parameter values u at which $\alpha(u)$ is *conj*- nondegenerate is an open set (it may be empty). This follows from definition and Proposition 4.1.6.

Corollary 4.2.14 *Let G be a semisimple Lie group, $(A, B) \in G \times G$ be an irrational pair. The family of all the pairs $(a, b) \in G \times G$ is conj- nondegenerate at (A, B).*

Proof The mapping $(a, b) \mapsto Conj(a, b)$ has full rank at (A, B), which is equal to n (Proposition 4.2.11). This implies the Corollary. $\Box$

For any real linear space (Lie algebra) $\mathfrak{g}$ we denote

$$\mathfrak{g}_{\mathbb{C}} \text{ its complexification,}$$

which is also a linear space (Lie algebra).

4.2.2 Semisimple Lie algebras and root decomposition

Definition 4.2.15 An element of a Lie algebra is called *regular*, if its adjoint has the minimal possible multiplicity of zero eigenvalue.

Definition 4.2.16 Let $\mathfrak{g}$ be a complex semisimple Lie group. A *Cartan subalgebra* associated to a regular element of $\mathfrak{g}$ is its centralizer : the set of the elements commuting with it.

It is well-known (see, [116], pp. 153, 159) that

- a) any Cartan subalgebra $\mathfrak{h}$ is a maximal commutative subalgebra ;
- b) all the Cartan subalgebras are conjugated ;
- c) the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is diagonalizable in an appropriate basis of $\mathfrak{g}$;
- d) the eigenvalues of the latter adjoint action are linear functionals on $\mathfrak{h}$, thus, elements of $\mathfrak{h}^*$, the nonidentically zero ones are called *roots* ;
- e) the roots are distinct and the corresponding eigenspaces are complex lines ;
- f) if α is a root, then so is $-\alpha$;
- g) for any root α the only roots complex-proportional to α are $\pm\alpha$;
- h) some roots form a complex basis in $\mathfrak{h}^*$ and moreover, an integer root basis in the following sense : each root is an integer linear combination of the basic roots ;
- i) the algebra $\mathfrak{g}$ is the direct sum (as a linear space) of $\mathfrak{h}$ and the root eigenlines.

Statement g) follows from the analogous statement in [116] (theorem 6 on p.159) for real-proportional roots and from statement h).

4.2.3 Proximal elements

Definition 4.2.17 A linear operator $\mathbb{R}^n \to \mathbb{R}^n$ is called *proximal*, if it has a unique complex eigenvalue (taken with multiplicity) of maximal modulus (then this eigenvalue is automatically real). An element of a Lie group is proximal, if its adjoint is.

Remark 4.2.18 The set of proximal operators (elements) is open.

Definition 4.2.19 A *maximal* $\mathbb{R}$- *split torus* in a semisimple Lie group G is a maximal connected Lie subgroup with a diagonalizable adjoint action on $\mathfrak{g}$ (which is automatically commutative). A semisimple Lie group is called *split* (see [116], p. 288), if some its maximal $\mathbb{R}$- split torus is a maximal connected commutative Lie subgroup.

Example 4.2.20 Each group $SL_n(\mathbb{R})$ is split : the diagonal matrices form a maximal $\mathbb{R}$- split torus. A typical diagonal matrix is a proximal element of $SL_n(\mathbb{R})$. The group $SO(3)$ is not split, has trivial maximal $\mathbb{R}$- split torus and no proximal elements. The group $SO(2,1)$ is not split and has one-dimensional maximal $\mathbb{R}$- split torus, whose nontrivial elements are proximal in $SO(2,1)$.

Lemma 4.2.21 *Let a semisimple Lie group contain a proximal element. Then each its maximal* $\mathbb{R}$-*split torus contains a proximal element.*

The proof of Lemma 4.2.21 is implicitly contained in [1] (p.25, proof of theorem 6.3).

Definition 4.2.22 An element g of a Lie group will be called *1- proximal,* if the operator $Ad_g - Id$ is proximal.

We use the following equivalent characterization of semisimple Lie groups with proximal elements.

Corollary 4.2.23 *A semisimple Lie group contains a proximal element, if and only if its unity component contains a 1- proximal element. In this case the 1- proximal elements form an open subset in* G *accumulating to the unity.*

In the proof of the corollary we use the following properties of the adjoint representation of a semisimple Lie group.

Proposition 4.2.24 *Let G be a connected semisimple Lie group. For any $x \in \mathfrak{g}$ ($g \in G$) and an eigenvalue λ of ad_x (Ad_g) the number $-\lambda$ (respectively, λ^{-1}) is also an eigenvalue of the corresponding adjoint with the same multiplicity, as λ.*

Proof It suffices to prove the statement of the proposition for the Lie algebra : this would imply its statement for any $g \in G$ close enough to 1 (belonging to an exponential chart), and then, for any $g \in G$ (the connectedness of G and the analytic dependence of the operator family Ad_g on $g \in G$). For any regular element $x \in \mathfrak{g}$ the nonzero eigenvalues of ad_x are split into pairs of opposite eigenvalues with equal multiplicities. This follows from the central symmetry of the root system of the complex Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$ containing x (see 4.2.2, statement f)). The regular elements are dense in $\mathfrak{g}$. This implies that the previous statement remains valid for any $x \in \mathfrak{g}$. This proves the proposition. □

Corollary 4.2.25 *Any 1- proximal element of a connected semisimple Lie group is proximal.*

Proof Let g be a 1- proximal element, $\lambda \in \mathbb{R}$ be the eigenvalue of $Ad_g - Id$ with maximal modulus (which is simple, and hence, nonzero). Then $(\lambda+1)^{\pm 1}$ are simple eigenvalues of Ad_g (by Proposition 4.2.24). We claim that $(\lambda+1)^{\pm 1}$ is the eigenvalue of Ad_g with maximal modulus, if $\lambda \in \mathbb{R}_{\pm}$. Indeed, it follows from definition (in both cases) that $(\lambda+1)^{\pm 1} \geq |\lambda|+1$. For any eigenvalue $\lambda' \neq \lambda$ of $Ad_g - Id$ one has $|\lambda'| < |\lambda|$ (1- proximality). This together with the previous and triangle inequalities implies that

$$(\lambda+1)^{\pm 1} \geq |\lambda|+1 > |\lambda'|+1 \geq |\lambda'+1|.$$

This proves the previous statement on the maximality of the eigenvalue $(\lambda+1)^{\pm 1}$ and thus, the proximality of Ad_g. Corollary 4.2.25 is proved. □

Proposition 4.2.26 *Let G be a semisimple Lie group, $T \subset G$ be a maximal $\mathbb{R}$- split torus. Let $g \in T$ be a proximal element of G. Then g is also 1- proximal.*

Proof The eigenvalues of Ad_g (which are real, since $Ad_T : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable) are positive, since this is true for $Ad_1 = Id$ and the torus T is connected. The nonunit eigenvalues are split into pairs of inverses (Proposition 4.2.24). Hence, we can order them as follows (distinct indices correspond to distinct (may be multiple) eigenvalues) :

$$0 < \lambda_1^{-1} < \lambda_2^{-1} < \cdots < \lambda_k^{-1} < 1 < \lambda_k < \cdots < \lambda_1. \tag{4.2.2}$$

The eigenvalue λ_1 is simple (proximality). One has

$$\lambda_1 - 1 > \lambda_1^{-1}(\lambda_1 - 1) = 1 - \lambda_1^{-1}, \text{ since } 0 < \lambda_1^{-1} < 1$$

by (4.2.2). This together with (4.2.2) implies that $\lambda_1 - 1$ is a simple eigenvalue of $Ad_g - Id$ with maximal modulus. Hence, the operator $Ad_g - Id$ is proximal. Proposition 4.2.26 is proved. □

Proof of Corollary 4.2.23. Let the unity component of G contain a 1- proximal element. Then this element is proximal (Corollary 4.2.25). Conversely, let G contain proximal elements. Let $T \subset G$ be a maximal $\mathbb{R}$- split torus, $g \in T$ be a proximal element of G (which exists by Lemma 4.2.21). Then g is 1- proximal (Proposition 4.2.26) and lies in the unity component of G.

Now let us prove the last statement of Corollary 4.2.23. To do this, consider the 1- parameter subgroup $\Gamma \subset T$ passing through the previous proximal element g. The elements $g^r \in \Gamma$, $r > 0$, are proximal, since Ad_g is proximal and any positive power of a proximal operator is also proximal. Therefore, they are 1- proximal (Proposition 4.2.26) and accumulate to 1. This together with Remark 4.2.18 proves the corollary. □

4.3 Proof of Theorems 4.1.1 and 4.1.33 for semisimple Lie groups with irreducible adjoint and proximal elements

Here and in Section 4.4 we prove Theorem 4.1.33, which deals with semisimple Lie groups having irreducible adjoint representation. For those Lie groups Theorem 4.1.1 follows from Theorem 4.1.33 (see 4.1.7). In the present section we treate the case of Lie group with proximal elements. The opposite case is treated in the next section.

4.3.1 Motivation and the plan of the proof

Let G be a semisimple Lie group with irreducible adjoint and proximal elements, $n = dimG$, $\alpha(u) = (a_1, \dots, a_M)(u)$ be a *conj*- nondegenerate at $u = 0$ family of M- ples of elements of G depending on parameter u (see Definition 4.2.12). Recall that the subgroup $< \alpha(0) > \subset G$ is dense. Without loss of generality we assume that

- the parameter space has the same dimension n, as $G : u \in \mathbb{R}^n$ (we can restrict our family to appropriate generically embedded copy of $\mathbb{R}^n$ in the parameter space, along which the family remains *conj*- nondegenerate).

We construct a sequence w_k of words in M elements such that there exists a sequence $u_k \in \mathbb{R}^n$ for which

$$w_k(\alpha(u_k)) = 1, \ u_k \to 0, \ \text{as } k \to \infty, \tag{4.3.1}$$

and the relations $w_k(\alpha(u)) = 1$ do not hold true identically in a neighborhood of 0. Then the mapping $\alpha(0) \mapsto \alpha(u)$ does not extend up to a group isomorphism $< \alpha(0) > \to < \alpha(u) >$ for arbitrarily small values of u. Indeed, the relations $w_k = 1$ hold true in the group $< \alpha(u) >$ for the values $u = u_k$ (which tend to 0), and do not hold for some other values of u (which can be chosen arbitrarily small as well). This will prove Theorem 4.1.33.

First let us motivate the proof of Theorem 4.1.33. A natural way to construct the previously mentioned words w_k is to achieve that $w_k(\alpha(0)) \to 1$. Then to guarantee the existence of a sequence $u_k \to 0$ of solutions to the equations $w_k(\alpha(u)) = 1$, we have to show that there exists a sequence $\delta_k \to 0$ such that $1 \in w_k(\alpha(D_{\delta_k}))$, whenever k is large enough. To do this, we have to prove an appropriate lower bound for derivatives of the mappings $w_k(\alpha(u))$ near 0 ; in particular, to show that certain derivatives will be greater than $\delta_k^{-1} dist(w_k(\alpha(0)), 1)$.

By density, we can always construct a sequence of words w_k so that $w_k(\alpha(0)) \to 1$. In the case, when $a_i(0)$ are close enough to unity, it suffices to take w_k to be a sequence of appropriate successive commutators

$$w_1 = []_1 = [a_1, a_2], \ w_2 = []_2 = [a_1, [a_1, a_2]], \dots$$

On the other hand, the derivatives of the corresponding mappings $w_k(\alpha(u))$ do not admit a satisfactory lower bound : the values at $\alpha(0)$ of the commutators converge exponentially to 1, and the previous derivatives (taken at 0) converge exponentially to zero.

In order to construct words w_k with large derivatives, we use the following observation. Fix a small $\Delta > 0$. Then $dist([]_k(\alpha(0)), 1) < \Delta$, whenever k is large enough. Consider all the powers $[]_k^m$ of the previous commutators. Put

$$m_k = \min\{m \in \mathbb{N}, \ dist([]_k^m(\alpha(0)), 1) \geq \Delta\}.$$

(The numbers m_k are well-defined provided that $[]_k(\alpha(0)) \neq 1$.) Then $\Delta \leq dist([]_k^{m_k}(\alpha(0)), 1) < 2\Delta$, whenever k is large enough, by definition, the previous inequality and the left invariance of the metric on G. We claim that if $a_1(0)$ and $a_2(0)$ are close enough to 1 and the family $\alpha(u)$ satisfies appropriate genericity assumption, then the derivative at 0 in certain directions of the mapping $u \mapsto []_k^{m_k}(\alpha(u)) \in G$ grows linearly in k, as that of the mappings ψ_k in the proof of Proposition 4.1.32.

In what follows we construct

- appropriate words $g_1, \ldots, g_n$, h, w and define recurrently the iterated commutators

$$w_{i0} = h,\ w_{ik} = g_i w_{i(k-1)} g_i^{-1} w_{i(k-1)}^{-1}, \tag{4.3.2}$$

- a sequence of collections

$$M_k = (m_{1k}, \ldots, m_{nk}),\ m_{ik} \in \mathbb{N},\ \text{and put}$$

$$\omega_k = w_{1k}^{m_{1k}} \ldots w_{nk}^{m_{nk}},\ w_k = w^{-1}\omega_k. \tag{4.3.3}$$

We show that the latter words w_k satisfy (4.3.1). To do this, we introduce the rescaled parameter

$$\widetilde{u} = ku,$$

as in Proposition 4.1.32, and show that

$$\omega_k(\alpha(k^{-1}\widetilde{u})) \to \Psi(\widetilde{u}),\ \text{as } k \to \infty;\ \Psi : \mathbb{R}^n \to G \text{ is a local diffeomorphism at } 0, \tag{4.3.4}$$

the previous convergence is uniform with derivatives on compact subsets in $\mathbb{R}^n$. Theorem 4.1.33 will be then deduced from (4.3.4) at the end of the subsection.

For a fixed $g \in G$ consider the corresponding commutator mapping

$$\phi_g : G \to G,\ \phi_g(y) = gyg^{-1}y^{-1}. \text{ One has } \phi_g(1) = 1,\ \phi_g'(1) = Ad_g - Id : \mathfrak{g} \to \mathfrak{g},$$

$$w_{ik}(\alpha(u)) = \phi^k_{g_i(\alpha(u))}(h(\alpha(u))). \tag{4.3.5}$$

For any 1- proximal element $g \in G$ (see Definition 4.2.22) denote

$$s(g) = \text{ the eigenvalue of } Ad_g - Id \text{ with maximal modulus, } L_g \subset \mathfrak{g} \text{ its eigenline.} \tag{4.3.6}$$

The function $s(g)$ is analytic on the (open) subset of 1- proximal elements, by the simplicity of the eigenvalue $s(g)$. Denote

$$\Pi = \{1\text{- proximal elements } g \in G \mid |s(g)| < 1\}. \tag{4.3.7}$$

Remark 4.3.1 Let G be an arbitrary semisimple Lie group with proximal elements. The above set Π is open and nonempty (Corollary 4.2.23).

The choice of the words g_j and h will be specified at the end of the subsection. It will be done so that

$$g_j(\alpha(0)) \in \Pi \text{ for any } j = 1, \ldots, n.$$

The following Proposition 4.3.2 describes the asymptotic behavior of the iterated commutators $\phi_g^k(y)$, as $k \to \infty$, for arbitrary $g \in \Pi$ and $y \in G$ close enough to 1. Using Proposition 4.3.2, we show (Corollary 4.3.3) that for appropriately chosen word h and arbitrary given $\varepsilon > 0$ one can choose appropriate exponents m_{jk} (which depend on g_j and ε, see (4.3.11)) so that the mapping sequence $\omega_k(\alpha(k^{-1}\widetilde{u}))$ converges to some mapping $\Psi(\widetilde{u})$, which depends only on g_j, h and ε. The mapping Ψ is explicitly given by formula (4.3.12) below. The main technical part of the proof of Theorem 4.1.33 is to show that one can adjust g_j, h and ε so that the limit Ψ be a local diffeomorphism at 0 (Lemmas 4.3.4, 4.3.6 and the Main Technical Lemma 4.3.5 below). Lemmas 4.3.4 and 4.3.6 easily follow from Lemma 4.3.5. Theorem 4.1.33 will be deduced from Lemma 4.3.6 and Proposition 4.3.2 at the end of the subsection. The proofs of Lemma 4.3.6 and Proposition 4.3.2 are omitted here. A sketch-proof of Lemma 4.3.5 will be given in 4.3.2.

Proposition 4.3.2 *Let G be a Lie group with proximal elements, Π be as in (4.3.7). There exist an open subset*

$$\Pi' \subset \Pi \times G, \ \Pi' \supset \Pi \times 1, \tag{4.3.8}$$

and a $\mathfrak{g}$- valued vector function $v_g(y)$ analytic in $(g, y) \in \Pi'$, $v_g(1) = 0$ (denote $dv_g : \mathfrak{g} \to \mathfrak{g}$ its differential in y at $y = 1$) such that for any $(g, y) \in \Pi'$ one has

$$v_g(y) \in L_g, \ dv_g|_{L_g} = Id : L_g \to L_g, \ \phi_g^k(y) = \exp(s^k(g)(v_g(y) + o(1))), \ \textit{as } k \to +\infty, \tag{4.3.9}$$

$s(g)$ and L_g are the same, as in (4.3.6). The latter "o" is uniform with derivatives in (g, y) on compact subsets in Π'.

Corollary 4.3.3 *Let G, n, M, $\alpha(u)$ be as at the beginning of the subsection, Π be as in (4.3.7), Π', v_g be as in Proposition 4.3.2. Let $g_1, \dots, g_n$, h be words in M elements such that*

$$(g_j(\alpha(0)), h(\alpha(0))) \in \Pi' \textit{ for any } j = 1, \dots, n. \textit{ Put}$$

$$s_j(u) = s(g_j(\alpha(u))), \ \widetilde{\nu}_j(u) = v_{g_j(\alpha(u))}(h(\alpha(u))) \in \mathfrak{g}, \ \nu_j = \widetilde{\nu}_j(0). \tag{4.3.10}$$

Let $\varepsilon > 0$. For any $k \in \mathbb{N}$ and $j = 1, \dots, n$ put

$$m_{jk} = [\varepsilon |s_j|^{-k}(0)]. \tag{4.3.11}$$

Let ω_k be the corresponding commutator power product (4.3.3). Then

$$\omega_k(\alpha(k^{-1}\widetilde{u})) \to \Psi(\widetilde{u}) = \exp(\varepsilon e^{(d \ln s_1(0))\widetilde{u}} \nu_1) \dots \exp(\varepsilon e^{(d \ln s_n(0))\widetilde{u}} \nu_n), \ \textit{as } k \to \infty, \tag{4.3.12}$$

uniformly with derivatives on compact subsets in $\mathbb{R}^n$.

Proof One has

$$w_{jk}^{m_{jk}}(\alpha(k^{-1}\widetilde{u})) \to \exp(\varepsilon e^{(d \ln s_j(0))\widetilde{u}} \nu_j) \tag{4.3.13}$$

uniformly with derivatives on compact sets in $\mathbb{R}^n$. Indeed, by (4.3.5) and (4.3.9), one has

$$w_{jk}^{m_{jk}}(\alpha(k^{-1}\widetilde{u})) = \exp(m_{jk} s_j^k(k^{-1}\widetilde{u})(\widetilde{\nu}_j(k^{-1}\widetilde{u}) + o(1))), \ \widetilde{\nu}_j(k^{-1}\widetilde{u}) \to \nu_j, \tag{4.3.14}$$

$$m_{jk} s_j^k(k^{-1}\widetilde{u}) \to \varepsilon e^{(d \ln s_j(0))\widetilde{u}}, \text{ since} \tag{4.3.15}$$

$$s_j^k(k^{-1}\widetilde{u}) = (s_j(0) + k^{-1}(ds_j(0))\widetilde{u} + o(k^{-1}))^k$$

$$= s_j^k(0)(1 + k^{-1}(d \ln s_j(0))\widetilde{u} + o(k^{-1}))^k = s_j^k(0) e^{(d \ln s_j(0))\widetilde{u}}(1 + o(1)) \tag{4.3.16}$$

and $m_{jk} s_j^k(0) \to \varepsilon$ by (4.3.11). Substituting (4.3.15) to (4.3.14) yields (4.3.13), which implies (4.3.12). The corollary is proved. □

Lemma 4.3.4 *Let G, n, $\alpha(u)$, M be as at the beginning of the subsection, Π be as in (4.3.7). There exists a collection $g_1, \dots, g_n$ of words in M elements such that $g_i(\alpha(0)) \in \Pi$ for all $i = 1, \dots, n$ and the system of n functions $s_i(u) = s(g_i(\alpha(u)))$ (which are well-defined in a neighborhood of 0) has the maximal rank n at 0. Moreover, given any collection $A_1, \dots, A_n \in \Pi$ one can achieve that in addition, the elements $g_i(\alpha(0))$ be arbitrarily close to A_i.*

For the proof of Theorem 4.1.33 in the general case, without the assumption that G has proximal elements, we use the following generalization of Lemma 4.3.4.

Lemma 4.3.5 (Main Technical Lemma). *Let G be an arbitrary semisimple Lie group with irreducible adjoint representation (not necessarily with proximal elements), $dimG = n$. Let $\alpha(u) = (a_1(u), \dots, a_M(u))$ be a conj- nondegenerate at 0 family of M- ples of its elements depending on a parameter $u \in \mathbb{R}^n$. Let $U \subset G$ be an arbitrary open subset, and let $\sigma : U \to \mathbb{R}$ be a smooth locally non-constant function. Then there exist n abstract words $g_i(a_1, \dots, a_M)$, $i = 1, \dots, n$, such that the system of n functions $s_i(u) = \sigma(g_i(\alpha(u)))$ is well-defined (locally near 0) and has the maximal rank n at 0. Moreover, for any given $A_1, \dots, A_n \in U$ one can achieve that in addition, the elements $g_i(\alpha(0)) \in G$ be arbitrarily close to A_i.*

Lemma 4.3.4 follows from Lemma 4.3.5 applied to $U = \Pi$ and the function $\sigma(g) = s(g)$.

Lemma 4.3.6 *Let G, n, M, $\alpha(u)$ be as at the beginning of the subsection, $\Pi' \subset G \times G$ be as in Proposition 4.3.2. There exist words $g_1, \dots, g_n$, h such that $(g_j(\alpha(0)), h(\alpha(0))) \in \Pi'$ for all j and for any $\varepsilon > 0$ small enough the corresponding mapping $\Psi(\widetilde{u})$ from (4.3.12) is a local diffeomorphism at 0.*

Proof of Theorem 4.1.33 modulo Proposition 4.3.2 and Lemmas 4.3.5 and 4.3.6. Let g_j, h, ε be as in Lemma 4.3.6, $s_j(u)$ be as in (4.3.10), m_{jk} be as in (4.3.11). Let ω_k be the corresponding commutator power product from (4.3.3), Ψ be the mapping from (4.3.12). Let $\delta > 0$ be such that $\Psi : \overline{D}_\delta \to \Psi(\overline{D}_\delta) \subset G$ be a diffeomorphism (it exists by Lemma 4.3.6). Let w be an arbitrary word such that

$$w(\alpha(0)) \in \Psi(D_\delta), \ w_k = w^{-1}\omega_k. \text{ Then}$$

$$w_k(\alpha(k^{-1}\widetilde{u})) \to \psi(\widetilde{u}) = w^{-1}(\alpha(0))\Psi(\widetilde{u}), \ \psi : \overline{D}_\delta \to \psi(\overline{D}_\delta) \subset G \text{ is a diffeomorphism,} \tag{4.3.17}$$
$$1 \in \psi(D_\delta)$$

(Corollary 4.3.3). Therefore, for any k large enough the image $w_k(\alpha(k^{-1}D_\delta))$ also contains 1, and hence, $w_k(\alpha(k^{-1}\widetilde{u}_k)) = 1$ for some $\widetilde{u}_k \in D_\delta$. Put

$$u_k = k^{-1}\widetilde{u}_k; \text{ one has } w_k(\alpha(u_k)) = 1, \ u_k \to 0.$$

The relation $w_k(\alpha(u)) = 1$, which holds for $u = u_k$, does not hold identically in $u \in D_{k^{-1}\delta'}$ for any $0 < \delta' \leq \delta$, because of the diffeomorphicity of the mappings $\widetilde{u} \mapsto w_k(\alpha(k^{-1}\widetilde{u}))$ on D_δ for large k (see (4.3.17); the convergence is uniform with derivatives on $\overline{D}_\delta$ there). Thus, the words w_k satisfy (4.3.1). This proves Theorem 4.1.33. □

4.3.2 Sketch-proof of the Main Technical Lemma

Denote $\hat{U} = \mathbb{R}^n$ the parameter u space under consideration. By assumption, the family $\alpha(u)$ is *conj*- nondegenerate. This together with the equality of the dimensions of G and $\hat{U}$ implies that the derivative along each nonzero vector $v \in T_0\hat{U}$ of the function $u \mapsto Conj(\alpha(u))$ is nonzero. (Fix a $v \in T_0\hat{U} \setminus 0$.) The derivatives along v of the mappings $u \mapsto w(\alpha(u))$ (where w is an arbitrary word) form a vector field on the dense subgroup $\Gamma =< \alpha(0) >\subset G$ (we extend it to 1 by 0). This vector field is well-defined (single-valued), if Γ is free. In general, if there are relations in Γ, it is single-valued, if and only if for any word w giving a relation (i.e., $w(\alpha(0)) = 1$) the corresponding mapping $u \mapsto w(\alpha(u))$ has zero derivative along v.

The maximal rank statement of Lemma 4.3.5 is equivalent to the statement that for any given $v \in T_0\hat{U} \setminus 0$ there exists an index j such that the corresponding vector at $g_j(\alpha(0))$ of the previous field is nonzero and transversal to the level hypersurface of the function σ. To prove that, we show that the previous vector field (if well-defined) is not Lipschitz at 1. Moreover, we show that for any line $\Lambda \subset \mathfrak{g}$ there exists a sequence of words $w_k(a_1, \dots, a_M)$, $w_k(\alpha(0)) \to 1$, as $k \to \infty$, such that

$$\frac{|\frac{dw_k(\alpha(u))}{dv}|}{dist(w_k(\alpha(0)), 1)} \to \infty, \ \text{as } k \to \infty, \tag{4.3.18}$$

the tangent line in $T_{w_k(\alpha(0))}G$ generated by the latter derivative tends to Λ. (4.3.19)

First we prove (by contradiction) that statement (4.3.18) holds true for some word sequence w_k. Suppose the contrary : the previous vector field on the dense subgroup $\Gamma \subset G$ is Lipschitz at 1. Then we show that it extends up to a vector field on the whole G that defines a flow of automorphisms of G. The latter automorphisms preserve conjugacy classes (semisimplicity). This contradicts the nonvanishing of the derivative along v of $Conj(\alpha(u))$.

Given any word sequence w_k satisfying (4.3.18), passing to a subsequence one can achieve that the tangent lines in $T_{w_k(\alpha(0))}G$ generated by the (big) derivatives from (4.3.18) converge to some line $\Lambda \subset \mathfrak{g}$, i.e., statement (4.3.19) holds true for this Λ. The union of all these possible limit lines Λ is closed and Ad_G- invariant. This follows by definition and the density of the subgroup $\Gamma \subset G$. We show that the latter union of limit lines is the whole $\mathfrak{g}$, by using the irreducibility of the adjoint.

The existence (for arbitrary Λ) of words satisfying (4.3.18) and (4.3.19) implies the following

Corollary 4.3.7 *Let G, n, $\alpha(u)$, $U \subset G$, $\sigma : U \to \mathbb{R}$ be the same, as in the Main Technical Lemma 4.3.5. Let $v \in T_0\mathbb{R}^n$, $v \neq 0$. For any $g \in U$ there exists a sequence of words $\widetilde{w}_k(a_1, \dots, a_M)$, $h_k = \widetilde{w}_k(\alpha(0)) \to g$, such that the derivatives $\frac{d\widetilde{w}_k(\alpha(u))}{dv}$ are transversal to the level hypersurfaces $\sigma = \sigma(h_k)$.*

Proof (sketch). It suffices to prove the statement of the corollary for any g belonging to a dense subset in U. We prove it for those $g \in U \cap \Gamma$ at which $d\sigma(g) \neq 0$. Inclusion $g \in \Gamma$ means that $g = w(\alpha(0))$ for some word w. If already the derivative $\nu = \frac{dw(\alpha(u))}{dv}$ is transversal to the hypersurface $\sigma = \sigma(g)$, then we put $\widetilde{w}_k = w$ and we are done. Now suppose that the latter derivative is tangent to the hypersurface $\sigma = \sigma(g)$. We fix an arbitrary line $\Lambda \subset \mathfrak{g}$ whose image in T_gG under the left multiplication by g is transversal to the same hypersurface. Let w_k be a word sequence satisfying (4.3.18) and (4.3.19) for this Λ. We show that the words $\widetilde{w}_k = ww_k$ satisfy the statements of the Corollary. The derivative $\widetilde{\nu}_k = \frac{d\widetilde{w}_k(\alpha(u))}{dv}$ is the sum of the two following vectors :

- the vector $\nu_k' \in T_{h_k}G$, which is the image of $\nu = \frac{dw(\alpha(u))}{dv}$ under the right multiplication by $w_k(\alpha(0))$;

- the vector $\nu_k \in T_{h_k}G$, which is the image of $\frac{dw_k(\alpha(u))}{dv}$ under the left multiplication by $w(\alpha(0))$.

An elementary calculation shows that $\frac{d\sigma}{d\nu_k'} = O(dist(h_k, g))$, as $k \to \infty$ (by construction : $\frac{d\sigma}{d\nu} = 0$), while the derivative $\frac{d\sigma}{d\nu_k}$ asymptotically dominates $O(dist(h_k, g))$ (this follows from (4.3.18) and (4.3.19)). Thus, the latter derivative dominates the former one and $\frac{d\sigma}{d\widetilde{\nu}_k} \neq 0$ for any k large enough. This proves the corollary. □

Proof of Lemma 4.3.5. Given a $\varepsilon > 0$ and $A_1, \dots, A_n \in U$, let us construct words $g_i(\alpha)$, $g_i(\alpha(0))$ being ε- close to A_i, such that the values $s_i(u) = \sigma(g_i(\alpha(u)))$, $i = 1, \dots, n$, are functions of joint rank n at 0. This will prove Lemma 4.3.5.

Given a tangent vector $v_1 \in T_0\hat{U} \setminus 0$, there exists a word g_1 (denote $s_1(u) = \sigma(g_1(\alpha(u))))$ such that $g_1(\alpha(0))$ is ε- close to A_1 and $\frac{ds_1(u)}{dv_1} \neq 0$ (*conj*- nondegeneracy and Corollary 4.3.7 applied to $g = A_1$). Take another vector $v_2 \neq 0$ tangent to the level hypersurface of the function s_1 at 0. Again applying the corollary to $v = v_2$, one can find a word g_2 with $g_2(\alpha(0))$ being ε- close to A_2 such that the derivative along v_2 of the function $s_2 : u \mapsto \sigma(g_2(\alpha(u)))$ does not vanish. Now take a vector $v_3 \neq 0$ tangent to the level surface of the vector function (s_1, s_2) and construct a word g_3 similarly etc. This yields the words g_i we are looking for : by construction, the system of functions $s_i : u \mapsto \sigma(g_i(\alpha(u)))$ has rank n at 0. Lemma 4.3.5 is proved. □

4.4 Case of semisimple Lie groups with irreducible adjoint and without proximal elements

In the case mentioned in the title of the section the proof (given below) of Theorem 4.1.33 is essentially the same, as before, but it becomes slightly more technical.

Everywhere below in this section, whenever the contrary is not specified, we consider that G is a semisimple Lie group with irreducible adjoint and no proximal elements. Let $\alpha(u) = (a_1(u), \dots, a_M(u))$ be a *conj*- nondegenerate family of M- ples of its elements. As in Section 4.3, we consider that $u \in \mathbb{R}^n$, $n = dimG$. We construct appropriate sequence of words w_r and a sequence $u_r \in \mathbb{R}^n$ such that

$$w_r(\alpha(u_r)) = 1, \ u_r \to 0, \ \text{as } r \to \infty, \tag{4.4.1}$$

and the relations $w_r(\alpha(u)) = 1$ do not hold identically in a neighborhood of 0. This will prove Theorem 4.1.33.

We construct appropriate words $g_1, \dots, g_n$, h, w, a collection

$$l = (l_1, \dots, l_n) \in \mathbb{Z}^n, \ \text{a sequence of numbers } k_r \in \mathbb{N}, \ k_r \to \infty, \ \text{as } r \to \infty,$$

a collection of sequences

$$m_{jr} \in \mathbb{N}, \ j = 1, \dots, n, \ r \in \mathbb{N}, \ \text{and put}$$

$$\omega_r = w_{1,k_r+l_1}^{m_{1r}} \dots w_{n,k_r+l_n}^{m_{nr}}, \ w_r = w^{-1}\omega_r, \tag{4.4.2}$$

where w_{j,k_r+l_j} are the iterated commutators given by the recurrent formula (4.3.2). We consider the rescaled parameter

$$\widetilde{u} = k_r u \text{ and show that}$$

$$\omega_r(\alpha(k_r^{-1}\widetilde{u})) \to \Psi(\widetilde{u}), \ \Psi : \mathbb{R}^n \to G \text{ is a local diffeomorphism at } 0, \tag{4.4.3}$$

the latter convergence is uniform with derivatives on compact sets in $\mathbb{R}^n$. This implies Theorem 4.1.33 analogously to the discussion at the end of Subsection 4.3.1. The implication is proved at the end of the present section.

In the proof of Theorem 4.1.33 we use Proposition 4.4.8 stated below. It describes the asymptotic behavior of iterated commutators

$$\phi_g^k(y) = [g \dots [g, y] \dots],$$

as $k \to \infty$. This is an analogue of Proposition 4.3.2 from Section 4.3. In the case under consideration the unity component of G contains no 1- proximal elements (for which Proposition 4.3.2 was formulated). We introduce so-called $\mathbb{C}$-*1-proximal elements* (see the next definition). We show that their set contains an open dense subset in the unity component (Proposition 4.4.1 and its Corollary 4.4.4, both stated below). We state Proposition 4.4.8 for the $\mathbb{C}$-1-proximal elements g such that the derivative $\phi'_g(1)$ is contracting. To do this, we show (Proposition 4.4.5 below) that for each $\mathbb{C}$-1-proximal element $g \in G$ there exists a unique $\phi'_g(1)$- invariant plane $L(g) \subset \mathfrak{g}$ equipped with a natural $\phi'_g(1)$- invariant complex structure such that the restriction $\phi'_g(1) : L(g) \to L(g)$ is multiplication by a complex eigenvalue $s(g)$ of the operator $\phi'_g(1) : \mathfrak{g} \to \mathfrak{g}$ with maximal modulus.

The words g_j will be chosen at the end of the subsection, in particular, so that each element $g = g_j(\alpha(0))$ be $\mathbb{C}$-1-proximal and $|s(g)| < 1$. For any collection of words g_j satisfying the latter statements and any given $\varepsilon > 0$, Proposition 4.4.9 and Corollary 4.4.10 (both stated below) provide sequences $k_r, m_{jr} \to \infty$ such that for any word h with $h(\alpha(0))$ close enough to the unity and any collection $l = (l_1, \dots, l_n) \in \mathbb{Z}^n$ the corresponding sequence of G- valued functions $\omega_r(\alpha(k_r^{-1}\widetilde{u}))$, see (4.4.2), converges to some mapping $\Psi : \mathbb{R}^n \to G$ uniformly with derivatives on compact sets in $\mathbb{R}^n$. The limit mapping Ψ is given explicitly by formula (4.4.11) below, which depends only on the words g_j, h, the collection $l \in \mathbb{Z}^n$ and ε. Lemma 4.4.11 stated below shows that one can adjust g_j, h and l so that Ψ be a local diffeomorphism at 0, whenever ε is small enough. This is the main technical part of the proof of Theorem 4.1.33. The proof of Lemma 4.4.11 uses the Main Technical Lemma from Section 4.3.

At the end of the section we deduce Theorem 4.1.33 from the technical statements listed above (Propositions 4.4.1, 4.4.8 and Lemma 4.4.11 ; their proofs are given in [50] and omitted here).

Proposition 4.4.1 *Let G be a connected semisimple Lie group. There exists a nonempty subset $U \subset G$ such that the subset $Ad_U \subset Ad_G \subset End(\mathfrak{g})$ is Zariski open in Ad_G and the adjoint of each $g \in U$ satisfies the following statements :*

1) the number of its nonunit complex eigenvalues is maximal and nonempty, and all they are simple ;

2) if there is a pair of distinct eigenvalues $\Lambda_1, \Lambda_2 \neq 1$ with $|\Lambda_1 - 1| = |\Lambda_2 - 1|$, then $\Lambda_1 = \overline{\Lambda}_2$.

Definition 4.4.2 An element g of a Lie group is called $\mathbb{C}$-*1-proximal*, if the operator $Ad_g - Id$ has a pair of simple nonreal complex-conjugated eigenvalues that are the unique eigenvalues with maximal modulus.

Proposition 4.4.3 *Any element of a semisimple Lie group whose adjoint satisfies the previous statements 1) and 2) is either 1- proximal (see Definition 4.2.22) or $\mathbb{C}$-1-proximal.*

Proof Let Ad_g satisfy 1) and 2), λ be its eigenvalue for which the modulus $|\lambda - 1|$ is the maximal possible. Then $\lambda - 1 \neq 0$ and λ is a simple eigenvalue (statement 1)). For any eigenvalue $\lambda' \neq \lambda, \bar{\lambda}$ one has $|\lambda - 1| > |\lambda' - 1|$ (statement 2)). Therefore, g is 1- proximal, if $\lambda \in \mathbb{R}$ and $\mathbb{C}$-1-proximal otherwise. Proposition 4.4.3 is proved. $\Box$

Corollary 4.4.4 *Let G be a semisimple Lie group without proximal elements. The set of $\mathbb{C}$- 1- proximal elements in G is open and contains a dense subset $U \subset G_0$ of its unity component G_0.*

Proof The openness of the set of $\mathbb{C}$-1-proximal elements follows from definition. The subset $U \subset G_0$ from Proposition 4.4.1 is open and dense (since Ad_U is nonempty and Zariski open in a smooth variety Ad_{G_0}, by Proposition 4.4.1). The set U consists of $\mathbb{C}$-1-proximal elements (Proposition 4.4.3 and absense of 1- proximal elements in G_0). Indeed, otherwise, a 1- proximal element of G_0 would be proximal (Corollary 4.2.25), - a contradiction to the conditions of Corollary 4.4.4. This proves Corollary 4.4.4. $\Box$

We use the following properties of the adjoint of a $\mathbb{C}$-1-proximal element.

Proposition 4.4.5 *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator with a pair of simple complex-conjugated eigenvalues $s, \bar{s} \notin \mathbb{R}$. There exists a unique A- invariant plane $L \subset \mathbb{R}^n$ whose complexification is the sum of the complex eigenlines corresponding to the eigenvalues s and $\bar{s}$. The plane L carries an A- invariant linear complex structure (i.e., a structure of complex line compatible with its real linear structure), unique up to complex conjugation. The restriction $A : L \to L$ acts by multiplication by either s or $\bar{s}$ in the latter complex structure (dependently on double choice of the complex structure).*

Proof By basic linear algebra, the previous plane L exists, unique and there exists a $\mathbb{R}$- linear nondegenerate operator $H : L \to \mathbb{C}$ such that $HAH^{-1}(z) = sz$. The H- pullback of the standard complex structure on $\mathbb{C}$ (or of its conjugate) is an A- invariant complex structure on L such that the restriction $A : L \to L$ acts by multiplication by s (respectively, $\bar{s}$). These are the only A- invariant linear complex structures on L. Or equivalently, the standard complex structure on $\mathbb{C}$ is the unique linear complex structure (up to complex conjugation) invariant under the multiplication by a number $s \in \mathbb{C} \setminus \mathbb{R}$. Indeed, each linear complex structure on a plane defines an ellipse centered at 0 (up to homothety) : the latter ellipse is an orbit of a vector under the multiplication by the complex numbers with unit modulus. Vice versa, an ellipse determines a linear complex structure uniquely up to complex conjugation. The only ellipse in $\mathbb{C}$ sent to a homothetic one by multiplication by a $s \in \mathbb{C} \setminus \mathbb{R}$ is a circle. This proves the previous uniqueness statement and Proposition 4.4.5. $\Box$

Definition 4.4.6 Let G be a Lie group, $g \in G$ be a $\mathbb{C}$-1-proximal element. Let $s(g)$ be an eigenvalue of $Ad_g - Id$ with the maximal modulus. Let $L(g) \subset \mathfrak{g}$ be the $Ad_g - Id$- (and hence, Ad_g-) invariant plane corresponding to the eigenvalues $s(g)$, $\overline{s(g)}$ (see Proposition 4.4.5). The corresponding $Ad_g - Id$-

invariant complex structure on $L(g)$, in which $Ad_g - Id : L(g) \to L(g)$ acts by multiplication by $s(g)$, will be called the $s(g)$- *complex structure.*

Proposition 4.4.7 *Let G be a Lie group, $V \subset G$ be a connected component of the subset of the $\mathbb{C}$-1-proximal elements (which is open by definition). The values $s(g)$, $\overline{s(g)}$ from Definition 4.4.6 yield two real-analytic complex-conjugated functions $s, \bar{s} : V \to \mathbb{C} = \mathbb{R}^2$.*

Proof The local real analyticity of the previous values follows from the simplicity of the eigenvalues $s(g)$, $\overline{s(g)}$. The global real analyticity (say, of $s(g)$) follows from the fact that its analytic extension along any closed loop in V does not change the analytic branch. Indeed, the result of analytic extension of $s(g)$ remains an eigenvalue of $Ad_g - Id$ with the maximal modulus, by definition and the previous local analyticity statement. Therefore, given a $g_0 \in V$ and a loop $\gamma \subset V$ based at g_0, the result of the analytic extension of $s(g)$ along γ is either $s(g_0)$, or $\overline{s(g_0)}$. In the latter case there exists a $g' \in \gamma$ where $s(g') \in \mathbb{R}$, by continuity. It follows from definition and the local analyticity that $s(g')$ is a double eigenvalue of $Ad_{g'} - Id$ with maximal modulus, - a contradiction to the $\mathbb{C}$-1- proximality. Proposition 4.4.7 is proved. □

In what follows, everywhere below in this section, we fix a real-analytic branch of the eigenvalue function $s(g)$ from Proposition 4.4.7, defined on the open set of all the $\mathbb{C}$-1-proximal elements. The corresponding family of planes $L(g) \subset \mathfrak{g}$ and the $s(g)$- complex structures on them (see the previous definition) also depend analytically on g. We define the multiplication of vectors in $L(g)$ by complex numbers in the sense of the $s(g)$- complex structure. Denote

$$\Pi_{\mathbb{C},1} = \{\mathbb{C} - 1 - \text{proximal elements } g \in G \text{ with } |s(g)| < 1\} \tag{4.4.4}$$

This is a nonempty open subset in G, by Corollary 4.4.4.

Proposition 4.4.8 *Let G be a Lie group such that $\Pi_{\mathbb{C},1} \neq \emptyset$, $s(g)$, $L(g)$ and the complex structures on the planes $L(g)$ be as above. There exists an open subset*

$$\Pi'_{\mathbb{C},1} \subset \Pi_{\mathbb{C},1} \times G, \ \Pi_{\mathbb{C},1} \times 1 \subset \Pi'_{\mathbb{C},1}, \tag{4.4.5}$$

and a $\mathfrak{g}$- valued vector function $v_g(y)$ analytic in $(g, y) \in \Pi'_{\mathbb{C},1}$, $v_g(1) = 0$ (denote $dv_g : \mathfrak{g} \to \mathfrak{g}$ its differential in y at $y = 1$) such that

$$v_g(y) \in L(g) \text{ for any } (g, y) \in \Pi'_{\mathbb{C},1}, \ dv_g|_{L(g)} = Id : L(g) \to L(g), \tag{4.4.6}$$

$$\phi_g^k(y) = \exp(s^k(g)v_g(y) + o(|s^k(g)|)), \text{ as } k \to \infty, \tag{4.4.7}$$

the latter "o" is uniform with derivatives on compact subsets in $\Pi'_{\mathbb{C},1}$.

Given a collection of words g_j, $j = 1, \dots, n$, with $g_j(\alpha(0)) \in \Pi_{\mathbb{C},1}$, we denote

$$\zeta_j = \arg s(g_j(\alpha(0))).$$

Proposition 4.4.9 *For any real vector $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$ there exists a sequence of numbers $k_r \in \mathbb{N}$, $k_r \to \infty$, as $r \to \infty$, such that*

$$k_r\zeta_j \to 0(mod 2\pi), \text{ as } r \to \infty, \text{ for any } j = 1, \dots, n. \tag{4.4.8}$$

Proof Consider ζ as an element of the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. The subgroup $<\zeta> \subset \mathbb{T}^n$ either is discrete, or accumulates to 0. In both cases there exists a sequence of numbers $k_r \in \mathbb{N}$, $k_r \to \infty$, such that $k_r\zeta \to 0$ in $\mathbb{T}^n$ (the latter statement is equivalent to (4.4.8)). In the second case this follows from definition. In the first case the group $<\zeta>$ is finite cyclic by compactness. Denote m its order, $k_r = rm$. Then $k_r\zeta = 0$ in $\mathbb{T}^n$ for all $r \in \mathbb{N}$. This proves Proposition 4.4.9. □

Corollary 4.4.10 *Let G, n, M, $\alpha(u)$ be as at the beginning of the Subsection, $\Pi_{\mathbb{C},1}$ be as in (4.4.4), $\Pi'_{\mathbb{C},1}$ be as in (4.4.5). Let $g_1,\dots,g_n,h$ be words in M elements such that*

$$(g_j(\alpha(0)),h(\alpha(0)))\in\Pi'_{\mathbb{C},1} \text{ for any } j=1,\dots,n. \tag{4.4.9}$$

Let $k_r\in\mathbb{N}$, $k_r\to\infty$, be a sequence satisfying (4.4.8) with $\zeta_j=\arg s(g_j(\alpha(0)))$. Let $\varepsilon>0$, put

$$m_{jr}=[\varepsilon|s|^{-k_r}(g_j(\alpha(0)))],\ \ s_j(u)=s(g_j(\alpha(u))),\ \ \nu_j=v_{g_j(\alpha(0))}(h(\alpha(0)))\in L(g_j(\alpha(0))), \tag{4.4.10}$$

see (4.4.6). Let $l=(l_1,\dots,l_n)\in\mathbb{Z}^n$ be an arbitrary collection of n integers, ω_r be the corresponding product (4.4.2) of iterated commutator powers. Then

$$\omega_r(\alpha(k_r^{-1}\widetilde{u}))\to\Psi(\widetilde{u})=\exp(\varepsilon s_1^{l_1}(0)e^{(d\ln s_1(0))\widetilde{u}}\nu_1)\dots\exp(\varepsilon s_n^{l_n}(0)e^{(d\ln s_n(0))\widetilde{u}}\nu_n), \tag{4.4.11}$$

as $r\to\infty$, uniformly with derivatives on compact sets in $\mathbb{R}^n$. (The multiplication of the vectors $\nu_j\in L(g_j(\alpha(0)))$ by complex numbers is defined in terms of the $s(g_j(\alpha(0)))$- complex structures on $L(g_j(\alpha(0)))$.)

Proof One has (as $r\to\infty$)

$$w_{j,k_r+l_j}(\alpha(k_r^{-1}\widetilde{u}))=\exp(s_j^{k_r+l_j}(k_r^{-1}\widetilde{u})\widetilde{\nu}_j(\widetilde{u})+o(|s_j^{k_r+l_j}(k_r^{-1}\widetilde{u})|)), \text{ where} \tag{4.4.12}$$

$$\widetilde{\nu}_j(\widetilde{u})=v_{g_j(\alpha(k_r^{-1}\widetilde{u}))}(h(\alpha(k_r^{-1}\widetilde{u})))\to\nu_j,$$

by definition and (4.4.7),

$$s_j^{k_r+l_j}(k_r^{-1}\widetilde{u})=s_j^{k_r+l_j}(0)e^{(d\ln s_j(0))\widetilde{u}}(1+o(1)), \text{ as in (4.3.16)},$$

$m_{jr}s_j^{k_r}(0)\to\varepsilon$ by (4.4.8) and (4.4.10). Hence, $w_{j,k_r+l_j}^{m_{jr}}(\alpha(k_r^{-1}\widetilde{u}))\to\exp(\varepsilon s_j^{l_j}(0)e^{(d\ln s_j(0))\widetilde{u}}\nu_j)$, as $r\to\infty$, by (4.4.12) and the latter asymptotics. This implies (4.4.11). □

Lemma 4.4.11 *Let G, n, M, $\alpha(u)$ be as at the beginning of the subsection. There exist words $g_1,\dots,g_n,h$ satisfying (4.4.9) and a $l=(l_1,\dots,l_n)\in\mathbb{Z}^n$ such that for any $\varepsilon>0$ small enough the corresponding mapping $\Psi(\widetilde{u})$ from (4.4.11) be a local diffeomorphism at 0.*

Proof of Theorem 4.1.33 modulo Propositions 4.4.1, 4.4.8 and Lemma 4.4.11. Let $g_1,\dots,g_n$, h, l, ε be as in Lemma 4.4.11, $s_j(u)=s(g_j(\alpha(u)))$, $\zeta_j=\arg s_j(0)$. Let $k_r\to\infty$ be a natural sequence satisfying (4.4.8). Let m_{jr} be the numbers from (4.4.10). Let ω_r be the corresponding iterated commutator power product (4.4.2), Ψ be the corresponding mapping from (4.4.11). Let $\delta>0$ be such that

$$\Psi:\overline{D}_\delta\to\Psi(\overline{D}_\delta)\subset G \text{ be a diffeomorphism.}$$

It exists by Lemma 4.4.11. Fix an arbitrary word w in M elements such that

$$w(\alpha(0))\in\Psi(D_\delta). \text{ Put } w_r=w^{-1}\omega_r.$$

For any r large enough the image $w_r(\alpha(k_r^{-1}D_\delta))$ contains 1. This follows from the convergence

$$w_r(\alpha(k_r^{-1}\widetilde{u}))\to\psi(\widetilde{u})=w^{-1}(\alpha(0))\Psi(\widetilde{u}) \tag{4.4.13}$$

(which takes place by definition and (4.4.11)) and the fact that

$$\psi:\overline{D}_\delta\to\psi(\overline{D}_\delta)\subset G \text{ is a diffeomorphism, and } 1\in\psi(D_\delta),$$

as at the end of Subsection 4.3.1. Therefore, for any r large enough there exists a parameter value

$$\widetilde{u}_r\in D_\delta\subset\mathbb{R}^n, \text{ put } u_r=k_r^{-1}\widetilde{u}_r, \text{ such that } w_r(\alpha(u_r))=1.$$

The sequence u_r satisfies (4.4.1). The relations $w_r(\alpha(u))=1$ do not hold identically in u for any r large enough, as at the end of 4.3.1. This proves Theorem 4.1.33 modulo Propositions 4.4.1, 4.4.8 and Lemma 4.4.11. □

4.5 A short proof of Theorem 4.1.1 for dense subgroups in $G = PSL_2(\mathbb{C})$

Let $A, B \in PSL_2(\mathbb{C})$ generate a free dense subgroup. We prove Theorem 4.1.1 by contradiction. Suppose there is a (simply connected) neighborhood $V \subset PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ of the pair (A, B) such that each pair $(a, b) \in V$ generates a free subgroup. Thus, each word $w(a, b)$ is a holomorphic function in $(a, b) \in PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ with values in $PSL_2(\mathbb{C})$; distinct words define holomorphic functions with disjoint graphs over V. Using holomorphic motion of the fixed points of the elements $w(a, b) \in PSL_2(\mathbb{C})$, we construct a nonstandard measurable almost complex structure on $\overline{\mathbb{C}}$ invariant under the action of $< A, B >$ (and hence, under the action of the whole group $PSL_2(\mathbb{C})$ by density). But the only measurable almost complex structure preserved under the action of $PSL_2(\mathbb{C})$ on $\overline{\mathbb{C}}$ is the standard complex structure, - a contradiction.

Remark 4.5.1 The author's initial proof of Theorem 4.1.1 in the case, when $G = PSL_2(\mathbb{C})$, followed a similar scheme (using the holomorphic motion of fixed points) but was longer than the one presented below. The final quasiconformal mapping and invariance argument, which simplified the proof essentially, is due to Étienne Ghys.

Recall that an element $b \in PSL_2(\mathbb{C})$ is called *elliptic*, if its action on $\overline{\mathbb{C}}$ is conjugated to a rotation. It is called *hyperbolic*, if it has two fixed points : one attracting and the other one repelling. Otherwise it is *parabolic*, i.e., has a unique fixed point and is conjugated to the translation. If b has two fixed points, then their multipliers are inverse to each other. The half-sum of their multipliers (denoted $\nu(b)$) is a holomorphic function $\nu : PSL_2(\mathbb{C}) \to \mathbb{C}$.

Proposition 4.5.2 *Let $V \subset PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ be an open set such that each pair $(a, b) \in V$ generates a free subgroup in $PSL_2(\mathbb{C})$. Then each element of the latter subgroup is hyperbolic.*

Proof Suppose the contrary : there exists a pair $(a, b) \in V$ and a nontrivial word w such that the multiplier of the transformation $w(a, b)$ at some its fixed point has unit modulus. This is equivalent to say that $\nu(w(a, b)) \in [-1, 1]$. There exists a pair $(c, d) \in PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ arbitrarily close to (a, b) (in particular, lying in V) such that the multiplier of $w(c, d)$ at some its fixed point be a root of unity, or equivalently, $\nu(w(c, d)) = \cos\theta$, $\theta \in \pi\mathbb{Q}$. This follows from the nonconstance of the holomorphic function $(c, d) \mapsto \nu(w(c, d))$ and openness of holomorphic mappings. (The function $\nu(w(c, d))$ is nonconstant on $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$, since $w(1, 1) = 1$ and the value of the word w on the generators of a Schottky group is hyperbolic.) By construction, the transformation $w(c, d)$ is elliptic of finite order, - a contradiction to the liberty of the group $< c, d >$. The proposition is proved. □

Thus, each element $w(a, b) \in PSL_2(\mathbb{C})$, $(a, b) \in V$, is hyperbolic, hence, its fixed points are analytic functions in $(a, b) \in V$. The graphs of the fixed point functions are disjoint. Indeed, otherwise, if two distinct hyperbolic elements of $PSL_2(\mathbb{C})$ have one common fixed point, then their commutator is parabolic : the latter fixed point is its unique fixed point. This contradicts the hyperbolicity of the commutator. If two hyperbolic elements have two common fixed points, then they commute, - a contradiction to the liberty.

For any $(a, b) \in V$ denote $Fix(a, b) \subset \overline{\mathbb{C}}$ the set of fixed points of all the elements of the group $< a, b >$. The set $Fix(A, B)$ is dense in $\overline{\mathbb{C}}$, since the subgroup $< A, B >$ is dense. The previous disjoint graphs of fixed point functons form a holomorphic motion over V of the sets $Fix(a, b)$, $(a, b) \in V$. They can be extended up to a global holomorphic motion : filling the whole product $V \times \overline{\mathbb{C}}$ by a union of disjoint graphs of holomorphic functions $V \to \overline{\mathbb{C}}$. This follows immediately from the density of $Fix(A, B)$ by the disjointness and elementary normal family argument (e.g., a version of Montel's theorem, see [88]).

Remark 4.5.3 The well-known Slodkowski theorem [108] says that any holomorphic motion in $D\times\overline{\mathbb{C}}$ of any subset of the Riemann sphere over unit disk D extends up to a holomorphic motion of the whole Riemann sphere. Here we do not use this theorem in full generality.

For any $(a,b)\in V$ denote $h_{a,b}:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ the mapping of the $\overline{\mathbb{C}}$- fiber $(a,b)\times\overline{\mathbb{C}}\subset V\times\overline{\mathbb{C}}$ to the fiber $(A,B)\times\overline{\mathbb{C}}$ defined by the holonomy of the previous holomorphic motion. In more detail, take any path in V from (a,b) to (A,B) and lift it to each one of the previous disjoint graphs in $V\times\overline{\mathbb{C}}$. By definition, the mapping $h_{a,b}$ sends the starting point of a lifted path to its end-point. The mapping $h_{a,b}$ does not depend on the choice of path by simple connectivity of V. It is a quasiconformal homeomorphism : any holomorphic motion has a quasiconformal holonomy [111]. The homeomorphism $h_{a,b}$ conjugates the actions on $\overline{\mathbb{C}}$ of the groups $<A,B>$ and $<a,b>$, since it conjugates them on the dense invariant subsets $Fix(A,B)$ and $Fix(a,b)$ in $\overline{\mathbb{C}}$, by construction. The quasiconformal homeomorphism $h_{a,b}$ transforms the standard complex structure on $\overline{\mathbb{C}}$ to a measurable almost complex structure (denoted by $\sigma(a,b)$). The latter structure is invariant under the action of the group $<A,B>$ (by definition and the previous conjugacy statement), and hence, under $PSL_2(\mathbb{C})$, by density. Now to prove the theorem, it suffices to show that for a generic pair (a,b) the almost complex structure $\sigma(a,b)$ is not standard.

For any $(a,b)\in V$ the elements a and b are hyperbolic with distinct fixed points; the latters form a quadruple denoted $Q(a,b)$ of points in $\overline{\mathbb{C}}$. If the cross-ratios of two quadruples $Q(a,b)$ and $Q(A,B)$ are distinct, then the quasiconformal homeomorphism $h_{a,b}$, which sends $Q(a,b)$ to $Q(A,B)$, is not conformal; hence, $\sigma(a,b)$ is not standard. This together with the discussion at the beginning of the section proves Theorem 4.1.1.

4.6 Sketch-proof of Theorem 4.1.29

For simplicity we sketch the proof of Theorem 4.1.29 only in the case, when G is a semisimple Lie group with irreducible adjoint and proximal elements. In the case, when G is the same but without proximal elements, the proof is analogous. In the case, when G is arbitrary semisimple Lie group, Theorem 4.1.29 is deduced from its statements in the previously mentioned cases and Proposition 4.2.9. In the general case Theorem 4.1.29 is then deduced from its statement for semisimple groups and the existence of the factorization $G\to G_{ss}$ (see 4.2.1).

Thus, we assume that G is semisimple, with irreducible adjoint and with proximal elements. Recall that G is $\varepsilon(x)$- approximable with bounded derivatives. Let $(A,B)\in G\times G$ be an irrational pair,

$$l_m=l_m(D_1)=l_m(A,B,D_1)$$

be the corresponding length majorant sequence for approximations by words in (A,B) on the unit ball $D_1\subset G$, see Definition 4.1.11. We show that there exist a sequence $w'_m(a,b)$ of nontrivial words, a sequence of pairs $(A_m,B_m)\in G\times G$, $(A_m,B_m)\to(A,B)$, and constants $c',c''>0$ (depending only on (A,B)) such that

$$w'_m(A_m,B_m)=1,\ |w'_m|\leq l'_m=c''l_m, \tag{4.6.1}$$

$$dist((A_m,B_m),(A,B))<\varepsilon(c'l'_m). \tag{4.6.2}$$

This means that the pair (A,B) is $\varepsilon(x)$- approximable by pairs with relations, and Theorem 4.1.29 with its Addendum then follow immediately.

For the proof of (4.6.1) and (4.6.2) we fix an arbitrary conj- nondegenerate family

$$\alpha(u)=(a(u),b(u))\in G\times G,\ u\in\mathbb{R}^n,\ n=dimG,\ \alpha(0)=(A,B).$$

As it was shown in Section 4.3 (see (4.3.17)), there exist a sequence of words w_k, a mapping $\psi:\mathbb{R}^n\to G$ and a $\delta>0$ such that

$$w_k(\alpha(k^{-1}\widetilde{u}))\to\psi(\widetilde{u}),\ \psi:\overline{D}_\delta\to\psi(\overline{D}_\delta)\subset G \text{ is a diffeomorphism},\ 1\in\psi(D_\delta), \tag{4.6.3}$$

the previous convergence is uniform with derivatives on compact sets in $\mathbb{R}^n$. Fix a $R>0$ such that

$$\psi(0) \in D_R = D_R(1) \subset G. \tag{4.6.4}$$

Let $l(R)$ be as in (4.1.5),

$$\Omega_{m,D_R},\ \widetilde{l}_m = l_m(D_R) = l_m + l(R),\ c_R = c(A,B,D_R)$$

be respectively the word collection and length majorant sequences and the constant, corresponding to the $\varepsilon(x)$- approximations on D_R by words in (A,B) with bounded derivatives, see Definition 4.1.11 and (4.1.5). For any k large enough one has $w_k(\alpha(0)) \in D_R$, by (4.6.3) and (4.6.4). We fix a sequence of words $\nu_{k,m} \in \Omega_{m,D_R}$ such that the elements $\nu_{k,m}(A,B)$ be $\varepsilon(c_R\widetilde{l}_m)$- approximants of $w_k(\alpha(0)) = w_k(A,B)$. We show that if we fix a k large enough, then the words

$$w'_m = \nu_{k,m}^{-1} w_k$$

satisfy statements (4.6.1) and (4.6.2). The existence of a pair sequence $(A_m,B_m) = \alpha(u_m)$, $u_m \in \mathbb{R}^n$, satisfying (4.6.1) and (4.6.2) is deduced from (4.1.5) and the following statements :

$$dist(w'_m(A,B),1) < \varepsilon(c_R\widetilde{l}_m) \text{ (by definition);} \tag{4.6.5}$$

$$\psi_{km}(\widetilde{u}) = w'_m(\alpha(k^{-1}\widetilde{u})) = (\nu_{k,m}^{-1}w_k)(\alpha(k^{-1}\widetilde{u})) \to \widetilde{\psi}(\widetilde{u}) = (\psi(0))^{-1}\psi(\widetilde{u}) \tag{4.6.6}$$

uniformly with derivatives on compact sets, as $k,m \to \infty$. Statement (4.6.6) follows from definition, (4.6.3) and the uniform boundedness of the derivatives of the words $\nu_{k,m}$ on one and the same neighborhood of (A,B) ($\varepsilon(x)$- approximability with bounded derivatives).

In more detail, fix a $k \in \mathbb{N}$ for which there exists a constant $K>0$ such that for any m large enough (dependently on k)

$$\psi_{km} : \overline{D}_\delta \to \psi_{km}(\overline{D}_\delta) \subset G \text{ is a diffeomorphism and } 1 = \widetilde{\psi}(0) \in \psi_{km}(D_\delta), \tag{4.6.7}$$

$$||(\psi'_{km}(x))^{-1}|| < K \text{ for any } x \in \overline{D}_\delta. \tag{4.6.8}$$

The existence of the previous k follows from (4.6.6). Put

$$u_m = k^{-1}\psi_{km}^{-1}(1),\ (A_m,B_m) = \alpha(u_m). \text{ By definition, } w'_m(A_m,B_m) = 1,$$

$$dist((A_m,B_m),(A,B)) = O(u_m) = O(dist(\psi_{rm}(0),1)) = O(dist(w'_m(A,B),1)) = O(\varepsilon(c_R\widetilde{l}_m)),$$

by (4.6.5). Thus, there exists a constant $C>1$ such that

$$dist((A_m,B_m),(A,B)) < C\varepsilon(c_R\widetilde{l}_m) \tag{4.6.9}$$

for any m large enough (that is, for which the previous pair (A_m,B_m) is well-defined). One has

$$|w'_m| \leq |\nu_{k,m}| + |w_k| \leq \widetilde{l}_m + |w_k| = l_m + \Delta,\ \Delta = |w_k| + l(R). \text{ Therefore,}$$

$$|w'_m| \leq l'_m = c''l_m,\ c'' = \max_m \frac{l_m+\Delta}{l_m},$$

$$dist((A_m,B_m),(A,B)) < C\varepsilon(c_R\widetilde{l}_m) < \varepsilon(c'l'_m),\ \text{where } c' = C^{-1}c_R \inf_m \frac{\widetilde{l}_m}{l'_m} = \frac{c_R}{Cc''}.$$

This proves (4.6.1), (4.6.2) and Theorem 4.1.29.

Chapitre 5

Restricted version of the infinitesimal Hilbert 16-th problem

5.1 Introduction : zeros of Abelian integrals

5.1.1 Restricted Infinitesimal Hilbert 16th Problem

The original Infinitesimal Hilbert 16th Problem is stated as follows. Consider a real polynomial H in two variables of degree $n+1$. The space of all such polynomials is denoted by $\mathcal{H}_n$.

Connected components of closed level curves of H are called *ovals* of H. Ovals form continuous families, see Fig. 5.1. Fix one family of ovals, say Γ, and denote by $\gamma(t)$ an oval of this family that belongs to the level curve $\{H = t\}$.

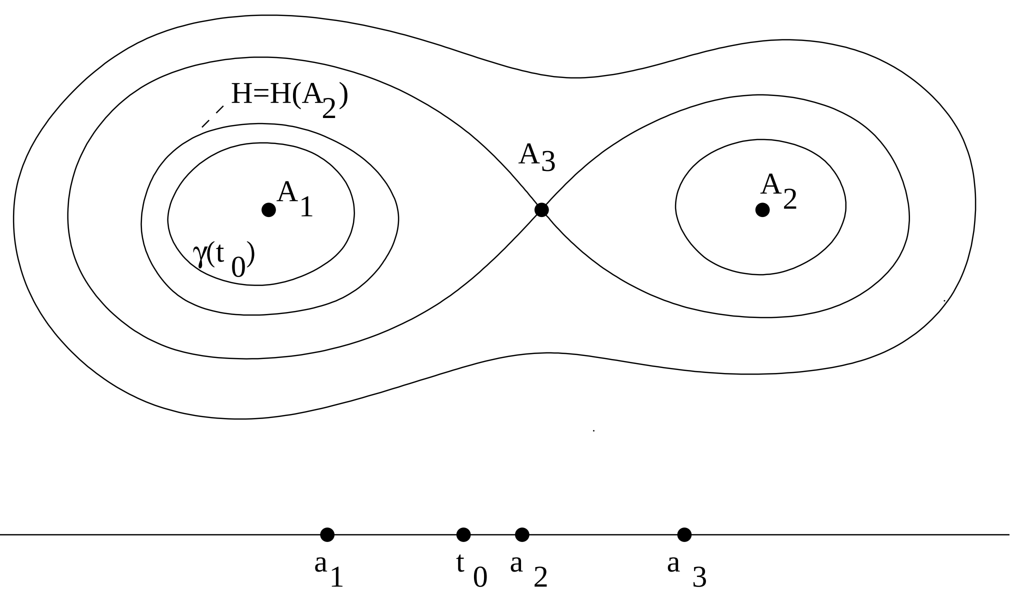

FIG. 5.1 – Families of ovals ; an oval around A_1 that belongs to the level curve $H = H(A_2)$ is distinguished.

Consider a polynomial one-form

$$\omega = Adx + Bdy$$

with polynomial coefficients $A(x,y)$ and $B(x,y)$ of degree at most n. The set of all such forms is denoted by $\Omega_n^{.}$ The main object to study below is the integral

$$I(t) = \int_{\gamma(t)} \omega. \tag{5.1.1}$$

Infinitesimal Hilbert 16th Problem . *Let H and ω be as above. Find an upper bound of the number of isolated real zeros of integral (5.1.1) for a polynomial $H \in \mathcal{H}_n$ and any family Γ of real ovals of H. The estimate should be uniform in ω and H, thus depending on n only.*

This problem stated more than 30 years ago is not yet solved. The existence of such a bound was proved by A.N.Varchenko [113] and A.G.Khovanskii [81]. A weaker version of the problem is called *restricted.* In order to formulate it we need the following

Definition 5.1.1 A polynomial $H \in \mathcal{H}_n$ is *ultra-Morse* provided that it has n^2 complex Morse critical points with pairwise distinct critical values, and the sum h of its higher order terms has no multiple linear factors.

Denote by $\mathcal{U}_n$ the set of all ultra-Morse polynomials in $\mathcal{H}_n$. The complement to this set is denoted by Σ_n and called *the discriminant set*. The integral (5.1.1) may be identically zero. The following theorem shows that for ultra-Morse polynomials this may happen by a trivial reason only.

Theorem 5.1.2 (Exactness theorem [61, 62, 102]) *Let H be a real ultra-Morse polynomial of degree higher than 2. Let the integral (5.1.1) be identically zero for some family of real ovals of the polynomial H. Then the form ω is exact : $\omega = df$.*

Denote by Ω_n^* the set of all non-exact polynomial one-forms from Ω_n.

Restricted version of the Infinitesimal Hilbert 16th Problem (RIHP) . *For any compact subset $\mathcal{K}$ of the set of ultra-Morse polynomials find an upper bound of the number of all real zeros of the integral (5.1.1) over the ovals of a polynomial $H \in \mathcal{K}$. The bound should be uniform with respect to $H \in \mathcal{K}$ and $\omega \in \Omega_n^*$. It may depend on n and $\mathcal{K}$ only.*

This problem is solved in papers [52, 53] (joint with Yu.S.Ilyashenko). The solution is based on the results of papers [46], [47] and [70]. Each one of the papers [46], [47],[70] is independent on the others. The paper [53] is the main one in the series. It contains the survey of results of all the four papers, as well as the solution of the RIHP.

Numerous results obtained during more than 30 years of the study of the infinitesimal Hilbert problem are presented in section 7 of a survey paper [69]. Partial solution of the RIHP (given in our preliminary, unpublished joint paper with Yu.S.Ilyashenko) was claimed in that survey paper. The paper [53] contains a complete solution to RIHP (modulo [46], [47], [70]). Its results with a brief proof were announced in [52].

The main results of the papers [46, 53, 70] are presented in this chapter.

5.1.2 Main results

To measure a gap between a compact set $\mathcal{K} \subset \mathcal{U}_n$ and the discriminant set Σ_n, let us first normalize ultra-Morse polynomials by an affine transformation in the target space. This transformation does not change the ovals of H, thus the number of zeros of the integral (5.1.1) remains unchanged.

Say that two polynomials G and H are equivalent iff

$$G = aH + b, \quad a > 0, \ b \in \mathbb{C}.$$

Definition 5.1.3 A polynomial is *balanced* if all its complex critical values belong to the closed disk of radius 2 centered at zero, and there is no smaller disk that contains all the critical values.

Remark 5.1.4 Any polynomial with at least two distinct complex critical values is equivalent to one and unique balanced polynomial. If the initial polynomial has real coefficients, then so does the corresponding balanced polynomial.

Let us define two positive functions on $\mathcal{U}_n$ such that at least one of them tends to zero as H tends to Σ_n. For any compact set $\mathcal{K} \subset \mathcal{U}_n$ the minimal values of these functions on $\mathcal{K}$ form a vector in $\mathbb{R}_+ \times \mathbb{R}_+$ that is taken as a size of the gap between $\mathcal{K}$ and Σ_n.

Definition 5.1.5 For any $H \in \mathcal{U}_n$ let $c_1(H)$ be n multiplied by the smallest distance between two lines in the zero locus of h, the higher order form of H. The distance between two lines is taken in sense of the Fubini-Study metric on the projective line $\mathbb{CP}^1$. Let $c'(H) = \min(c_1(H), 1)$.

Denote by $\mathcal{V}_n$ the set of all polynomials with more than one critical value and more than one line in the locus of the higher order homogeneous form. By Definition 5.1.1, $\mathcal{U}_n \subset \mathcal{V}_n$.

Definition 5.1.6 For any $H \in \mathcal{V}_n$, let G be the balanced polynomial equivalent to H. Let $c_2(H)$ be the minimal distance between two critical values of G multiplied by n^2. Let $c''(H) = \min(c_2(H), 1)$.

Note that inequality $c'(H)c''(H) > 0$ is equivalent to the statement that H is ultra-Morse.

In what follows, we deal with balanced ultra-Morse polynomials only. This may be done without loss of generality : any ultra-Morse polynomial is equivalent to a balanced one ; equivalent polynomials have the same number of zeros of the integral (5.1.1) over the corresponding families of ovals.

Theorem A. [53] *Let H be a real ultra-Morse polynomial of degree $n+1$. Let $\Gamma = \{\gamma(t)\}$ be an arbitrary continuous family of real ovals of H. There exists a universal positive c such that the integral (5.1.1) has at most $(1 - \log c'(H))e^{\frac{c}{c''(H)}n^4}$ isolated zeros.*

Appendix. The statement of Theorem A holds with $c = 5.000$.

An approach to the Infinitesimal Hilbert 16th Problem itself presented below motivates the following complex counterpart of Theorem A, namely, Theorem B that gives an estimate of the number of zeros of the integral (5.1.1) in the complex domain. Consider an ultra-Morse polynomial H and let

$$\nu = \nu(H) := \frac{c''(H)}{4n^2}. \tag{5.1.2}$$

Fix any real noncritical value t_0 of H,

$$|t_0| < 3,$$

whose distance to the complex critical values of H is no less than ν. Consider a real oval $\gamma_0 \subset \{H = t_0\}$. We suppose that such an oval exists. Let $a = a(t_0) < t_0 < b(t_0) = b$ (or $a(H, t_0),\ b(H, t_0)$ for variable H) be the nearest real critical values of H to the left and to the right from t_0 respectively ; or $-\infty, +\infty$ if there are none. Denote by $\sigma(t_0)$ the interval $(a(t_0), b(t_0))$ and let $\Gamma(\gamma_0)$ be the continuous family of ovals that contains γ_0 :

$$\Gamma(\gamma_0) = \{\gamma(t)\ |t \in \sigma(t_0),\ \gamma(t_0) = \gamma_0\}. \tag{5.1.3}$$

The following cases for $(a, b) = \sigma(t_0)$ are possible :

$$(a, b),\ b > a, -\infty < a < b < +\infty;\ (a, +\infty);\ (-\infty, b).$$

If a is finite, and $\lim\ \mathrm{top}\ _{t\to a}\gamma(t)$ contains a saddle critical point of H, then a is a logarithmic branch point of I. If $\lim\ \mathrm{top}\ _{t\to a}\gamma(t)$ is a singleton, or contains no critical point of H, then a is called an apparent singularity. The same for b.

Denote by $B = B_H$ the set of all noncritical values of H :

$$B = \mathbb{C} \setminus \{a_1 \ldots, a_\mu\},\ \mu = n^2,\ a_j \text{ are the complex critical values of } H.$$

Let W be the universal cover over B with the base point t_0 and the projection

$$\pi : W \to B \subset \mathbb{C}.$$

For any $t \in \mathbb{C}$ denote

$$S_t = \{H = t\} \subset \mathbb{C}^2.$$

Definition 5.1.7 Any point $\hat{t} \in W$ is represented by a class $[\lambda]$ of curves in B starting at t_0 and terminating at $t = \pi\hat{t}$; all the curves of the class are homotopic on B. Any cycle γ from $H_1(S_{t_0}, \mathbb{Z})$ may be continuously extended over λ as an element of the homology groups of level curves of H; the resulting cycle $\gamma(\hat{t})$ from $H_1(S_t, \mathbb{Z})$ is called an *extension* of γ corresponding to $\hat{t}$.

This construction allows us to extend the integral (5.1.1) to W : for any $\hat{t} \in W$,

$$I(\hat{t}) = \int_{\gamma(\hat{t})} \omega. \tag{5.1.4}$$

For any $0 < r \leq \nu$ denote by $a + re^{i\varphi} \in W$ a point represented by a curve $\Gamma_1\Gamma_2 \subset B$, where Γ_1 is an oriented segment from t_0 to $t_1 = a + r \in \sigma(t_0)$, $\Gamma_2 = \{a + re^{i\theta} \mid \theta \in [0, \varphi]\}$; Γ_2 is oriented from t_1 to t. In the same way $b - re^{i\varphi} \in W$ is defined. Let

$$\Pi(a) = \{a + re^{i\varphi} \in W \mid 0 < r \leq \nu, |\varphi| \leq 2\pi\}, \text{ for } a \neq -\infty \tag{5.1.5}$$

$$\Pi(b) = \{b - re^{i\varphi} \in W \mid 0 < r \leq \nu, |\varphi| \leq 2\pi\}, \text{ for } b \neq +\infty$$

Let

$$D(l, a) = \{a + re^{i\varphi} \in W \mid a + re^{\frac{i\varphi}{l}} \in \Pi(a)\}$$

$$D(l, b) = \{b - re^{i\varphi} \in W \mid b - re^{\frac{i\varphi}{l}} \in \Pi(b)\},$$

$$D(l, a) = \emptyset, \text{ if } a = -\infty; \; D(l, b) = \emptyset, \text{ if } b = +\infty.$$

Let $DP_R = DP_R(H, t_0)$ be the disk of radius R in the Poincaré metric of W centered at t_0.

For any real polynomial H, the choice of a cycle γ_0 determines a family of ovals (5.1.3) over which the integral (5.1.1) is taken. When we want to specify this choice we write I_{H,γ_0} or I_H instead of I. The integral I_{H,γ_0} may be analytically extended not only as a function of $\hat{t} \in W$, but also as a function of H.

An analytic extension of the integral I to W is denoted by the same symbol I. For any positive R and natural l denote by $G = G(l, R, H, t_0)$ the domain

$$G = DP_R(H, t_0) \;\cup\; D(l, a(H, t_0)) \;\cup\; D(l, b(H, t_0)), \text{ see Figure 5.2.}$$

Theorem B. [53] *For any real ultra-Morse polynomial H, any real oval γ_0 of H, any natural l and any positive $R > \frac{288n^4}{c''(H)}$, the number of zeros of the integral I_{H,γ_0} in $G = G(l, R, H, t_0)$, where $t_0 = H \mid \gamma_0$, is estimated as follows :*

$$\#\{\hat{t} \in G(l, R, H, t_0) | I_{H,\gamma_0}(\hat{t}) = 0\} \leq (1 - \log c'(H)) \cdot \left(e^{7R} + A^{4800} e^{\frac{481l}{c''(H)}}\right), \; A = e^{\frac{n^4}{c''(H)}}. \tag{5.1.6}$$

The lower bound on R in the statement of the theorem is motivated by the remark in Subsection 5.2.4 below.

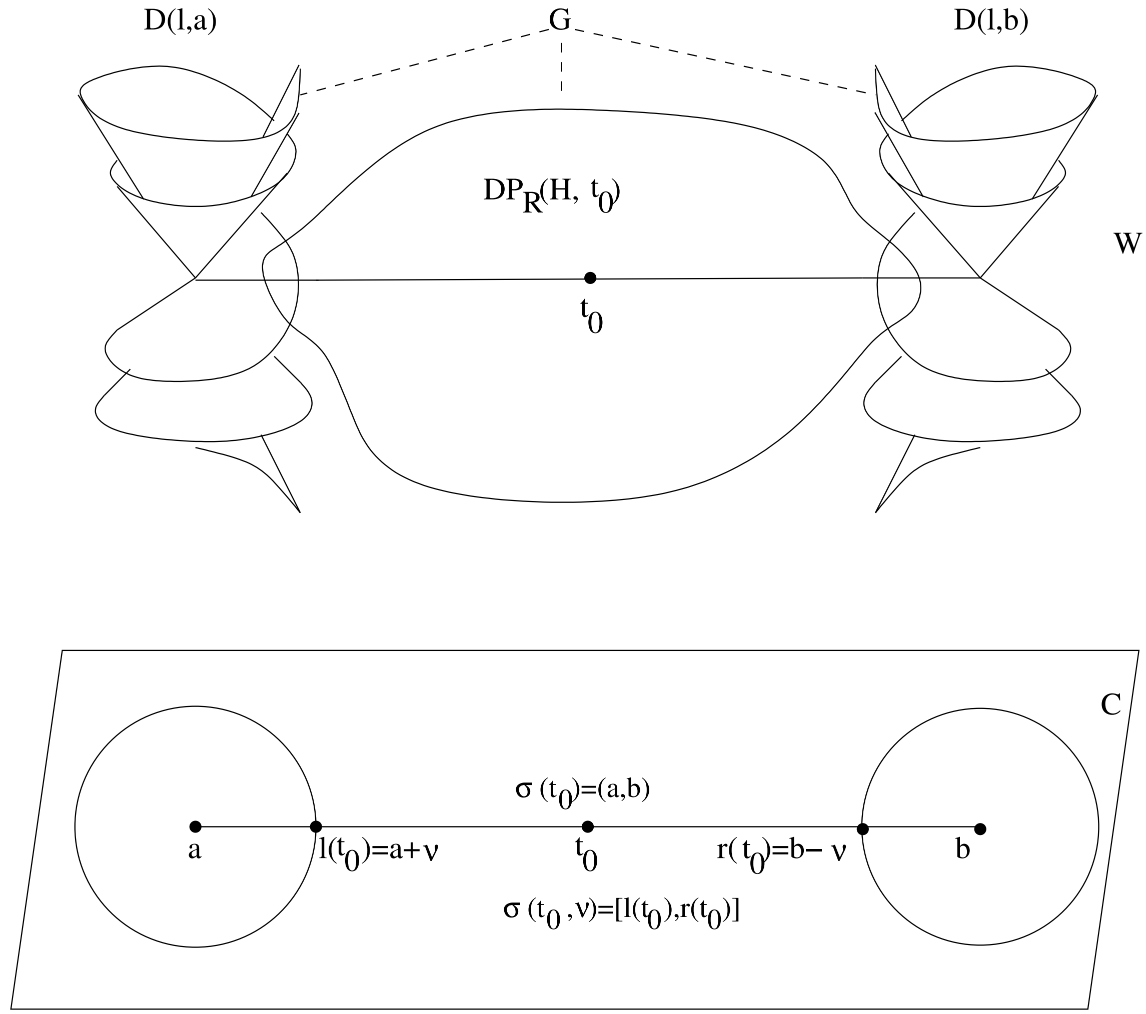

FIG. 5.2 – The domains $DP_R(H, t_0), D(l, a), D(l, b) \subset W$; the domain G is their union

5.1.3 An approach to a solution of the Infinitesimal Hilbert 16th Problem

Conjecture (Yu.S.Ilyashenko). *For any n there exist $\delta(n), l(n), R(n)$ with the following property. Let H_0 be an arbitrary real polynomial from $\mathcal{H}_n$, t_0 be its real noncritical value and γ_0 be a real oval of H_0 that belongs to $\{H_0 = t_0\}$ (we suppose that such an oval exists). Let I_H be the integral (5.1.1). The integral I_H depends on H as a parameter. Let $t_1 \in \sigma(t_0), I_{H_0}(t_1) = 0$ and $t(H)$ be a germ of an analytic function defined by the equation $I_H(t(H)) \equiv 0,\ t(H_0) = t_1$. The required property is the following. There exists a path $\lambda \subset \mathcal{H}_n$ depending on H_0 only starting at H_0 and ending at some $H_1 \in \mathcal{H}_n$ such that :*

$$c'(H_1) \geq \delta(n),\ c''(H_1) \geq \delta(n); \tag{5.1.7}$$

the analytic extension $t(H_1)$ of the function $t(H)$ along λ starting at the value t_1 belongs to the domain $G(l(n), R(n), H_1, t_0)$.

The conjecture above implies the solution of the Infinitesimal 16th Problem. Indeed, suppose that the conjecture is true. Let $N(n)$ be the right hand side of the inequality (5.1.6) with $c'(H)$ and $c''(H)$ replaced by $\delta(n)$; R and l replaced by $R(n)$ and $l(n)$ respectively. Then the number of real zeros of integral I_{H_0} can not exceed $N(n)$. If not, any of real zeros of I_{H_0} would be extended along λ up to a zero of an integral I_{H_1} located in $G = G(l(n), R(n), H_1, t_0)$, for some polynomial H_1 satisfying inequalities (5.1.7). Thus the number of zeros of the integral I_{H_1} in G will exceed $N(n)$. But Theorem B implies that the number of zeros of H_1 in G is no greater than $N(n)$, a contradiction.

The chapter is structured as follows. In Section 5.2 we present the main ideas of the proof of Theorem A. Section 5.2 contains also a survey of the previous investigations and describes some results of [46] ; these results may be called "quantitative algebraic geometry". Moreover, we prove in this section a part of Theorem A, namely, Theorem $A1$, modulo the Main Lemma. In Section 5.3 we sketch the proof of the Main Lemma. The proof relies upon two statements : formula for the determinant of periods, and upper estimates of Abelian integrals provided by quantitative algebraic geometry. These two statements are proved in two separate papers, [47] and [46] respectively. The proof of Theorem A (modulo Theorems A1 and A2) will be given in Subsection 5.2.5. Theorem A2 is written in [53] and is due to Yu.S.Ilyashenko. It will be proved in Section 5.4. The Main Lemma is an important tool for both Theorems A and B.

5.2 Main ideas of the proof and survey of the related results

5.2.1 Historical remarks

A survey of the history of the Infinitesimal Hilbert 16th Problem may be found in [69], and we will not repeat it here. In particular, a much weaker version of Theorem A is claimed there as Theorem 7.7. The first solution to restricted Hilbert problem was suggested in [100]. An explicit upper bound for the same numbers of zeros as in Theorem A was suggested there as a tower of four exponents with coefficients "that may be explicitly written following the proposed constructive solution." It is unclear how much efforts is needed to write these constants down. Moreover, exponential of a polynomial presented in Theorem A is much simpler (though still very excessive) than the tower of four exponentials.

The result of [100] is a crown of a series of papers [97] - [99]. Solution to the restricted version of the Infinitesimal Hilbert 16th Problem presented there is only one application of a vast theory. This theory presents an upper bound of the number of zeros of solutions to linear systems of differential equations. Similar results for components of vector solutions to linear systems are obtained. Abelian integrals are considered as solutions to Picard-Fuchs equations. Using the above-mentioned theory, A. Grigoriev [55, 56] have proved another upper bound for the number of zeros of Abelian integrals in domains distant from the critical values. His estimate is given by double exponent of the sum of two terms : a power of the degree of the hamiltonian and a constant term. The latter power is universal : its exponent is a constant independent on the hamiltonian and the form. The previous constant term depends on the minimal gap between the domain under consideration and the critical values. In difference to our result, Grigoriev's bound depends only on the latter gap and does not depend on the higher terms of the hamiltonian.

On the contrary, our presentation is focused on the study of Abelian integrals given by formula (5.1.1) "as they are" and not as solutions of differential equations.

5.2.2 Quantitative algebraic geometry

Everywhere below for any $r > 0$ and $w \in \mathbb{C}$ we denote

$$D_r(w) = \{|z - w| < r\} \subset \mathbb{C},\ D_r = D_r(0).$$

Our main tool is Growth-and-Zeros theorem for holomorphic functions stated in the next subsection. It requires, in particular, an upper bound of the integral under consideration. We fix an integrand, say $w = x^k y^{n-k} dx$. Depending on a scale in $\mathbb{C}^2$, a cycle γ in the integral $\int_\gamma \omega$ may be located in a small or in a large ball. According to this, the integrand will be small or large. We want to estimate the integral at a certain point of the universal cover W represented by an arc that connects a base point t_0 with some point, say t, with $|t| \leq 3$. To make this restriction meaningful, the scale in the range of the polynomial should be chosen; in other words, the polynomial should be balanced. The argument above shows that it should be also *rescaled* in sense of the following definitions.

Definition 5.2.1 The *norm* of a homogeneous polynomial h is the maximal value of its modulus on the unit sphere; this norm is denoted by $\|h\|_{\max}$.

Definition 5.2.2 A balanced polynomial $H \in \mathbb{C}[x, y]$ is *rescaled* provided that the norm of its higher order form h equals one : $||h||_{\max} = 1$, and the origin is a critical point for H. Briefly, a balanced rescaled polynomial will be called *normalized*.

Remark 5.2.3 Any ultra-Morse polynomial may be transformed to a normalized one by homotheties and shifts in the source and target spaces (not in the unique way). The functions c' and c'' remain unchanged under such transformations.

Definition 5.2.4 We say that the topology of a complex level curve $S_t = H^{-1}(t)$ of a polynomial $H \in \mathcal{H}_n$ is *located in a bidisk*

$$D_{X,Y} = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq X,\ |y| \leq Y\}$$

provided that the difference $S_t \setminus D_{X,Y}$ consists of $n + 1 = \deg H$ punctured topological disks, and the restriction of the projection $(x, y) \mapsto x$ to any of these disks is a biholomorphic map onto $\{x \in \mathbb{C}| X < |x| < \infty\}$.

Theorem C [46]. *For a normalized polynomial, the Hermitian basis in $\mathbb{C}^2$ may be so chosen that the topology of all level curves S_t for $|t| \leq 5$ will be located in a bidisk $D_{X,Y}$ with*

$$X \leq Y \leq (c'(H))^{-14n^3} n^{65n^3} = R_0.$$

This theorem is of independent interest, providing one of the first results in *quantitative algebraic geometry.* On the other hand, it implies upper estimates of Abelian integrals used in the proof of Theorem A and required by the Growth-and-Zeros theorem below.

In the rest of this section, we describe the main ideas of the proof of a simplified version of Theorem A, namely Theorem A1 stated below. It provides an upper bound for the number of zeros of the integral (5.1.1) on a real segment that is ν-distant from critical values of H and belongs to the disk $\overline{D}_3$, thus being distant from infinity; recall that $\nu = \nu(H)$ is given by (5.1.2).

Together with the use of Theorem A1, we get an estimate of the number of zeros of the integral I_{H,γ_0} near the endpoints of $\sigma(t_0)$, as well as near infinity (Theorem A2 stated in 5.2.5). Together with Theorem A1, this completes the proof of Theorem A. The tools used in the proof of Theorem A2 include Petrov method and a so called KRY theorem. The latter one is a recent result in one-dimensional complex analysis [82, 105]. Its improved version is proved by Yu.S.Ilyashenko in a separate paper [70]. In this form it provides a powerful tool to estimate the number of zeros of analytic functions near logarithmic singularities.

5.2.3 Growth-and-Zeros Theorem for Riemann surfaces

The idea of the proof of Theorem $A1$ is to consider an analytic extension of the integral (5.1.1) to the complex domain and to make use of the following Growth-and-Zeros theorem. The symbol diam_{int} used in the statement of the theorem denotes the intrinsic diameter, see Definition 5.2.6 below. We need a notion of a π-gap between a set and its subset on a Riemann surface.

Definition 5.2.5 Let W be a Riemann surface, $\pi : W \to \mathbb{C}$ be a holomorphic function (called projection) with non-zero derivative. Consider the metric on W lifted from $\mathbb{C}$ by projection π. Let $U \subset W$ be a connected domain, and $K \subset U$ be a compact set. For any $p \in U$ let $\varepsilon(p, \partial U)$ be the supremum of radii of disks centered at p, located in U and such that π is bijective on these disks. The π-gap between K and ∂U, is defined as

$$\pi\text{-gap } (K, \partial U) = \min_{p \in K} \varepsilon(p, \partial U).$$

Growth-and-zeros theorem. *Let W, π be the same as in Definition 5.2.5. Let $U \subset W$ be a domain conformally equivalent to a disk. Let $K \subset U$ be a path connected compact subset of U (different from a single point). Suppose that the following two assumptions hold :*

Diameter condition :

$$diam_{\ int} K \le D;$$

Gap condition :

$$\pi\text{-}gap(K, \partial U) \ge \varepsilon.$$

Let I be a bounded holomorphic function on $\overline{U}$. Then

$$\#\{z \in K | I(z) = 0\} \le e^{\frac{2D}{\varepsilon}} \log \frac{\max_{\overline{U}} |I|}{\max_K |I|} \tag{5.2.1}$$

The definition of the intrinsic diameter is well known ; yet we recall it for the sake of completeness.

Definition 5.2.6 The *intrinsic distance* between two points of a path connected set in a metric space is the infinum of the lengths of paths in K that connect these points (if exists). The *intrinsic diameter* of K is the supremum of intrinsic distances between two points taken over all the pairs of points in K.

Definition 5.2.7 The second factor in the right-hand side of (5.2.1) is called the Bernstein index of I with respect to U and K and denoted $B_{K,U}(I)$:

$$B_{K,U}(I) = \log \frac{M}{m},\ M = \sup_U |I|,\ m = \max_K |I|. \tag{5.2.2}$$

Proof of the Growth-and-Zeros theorem. The above theorem is proved in [74] for the case when $W = \mathbb{C}, \pi = Id$. In fact, in [74] another version of (5.2.1) is proved with (5.2.1) replaced by

$$\#\{z \in K | I(z) = 0\} \le B_{K,U}(I) e^{\rho}, \tag{5.2.3}$$

where ρ is the diameter of K in the Poincaré metric of U. In this case it does not matter whether U belongs to $\mathbb{C}$ or to a Riemann surface.

Proposition 5.2.8 *Let K, U be two sets in the Riemann surface W from Definition 5.2.5, and let the Diameter and Gap conditions from the Growth-and-Zeros theorem hold. Then the diameter of K in the Poincaré metric of U admits the following upper estimate :*

$$\rho \le 2D/\varepsilon. \tag{5.2.4}$$

Proof Denote by $|v|_{PU}$ the length of a vector v in sense of the Poincaré metric of U. By the monotonicity property of the Poincaré metric, the length $|v|_{PU}$ of any vector v attached at any point $p \in K$ is no greater than two times the Euclidean length of v divided by the π-gap between K and ∂U. This implies (5.2.4) □

Together with (5.2.3), this proves (5.2.1). □

5.2.4 Theorem A1 and Main Lemma

In what follows, H will be an ultra-Morse polynomial unless the converse is stated. Consider a normalized polynomial H. Let a_j be its complex critical values, $j = 1, \ldots, n^2$; ν, t_0, W and π be the same as in 5.1.2. Let I be the integral (5.1.1) as in Theorem A (well defined for $t = t_0$). It admits an analytic extension to W, which will be denoted by the same symbol I.

Let $a = a(t_0), b = b(t_0)$ be the same as in 5.1.2, and ν be from (5.1.2). Let

$$l(t_0) = \begin{cases} a + \nu \text{ for } a \neq -\infty \\ -3 \text{ for } a = -\infty, \end{cases}$$

$$r(t_0) = \begin{cases} b - \nu \text{ for } b \neq +\infty \\ 3 \text{ for } b = +\infty. \end{cases}$$

Let

$$\sigma(t_0, \nu) = [l(t_0), r(t_0)], \text{ see Figure 5.2.}$$

We identify $\sigma(t_0, \nu) \subset \mathbb{C}$ with its lift to W that contains t_0.

Theorem A1. *In the assumptions stated at the beginning of the subsection, for any complex form* $\omega \in \Omega_n^*$,

$$\#\{t \in \sigma(t_0, \nu) \mid I(t) = 0\} < (1 - \log c'(H)) A^{578}, \; A = e^{\frac{n^4}{c''(H)}}. \tag{5.2.5}$$

This theorem is an immediate corollary of the Growth-and-Zeros theorem and the Main Lemma stated below. Let

$$L^{\pm}(t_0) = \begin{cases} \{a + \nu e^{\pm i\varphi} \in W \mid \varphi \in [0, 2\pi]\} \text{ for } a \neq -\infty \\ \{-3e^{\pm i\varphi} \in W \mid \varphi \in [0, 2(n+1)\pi]\}, \text{ for } a = -\infty, \end{cases} \tag{5.2.6}$$

$$R^{\pm}(t_0) = \begin{cases} \{b - \nu e^{\pm i\varphi} \in W \mid \varphi \in [0, 2\pi]\} \text{ for } b \neq +\infty \\ \{+3e^{\pm i\varphi} \in W \mid \varphi \in [0, 2(n+1)\pi]\}, \text{ for } b = +\infty, \end{cases} \tag{5.2.7}$$

$$\Gamma_a = L^+(t_0) \cup L^-(t_0), \; \Gamma_b = R^+(t_0) \cup R^-(t_0), \; \Sigma = \Gamma_a \cup \Gamma_b \cup \sigma(t_0, \nu).$$

Main Lemma. *Let H be a normalized polynomial of degree $n+1 \geq 3$ with critical values $a_j : \; j = 1, ..., n^2$, ω be a complex polynomial 1-form of degree no greater than n. Let W, ν, Σ be the same as at the beginning of this subsection. Then there exists a path connected compact set $K \subset W$, $K \supset \Sigma$, $\pi K \subset \overline{D_3}$, with the following properties :*

$$diam_{int} K < 36n^2; \tag{5.2.8}$$

$$dist(\pi K, a_j) \geq \nu \text{ for any } j = 1, ..., n^2. \tag{5.2.9}$$

Moreover, let U be the minimal simply connected domain in W that contains the $\nu/2$ neighborhood of K. Then the Bernstein index of the integral (5.1.1) *admits the following upper bound :*

$$B_{K,U}(I) < (1 - \log c'(H)) A^2. \tag{5.2.10}$$

The proof of the Main Lemma is sketched in Section 5.3. This Lemma is used also in the estimate of the number of zeros of the integral in the intervals $(a, l(t_0))$, $(r(t_0), b)$. In fact, a much better estimate for the Bernstein index holds :

$$B_{K,U}(I) < \frac{2700n^{18}}{c''(H)} - 30n^6 \log c'(H) := B(n, c', c''). \tag{5.2.11}$$

Together with the elementary inequality

$$B(n, c', c'') < (1 - \log c')A^2, \tag{5.2.12}$$

it implies (5.2.10).

Proof of Theorem A1. Let us apply Growth-and-Zeros theorem to the function I in the domain U in order to estimate the number of zeros of I in K; note that $K \supset \sigma(t_0, \nu)$. The intrinsic diameter of K is estimated from above by (5.2.8). The gap condition for U and K has the form

$$\pi - \text{gap }(K,\ \partial U) = \varepsilon = \frac{\nu}{2} = \frac{c''}{8n^2}$$

by the definition of U. Hence,

$$e^{\frac{2D}{\varepsilon}} < e^{\frac{72n^2}{c''}8n^2} = A^{576}.$$

The Bernstein index $B_{K,U}(I)$ is estimated from above in (5.2.10). By Growth-and-Zeros theorem

$$\#\{t \in \sigma(t_0) \mid I(t) = 0\} < B_{K,U}(I)A^{576} < (1 - \log c')A^{578}.$$

This proves (5.2.5). □

The following remark motivates the restriction on R in Theorem B.

Remark 5.2.9 Let K be the set from the Main Lemma, $\rho_W K$ be its diameter in the Poincaré metric of W. Then

$$\rho_W K < (c'')^{-1}288n^4. \tag{5.2.13}$$

Indeed, $\rho_W K$ is no greater than the ratio of the double intrinsic diameter of K divided by its minimal distance to the critical values of H (Proposition 5.2.8). Together with (5.2.8) and (5.2.9) this implies (5.2.13). On the other hand, in the proof of Theorem B, we apply Growth-and-Zeros theorem in the case, when the Poincaré disk $DP_R(H, t_0)$ is large enough, namely, contains the set K. Hence, the maximum of $|I|$ over the disk is no less than $\max |I|$ over K. The latter maximum is estimated from below in the proof of the Main Lemma.

5.2.5 Theorem A2 and proof of Theorem A

Theorem A2. *Let H, $t_0, a = a(t_0), b = b(t_0)$, $l(t_0)$, $r(t_0)$ be the same as in the previous subsection. Let ω be a real 1- form in Ω^*_n. Then, in assumptions of Theorem A1,*

$$\#\{t \in (a, l(t_0)) \cup (r(t_0), b) \mid I(t) = 0\} < (1 - \log c')A^{4800} \tag{5.2.14}$$

Proof of Theorem A. By Theorems A1 and A2

$$\#\{t \in (a, b),\ I(t) = 0\} < (1 - \log c')A^{578} + (1 - \log c')A^{4800} < 2(1 - \log c')A^{4800}. \tag{5.2.15}$$

This implies the estimate of the number of zeros given by Theorem A on the interval (a, b).

Let $\sigma' \subset \mathbb{R}$ be the maximal interval of continuity of the family Γ of real ovals that contains γ_0. Then σ' is bounded by a pair of critical values, at most one of them may be infinite. In general, the interval σ' may contain critical values (see Fig. 5.1, which presents a possible arrangement of level curves of H in this case : A_1, A_2, A_3 are critical points of H, $a_j = H(A_j)$, $a_2 \in \sigma' = (a_1, a_3)$, $t_0 \in (a_1, a_2)$). In this case $\sigma' \neq (a, b) = (a_1, a_2)$. Let us estimate the number of zeros on σ'. The interval σ' is split into at most n^2 subintervals bounded by critical values. On each subinterval the number of zeros of I is estimated by (5.2.15), as before. Therefore, the number of zeros of I on σ' is less than $2n^2(1 - \log c')A^{4800} < (1 - \log c')A^{4801}$. This proves Theorem A. □

5.3 An upper bound for the number of zeros on a real segment distant critical values. Proof of the Main Lemma

In this section we prove the Main Lemma (modulo technical details) and hence Theorem A1. We also prove the Modified Main Lemma, see Subsection 5.3.8 below, and prepare necessary tools for the proof of Theorem A2.

5.3.1 The plan of the proof of the Main Lemma

The proof of the Main Lemma is based on the following idea. The integral (5.1.1) is extended onto the universal cover W of the set of noncritical values of the real ultra-Morse polynomial H; the base point of this cover belongs to $(-3,3)$. The upper estimate of the Bernstein index of this integral in the pair of domains U, K requires an upper bound of the maximal modulus of the integral in $\overline{U}$, and a lower bound in K. When we consider these maxima instead of their ratio, we have to normalize the form ω, multiplying it by an appropriate complex factor.

Definition 5.3.1 A polynomial 1-form is normalized if the maximal magnitude of its coefficients equals 1, and some coefficients equal 1.

The upper bound of the integral is provided by the quantitative algebraic geometry. The main difficulty is to obtain the lower bound. For this we consider μ^2 integrals instead of a single one; recall that $\mu = n^2$. Namely, we introduce a special set of μ monomial 1- forms $\omega_i, i = 1, \dots, \mu$ and a special set of vanishing cycles on the level curves $S_t = \{H = t\} : \delta_1(t), \dots, \delta_\mu(t)$. The matrix $\mathbb{I}(t)$ with the entries $I_{ij}(t) = \int_{\delta_j(t)} \omega_i$ is called *a matrix of periods.* The determinant $\Delta(t) = \det \mathbb{I}(t)$ is single-valued. The first step is to evaluate this determinant and to provide a lower bound for $\Delta(t)$ when t is distant from the critical values of H. This is done in [47] and [46]. The second step is to give an upper estimate for the entries of $\mathbb{I}$. This estimate is based on the results of [46] (see Theorem C stated in 5.2.2). The main step is to construct the set $K \subset W$. This set is constructed in such a way that the assumption "$m := \max_K |I|, I(t) = \int_{\gamma(t)} \omega$, is small" implies that *all the integrals* $\int_{\delta_j(t_0)} \omega,\ j = 1, ..., \mu$ are small. This implication makes use of the Picard-Lefschets theorem, and the connectedness of the intersection graph of the special system of vanishing cycles.

The implication above is used in the following way. For a normalized form ω, one may replace some row of the matrix $\mathbb{I}$ by the row $\int_{\delta_1(t)} \omega, \dots, \int_{\delta_\mu(t)} \omega$ without changing the main determinant. All the entries of $\mathbb{I}$ are estimated from above; the determinant of $\mathbb{I}$ is estimated from below. This implies that none of the rows of $\mathbb{I}$ may be too small, and thus provides a lower bound for m. The domain U is chosen as a slightly modified ε-neighborhood of K for appropriate ε. The upper estimate of $M = \max_{\overline{U}} |I|$ is obtained by quantitative algebraic geometry [46], as the upper bound of the $|I_{ij}||_U$ above, and a Geometric lemma stated in 5.3.4. Upper estimate of M and lower bound for m imply an upper estimate of the Bernstein index $B_{U,K}(I)$ and thus prove the Main Lemma.

5.3.2 Special set of vanishing cycles and modified Main Lemma

All along this section H is a real normalized ultra-Morse polynomial of degree $n + 1 \geq 3,\ \mu = n^2;\ a_1, \dots, a_\mu$ are the critical values of $H,\ \nu$ is the same as in (5.1.2), $\varepsilon = \nu/2$. For t close to $a_j,\ \delta_j(t)$ is a local vanishing cycle corresponding to a_j on a level curve

$$S_t = \{H = t\}.$$

Recall the definition of this cycle.

Consider an ultra-Morse polynomial in $\mathbb{C}^2$ having a (Morse) critical point with a critical value a. An intersection of a level curve of this function corresponding to a value close to a with an appropriate neighborhood of the critical point is diffeomorphic to an annulus. This follows from the Morse lemma. The annulus above may be called a local level curve corresponding to the a critical value a.

Definition 5.3.2 A generator of the first homology group of the local level curve corresponding to a is called a *local vanishing cycle* corresponding to a.

A local vanishing cycle is well defined up to change of orientation.

A path $\alpha_j : [0,1] \to \mathbb{C}$ is called *regular* provided that

$$\alpha_j(0) = t_0,\ \alpha_j(1) = a_j,\ \alpha_j[0,1) \subset B \tag{5.3.1}$$

Definition 5.3.3 Let α_j be a regular path, $s \in [0,1]$ be close to 1, $\delta_j(t)$, $t = \alpha_j(s)$, be a local vanishing cycle on S_t corresponding to a_j. Consider the extension of δ_j along the path α up to a continuous family depending on s of cycles $\delta_j(\alpha_j(s))$ in complex level curves $H = \alpha_j(s)$. The homology class $\delta_j = \delta_j(t_0) \in H_1(S_{t_0}, \mathbb{Z})$ (corresponding to $s = 0$) is called a *cycle vanishing along* α_j.

Definition 5.3.4 Consider a set of regular paths $\alpha_1, \dots, \alpha_\mu$, see (5.3.1). Suppose that these paths are not pairwise and self intersected. Then the set of cycles $\delta_j \in H_1(S_{t_0}, \mathbb{Z})$ vanishing along α_j, $j = 1. \dots, \mu$, is called a *marked set of vanishing cycles* on the level curve $H = t_0$.

Recall that the *intersection graph* of a set of cycles in $H_1(S_t, \mathbb{Z})$ is the graph whose vertices are the elements of the set ; two vertices are connected by an edge, if and only if the corresponding cycles have nonzero intersection index.

Theorem 5.3.5 *[9] Let H be a ultra-Morse polynomial. For any noncritical value t any marked set of vanishing cycles in $H_1(S_t, \mathbb{Z})$ is a basis in the same homology group and has a connected intersection graph.*

Recall that $W = W(t_0, H)$ is the universal cover over the set of noncritical values of H with the base point t_0 and the projection $\pi : W \to \mathbb{C}$.

Let $\delta_1, ..., \delta_\mu$ be a marked set of vanishing cycles. For any cycle δ_l from this set, consider an integral

$$I_l(t) = \int_{\delta_l(t)} \omega,$$

over local vanishing cycles, for t close to a_l. This integral is holomorphic at a_l, and takes zero value at a_l. Denote by W_l the Riemann surface of the analytic extension of this integral. Note that the Riemann surface W_l contains the disk $D_\nu(a)$.

Lemma 5.3.6 (Modified Main Lemma). *The Main Lemma from Subsection 5.2.4 holds true provided that the real oval $\gamma(t)$ of integration (1.1) is replaced by a local vanishing cycle $\delta_l(t)$ close to the corresponding critical value a_l, W is replaced by W_l and Σ is replaced by the disk $\overline{D}_\nu(a_l)$.*

This lemma is proved in 5.3.8.

5.3.3 Matrix of periods

Consider and fix an arbitrary marked set of vanishing cycles δ_j, $j = 1, \dots, \mu$. For any $\hat{t} \in W$, let $\delta_j(\hat{t})$ be the extension of δ_j corresponding to $\hat{t}$ (as in Definition 5.1.7).

Definition 5.3.7 Consider a set Ω of μ forms ω_j of the type

$$\omega_i = yx^k y^l dx,\ k, l \geq 0,\ k + l \leq 2n - 2, \tag{5.3.2}$$

(k, l) depends on i, such that all the forms with $k + l \leq n - 1$ are included in the set, and the number of forms with monomials of degree $2n - k$ equals k for $1 \leq k \leq n$. In what follows, such a set is called *standard*.

A *matrix of periods* $\mathbb{I} = (I_{ij}),\ 1 \le i \le \mu,\ 1 \le j \le \mu$, is the matrix function defined on W by the formula :

$$I_{ij}(\hat{t}) = \int_{\delta_j(\hat{t})} \omega_i,\ \mathbb{I}(\hat{t}) = (I_{ij}(\hat{t})) \tag{5.3.3}$$

where $\delta_j,\ j = 1, \ldots, \mu$, form a marked set of vanishing cycles; $\{\omega_i \mid i = 1, \ldots, \mu\}$ is a standard set of forms (5.3.2).

When we want to specify dependence on H, we write $\mathbb{I}(\hat{t}, H)$ instead of $\mathbb{I}(\hat{t})$.

5.3.4 Upper estimates of integrals

Denote by $|\lambda|$ the length of a curve λ, and by $U^\varepsilon(A)$ the ε-neighborhood of a set A.

The main result of the quantitative algebraic geometry that we need is the following

Theorem 5.3.8 *Let δ_j be a vanishing cycle from a marked set, see Definition 5.3.4, corresponding to a curve $\alpha_j, |\alpha_j| \le 9$ (recall that $|t_0| \le 3$). Let $\lambda \subset B$ be a curve starting at t_0 (denote by t its endpoint) such that*

$$|\lambda| \le 36n^2 + 1,\ \ |t| \le 5. \tag{5.3.4}$$

Let the curve $\alpha_j \cap U^\varepsilon(a_j)$ be a connected arc of α_j, and the curves $\alpha_j \setminus U^\varepsilon(a_j)$ and λ have an empty intersection with ε-neighborhoods of the critical values a_k, where $\varepsilon = \nu/2,\ \nu$ is from (5.1.2). Let ω be a form (5.3.2), $\hat{t} \in W$ corresponds to $[\lambda]$, and $\delta_j(\hat{t})$ be the extension of δ_j to $\hat{t}$. Then

$$\left|\int_{\delta_j(\hat{t})} \omega\right| < 2^{\frac{2600n^{16}}{c''(H)}} (c'(H))^{-28n^4} := M_0 \tag{5.3.5}$$

This result is based on Theorem C from 5.2.2. Both results are proved in the paper [46].

We have to give an upper bound of the integral not over a vanishing cycle, but over a real oval. The following lemma shows that the real oval is always a linear combination of some (at most μ) vanishing cycles with coefficients ± 1.

Lemma 5.3.9 (Geometric lemma). *Let H be a real ultra-Morse polynomial and γ be a real oval of H. Let $H|_\gamma = t_0$. Denote by s the number of critical points of H located inside γ in the real plane. Let $a_1, \ldots, a_s$ be the corresponding critical values. Let $\alpha_j,\ j = 1, \ldots, s$, be nonintersecting and nonself-intersecting paths that connect t_0 with these critical values and satisfy assumption (5.3.1). Moreover, suppose that all these paths belong to the upper halfplane and for any a_j (which is real), an open domain bounded by a path α_j and a real segment (connecting the endpoints of α_j) contains no critical value of H (see Figures 5.3 and 5.4). Let δ_j be the vanishing cycles that correspond to the paths α_j. Then*

$$[\gamma] = \Sigma_1^s \varepsilon_j \delta_j,\ \ \text{where } \varepsilon_j = \pm 1. \tag{5.3.6}$$

A proof of Lemma 5.3.9 (given in [53] and omitted here) is based on Picard-Lefschetz theorem [9].

Upper estimates of the integrals of monomial forms over vanishing cycles are provided by Theorem 5.3.8. When we replace a monomial form by a polynomial one, the following changes are needed. Let $\omega \in \Omega_n^*$ be the form in the integral I. There exists another form of type

$$\omega' = \sum_{k+l \le n-1} a_{kl} x^k y^{l+1} dx, \tag{5.3.7}$$

such that the difference $\omega - \omega'$ is exact. We may replace the form ω by ω' in (5.1.1); the integral I will be preserved. Moreover, we can replace the form ω' by a normalized form $\alpha\omega', \alpha \in \mathbb{C}$, see Definition 5.3.1. Hence, we may assume that the form ω in the integral I has the type (5.3.7) and is normalized from the very beginning. When we replace a polynomial form by a normalized one, the previous upper bound of the integral should be multiplied by the number of monomials, namely, by $\frac{n(n+1)}{2}$. When the vanishing cycle is replaced by a real one, the integral is replaced by a sum of $s \le n^2$ integrals over vanishing cycles, by the Geometric Lemma. This results in another multiplication by n^2.

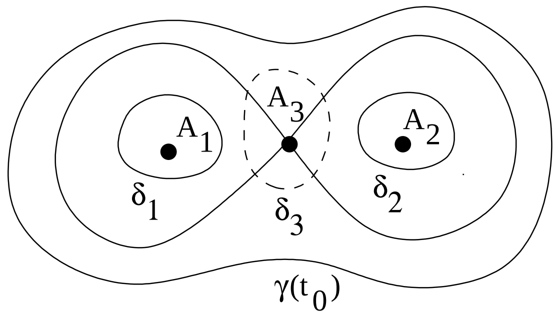

FIG. 5.3 – The cycle $\gamma = \gamma(t_0)$ and local vanishing cycles $\delta_j = \delta_j(t_j)$; the points t_j close to a_j are marked at Fig.4.

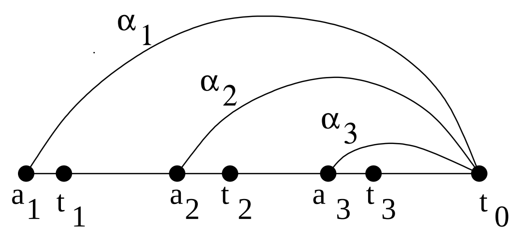

FIG. 5.4 – The paths for the extension of the local vanishing cycles $\delta_j(t)$.

Corollary 5.3.10 *In the condition of Theorem 5.3.8 let H be a real polynomial, $\gamma(\hat{t})$ be the extension to $\hat{t}$ of a real oval, ω be a normalized form* (5.3.7). *Then*

$$|I_{\gamma(\hat{t})}\omega| \leq \frac{n^3(n+1)}{2} M_0. \tag{5.3.8}$$

5.3.5 Determinant of periods

The determinant of the matrix of periods (5.3.3) is called the *determinant of periods.* It appears that this determinant is single-valued on B, thus depending not on a point of the universal cover W, but rather on the projection of this point to B. Let

$$\Delta(t) = \det \mathbb{I}(\hat{t}), \; t = \pi\hat{t}.$$

The main determinant is single-valued ; this follows from the Picard-Lefschetz theorem. Indeed, a circuit around one critical value adds the multiple of the correspondent column to some other columns of the matrix of periods. Thus the determinant remains unchanged.

When we want to specify the dependence of the main determinant on H, we write $\Delta_H(t)$. This function is a polynomial in t, and an algebraic function in the coefficients of H. The formula for the main determinant (with ω_i of appropriate degrees) with a sketch of the proof was claimed by A.Varchenko [114] ; this formula is given up to a constant factor not precisely determined. The complete answer (under the same assumption on the degrees of ω_i) is obtained in [47], with the latter constant factor calculated explicitly. Moreover, the following lower estimate holds :

Theorem 5.3.11 *For any normalized ultra-Morse polynomial H, the tuple Ω of standard forms (5.3.2) may be so chosen that for any $t \in \mathbb{C}$ lying outside the $\nu = \frac{c''}{4n^2}$- neighborhoods of the cri-*

tical values of H the following lower estimate holds :

$$|\Delta(t,H)| \geq (c'(H))^{6n^3} (c''(H))^{n^2} n^{-62n^3} := \Delta_0 \tag{5.3.9}$$

This result is proved in [46] with the use of the explicit formula for the Main Determinant mentioned before, and results of the quantitative algebraic geometry.

5.3.6 Construction of the set K

We can now pass to the construction of the set K mentioned in the Main Lemma. We first construct a smaller set K'.

Lemma 5.3.12 (Construction lemma). *Let $\gamma \subset S_{t_0}$ be a real oval of an ultra-Morse polynomial. Then there exist :*

a set of regular paths $\alpha_j,\ j=1,\dots,\mu$, (see (5.3.1)), such that $|\alpha_j| \leq 9$, and the paths α_j are not pairwise and self intersected;

a path connected set $K' \subset W,\ t_0 \in K',\ \pi K' \subset D_3$, such that for any cycle $\delta_j \in H_1(S_{t_0},\mathbb{Z})$ vanishing along α_j there exist two points $\tau_1,\tau_2 \in K' \cap \pi^{-1}(t_0)$ with the property

$$[\gamma(\tau_1)] - [\gamma(\tau_2)] = l_j[\delta_j],\ l_j \in \mathbb{Z}\setminus 0. \tag{5.3.10}$$

Moreover,

$$diam_{int} K' < 19n^2, \tag{5.3.11}$$

and $\pi K'$ is disjoint from the ν-neighborhoods of the critical values $a_j,\ j=1,\dots,\mu$.

The next modification of this lemma will be used in the proof of the Modified Main Lemma.

Lemma 5.3.13 (Construction lemma for vanishing cycles). *Construction lemma holds true if $\gamma \subset S_{t_0}$ is replaced by any vanishing cycle $\delta_l = \delta_l(t_0)$ from an arbitrary marked set of vanishing cycles, and W is replaced by W_l (see 5.3.2). In the conclusion, (5.3.10) should be replaced by*

$$[\delta_l(\tau_1)] - [\delta_l(\tau_2)] = l_j[\delta_j(t_0)],\ \textit{for } j \neq l,\ l_j \in \mathbb{Z}\setminus 0.$$

Both lemmas are purely topological. Their proof is given in [53] and omitted here. It is based on Picard-Lefschetz theorem [9] and the connectivity of the intersection graph of marked basis of vanishing cycles (Theorem 5.3.5). In what follows we deduce the Main Lemma from Lemma 5.3.12 and Theorems 5.3.8, 5.3.11.

Corollary 5.3.14 (of Lemma 5.3.12). *For any form ω (not necessarily of type (5.3.2)) and any marked set of vanishing cycles consider the vector function*

$$\mathbb{I}_\omega : W \to \mathbb{C}^\mu,\ \hat{t} \mapsto \left(\int_{\delta_1(\hat{t})} \omega, \dots, \int_{\delta_\mu(\hat{t})} \omega\right). \tag{5.3.12}$$

Let $||\cdot||$ denote the Euclidean length in $\mathbb{C}^\mu$. Then

$$m_0 := \max_{\hat{t} \in K' \cap \pi^{-1}(t_0)} |I(\hat{t})| \geq \frac{1}{2n} ||\mathbb{I}_\omega(t_0)||. \tag{5.3.13}$$

Proof Consider a component of the vector $\mathbb{I}_\omega(t_0)$ with the largest magnitude. Let its number be j. Then

$$\left| \int_{\delta_j(t_0)} \omega \right| \geq \frac{1}{n} ||\mathbb{I}_\omega(t_0)||. \tag{5.3.14}$$

By Lemma 5.3.12, there exist $\tau_1, \tau_2 \in K' \cap \pi^{-1}(t_0)$ such that

$$I(\tau_1) - I(\tau_2) = l_j \int_{\delta_j(t_0)} \omega,\ l_j \in \mathbb{Z} \setminus 0.$$

Hence, at least one of the integrals $I(\tau_l)$ in the left hand side, say $I(\tau_l),\ l \in \{1,2\}$, admits a lower estimate :

$$|I(\tau_l)| \geq \frac{1}{2} \left| \int_{\delta_j(t_0)} \omega \right|. \tag{5.3.15}$$

Together with (5.3.14) this proves the corollary. □

Let us now take

$$K = K' \cup \Sigma, \Sigma = \sigma(t_0) \cup L^{\pm}(t_0) \cup R^{\pm}(t_0), \tag{5.3.16}$$

see (5.2.6), (5.2.7).

In the following section we will check that this K satisfies the requirements of the Main Lemma.

5.3.7 Proof of the Main Lemma

Let us take K as in (5.3.16). Let ν be the same as in (5.1.2). Let U be the smallest simply connected set that contains the ε-neighborhood of $K,\ \varepsilon = \nu/2$. Then (5.2.8) follows from (5.3.11), (5.3.16). The last statement of Lemma 5.3.12 implies (5.2.9).

Let us now check (5.2.10), that is, estimate from above the Bernstein index $B_{K,U}(I)$ for the integral (5.1.1).

Let the form ω in the integral (5.1.1) be normalized, and let, as before, $M = \max_{\bar{U}} |I|,\ m = \max_K |I|$. By Corollary 5.3.10,

$$M \leq \frac{n^3(n+1)}{2} M_0 := M_0'$$

where M_0 is from (5.3.5). Let us now estimate m from below, following the ideas presented at the beginning of the section.

Let in (5.3.7) $|a_{k_0 l_0}| = 1,\ \omega_i = y x^{k_0} y^{l_0} dx$. Without loss of generality we may assume that $a_{k_0 l_0} = 1$. Let us now replace the ith row of the matrix $\mathbb{I}$ by the vector $\mathbb{I}_\omega$. This transformation is equivalent to adding a linear combination of rows of $\mathbb{I}$ to the ith row, so the determinant $\Delta(t_0)$ remains unchanged.

By Theorem 5.3.8 and (5.2.8), all the entries in other rows are estimated from above by M_0, see (5.3.5). (The corresponding paths α_j used in the construction of K are chosen as in Lemma 5.3.12, so, the inequality $|\alpha_j| \leq 9$ of Theorem 5.3.8 holds true.) Hence, all the vector-rows except for the ith one have the length at most nM_0. By (5.3.13), the ith row has the length at most $2nm_0$. We can now obtain a lower bound for m. Indeed, $m \geq m_0$. On the other hand,

$$\Delta_0 \leq |\Delta(t_0)| \leq 2m_0 M_0^{\mu-1} n^{\mu}, \mu = n^2,$$

where Δ_0 is the same as in (5.3.9). Therefore,

$$m \geq m_0 \geq \frac{1}{2} \Delta_0 M_0^{1-\mu} n^{-\mu}. \tag{5.3.17}$$

We can now estimate $B_{K,U}(I)$ from above. Indeed,

$$B_{K,U}(I) = \log M - \log m \geq \log M_0' - \log m_0.$$

Elementary estimates (together with (5.3.17)) imply that

$$\log M_0' - \log m_0 > (1 - \log c')A^2. \tag{5.3.18}$$

This proves the Main Lemma.

5.3.8 Modified Main Lemma and zeros of integrals over (complex) vanishing cycles

Proof of the Modified Main Lemma. The arguments of the previous subsection work almost verbatim. The previous corollary for the integral $I = I_l$ taken over δ_l instead of γ, is stated and proved in the same way.

Let K' be the same as in Lemma 5.3.13. Instead of (5.3.16), let

$$K = K' \cup \alpha_l \cup \overline{D}_\nu(a_l).$$

Let U be the smallest simply connected set that contains the ε-neighborhood of K.

By Theorem 5.3.8,

$$\max_{\overline{V}} |I_l| \leq M_0, \text{ where } V = U \setminus D_\nu(a_l).$$

But I_l is holomorphic in $D_\nu(a_l)$. Hence, by the maximum modulus principle, the previous inequality holds in U instead of V. After that, the rest of the arguments of the previous subsection work. This proves the Modified Main Lemma. □

The following theorem will be used in the next section.

Theorem 5.3.15 *The number of zeros of the integral I_l in the disk $D_\nu(a_l)$ satisfies the inequality :*

$$\#\{\hat{t} \in D_\nu(a_l) \mid I_l(\hat{t}) = 0\} \leq (1 - \log c'(H))A^{578}. \tag{5.3.19}$$

The proof is the same as for Theorem A1, section 5.2.4.

5.4 Estimates of the number of zeros of Abelian integrals near critical values

In this section we give a proof (due to Yu.S.Ilyashenko) of Theorem A2, see 5.2.5. Together with Theorem A1 (whose proof was given in Section 5.2), Theorem A2 implies Theorem A.

We split the proof of Theorem A2 into three cases : 1) $a, b \neq \infty$; 2) $a = -\infty$; 3) $b = +\infty$. First we prove Theorem A2 in Case 1 (Subsections 5.4.1-5.4.5). Cases 2 and 3 are treated in 5.4.6.

5.4.1 Argument principle, KRY theorem and Petrov's method

All the three cases are treated in a similar way. We want to apply the argument principle.

The estimates near infinity are based on the argument principle only. The estimates near finite critical points use the Petrov's method that may be considered as a generalization of the argument principle for multivalued functions. The increment of the argument is estimated through the Bernstein index of the integral, bounded from above in the previous sections. The relation between these two quantities is the subject of the Khovanskii-Roitman-Yakovenko (KRY) theorem and Theorem 5.4.3 stated below. It seems surprising that these theorems were not discovered in the classical period of the development of complex analysis. The latter theorem is proved in [70] ; the proof is based on the KRY theorem and methods of [82] and [105].

At this spot we begin the proof of Theorem A2 in case 1. Recall the statement of the theorem in case 1.

Theorem A2 (Case 1). *Let $a \neq \infty, b \neq \infty$. Then*

$$\#\{t \in (a, l(t_0)) \cup (r(t_0), b) \mid I(t) = 0\} < (1 - \log c')e^{\frac{4700}{c''}n^4},$$

where $l(t_0)$ and $r(t_0)$ are the same as at the beginning of 5.2.4.

We will prove that

$$\#\{t \in (a, l(t_0)) \mid I(t) = 0\} < \frac{1}{2}(1 - \log c')e^{\frac{4700}{c''}n^4}. \tag{5.4.1}$$

Similar estimate for $(r(t_0), b)$ is proved in the same way. These two estimates imply Theorem A2.

Let $\Pi = \Pi(a)$ be the same as in (5.1.5), namely

$$\Pi = \{t \in W \mid 0 < |t - a| \le \nu, |\arg(t - a)| \le 2\pi\}.$$

Lemma 5.4.1 *Inequality (5.4.1) holds provided that in (5.4.1) the interval* $(a, l(t_0))$ *is replaced by* Π.

Lemma 5.4.1 implies (5.4.1) because $(a, l(t_0)) \subset \Pi$. Let

$$\Pi_\psi = \{t \in \Pi \mid \psi \le |t - a| \le \nu\}$$

Lemma 5.4.2 *Lemma 5.4.1 holds provided that in (5.4.1) the domain* Π *is replaced by* Π_ψ.

Lemma 5.4.2 implies Lemma 5.4.1, because

$$\Pi = \cup_{\psi > 0} \Pi_\psi.$$

The proof of Lemma 5.4.2 occupies this and the next four Subsections. We have

$$\partial \Pi_\psi = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4.$$

As sets, the curves Γ_j are defined by the formulas below; the orientation is defined separately :

$$\Gamma_1 = \{t \mid |t - a| = \nu, |arg(t - a)| \le 2\pi\} = \Gamma_a$$

$$\Gamma_3 = \{t \mid |t - a| = \psi, |arg(t - a)| \le 2\pi\}$$

$$\Gamma_{2,4} = \{t \mid \psi \le |t - a| \le \nu, arg(t - a) = \pm 2\pi\}.$$

The curve Γ_1 is oriented counterclockwise, Γ_2 is oriented from the right to the left, Γ_3 is oriented clockwise, Γ_4 is oriented from the left to the right.

Let $\#\{t \in (a + \psi, l(t_0)) \mid I(t) = 0\} = N_\psi$. Denote by $R_\Gamma(f)$ the increment of the argument of a holomorphic function f along a curve Γ (R of Rouchet),

$$V_\Gamma(f) = \text{ the variation of the argument of } f \text{ along } \Gamma. \text{ Obviously, } \mid R_\Gamma(f) \mid \le V_\Gamma(f).$$

In assumption that $I \ne 0$ on $\partial \Pi_\psi$, the argument principle implies that

$$N_\psi \le \frac{1}{2\pi} R_{\partial \Pi_\psi}(I) = \frac{1}{2\pi} \sum_1^4 R_{\Gamma_j}(I). \tag{5.4.2}$$

The first term in this sum is estimated by the modified KRY theorem, the second and the forth one by the Petrov method, the third one by the Mardesic theorem. The case when the above assumption fails is treated in 5.4.3.

5.4.2 Bernstein index and variation of argument

The first step in establishing a relation between variation of argument and the Bernstein index was done by the following KRY theorem.

Let U be a connected and simply connected domain in $\mathbb{C}$, $\Gamma \subset U$ be a (nonoriented compact) curve, f be a bounded holomorphic function on U.

KRY theorem, [82] *For any tuple $U, \Gamma \subset U$ as above and a compact set $K \subset U$ there exists a geometric constant $\alpha = \alpha(U, K, \Gamma)$, such that*

$$V_\Gamma(f) \leq \alpha B_{K,U}(f).$$

In [82] an upper estimate of the Bernstein index through the variation of the argument along $\Gamma = \partial U$ is given; we do not use this estimate. On the contrary, we need an improved version of the previous theorem with α explicitly written and U being a domain on a Riemann surface. These two goals are achieved in the following theorem.

Let $|\Gamma|$ be the length, and $\kappa(\Gamma)$ be the total curvature of a curve on a surface endowed with a Riemann metric.

Theorem 5.4.3 *[70] Let $\Gamma \Subset U'' \Subset U' \Subset U \subset W$ be respectively a curve, and three open sets in a Riemann surface W. Let $f : U \to \mathbb{C}$ be a bounded holomorphic function, $f|_\Gamma \not\equiv 0$. Let $\pi : W \to \mathbb{C}$ be a projection which is locally biholomorphic, and the metric on W be a pullback of the Euclidean metric in $\mathbb{C}$. Let $\varepsilon < \frac{1}{2}$ and the following gap conditions hold :*

$$\pi\text{-}gap\ (\Gamma, U'') \geq \varepsilon,\ \ \pi\text{-}gap\ (\overline{U''}, U') \geq \varepsilon,\ \ \pi\text{-}gap\ (\overline{U'}, U) \geq \varepsilon. \tag{5.4.3}$$

Let $D > 1$ and the following diameter conditions hold :

$$diam\ _{int}U'' \leq D,\ \ diam\ _{int}U' \leq D \tag{5.4.4}$$

Then

$$V_\Gamma(f) \leq B_{U'',U}(f)\left(\frac{|\Gamma|}{\varepsilon} + \kappa(\Gamma) + 1\right)e^{\frac{5D}{\varepsilon}}. \tag{5.4.5}$$

Recall that intrinsic diameter and π-gap are defined in 5.2.3.

We can now estimate from above the first term in the sum (5.4.2). The estimate works in both cases when a is finite or infinite.

Lemma 5.4.4 *Let H be a normalized polynomial of degree $n + 1 \geq 3$. Let I be the same integral as in (5.1.1). Let K be a compact set mentioned in the Main Lemma, and $\Gamma_1 = \Gamma_a$ be the same as in this lemma (a may be infinite). Then*

$$V_{\Gamma_1}(I) < (1 - \log c'(H))A^{4600},\ \ A = e^{\frac{n^4}{c''}}. \tag{5.4.6}$$

In what follows, we write c', c'' instead of $c'(H), c''(H)$.

The lemma follows easily from Theorem 5.4.3 and the Main Lemma, see [53] for more detail.

Remark 5.4.5 Lemma 5.4.4 remains valid if in its hypothesis the integral I is replaced by an integral J over the cycle vanishing at the critical value a of H. The proof of this modified version of Lemma 5.4.4 repeats that of the original one with the following change : we use the Modified Main Lemma instead of the Main Lemma.

Corollary 5.4.6 *Suppose that the integral J with a real integrand ω is taken over a local vanishing cycle δ_t corresponding to the real critical value a. Then the number of zeros of J in the disk centered at a of radius $\nu = \frac{c''}{4n^2}$ admits the following upper estimate :*

$$N_J := \#\{t \in \mathbb{C} \mid |t-a| < \nu, J(t) = 0\} < \frac{1}{2\pi}(1 - \log c')A^{4600} \tag{5.4.7}$$

This follows from the modified Lemma 5.4.4 and the argument principle.

5.4.3 Application of the Petrov's method

The Petrov's method applied below is based on the remark that the magnitude of the increment of the argument of a nonzero function along an oriented curve is no greater than the number of zeros of the imaginary part of this function increased by 1 and multiplied by π. Indeed, at any half circuit around zero, a planar curve crosses an imaginary axis at least once. The method works when the imaginary part of a function appears to be more simple than the function itself.

Let $\delta(t) \in H_1(t)$ be the local vanishing cycle at the point a. Let ω be the same real form as in integral (5.1.1). Let J be the germ of integral $J(t) = \int_{\delta(t)} \omega$ along the cycle $\delta(t)$, which is a local vanishing cycle at $t = a$. Note that J is single-valued in any simply connected neighborhood of a that contains no other critical values of H. Let $l_0 = (\gamma(t), \delta(t)) \neq 0$ be the intersection index of the cycles $\gamma(t)$ and $\delta(t)$. As the cycle $\gamma(t)$ is real and H is ultra-Morse, l_0 may take values $\pm 1, \pm 2$ only. This is implied by the following lemma.

Lemma 5.4.7 *Consider a maximal family of real ovals that contains $\gamma(t_0)$. The union of the ovals of the family forms an open domain. The boundary of this domain consists of one or two connected components. Any of these components belongs to a critical level of H and contains a unique critical point. Fix any of these critical points and denote by δ the corresponding local vanishing cycle. Then the cycle δ may be extended to a cycle $\delta(t_0)$ that belongs to a marked set of vanishing cycles constructed above. Moreover,*

$$(\delta(t_0), \gamma(t_0)) \neq 0, \text{ more precisely, it is equal to } \pm 1, \ \pm 2.$$

Let

$$\Gamma_0 = \{t \in \mathbb{R} \mid te^{2\pi i} \in \Gamma_2\}.$$

Then by the Picard-Lefschetz theorem

$$I\mid_{\Gamma_2} = (I + l_0 J)\mid_{\Gamma_0}, \ I\mid_{\Gamma_4} = (I - l_0 J)\mid_{\Gamma_0}.$$

Proposition 5.4.8 *The integral J is purely imaginary on the real interval (a, b).*

Proof Recall that the form ω and the polynomial H are real. Then

$$J(t) = -\overline{J(\overline{t})}.$$

Indeed, $\omega = Q(x, y)dx$. The involution $\mathbf{i} : (x, y) \mapsto (\overline{x}, \overline{y})$ brings the integral $J(t) = \int\limits_{\delta(t)} Qdx$ to $\int\limits_{\mathbf{i}\delta(t)} \overline{Q}d\overline{x} = \int\limits_{-\delta(\overline{t})} \overline{Q}d\overline{x} = -\overline{\int_{\delta(\overline{t})} Qdx} = -\overline{J(\overline{t})}$. On the other hand, for real t we have $t = \overline{t}$ and thus, $J(t) = J(\overline{t})$. Hence, $J(t) = -\overline{J(t)}$ for $t \in (a, b)$. This implies Proposition 5.4.8. $\square$

Corollary 5.4.9 *Let, as above, $l_0 \neq 0$ be the intersection index of the cycles $\gamma(t)$ and $\delta(t)$. Then*

$$Im I\mid_{\Gamma_{2,4}} = \pm l_0 J\mid_{\Gamma_0}.$$

Proof This follows from Proposition 5.4.8, Picard-Lefschetz theorem and the reality of I on Γ_0. □

Suppose first that I has no zeros on Γ_2 and Γ_4. Then

$$\left|R_{\Gamma_{2,4}}(I)\right| \leq \pi(1+N), \text{ where } N = \#\{t \in \Gamma_0 \mid J(t) = 0\}. \tag{5.4.8}$$

Obviously, $N \leq N_J$, see (5.4.7). The right hand side of this inequality is already estimated from above in Corollary 5.4.6. Hence,

$$\left|R_{\Gamma_{2,4}}(I)\right| \leq \pi + \frac{1}{2}(1 - \log c')A^{4600}.$$

Suppose now that I has zeros on Γ_2 (hence on Γ_4, by Proposition 5.4.8). Indeed, its real part is the same at the corresponding points of $\Gamma_2, \Gamma_0, \Gamma_4$, and the imaginary parts of $I|_{\Gamma_2}$ and $I|_{\Gamma_4}$ are opposite at these points. In this case we replace the domain Π_ψ by Π'_ψ defined as follows.

The curves $\Gamma_{2,4}$ should be modified. A small segment of Γ_2 centered at zero point of I that contains no other zeros of J, should be replaced by an upper half-circle having this segment as a diameter and containing no zeros of J. A similar modification should be done for Γ_4 making use of lower half-circles. Denote the modified curves by $\Gamma'_{2,4}$. Let Π'_ψ be the domain bounded by the curve

$$\partial \Pi'_\psi = \Gamma_1 \Gamma'_2 \Gamma_3 \Gamma'_4. \tag{5.4.9}$$

It contains Π_ψ, and we will estimate from above the number of zeros of I in Π'_ψ still using the argument principle. The increment of $\arg I$ along Γ_1 is already estimated in 5.4.2. Here we give an upper bound for the increment of $\arg I$ along $\Gamma'_{2,4}$. The increment along Γ_3 is estimated in the next subsection.

Proposition 5.4.10 *Let N be the same as in (5.4.8). Then*

$$\mid R_{\Gamma'_{2,4}}(I) \mid \leq \pi(2N+1). \tag{5.4.10}$$

Proof We will prove the proposition for Γ'_2; the proof for Γ'_4 is the same. Let I have zeros $b_j \in \Gamma_2,\ j = 1, ..., k$, the number of occurrences of b_j in this list equals its multiplicity. Note that

$$\text{Im } I|_{\Gamma_2} = l_0 J \tag{5.4.11}$$

Hence, at the points $b_j,\ J$ has zeros of no less multiplicity than I. Hence, the total multiplicity k' of zeros of J at the points $b_j \in \Gamma_2,\ j = 1, ..., k$, is no less than k. Let J have s zeros on Γ'_2. We have : $k' \geq k,\ s \leq N - k' \leq N - k$. Let $\sigma_1, ..., \sigma_q,\ q \leq k+1$, be the open intervals, the connected components of the difference of Γ'_2 and the half-circles constructed above. Let s_j be the number of zeros of J on $\sigma_j,\ \sum_1^q s_j = s$. Let

$$R_j = R_{\sigma_j}(I).$$

Then

$$R_j \leq \pi(s_j + 1).$$

Hence,

$$\mid R_{\Gamma'_2}(I) \mid \leq \pi(k + \sum_1^q (s_j+1)) \leq \pi(2k+1+s) \leq \pi(2k'+1+s) \leq \pi(2N+1), \tag{5.4.12}$$

where $N \leq N_J < \frac{1}{2\pi}(1 - \log c')A^{4600}$, see (5.4.7). □

5.4.4 Application of the Mardesic theorem

Proposition 5.4.11 *Let I be the integral (5.1.1), and Γ_3 be the same as in Subsection 5.4.1. Then for ψ small enough,*

$$|R_{\Gamma_3}(I)| \leq \pi(4n^4 + 1). \tag{5.4.13}$$

Proof Let J and l_0 be the same as in the previous subsection. Let $a = 0$, and $I(e^{2\pi i}t)$ means the result of the analytic extension of I from a value $I(t)$ along a curve $e^{2\pi i\varphi}t,\ \varphi \in [0, 1]$. By the Picard-Lefshetz theorem, for small t

$$I(e^{2\pi i}t) = I(t) + l_0 J(t).$$

Consider the function

$$Y(t) = I(t) - l_0 \frac{\log t}{2\pi i} J(t).$$

This function is single-valued because the increments of both terms I and $l_0 \frac{\log t}{2\pi i} J(t)$ under the analytic extension over a circle centered at 0 cancel. The function I is bounded along any segment ending at zero, and J is holomorphic at zero, with $J(0) = 0$. Hence, Y is holomorphic and grows no faster than $\log|t|$ in a punctured neighborhood of zero. (In fact, it is bounded in the latter neighborhood : $|J(t)\log t| \leq c|t||\log t| \to 0$, as $t \to 0$.) By the Removable Singularity Theorem, it is holomorphic at zero. Hence,

$$I(t) = Y(t) + l_0 \frac{\log t}{2\pi i} J(t) \tag{5.4.14}$$

with Y and J holomorphic. We claim that *the increment of the argument of I along Γ_3 for ψ small is bounded from above through $\mathrm{ord}_0 J$, the multiplicity of zero of J at zero.* The latter order is estimated from above by the following theorem by Mardesic :

Theorem 5.4.12 *[90]. The multiplicity of any zero of the integral I (or J) taken at a point where the integral is holomorphic does not exceed n^4.*

The function (5.4.14) is multivalued. The proof of Proposition 5.4.11 is based on the following simple remark. Let f_1, f_2 be two continuous functions on a segment $\sigma \subset \mathbb{R}$, and $|f_1| \geq 2|f_2|$. Then $|R_\sigma(f_1 + f_2)| \leq |R_\sigma(f_1)| + \frac{\pi}{3}$. Indeed, the value $R_\sigma(f_1 + \varepsilon f_2)$ cannot change more than by $\frac{\pi}{3}$, as ε ranges over the segment $[0, 1]$.

To complete the proof of Proposition 5.4.11, we need to consider three cases. Let $\nu = \mathrm{ord}_0 Y,\ \mu = \mathrm{ord}_0 J,\ f(\varphi) = Y(\psi e^{2\pi i\varphi}),\ g(\varphi) = l_0 \left(J \frac{\log t}{2\pi i}\right)(\psi e^{2\pi i\varphi})$. Note that $\mu \leq n^4$.

Case (i) : $\nu < \mu$. Then, for ψ small, $2|g| \leq |f|$. By the previous remark, applied to $f_1 = f,\ f_2 = g$, we get

$$|R_{\Gamma_3}(I)| \leq \pi(4\nu + 1) \leq \pi(4\mu + 1) \leq \pi(4n^4 + 1).$$

Case (ii) : $\nu = \mu$. Then, for ψ small, $2|f| \leq |g|$, because of the logarithmic factor in g. In the same way as before, we get

$$|R_{\Gamma_3}(I)| \leq \pi(4\mu + 1) \leq \pi(4n^4 + 1).$$

Case (iii) : $\nu > \mu$. In the same way, as in Case (ii), we get (5.4.13). □

5.4.5 Proof of Theorem A2 in case 1 (endpoints of the interval considered are finite)

Proof It is sufficient to prove Lemma 5.4.2. We prove a stronger statement

$$N(I, \Pi'_\psi) := \#\{t \in \Pi'_\psi \mid I(t) = 0\} < \frac{1}{2}(1 - \log c')A^{4600} \tag{5.4.15}$$

By the argument principle

$$2\pi N(I, \Pi'_\psi) \leq V(\Gamma_1) + |\ R_{\Gamma'_2}(I)\ | + |\ R_{\Gamma_3}(I)\ | + |\ R_{\Gamma'_4}(I)\ | \tag{5.4.16}$$

The first term in the r.h.s is estimated in (5.4.6). The second and the fourth terms are estimated from above in (5.4.10) (the N in the r.h.s. of (5.4.10) is estimated from above by N_J, see (5.4.7)). The third term is estimated in (5.4.13). Altogether this proves (5.4.15), hence, Lemma 5.4.2 and implies a stronger version of (5.4.1) :

$$N(I, \Pi'_\psi) < \frac{1}{2}(1 - \log c')A^{4600}.$$

This proves Theorem A2 in case 1. □

5.4.6 Proof of Theorem A2 in Case 2 (near an infinite endpoint)

Here we prove Theorem A2 for a segment with one endpoint, say, b, infinite.

Proposition 5.4.13 *The integral I has an algebraic branching point at infinity of order $n+1$.*

Proof of Theorem A2 near infinity. We consider the case $b = +\infty$ only; the case $a = -\infty$ is treated in the same way. Let W_I be the Riemann surface of the integral I. Let $\Gamma \subset W_I$ be the degree $n+1$ cover of the circle $|t| = 3$ with the base point $t_1 = +3$. This is a closed curve on W_I. This curve is a boundary of a domain on W_I that covers $n+1$ times a neighborhood of infinity on the Riemann sphere. Let us denote this domain by W_I^∞. We will estimate from above

$$N_\infty = \{t \in W_I^\infty \mid I(t) = 0\}.$$

This will give an upper estimate to the number of zeros of I on $\sigma^+ = (3, +\infty)$ because $\sigma^+ \subset W_I^\infty$. We will use the argument principle in the form

$$N_\infty \leq \frac{1}{2\pi} V_\Gamma(I) + n + 1. \tag{5.4.17}$$

This follows from the argument principle and the fact that the infinity is the only pole of $I|_{W_I^\infty}$, and its order is at most $n+1$. The latter bound on the order follows from the condition that the 1- form under the integral (1.1) has degree at most n, and the fact that the integration oval $\gamma(t)$ has size (and length) of the order $O(|t|^{\frac{1}{n+1}})$, as $t \to \infty$, $t \in \mathbb{R}$.

The variation in the right hand side will be estimated by Theorem 5.4.3. To apply this theorem we need to define all the entries like in the previous subsection.

We have : $\Gamma = \partial W_I^\infty$. Without loss of generality we consider that $I|_\Gamma \neq 0$ (one can achieve this by slight contraction of the circle $|t| = 3$). Let K be the same as in the Main Lemma. Denote by U_0 the set U from that Lemma : both K and U_0 are taken projected to the Riemann surface of the integral I. By (5.2.7), $K \supset \Gamma$. Let $\varepsilon = \frac{\nu}{6} = \frac{c''}{24n^2}$, U'', U', U be respectively the minimal simply connected domain containing ε-, 2ε-, 3ε- neighborhood of K. One has $K, \Gamma \Subset U'' \Subset U' \Subset U$. Then U coincides with the projection of U_0 to W_I (up to filling holes, if there are any). Therefore, $\max_{\overline{U_0}} |I| = \max_{\overline{U}} |I|$ (the maximum principle). Hence,

$$B_{U'',U}(I) \leq B_{K,U_0}(I) < (1 - \log c')A^2.$$

The latter inequality is (5.2.10). This provides the estimate of the Bernstein index from inequality (5.4.5) in Theorem 5.4.3. Other ingredients are the following.

By (5.2.8), the diameter condition (5.4.4) holds with

$$D = 36n^2 + 1.$$

The gap condition (5.4.3) for Γ , U'', U', U holds with the above $\varepsilon = \frac{c''}{24n^2}$. Hence,

$$e^{\frac{5D}{\varepsilon}} \leq A^{4600}$$

Moreover,

$$| \Gamma | = 6\pi(n+1), \ | \kappa(\Gamma) | = 2\pi(n+1).$$

Altogether, by Theorem 5.4.3, this implies :

$$V_\Gamma(I) \leq (1 - \log c')C(n, c'')A^{4602},$$

with $C(n, c'') = \frac{6\pi(n+1)}{\varepsilon} + 2\pi(n+1) + 1 < A^{90}$. Together with (5.4.17) this proves Theorem A2, Case 2. $\square$

Chapitre 6

Confluence of singular points and Stokes phenomena

6.1 Introduction : Stokes phenomena and main results

6.1.1 Brief statements of results, plan of the chapter and historical remarks

Consider a linear analytic ordinary differential equation

$$\dot{z} = \frac{A(t)}{t^{k+1}} z, \quad z \in \mathbb{C}^n, \ |t| \leq 1, \ k \in \mathbb{N} \tag{6.1.1}$$

with a nonresonant irregular singularity of order (the Poincaré rank) k at 0 (or briefly, an irregular equation). This means that $A(t)$ is a holomorphic matrix function such that the matrix $A(0)$ has distinct eigenvalues (denote them by λ_i). Then the matrix $A(0)$ is diagonalizable, and without loss of generality we suppose that it is diagonal.

Definition 6.1.1 Two equations of type (6.1.1) are analytically (formally) equivalent, if there exists a change $z = H(t)w$ of the variable z, where $H(t)$ is a holomorphic invertible matrix function (respectively, a formal invertible matrix power series), that transforms one equation into the other.

The analytic classification of irregular equations (6.1.1) is well known [8, 10, 71, 75, 107] : the complete system of invariants for analytic classification consists of a formal normal form (6.1.4) and Stokes operators (6.1.6) defined in Subsection 6.1.2 ; the latter are linear operators acting in the solution space of (6.1.1) comparing appropriate "sectorial canonical solution bases".

On the other hand, an irregular equation (6.1.1) can be regarded as a result of *confluence of Fuchsian singular points* (recall that a Fuchsian singular point of a linear equation is a first order pole of its right-hand side). Namely, consider a deformation

$$\dot{z} = \frac{A(t,\varepsilon)}{f(t,\varepsilon)} z, \qquad f(t,\varepsilon) = \prod_{i=0}^{k} (t - \alpha_i(\varepsilon)), \tag{6.1.2}$$

of equation (6.1.1) that splits the irregular singular point 0 of the nonperturbed equation into $k+1$ Fuchsian singularities $\alpha_i(\varepsilon)$ of the perturbed equation, i.e., $\alpha_i(\varepsilon) \neq \alpha_j(\varepsilon)$ for $i \neq j$. The family (6.1.2) depends on a parameter $\varepsilon \in \mathbb{R}_+ \cup 0$, $f(t,0) \equiv t^{k+1}$, $A(t,0) \equiv A(t)$.

The *monodromy group* of a Fuchsian equation acts linearly in its solution space by analytic extensions of solutions along closed loops. The analytic equivalence class of a Fuchsian equation is completely determined by the local types of its singularities and the action of its monodromy group.

Everywhere in what follows we denote by M_i the monodromy operator of the perturbed equation (6.1.2) along a loop going around the singular point α_i (the choice of the corresponding loops will be specified later). The monodromy group of the perturbed equation is generated by appropriately chosen operators M_i.

In 1984, V. I. Arnold proposed the following question. Consider a generic deformation (6.1.2). Is there an operator

$$M_{i_1}^{d_1} \dots M_{i_l}^{d_l} \tag{6.1.3}$$

from the monodromy group of the perturbed equation that converges to a Stokes operator of the nonperturbed equation ?

A version of this question was proposed independently by J.-P. Ramis in 1988.

It appears that already in the simplest case of dimension 2 and Poincaré rank $k = 1$ generically *each operator from the monodromy group* (*except for that along a circuit around both singularities* (*and its powers*)) *tends to infinity*, and none tends to a Stokes operator. In other terms, no word (6.1.3) with $d_i \in \mathbb{Z}$ tends to a Stokes operator. But if $k = 1$, then appropriate words (6.1.3) with *noninteger* powers d_i tend to Stokes operators (Theorem 6.2.12 in Subsection 6.2.2). The last two statements are proved in [42].

The previous question and its nonlinear analogue for parabolic mappings were studied by J.-P. Ramis, B. Khesin, A. Duval, C. Zhang and J. Martinet (see the historical overview in Subsection 6.1.3 and that of recent results below). It was proved by the author [38] in the general case that appropriate branches of the eigenfunctions of the monodromy operators M_i of the perturbed equation tend to appropriate canonical solutions of the nonperturbed equation (Theorem 6.2.5). In the case of Poincaré rank $k = 1$ this implies (Corollary 6.2.6 stated in the two-dimensional case) that Stokes operators of the nonperturbed equation are limits of transition operators between appropriate eigenbases of the monodromy operators M_i. This corollary has a generalization for higher Poincaré rank and dimension [38]. These results are also extended to a generic resonant case [40].

The conjecture saying that Stokes operators are limit transition operators between monodromy eigenbases of the perturbed equation was first proposed by A. A. Bolibrukh in 1996.

Nonlinear analogues of the previous statements for parabolic mappings (i.e., one-dimensional conformal mappings tangent to identity) and their Écalle-Voronin moduli, saddle-node singularities of two-dimensional holomorphic vector fields and their Martinet-Ramis invariants (sectorial central manifolds in higher dimensions) were obtained by the author in [39] (see Theorem 6.4.17 in Section 6.4 for two-dimensional saddle-nodes). Generalizations and other versions of the statement on parabolic mappings were later obtained in the paper [91] by P. Mardesic, R. Roussarie, C. Rousseau, and in two papers by the following authors : (1) X. Buff and Tan Lei (unpublished) ; (2) A. Douady, F. Estrada, P. Sentenac [24].

In Subsection 6.1.2 we recall the analytic classification of irregular equations (6.1.1) and the definitions of sectorial canonical solution bases and Stokes operators. Subsection 6.1.3 contains a survey of previous results.

In Subsection 6.2.1 we state the results on the representation of Stokes operators as limit transition operators between monodromy eigenbases (Theorem 6.2.5 and Corollary 6.2.6). In Subsection 6.2.2 we state Theorem 6.2.12 on convergence of appropriate word (6.1.3) to a Stokes operator. Its proof is given in Section 6.3.

In Section 6.4 we state the results from [39] concerning two-dimensional saddle-nodes. One of them (Corollary 6.4.22) is used in the proof of Theorem 6.2.5 given in Subsection 6.4.3. Corollary 6.4.22 is proved in Subsection 6.4.4.

6.1.2 Analytic classification of irregular equations. Canonical solutions and Stokes operators

Let (6.1.1) be an irregular equation.

One can ask the following question : is it true that the variables $z = (z_1, \dots, z_n)$ in the equation can be separated, more precisely, that (6.1.1) is analytically equivalent to a direct sum of one-dimensional linear equations, i.e., a linear equation with a diagonal matrix function on the right-hand side? Generically, the answer is "no". At the same time any irregular equation (6.1.1) is formally equivalent to a unique direct sum of the type

$$\dot{w}_i = \frac{b_i(t)}{t^{k+1}}\, w_i, \quad i = 1, \dots, n, \tag{6.1.4}$$

where $b_i(t)$ are polynomials of degree at most k, $b_i(0) = \lambda_i$. The normalizing series bringing (6.1.1) to (6.1.4) is unique up to left multiplication by a constant diagonal matrix. The system (6.1.4) is called the *formal normal form* of (6.1.1) [8, 10, 71, 75, 107].

Generically the normalizing series diverges. At the same time there exists a finite covering $\bigcup_{j=0}^{N} S_j$ of a punctured neighborhood of zero in the t-line by radial sectors S_j (i.e., those with the vertex at 0) that have the following property. There exists a unique change of variables $z = H_j(t)w$ over each S_j that transforms (6.1.1) to (6.1.4), where $H_j(t)$ is an analytic invertible matrix function on S_j that can be C^∞-smoothly extended to the closure $\overline{S}_j$ of the sector so that its asymptotic Taylor series at 0 coincides with the normalizing series. The preceding statement on existence and uniqueness of sectorial normalization holds in any good sector (see the two following Definitions); the covering consists of good sectors [8, 10, 71, 75, 107].

Case $k = 1$, $n = 2$, $\lambda_1 - \lambda_2 \in \mathbb{R}$.

Definition 6.1.2 A sector in $\mathbb{C}$ with the vertex at 0 is said to be *good*, if it contains only one imaginary semiaxis $i\mathbb{R}_\pm$, and its closure does not contain the other one (see Fig. 6.1).

General case.

Definition 6.1.3 Let $k \in N$, $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ be an n-tuple of distinct numbers, t be the coordinate on $\mathbb{C}$. For a given pair $\lambda_i \neq \lambda_j$ the rays in $\mathbb{C}$ starting at 0 and forming the set $\mathrm{Re}((\lambda_j - \lambda_i)/t^k) = 0$ are called the (k, Λ)-*imaginary dividing rays* corresponding to the pair (λ_i, λ_j). A radial sector is said to be (k, Λ)-*good*, if for any pair (λ_i, λ_j), $j \neq i$, it contains exactly one corresponding imaginary dividing ray and so does its closure.

Remark 6.1.4 In the case, when $k = 1$, $n = 2$, $\lambda_1 - \lambda_2 \in \mathbb{R}$, the imaginary dividing rays are the imaginary semiaxes, and the notions of "good" sector and (k, Λ)-good sector coincide.

Remark 6.1.5 The ratio $\frac{w_i}{w_j}(t)$ of solutions of equations from (6.1.4) tends either to zero or to infinity, as t tends to zero along a ray distinct from the imaginary dividing rays corresponding to the pair (λ_i, λ_j). Its limit changes exactly when the ray under consideration jumps over one of the latter imaginary dividing rays.

Consider a covering $\bigcup_{j=0}^{N} S_j$ of a punctured neighborhood of zero by good (or (k, Λ)-good) sectors numbered counterclockwise, and put $S_{N+1} = S_0$. The standard splitting of the normal form (6.1.4) into the direct sum of one-dimensional equations defines a canonical base in its solution space (uniquely up to multiplication of the base functions by constants) with a diagonal fundamental matrix. Denote the latter fundamental matrix by

$$W(t) = \mathrm{diag}(w_1, \dots, w_n).$$

Together with the normalizing changes H_j in S_j, it defines the canonical bases $(f_{j1}, \dots, f_{jn})$ in the solution space of (6.1.1) in the sectors S_j with the fundamental matrices

$$Z^j(t) = H_j(t)W(t), \quad j = 0, \dots, N+1, \tag{6.1.5}$$

where for any $j = 0, \ldots, N$ the branch ("with index $j+1$") of the fundamental matrix $W(t)$ in S_{j+1} is obtained from that in S_j by the counterclockwise analytic extension for any $j = 0, \ldots, N$. (We put $S_{N+1} = S_0$. The corresponding branch of W "with index $N+1$" is obtained from that "with index 0" by right multiplication with the monodromy matrix of the formal normal form (6.1.4).) In the connected component of the intersection $S_j \cap S_{j+1}$ there are two canonical solution bases coming from S_j and S_{j+1}. Generically, they do not coincide. The transition between them is defined by a constant matrix C_j :

$$Z^{j+1}(t) = Z^j(t)C_j. \tag{6.1.6}$$

The transition operators (matrices C_j) are called *Stokes operators* (*matrices*) (see [8, 10, 71, 75, 107]). The nontriviality of Stokes operators yields the obstruction to analytic equivalence of (6.1.1) and its formal normal form (6.1.4).

Remark 6.1.6 The Stokes matrices (6.1.6) are well defined up to simultaneous conjugation by one and the same diagonal matrix.

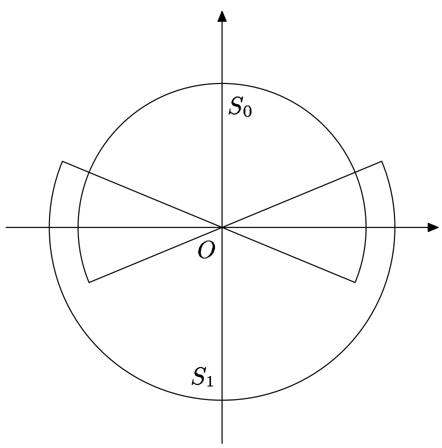

FIG. 6.1 – Case $\lambda_1 - \lambda_2 \in \mathbb{R}_+$. A covering by two good sectors

Example 6.1.7 Let $k = 1$, $n = 2$. In this case we may assume without loss of generality that $\lambda_1 - \lambda_2 \in \mathbb{R}_+$ (one can achieve this by linear change of the time variable). Then the above covering consists of two sectors S_0 and S_1 (Fig. 6.1). The former contains the positive imaginary semiaxis and its closure does not contain the negative one; the latter has the same properties with respect to the negative (respectively, positive) imaginary semiaxis. There are two components of the intersection $S_0 \cap S_1$. So, in this case we have a pair of Stokes operators. *The Stokes matrices* (6.1.6) *are unipotent :* the one corresponding to the left intersection component is lower-triangular; the other one is upper-triangular [8, 10, 71, 75, 107].

Remark 6.1.8 Stokes operators of an irregular equation (6.1.1) with a diagonal matrix in the right-hand side are identity operators. In this case, (6.1.1) is analytically equivalent to its formal normal form. In general, *two irregular equations are analytically equivalent, if and only if they have the same formal normal form and the corresponding Stokes matrix tuples are obtained from each other by simultaneous conjugation by one and the same diagonal matrix*, cf. the previous remark. Thus, the formal normal form and the Stokes matrix tuple taken up to the previous conjugation present the complete system of invariants for analytic classification of irregular equations (see [8, 10, 71, 75, 107]).

6.1.3 Previous results

Earlier, in 1919, R. Garnier [31] had studied some particular deformations of some class of linear equations with nonresonant irregular singularity. He obtained some analytic classification invariants for these equations by studying their deformations. The complete system of analytic classification invariants (Stokes operators and formal normal form) for general irregular nonresonant differential equations was obtained later in the 70's in the papers by Jurkat, Lutz, Peyerimhoff [75], Sibuya [107] and Balser, Jurkat, Lutz [10]. Later Jurkat, Lutz and Peyerimhoff extended their results to some resonant cases [76]. In 1985, J.-P. Ramis proved that the Stokes operators and the monodromy operators of a linear ordinary differential equation belong to its Galois group ([103], see also [71]). In 1989 he considered the classical confluenting family of hypergeometric equations and proved convergence of appropriate branches of monodromy eigenfunctions of the perturbed equation to canonical solutions of the nonperturbed one by direct calculation [104]. In the late 80's, B. Khesin also proved a version of this statement, but his result was not published. In 1991, A. Duval [25] proved this statement for the biconfluenting family of hypergeometric equations (where the nonperturbed equation is equivalent to Bessel's equation) by direct calculation. In 1994, C. Zhang [119] had obtained the expression of Garnier's invariants via Stokes operators (for the class of irregular equations considered by Garnier).

The analytic classification of germs of parabolic mappings was obtained separately by J. Écalle [26] and S. M. Voronin [117]. The orbital analytic classification of germs of two-dimensional saddle-node holomorphic vector fields was obtained by J. Martinet and J.-P. Ramis in their joint paper [93]. The analytic classification of two-dimensional saddle-nodes of multiplicity two was recently obtained in the joint paper [118] by S. M. Voronin and Yu. I. Meshcheryakova.

A particular case of the result from [39] concerning parabolic mappings (analogous to the previously mentioned statements on linear equations) was obtained by J. Martinet [92]. For other related results concerning parabolic mappings see also [91] and the references therein.

6.2 Main results. Stokes operators and limit monodromy

In the present section we formulate the statements expressing the Stokes operators as limit transition operators between monodromy eigenbases of the confluenting Fuchsian equation (Theorem 6.2.5 and Corollary 6.2.6) and as limits of some words (6.1.3) of noninteger powers of monodromy operators (Theorem 6.2.12).

6.2.1 Stokes operators as limit transition operators between monodromy eigenbases

We formulate the result from the title of this subsection only in the case when $k = 1$, $n = 2$ (see [38] in the general case). Let λ_i, $i = 1, 2$, be the eigenvalues of the matrix $A(0)$. Without loss of generality we assume that $\lambda_1 - \lambda_2 \in \mathbb{R}_+$: one can achieve this by linear change of the time variable.

We consider a deformation of (6.1.1),

$$\dot{z} = \frac{A(t,\varepsilon)}{f(t,\varepsilon)}z, \quad f(t,\varepsilon) = (t-\alpha_0(\varepsilon))(t-\alpha_1(\varepsilon)), \quad f(t,0) \equiv t^2, \quad A(t,0) = A(t), \tag{6.2.1}$$

where $A(t,\varepsilon)$ and $f(t,\varepsilon)$ depend continuously on a parameter $\varepsilon \geq 0$ so that $\alpha_0(\varepsilon) \neq \alpha_1(\varepsilon)$ for $\varepsilon > 0$. Without loss of generality we assume that $\alpha_0 + \alpha_1 \equiv 0$. We formulate the statement from the title of the subsection for a generic deformation (6.2.1), see the following Definition.

Definition 6.2.1 A family of quadratic polynomials $f(t,\varepsilon)$ depending continuously on a nonnegative parameter ε, $f(t,0) \equiv t^2$, with roots $\alpha_i(\varepsilon)$, $i = 0, 1$, $\alpha_0 + \alpha_1 \equiv 0$, is said to be *generic*, if $\alpha_0(\varepsilon) \neq \alpha_1(\varepsilon)$ for $\varepsilon \neq 0$, and the line passing through $\alpha_0(\varepsilon)$ and $\alpha_1(\varepsilon)$ intersects the real axis at an angle bounded away from 0 uniformly in ε. A family (6.2.1) of *linear equations* is said to be *generic*, if the corresponding family of polynomials $f(t,\varepsilon)$ is generic.

Recall the following :

Definition 6.2.2 A singular point t_0 of a linear analytic ordinary differential equation $\dot{z} = \frac{B(t)}{t-t_0} z$ is said to be *Fuchsian*, if it is a first order pole of the right-hand side (i.e., the corresponding matrix function $B(t)$ is holomorphic at t_0). The *characteristic numbers* of a Fuchsian singularity are the eigenvalues of the corresponding residue matrix $B(t_0)$ (which are equal to the logarithms divided by $2\pi i$ of the eigenvalues of the corresponding monodromy operator).

Remark 6.2.3 A family (6.2.1) of linear equations is generic if and only if the difference of the characteristic numbers at $\alpha_0(\varepsilon)$ (or equivalently, at $\alpha_1(\varepsilon)$) of the perturbed equation is not real for small ε and, moreover, has argument bounded away from $\pi\mathbb{Z}$ uniformly in ε small enough. The latter condition implies that the monodromy operator of the perturbed equation around each singular point α_i has distinct eigenvalues (moreover, their moduli are distinct), and hence, a well-defined eigenbase in the solution space (for small ε).

The singularities of the perturbed equation from a generic family have imaginary parts of constant (and opposite) signs (by definition). Without loss of generality we assume in what follows that

$$\operatorname{Im}\alpha_0 > 0, \quad \operatorname{Im}\alpha_1 < 0 \quad \text{(see Fig. 6.2).}$$

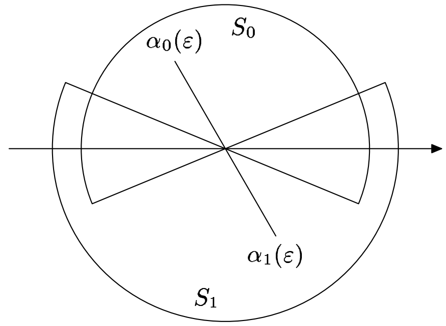

FIG. 6.2 – Two generically confluenting singularities

Definition 6.2.4 Let (6.2.1) be a generic family of linear equations (see the previous definition) whose singularity families satisfy the previous inequalities. Let S_j, $j = 0, 1$, be a pair of good sectors in the t-line such that $\alpha_j(\varepsilon) \in S_j$, $j = 0, 1$, $i\mathbb{R}_+ \subset S_0$, $i\mathbb{R}_- \subset S_1$ (see Fig. 6.1). The sector S_j is said to be *the sector associated to the singularity family* α_j, $j = 0, 1$.

We show that appropriate branches of the eigenfunctions of the monodromy operator M_i around α_i of the perturbed equation converge to canonical solutions of the nonperturbed equation in the corresponding sector S_i. This will imply the statement from the title of this subsection.

To formulate the latter statement precisely, consider the auxiliary domain

$$S_i' = S_i \setminus [\alpha_0(\varepsilon), \alpha_1(\varepsilon)], \tag{6.2.2}$$

which is simply-connected, and the canonical branches of the monodromy eigenfunctions on the domain S_i'. In more detail, consider a small circle going around α_i, and take a base point on it outside the segment $[\alpha_0(\varepsilon), \alpha_1(\varepsilon)]$. In the space of local solutions of the perturbed equation at the base point consider the monodromy operator M_i acting by the analytic extension of a solution along the circle from the base point to itself in the counterclockwise direction. The eigenfunctions of M_i have well-defined branches (up to multiplication by constants) in the corresponding disc with the segment

$[\alpha_0(\varepsilon), \alpha_1(\varepsilon)]$ deleted. Their immediate analytic extension yields their canonical branches on S_i'. In other terms, we identify the space of local solutions with the space of solutions on S_i' by immediate analytic extension, consider M_i as an operator acting in the latter space and take its eigenfunctions.

The canonical basic solutions of the nonperturbed equation are numbered by the indices 1 and 2, which correspond to the eigenvalues λ_1, λ_2 of $A(0)$. To state the results previously mentioned, let us define an analogous numbering of the monodromy eigenfunctions at $\alpha_i(\varepsilon)$. The monodromy eigenfunctions are numbered by the eigenvalues of the corresponding residue matrix. The latter eigenvalues are proportional to those of the matrix $A(\alpha_i(\varepsilon), \varepsilon)$, which tend to λ_1 and λ_2, as $\varepsilon \to 0$. This induces the numbering of the monodromy eigenfunctions with the indices 1 and 2 corresponding to the limit eigenvalues λ_1 and λ_2.

Theorem 6.2.5 *Let* (6.2.1) *be a generic family of linear ordinary differential equations* (*see Definition* 6.2.1), $\alpha_i(\varepsilon)$ *its singularity family, let* S_i *be the corresponding sector* (*see the previous definition*), *and* S_i' *the domain* (6.2.2). *Consider the eigenbase on* S_i' *of the monodromy operator of the perturbed equation around* $\alpha_i(\varepsilon)$. *The appropriately normalized eigenbase* (*by multiplication of the basic functions by constants*) *converges to the canonical solution base* (6.1.5) *on* S_i *of the nonperturbed equation.*

Corollary 6.2.6 *Let* (6.2.1) *be a generic linear equation family* (*see Definition* 6.2.1), α_i *its singularity families, let* S_i *be the corresponding sectors* (*see the previous definition*) *chosen to cover a punctured neighborhood of zero, and* S_i' *the corresponding domains* (6.2.2). *Let* C_0, C_1 *be the corresponding Stokes matrices* (6.1.6) *of the nonperturbed equation in the left* (*respectively, right*) *component of the intersection* $S_0 \cap S_1$. *Consider the eigenbase on* S_i' *of the monodromy operator of the perturbed equation around* $\alpha_i(\varepsilon)$. *Denote by* $Z_\varepsilon^i(t)$ *the fundamental matrix of this eigenbase. Let* $C_0(\varepsilon)$ $(C_1(\varepsilon))$ *be the transition matrix between the monodromy eigenbases* $Z_\varepsilon^i(t)$, $i = 0, 1$, *in the left* (*respectively, right*) *component of the intersection* $S_0' \cap S_1'$:

$$\begin{aligned} Z_\varepsilon^1(t) &= Z_\varepsilon^0(t) C_0(\varepsilon) \quad \textit{for} \quad \operatorname{Re} t < 0; \\ Z_\varepsilon^0(t) &= Z_\varepsilon^1(t) C_1(\varepsilon) \quad \textit{for} \quad \operatorname{Re} t > 0. \end{aligned} \tag{6.2.3}$$

For any $i = 0, 1$ *and appropriately normalized monodromy eigenbases* Z_ε^j, $j = 0, 1$ (*the normalization of* Z_ε^0 (*only*) *depends on the choice of* i), $C_i(\varepsilon) \to C_i$ *as* $\varepsilon \to 0$.

Remark 6.2.7 Theorem 6.2.5 and Corollary 6.2.6 extend to the general case of arbitrary Poincaré rank k and dimension n [38], as do the notions of a generic family of linear equations and a sector associated to a singularity family. The statement of Corollary 6.2.6 in the case of $k = 1$ and arbitrary n remains the same. But for higher k (when the number $k + 1$ of transition matrices is less than that of Stokes matrices) it says that appropriate products of subsequent Stokes matrices (not all the Stokes matrices themselves) are limit transition matrices between appropriate branches of monodromy eigenbases. These limit products of Stokes matrices cover all the Stokes matrices. On the other hand, each element of a Stokes matrix in a limit product can be expressed as a polynomial in the product elements; so, all the Stokes matrices can be recovered from the limit transition matrices.

6.2.2 Stokes operators as limits of commutators of appropriate powers of the monodromy operators

The Stokes and monodromy operators act in different linear spaces : in the solution spaces of the nonperturbed (respectively, perturbed) equations. To formulate the statement from the title of the subsection, let us first identify these solution spaces and specify the loops defining the monodromy operators.

Let (6.2.1) be a generic family of linear equations. Take the “base point”

$$t_0 = -\frac{1}{2}.$$

Remark 6.2.8 The space of local solutions of a linear equation at a nonsingular point $t_0 \in \mathbb{C}$ is identified with the space of initial conditions at t_0 (which is common for the nonperturbed and the perturbed equations). This identifies the solution spaces of the latter. The space thus obtained will be denoted by H_{t_0}.

Remark 6.2.9 Let (6.1.1) be an irregular equation with $k = 1$, $n = 2$, $\lambda_1 - \lambda_2 \in \mathbb{R}$, and let S_0, S_1 be good sectors covering a punctured neighborhood of zero in the t-line, both containing $\mathbb{R}_-$ and $\mathbb{R}_+$ (see Fig. 6.1). Let C_0, C_1 be the Stokes operators (6.1.6) corresponding to the left (respectively, right) intersection component of the sectors. The operator C_0 (C_1) is well defined in the space H_{t_0} of local solutions of (6.1.1) at any point $t_0 \in \mathbb{R}_-$ (respectively, $t_0 \in \mathbb{R}_+$).

Now let us define the monodromy operators acting in the previous space H_{t_0}.

Definition 6.2.10 Let (6.2.1) be a generic family of linear equations, $\alpha_i(\varepsilon)$, $i = 0, 1$, be its singularity families. Fix a point $t_0 \in \mathbb{R}$ (independent of ε). Let l_i be a small circle centered at $\alpha_i(\varepsilon)$ whose closed disc is disjoint from $-\alpha_i(\varepsilon)$, $a_i = [t_0, \alpha_i] \cap l_i$, with the segment $[t_0, a_i]$ oriented from t_0 to a_i. Consider the closed path $\psi_i = [t_0, a_i] \circ l_i \circ [t_0, a_i]^{-1}$, $i = 0, 1$, which starts and ends at t_0 (see Fig. 6.3). Define $M_i : H_{t_0} \to H_{t_0}$ to be the corresponding monodromy operator of the perturbed equation.

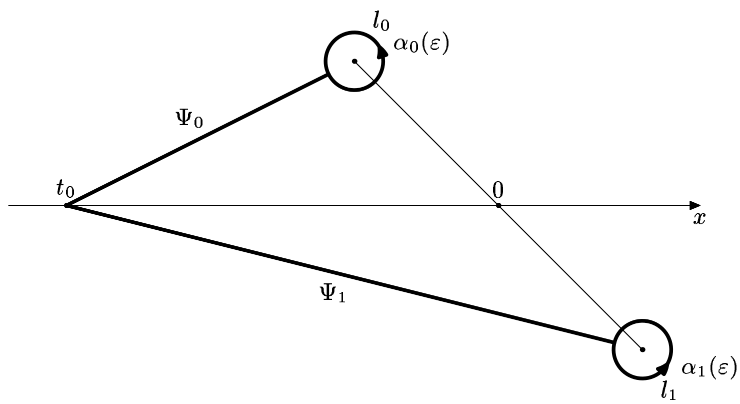

FIG. 6.3 – The loops for the monodromy operators

We show that commutators of appropriate noninteger powers of the operators M_i (see the following definition) tend to the Stokes operators.

Definition 6.2.11 Let $d \in \mathbb{R}$, and let $M : H \to H$ be a linear operator in a finite-dimensional linear space having distinct eigenvalues. The *d-th power of M* is the operator having the same eigenlines as M, whose corresponding eigenvalues are some values of d-th powers of those of M.

Theorem 6.2.12 *Let* (6.2.1) *be a generic family of linear equations* (*see Definition* 6.2.1) *and* $\alpha_i(\varepsilon)$, $i = 0, 1$, *its singularity families. Let* $t_0 = \pm 1/2$, H_{t_0} *the corresponding local solution space* (*see Remark* 6.2.8). *Let* $M_i : H_{t_0} \to H_{t_0}$ *be the corresponding monodromy operators from Definition* 6.2.10. *Let* S_i, $i = 0, 1$, *be the corresponding associated sectors* (*see Definition* 6.2.4) *forming a covering of a punctured neighborhood of zero, and let* C_0, C_1 *be the Stokes operators* (6.1.6) *of the nonperturbed equation corresponding to the left* (*respectively, right*) *component of the intersection* $S_0 \cap S_1$ (*acting in the spaces* $H_{-1/2}$ *and* $H_{1/2}$ *respectively, see Remark* 6.2.9). *Then for any pair of numbers* $d_0, d_1 > 0$ *such that* $d_0 + d_1 < 1$

$$\begin{aligned} &M_1^{-d_1} M_0^{d_0} M_1^{d_1} M_0^{-d_0} \to C_0 \quad \textit{in the space} \quad H_{-1/2}, \\ &M_0^{-d_0} M_1^{d_1} M_0^{d_0} M_1^{-d_1} \to C_1 \quad \textit{in the space} \quad H_{1/2} \quad \textit{as} \quad \varepsilon \to 0. \end{aligned} \tag{6.2.4}$$

Theorem 6.2.12 is proved in the next section.

Remark 6.2.13 The statements of Theorem 6.2.12 imply the same statements in any space H_{t_0}, $\mathrm{Re}\, t_0 < 0$ (respectively, $\mathrm{Re}\, t_0 > 0$). Theorem 6.2.12 extends to the case of $k = 1$ and arbitrary dimension [42].

6.3 Convergence of the commutators to Stokes operators. Proof of Theorem 6.2.12

6.3.1 Projectivization. The plan of the proof of Theorem 6.2.12

Let us prove convergence of the first commutator in (6.2.4); the proof of the convergence of the other commutator is analogous.

Thus, from now on, we put $t_0 = -1/2$.

For the proof of Theorem 6.2.12 we consider the projectivization of the space $H_{t_0} = \mathbb{C}^2$. The projectivizations of the monodromy and Stokes operators are Möbius transformations $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (denote by $m_i : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the projectivizations of the monodromy operators M_i, and by σ the projectivization of the Stokes operator C_0).

Let $d_0, d_1 > 0$, $d_0 + d_1 < 1$. Denote

$$m_i' = m_i^{d_i}.$$

For the proof of (6.2.4) we show (below and in subsections 6.3.2, 6.3.3) that

$$(m_1')^{-1} m_0' m_1' (m_0')^{-1} \to \sigma \quad \text{as} \quad \varepsilon \to 0. \tag{6.3.1}$$

This means that the commutator (6.2.4) multiplied by an appropriate constant (depending on the parameter) converges to C_0. The commutator (6.2.4) has unit determinant, as does any commutator and the operator C_0 (which is unipotent, see Example 6.1.7). This together with (6.3.1) implies that its limit exists and is equal to either C_0 or $-C_0$. The fact that it is really equal to C_0 will be proved in subsection 6.3.4.

To sketch the proof of (6.3.1), let us first recall the following :

Definition 6.3.1 ([6]) A Möbius transformation is said to be *hyperbolic*, if it has two fixed points one of which is attracting (then the other is repelling). It is said to be *parabolic*, if it has only one fixed point. (Otherwise, it is said to be *elliptic*.)

In what follows, we represent hyperbolic and parabolic transformations by figures as follows. The Riemann sphere $\overline{\mathbb{C}}$ will be drawn in the form of a circle. A hyperbolic transformation with fixed points a and b, a being repelling, will be represented by marking a and b at the circle (representing $\overline{\mathbb{C}}$) and an oriented segment going from a to b (see Fig. 6.4(a)). A parabolic transformation with fixed point a, sending b to c, will be represented by marking the points a, b, c and the arrow from b to c on the circular arc joining them and disjoint from the fixed point a (see Fig. 6.4(b)).

Remark 6.3.2 The projectivization of a Stokes operator of an irregular equation is parabolic, since a Stokes operator is unipotent (see Example 6.1.7). The projectivization of a two-dimensional linear operator having eigenvalues with distinct modulus is hyperbolic : its repelling fixed point corresponds to the eigenfunction with the eigenvalue of the smallest modulus; its multiplier at the repelling fixed point is equal to the ratio of the eigenvalues. Each monodromy operator M_i from Theorem 6.2.12 has eigenvalues of distinct moduli (see Remark 6.2.3), so, *its projectivization m_i is hyperbolic.*

For the proof of (6.3.1) we state and prove its analogue (Lemma 6.3.11 below) for commutators of families of hyperbolic transformations generalizing $m_i' = m_i'(\varepsilon)$. To do this and to motivate the proof, let us first describe the arrangement of the fixed points of m_0, m_1 and σ.

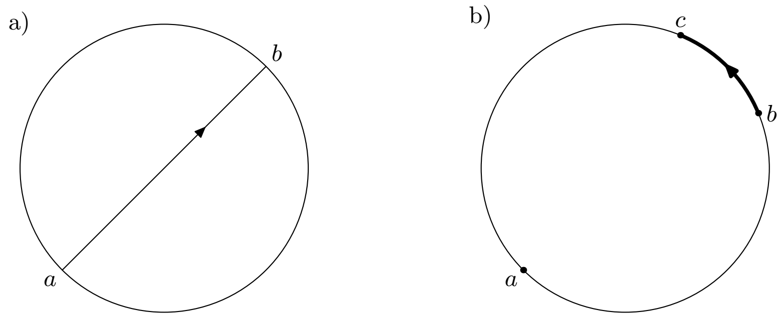

FIG. 6.4 – Hyperbolic and parabolic transformations

Proposition 6.3.3 *Let* (6.1.1) *be a two-dimensional irregular equation, λ_1, λ_2 be the eigenvalues of the corresponding matrix $A(0)$, and $\lambda_1 - \lambda_2 > 0$. Let S_0, S_1 be the sectors from Example* 6.1.7 *(see Fig.* 6.1*), C_0 be the Stokes operator* (6.1.6) *corresponding to the left component of their intersection, and let σ be the projectivization of C_0. Let f_{i1}, f_{i2} be the canonical solutions of* (6.1.1) *on the sectors S_i, $i = 0, 1$, p_{i1}, p_{i2} be their projectivizations. Then σ is a parabolic transformation with the fixed point p_{02},*

$$p_{02} = p_{12}, \quad \sigma(p_{02}) = p_{02}, \quad \sigma(p_{01}) = p_{11}, \quad (\textit{see Fig. } 6.5(\mathrm{b})).$$

Proposition 6.3.3 follows from the definition, the unipotence and the lower triangularity of the Stokes matrix C_0 (see Example 6.1.7).

Proposition 6.3.4 *Let* (6.2.1) *be a generic family of linear equations, $t_0 \in \mathbb{R}$, M_i be the monodromy operators of the perturbed equation from Definition* 6.2.10*, $f_{i1,\varepsilon}$, $f_{i2,\varepsilon}$ be their basic eigenfunctions, and λ_{i1}, λ_{i2} the corresponding eigenvalues. Then*

$$\mu_0 = \frac{\lambda_{01}}{\lambda_{02}} \to \infty, \quad \mu_1 = \frac{\lambda_{12}}{\lambda_{11}} \to \infty, \quad as \quad \varepsilon \to 0. \tag{6.3.2}$$

Corollary 6.3.5 *Under the conditions of Proposition* 6.3.4*, let $m_i : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the projectivizations of M_i, $p_{ij,\varepsilon}$ be those of $f_{ij,\varepsilon}$. Then m_i are hyperbolic transformations with fixed points $p_{i1,\varepsilon}$, $p_{i2,\varepsilon}$. More precisely, $p_{02,\varepsilon}$ is the repelling point of m_0, $p_{11,\varepsilon}$ is that of m_1 (see Fig* 6.5(a)*), the corresponding multipliers are equal to μ_0, μ_1, see* (6.3.2) *: they tend to infinity. Let S_i be the sectors associated to the singularities α_i of the perturbed equation (see Definition* 6.2.4*), p_{ij} be the projectivizations of the canonical sectorial solutions on S_i of the nonperturbed equation. Then*

$$p_{ij,\varepsilon} \to p_{ij} \quad as \quad \varepsilon \to 0 \quad (\textit{see Fig. } 6.5(\mathrm{b})). \tag{6.3.3}$$

Statement (6.3.3) follows from Theorem 6.2.5.

To motivate the proofs of the convergence of the commutators in (6.2.4) and (6.3.1), consider the simplest case, where in the family of equations (6.2.1) the matrix function family $A(t, \varepsilon)$ is lower-triangular. Then the line $z_1 = 0$ is invariant for each equation of the family. This implies that the monodromy operators M_0 and M_1 have a common eigenfunction (whose graph lies in the invariant line $z_1 = 0$) and their projectivizations m_i have the common fixed point $p_{02,\varepsilon} = p_{12,\varepsilon}$, repelling for m_0 and attracting for m_1 (see Fig. 6.6(a) below). In this case not only does the commutator in (6.3.1) converge : it does so with arbitrary powers $m_i^{d_i}$, $d_i > 0$, in particular, $m_1^{-1} m_0 m_1 m_0^{-1} \to \sigma$. This is implied by (6.3.2), (6.3.3) and a more general Proposition 6.3.6 stated below. To formulate it, let us introduce the following notation :

$$h_{a,b,\nu} : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \tag{6.3.4}$$

is the hyperbolic transformation of the Riemann sphere fixing points $a, b \in \overline{\mathbb{C}}$; a is repelling with the multiplier ν.

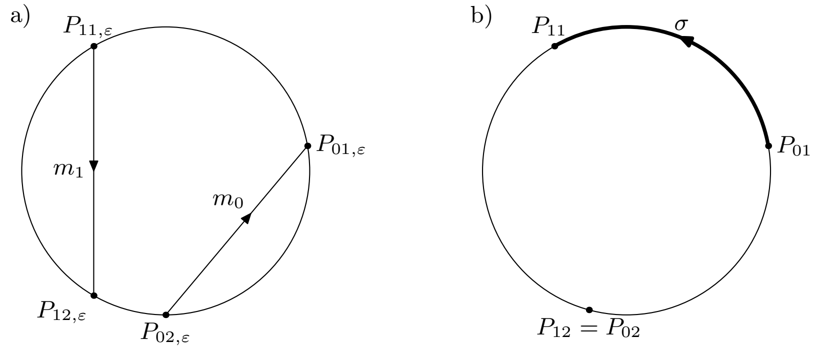

FIG. 6.5 – The projectivizations of the monodromy and Stokes operators

Proposition 6.3.6 *Let p, p_{01}, p_{11} be three distinct points of the Riemann sphere, and let $\sigma : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be the parabolic transformation fixing p and sending p_{01} to p_{11}. Consider three arbitrary families of points $a, b_0, b_1 \in \overline{\mathbb{C}}$ converging to p_{ij} (see Fig.* 6.6) *:*

$$a \to p, \qquad b_0 \to p_{01}, \qquad b_1 \to p_{11}.$$

Then in the notation (6.3.4)

$$h^{-1}_{b_1,a,\nu_1} h_{a,b_0,\nu_0} h_{b_1,a,\nu_1} h^{-1}_{a,b_0,\nu_0} \to \sigma \quad \textit{as} \quad (a, b_0, b_1) \to (p, p_{01}, p_{11}), \ \nu_0, \nu_1 \to \infty.$$

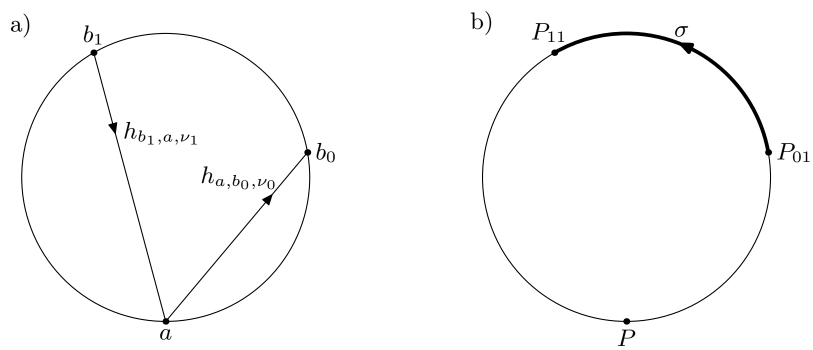

FIG. 6.6 – Degenerating hyperbolic transformations with a common fixed point

The proof of Proposition 6.3.6 is straightforward and can be done by hand (e.g., multiplying the (triangular) matrices of the h' s explicitly). It is omitted to save space.

In the previous case of the lower-triangular matrix $A(t, \varepsilon)$ the families m_i of hyperbolic transformations (and also $m_i^{d_i}$ with arbitrary $d_i > 0$) satisfy the conditions of Proposition 6.3.6 by (6.3.2), (6.3.3). This together with the proposition implies (6.3.1).

In the general case, the transformations m_i have distinct fixed points : $p_{02,\varepsilon} \neq p_{12,\varepsilon}$. On the other hand, the latter fixed points are confluent to the fixed point $p = p_{02}$ of σ. For the proof of (6.3.1) in the general case we show first that the distance $\mathrm{dist}(p_{02,\varepsilon}, p_{12,\varepsilon})$ is not too large : it decreases as $O(\mu_1^{-1})$ (Corollary 6.3.8). Then we state and prove a generalization (Lemma 6.3.11) of Proposition 6.3.6 for families of hyperbolic transformations h_{a_0,b_0,ν_0}, h_{b_1,a_1,ν_1} that have no common fixed point, but confluenting families of fixed points $a_0, a_1 \to p$ such that the distance $\mathrm{dist}(a_0, a_1)$ between them decreases fast enough, more precisely, as $o(|\nu_0\nu_1|^{-1})$. We apply Lemma 6.3.11 to the hyperbolic transformations $m_i' = m_i^{d_i}$ and $\nu_i = \mu_i^{d_i}$. To show the possibility of applying Lemma 6.3.11 to m_i', it suffices to prove that $\operatorname{dist}(p_{02,\varepsilon}, p_{12,\varepsilon}) = o(|\nu_0\nu_1|^{-1})$. This is the place where we use the inequalities on the exponents d_i from Theorem 6.2.12.

To estimate the distance $\operatorname{dist}(p_{02,\varepsilon}, p_{12,\varepsilon})$, we use the following

Lemma 6.3.7 *Let* (6.2.1) *be a generic family of linear equations* (*see Definition* 6.2.1), α_i *be its singularity families,* S_i *be the corresponding sectors* (*see Definition* 6.2.4) *chosen to cover a punctured neighborhood of zero,* S_i' *be the corresponding domains from* (6.2.2). *Let* C_0, C_1 *be the Stokes matrices* (6.1.6) *of the nonperturbed equation* (*corresponding to the left* (*respectively, right*) *component of the intersection* $S_0 \cap S_1$),

$$C_0 = \begin{pmatrix} 1 & 0 \\ c_0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \quad (\textit{see Example } 6.1.7). \tag{6.3.5}$$

Let M_i *be the monodromy operator of the perturbed equation around* $\alpha_i(\varepsilon)$ *acting in the space of solutions on* S_i'. *Let* Z_ε^i *be* (*the fundamental matrix of*) *its eigenbase. Let* $C_0(\varepsilon)$ *be the transition matrix* (6.2.3) *between the bases* Z_ε^i *that converges to* C_0, *as* $\varepsilon \to 0$, *see Corollary* 6.2.6 (*we consider the transition in the left component of the intersection* $S_0' \cap S_1'$) :

$$C_0(\varepsilon) = \begin{pmatrix} 1 + o(1) & u(\varepsilon) \\ c_0 + o(1) & 1 + o(1) \end{pmatrix}, \quad u(\varepsilon) \to 0.$$

Let $\lambda_{11}, \lambda_{12}$ *be the eigenvalues of* M_1 *at* $\alpha_1(\varepsilon)$, $\mu_1 = \lambda_{12}/\lambda_{11}$ *be the corresponding multiplier of its projectivization. Then the upper triangular element* $u(\varepsilon)$ *of the matrix* $C_0(\varepsilon)$ *has the asymptotics*

$$u(\varepsilon) = (-c_1 + o(1))\mu_1^{-1} \quad \textit{as} \quad \varepsilon \to 0, \tag{6.3.6}$$

where c_1 *is the upper triangular element of the Stokes matrix* C_1 *in* (6.3.5).

Lemma 6.3.7 is proved in subsection 6.3.2.

Corollary 6.3.8 *Let* (6.2.1) *be a generic family of linear equations,* $t_0 = -1/2$, M_i *be the monodromy operators from Definition* 6.2.10, m_i *be their projectivizations,* $p_{02,\varepsilon}$ *be the repelling fixed point of* m_0, $p_{12,\varepsilon}$ *be the attracting fixed point of* m_1, *and let* μ_1^{-1} *be the multiplier of the latter attracting fixed point. Then*

$$\operatorname{dist}(p_{02,\varepsilon}, p_{12,\varepsilon}) = O(\mu_1^{-1}) \quad \textit{as} \quad \varepsilon \to 0.$$

Remark 6.3.9 The multipliers of a hyperbolic transformation at its fixed points are inverse. In particular, in the preceding corollary, μ_1 is the multiplier of m_1 at its repelling fixed point $p_{11,\varepsilon}$.

Proposition 6.3.10 *Let* M_i *be the monodromy operators from Definition* 6.2.10, m_i *their projectivizations, and* μ_i *the multipliers at their repelling fixed points. Then*

$$|\mu_0| = |\mu_1|^{1+o(1)} \quad \textit{as} \quad \varepsilon \to 0.$$

Proof Recall that appropriate logarithms of the eigenvalues of the monodromy operators around singularities are equal to $2\pi i$ times the corresponding eigenvalues of the residue matrices (i.e., the characteristic numbers). The characteristic numbers at $\alpha_0(\varepsilon)$ are equal to $-(1 + o(1))$ times those at $\alpha_1(\varepsilon)$. This together with (6.3.2) implies that $\ln|\mu_0| = (1 + o(1))\ln|\mu_1|$, which proves Proposition 6.3.10. □

As is shown below, (6.3.1) is implied by Corollary 6.3.8, Proposition 6.3.10, the inequalities on d_i from Theorem 6.2.12, and the following lemma.

Lemma 6.3.11 *Let p, p_0, p_1 be three distinct points of the Riemann sphere, and $\sigma : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ the parabolic transformation fixing p and sending p_0 to p_1. Consider four arbitrary families of points $a_0, a_1, b_0, b_1 \in \overline{\mathbb{C}}$ converging to p, p_0 and p_1 (see Fig. 6.7) :*

$$a_0, a_1 \to p, \qquad b_0 \to p_0, \qquad b_1 \to p_1.$$

Then in the notations (6.3.4)

$$h^{-1}_{b_1,a_1,\nu_1} h_{a_0,b_0,\nu_0} h_{b_1,a_1,\nu_1} h^{-1}_{a_0,b_0,\nu_0} \to \sigma, \tag{6.3.7}$$

as $a_0, a_1 \to p$, $(b_0, b_1) \to (p_0, p_1)$, $\nu_0, \nu_1 \to \infty$ so that $\mathrm{dist}(a_0, a_1) = o(|\nu_0\nu_1|^{-1})$.

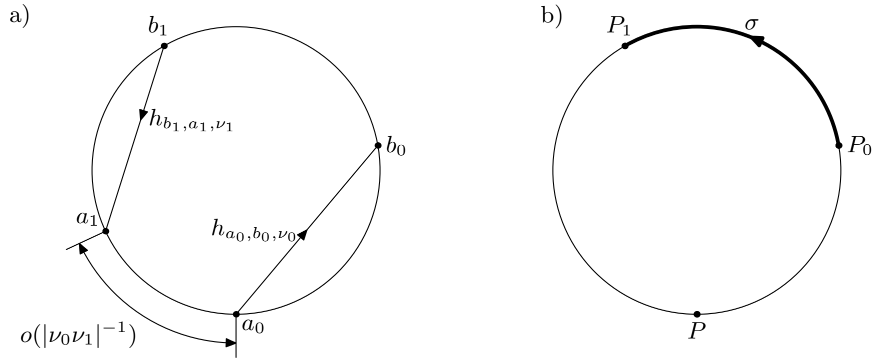

FIG. 6.7 – Degenerating hyperbolic transformations with a pair of rapidly confluenting fixed points

Lemma 6.3.11 is proved in subsection 6.3.3.

Proof of (6.3.1) Let us show that the families of hyperbolic transformations $m'_i = m_i^{d_i}$ satisfy the conditions of Lemma 6.3.11. Their fixed points converge to p_{ij} by (6.3.3). Their multipliers at the repelling fixed points are equal to $\nu_i = \mu_i^{d_i}$. Now it suffices to prove the last asymptotic formula in (6.3.7) saying in our case that $\mathrm{dist}(p_{02,\varepsilon}, p_{12,\varepsilon}) = o(|\mu_0^{d_0}\mu_1^{d_1}|^{-1})$. The latter formula follows from Corollary 6.3.8, Proposition 6.3.10, positivity of the powers d_i and the inequality $d_0 + d_1 < 1$ from the conditions of Theorem 6.2.12. This together with Lemma 6.3.11 proves (6.3.1). □

6.3.2 The upper triangular element of the transition matrix. Proof of Lemma 6.3.7

The transition matrix $C_0(\varepsilon)$, which converges to the Stokes matrix C_0, $Z^1_\varepsilon = Z^0_\varepsilon C_0(\varepsilon)$, compares the monodromy eigenbases Z^0_ε and Z^1_ε in the left component of the intersection $S'_0 \cap S'_1$, in particular, on a real interval in $\mathbb{R}_-$. It is not changed when we extend the basic functions analytically from $\mathbb{R}_-$ to $\mathbb{R}_+$ along the real line. Denote by $Z^i_{\varepsilon,+}$ the corresponding branch on $\mathbb{R}_+$ of the extended fundamental matrix Z^i_ε, $i = 0, 1$. By construction, $Z^0_{\varepsilon,+}$ is obtained from $Z^0_\varepsilon|_{\mathbb{R}_+}$ by applying the monodromy operator M_0 ; $Z^1_{\varepsilon,+}$ is obtained from $Z^1_\varepsilon|_{\mathbb{R}_+}$ by applying the inverse monodromy operator M_1^{-1} :

$$Z^1_{\varepsilon,+} = Z^1_\varepsilon|_{S'_1} M_1^{-1}; \quad \text{the matrix } M_1 \text{ is diagonal} \tag{6.3.8}$$

On the other hand, we can choose a renormalization of the eigenbase $Z^0_{\varepsilon,+}$ by multiplication of the basic functions by constants (i.e., changing it to $Z^0_{\varepsilon,+}\Lambda(\varepsilon)$, $\Lambda(\varepsilon) = \mathrm{diag}(l_1(\varepsilon), l_2(\varepsilon))$ so that in the right component of the intersection $S'_0 \cap S'_1$ the transition matrix $C_1(\varepsilon)$ between $Z^0_{\varepsilon,+}\Lambda(\varepsilon)$ and Z^1_ε tends to the Stokes matrix C_1 :

$$Z^0_{\varepsilon,+}\Lambda(\varepsilon) = Z^1_\varepsilon|_{S'_1} C_1(\varepsilon), \quad C_1(\varepsilon) \to C_1.$$

Substituting (6.3.8) and (6.2.3) in the latter formula yields

$$C_0(\varepsilon) = \Lambda(\varepsilon) C_1^{-1}(\varepsilon) M_1^{-1}. \tag{6.3.9}$$

The matrices $C_i(\varepsilon)$ tend to the Stokes matrices C_i, which are unipotent. The matrices $\Lambda(\varepsilon)$, M_1 are diagonal and depend on ε. This implies that

$$\Lambda(\varepsilon) = M_1(1 + o(1)) \quad \text{as} \quad \varepsilon \to 0.$$

This together with (6.3.9) implies (6.3.6).

6.3.3 Commutators of hyperbolic transformations with close fixed points. Proof of Lemma 6.3.11

Lemma 6.3.11 can be proved "by hand" by multiplying explicitly the matrices of the hyperbolic transformations in the commutator (6.3.7).

Denote the latter commutator by Δ. For the proof of Lemma 6.3.11 it suffices to show that

$$\begin{gathered} \Delta(a_0) \to p, \\ \Delta'(a_0) \to 1, \qquad \Delta(b_0) \to p_1 : \end{gathered} \tag{6.3.10}$$

these statements imply that Δ does not tend to infinity and each of its limit points is a Möbius transformation having fixed point p with unit multiplier and sending p_0 to p_1 (thus, coinciding with σ), hence $\Delta \to \sigma$.

Let us prove (6.3.10) (the proof of the other two statements is analogous). Recall the last asymptotic condition from Lemma 6.3.11 :

$$\operatorname{dist}(a_0, a_1) = o(|\nu_0\nu_1|^{-1}). \tag{6.3.11}$$

Consider the orbit of the point a_0 under consecutive hyperbolic transformations forming the commutator (6.3.7). Applying $h^{-1}_{a_0,b_0,\nu_0}$ does not move a_0. Applying h_{b_1,a_1,ν_1} moves a_0 to a point (denoted by a_0') close to a_1 ; more precisely,

$$\operatorname{dist}(a_0', a_1) = \nu_1^{-1} \operatorname{dist}(a_0, a_1)(1 + o(1)) = o(\nu_0^{-1}\nu_1^{-2}) \tag{6.3.12}$$

(by (6.3.11)). Put

$$a_0'' = h_{a_0,b_0,\nu_0} a_0', \qquad a_0''' = h^{-1}_{b_1,a_1,\nu_1} a_0''.$$

For the proof of (6.3.10) it suffices to show that

$$a_0''' \to p, \quad \text{or equivalently}, \quad \operatorname{dist}(a_0''', a_1) \to 0. \tag{6.3.13}$$

By (6.3.11), (6.3.12),

$$\operatorname{dist}(a_0', a_0) = o(|\nu_0\nu_1|^{-1}).$$

Applying h_{a_0,b_0,ν_0} to a_0' yields : $\operatorname{dist}(a_0'', a_0) = o(\nu_1^{-1}) \to 0$, hence by (6.3.11),

$$\operatorname{dist}(a_0'', a_1) = o(\nu_1^{-1}).$$

Applying $h^{-1}_{b_1,a_1,\nu_1}$ to a_0'' and using the previous formula yields $\operatorname{dist}(a_0''', a_1) \to 0$. This proves (6.3.13) and (6.3.10).

6.3.4 Convergence of projectivizations versus convergence of linear operators. The end of the proof of Theorem 6.2.12

We have already proved that the projectivization of the first commutator in (6.2.4) converges to that of the Stokes matrix C_0. Let us show that the commutator itself converges to C_0. This is implied by Lemma 6.3.11 and the following :

Proposition 6.3.12 *Under the conditions of Lemma* 6.3.11 *consider two-dimensional linear operators whose projectivizations are the hyperbolic transformations $h_{x,y,\nu}$ from the commutator* (6.3.7). *Then the corresponding commutator of linear operators converges to a unipotent operator.*

Proof The transformation σ is parabolic ; thus, it is the projectivization of a (unique) unipotent operator (denote that operator by C). The convergence of projectivizations means that the commutator of the linear operators under consideration multiplied by appropriate constant converges to C. The commutator has unit determinant, as a commutator, and so does C. Therefore, the commutator converges either to C, or to $-C$. Let us show that it converges to C.

Let a_0, a_1 be the confluenting fixed points of the hyperbolic transformations. In the case where $a_0 \equiv a_1$, this statement holds by definition : the operators in the commutator have a common eigenline, hence, the corresponding eigenvalue of the commutator is equal to 1, not -1, so, the limit is C.

In the general case we can consider without loss of generality that the families of points a_0, a_1 meet infinitely many times while confluenting. The commutators of linear operators corresponding to the meeting places tend to C by the previous statement. This proves the proposition. □

Thus, by Lemma 6.3.11 and the above proposition, the commutator (6.2.4) converges to a unipotent operator whose projectivization is the same as that of the Stokes operator C_0, which is also unipotent. Hence, the limit operator coincides with C_0. This finishes the proof of Theorem 6.2.12.

6.4 Nonlinear analogues and proof of Theorem 6.2.5

In the present section we state the nonlinear analogues of Theorem 6.2.5 and Corollary 6.2.6 for two-dimensional saddle-node holomorphic vector fields and their Martinet-Ramis moduli (subsection 6.4.2). We consider a two-dimensional holomorphic vector field with an elementary degenerate singular point (saddle-node). We study its generic deformation under which the degenerate singularity of the nonperturbed field splits into nondegenerate linearizable singularities of the perturbed field. The Martinet–Ramis invariant (of the orbital analytic classification) of the nonperturbed field is expressed in terms of the limit transition functions between the linearizing charts of the singularities of the perturbed field in [39]. Here we state this result only in the case of multiplicity two (see [39] for its statement for higher multiplicities). The linearizing charts determine the canonical first integrals of the perturbed field. Theorem 6.4.17 says that appropriate branches of the canonical first integrals of the perturbed field converge to appropriate sectorial canonical integrals of the nonperturbed field. This implies that the components of the Martinet-Ramis invariant are the limit transition functions between the canonical integrals of the perturbed field (Corollary 6.4.18).

The main result on saddle-nodes (Theorem 6.4.17) implies Corollary 6.4.22 saying that the "horizontal" separatrices of the perturbed field converge to the sectorial central manifolds (zeros of canonical integrals) of the nonperturbed field.

The main result on linear equations (Theorem 6.2.5) is related to its nonlinear analogue for saddle-nodes. Namely, the projectivization transforms the nonperturbed linear equation (6.1.1) to a holomorphic vector field on $\overline{\mathbb{C}} \times \{|t| < 1\}$ having two saddle-node singularities. A generic deformation of (6.1.1) is transformed to a generic deformation of the pair of saddle-nodes. It appears that Theorem 6.2.5 reformulated in terms of the projectivization follows from the previously mentioned Corollary 6.4.22 on the convergence of the horizontal separatrices of generically perturbed saddle-nodes.

The previously mentioned results concerning saddle-nodes are stated in Subsection 6.4.2. Theorem 6.2.5 and Corollary 6.4.22 are proved in Subsections 6.4.3 and 6.4.4, respectively.

The basic definitions (canonical first integrals and Martinet-Ramis moduli of saddle-nodes), which may be found in [67, 93], are recalled in Subsection 6.4.1.

6.4.1 Two-dimensional saddle-node singularities and their Martinet-Ramis invariants

Definition 6.4.1 We say that an isolated singular point of a holomorphic vector field is of *complex saddle-node* type, if the corresponding linearization operator has exactly one zero eigenvalue.

Definition 6.4.2 Two holomorphic vector fields are said to be *orbitally analytically equivalent*, if there exists a biholomorphic diffeomorphism of the corresponding phase spaces that maps the complex phase curves of the first vector field into the phase curves of the second one. Orbital analytic equivalence of germs of holomorphic vector fields is defined similarly. The formal orbital equivalence of germs is defined analogously with a formal diffeomorphism, i.e., a two-dimensional formal power series invertible under composition. More precisely, two germs are said to be *formally orbitally equivalent*, if there exists a formal diffeomorphism transforming the first germ to the second one multiplied by a formal nonzero function, i.e., a formal power series with nonzero free term.

Remark 6.4.3 Any germ of a holomorphic vector field in $(\mathbb{C}^2, 0)$ with a saddle-node singularity at the origin is orbitally analytically equivalent to the germ at the origin of a vector field of the form

$$\begin{cases} \dot{p} = p + O(|p|^2 + |t|^{k+1}), \\ \dot{t} = t^{k+1}. \end{cases} \tag{6.4.1}$$

Definition 6.4.4 Let S be a radial sector on a complex line with coordinate t. For any $r > 0$, we set $S^r = S \cap \{|t| < r\}$.

One can ask the following question : Is it possible to separate variables in the differential equation corresponding to the vector field (6.4.1) or, more precisely, is it true that the germ of (6.4.1) is locally orbitally analytically equivalent to the germ of a field corresponding to a differential equation with separated variables? Generally, this question has a negative answer. At the same time, the answer is positive for the formal equivalence. Namely, any saddle-node field (6.4.1) is formally orbitally equivalent to a unique vector field of the form

$$\begin{cases} \dot{\tilde{p}} = \tilde{p}(1 + \lambda t^k), \\ \dot{t} = t^{k+1}, \end{cases} \qquad \lambda \in \mathbb{C}. \tag{6.4.2}$$

The corresponding vector field (6.4.2) is called the *formal normal form* of (6.4.1) (see [67, 93]).

Generically, the normalizing power series is divergent. On the other hand, there are neighborhoods U_p and U_t of the origin on the axes p and t, respectively, and a covering of the punctured neighborhood U_t by $2k$ radial sectors S_j (i.e., sectors with vertex at the origin), $j = 0, \ldots, 2k-1$, possessing the following property : for appropriate $r > 0$ in each of the domains $\widetilde{S}_j = U_p \times S_j^r$, there is a holomorphic coordinate transformation

$$\widetilde{H}_j : (p, t) \mapsto (\tilde{p} = H_j(p, t), t) \tag{6.4.3}$$

that transforms (6.4.1) to its normal form (6.4.2) ; furthermore, at the origin $H_j(p, t)$ possesses an asymptotic power series in z and t coinciding with the normalizing series (see [67, 93]).

The "nontriviality" of the transition from one normalizing chart (6.4.3) to another (over the intersection of the sectors of the covering) gives rise to an obstruction for orbital analytic equivalence between the vector field (6.4.1) and its formal normal form (6.4.2), and is called the *nonlinear Stokes phenomenon*. This obstruction is the nontriviality of the Martinet-Ramis invariant. We now give its definition. To this end, consider the canonical first integral

$$I(\tilde{p}, t) = \tilde{p} t^{-\lambda} \exp\left\{\frac{1}{kt^k}\right\}$$

of the formal normal form (6.4.2). The integral I, together with the sectorial normalizing coordinate transformations $\widetilde{H}_j$, induces the first integrals

$$I_j = I \circ \widetilde{H}_j \tag{6.4.4}$$

of (6.4.1) over the sectors S_j (more precisely, in the domains $\widetilde{S}_j^r$). These integrals are called the *sectorial canonical integrals*. We set $S_{2k} = S_0$, $\widetilde{H}_{2k} = \widetilde{H}_0$. In the definitions of all the integrals $I_j = I \circ \widetilde{H}_j$, $j = 0, \dots, 2k$, we choose the branches of the (multivalued) function I so that for each $j \leq 2k-1$ its branch over S_{j+1} (corresponding to the index $j+1$) be the analytic extension of its branch over S_j when moving counterclockwise in the t-plane. We introduce $2k$ transition functions $\phi_j(\tau)$, $j = 0, \dots, 2k-1$, comparing the canonical integrals I_j and I_{j+1} over components of the intersections of the corresponding sectors S_j and S_{j+1} :

$$I_{j+1} = \phi_j \circ I_j. \tag{6.4.5}$$

Remark 6.4.5 The system of functions ϕ_j in (6.4.5) is determined uniquely up to conjugation by multiplication by a constant, i.e., up to transformations of the form

$$\phi_j(\tau) \mapsto c\phi_j(c^{-1}\tau), \quad \text{where} \quad c \in \mathbb{C} \setminus 0 \text{ does not depend on } j. \tag{6.4.6}$$

The vector field (6.4.1) is orbitally analytically equivalent to its formal normal form (6.4.2) if and only if $\phi_j(\tau) \equiv \tau$ for all j. More generally, two germs of vector fields of the form (6.4.1) are orbitally analytically equivalent if and only if they have the same formal normal form and the corresponding systems of functions ϕ_j from (6.4.5) are obtained one from the other by applying successively a transformation of the form (6.4.6) and a cyclic shift of order k of the $2k$ indices j (see [67, 93]).

Example 6.4.6 Consider the case of multiplicity two, i.e., when $k = 1$ in (6.4.1), (6.4.2). Then the previous covering consists of the same two good sectors S_0 and S_1, as in subsection 6.1.2, in the case of linear equations (see Example 6.1.7 and Fig. 6.1). The previous collection $\{\phi_j\}$ consists of two functions ϕ_0 and ϕ_1. The function $\phi_1(\tau)$ is holomorphic on $\mathbb{C}$ and has the form $\phi_1(\tau) = \tau + c_1$, $c_1 \in \mathbb{C}$. The function $\phi_0(\tau)$ is holomorphic in a neighborhood of the origin and has unit derivative at $0 : \phi_0(\tau) = \tau + o(\tau)$, as $\tau \to 0$.

Definition 6.4.7 The equivalence class of a collection of functions ϕ_j in (6.4.5) under transformations (6.4.6) (and cyclic shifts of order k of the indices, if $k > 1$) is called *the Martinet-Ramis orbital analytic classification invariant* of the vector field (6.4.1).

6.4.2 Confluence of singular points and Martinet-Ramis invariant

We state the result on expressing the Martinet-Ramis invariant via limit transitions between linearizing charts only in the case of multiplicity two, i.e., $k = 1$ (its statement in the general case may be found in [39]). To do this, we introduce some notations and recall the theorem on linearizability of a generic nondegenerate singular point of a two-dimensional holomorphic vector field.

Definition 6.4.8 A singular point of a holomorphic vector field is said to be *linearizable* if the corresponding germ of the field is orbitally analytically equivalent to its linear part.

Definition 6.4.9 The *characteristic number* of a two-dimensional holomorphic vector field at its singular point is the ratio of the eigenvalues of the corresponding linearization operator.

Theorem 6.4.10 ([8]) *A singular point of a two-dimensional holomorphic vector field with a finite nonreal characteristic number is linearizable.*

Definition 6.4.11 A singular point of a two-dimensional holomorphic vector field is said to be *typical* if in suitable coordinates the corresponding linear part has the form

$$\begin{cases} \dot{p} = \lambda p, \\ \dot{t} = \mu t, \end{cases} \qquad |\lambda| > |\mu|, \qquad \frac{\mu}{\lambda} \notin \mathbb{R}.$$

The *canonical integral* of this linear vector field is its first integral $pt^{-\lambda/\mu}$. The *canonical integral* of a two-dimensional vector field at its typical singular point is obtained from the canonical integral of the corresponding linear part by applying the linearizing coordinate transformation.

Remark 6.4.12 If a singular point of a two-dimensional holomorphic vector field is typical in the sense of the preceding definition, then it is linearizable (Theorem 6.4.10). The corresponding canonical integral is determined uniquely up to a constant factor.

We consider the following continuous one-parameter deformation (depending on the parameter $\varepsilon \geq 0$) of the saddle-node (6.4.1) (which corresponds to $\varepsilon = 0$) in the class of holomorphic vector fields :

$$\begin{cases} \dot{p} = p(1 + R(p,t,\varepsilon)) + g(t,\varepsilon)f(t,\varepsilon), \\ \dot{t} = f(t,\varepsilon), \end{cases} \qquad f(t,\varepsilon) = (t - \alpha_0(\varepsilon))(t - \alpha_1(\varepsilon)), \tag{6.4.7}$$

$$f(t,0) = t^2, \qquad R(0,0,0) = 0, \qquad \alpha_0 + \alpha_1 \equiv 0,$$

where g and R are continuous families of holomorphic functions. Assume that the degenerate singular point 0 of the nonperturbed field splits into two typical singularities $(0, \alpha_i(\varepsilon))$ of the perturbed field, $\alpha_i(\varepsilon) \neq \alpha_l(\varepsilon)$ for $i \neq l, \varepsilon \neq 0$. For a generic deformation (6.4.7) (see the next definition) we shall express the Martinet-Ramis invariant of the nonperturbed field in terms of the limit transition functions comparing the canonical integrals of the perturbed field.

Remark 6.4.13 When we restrict ourselves to deformations of the type (6.4.7) only, we do not loose generality (see [39]).

Definition 6.4.14 A vector field family (6.4.7) is said to be a *generic saddle-node family*, if the corresponding family of polynomials $f(t,\varepsilon)$ is generic (see Definition 6.2.1).

Remark 6.4.15 Suppose that (6.4.7) is a generic saddle-node family. Then the arguments of the characteristic numbers of the singular points of the perturbed vector field are uniformly bounded away from $\pi\mathbb{Z}$ for all sufficiently small values of the parameter. In particular, for small ε the singular points of the perturbed field are typical : one eigenvalue of the corresponding linearized operator tends to zero and the other eigenvalue tends to one. Thus the corresponding canonical integrals (see Definition 6.4.11) are well defined for small $\varepsilon \neq 0$. Conversely, if the characteristic numbers of the perturbed field in a continuous family of vector fields (6.4.7) satisfy the above estimate, then the families $f(t,\varepsilon)$ and (6.4.7) are generic.

Recall that the roots $\alpha_i(\varepsilon)$ of a generic family $f(t,\varepsilon)$ of polynomials have imaginary parts of constant sign. Without loss of generality we assume that $\operatorname{Im}\alpha_0 > 0$, then $\operatorname{Im}\alpha_1 < 0$.

A sector in the t-line associated to a root family $\alpha_i(\varepsilon)$ of $f(t,\varepsilon)$, $i = 0, 1$, is defined in the same way as in Definition 6.2.4.

For a typical family (6.4.7), we shall show that a branch of the appropriately normalized canonical integral of the perturbed field at the singular point $(0, \alpha_i(\varepsilon))$ converges to the sectorial integral of the nonperturbed field over the corresponding sector S_i.

Definition 6.4.16 Suppose that V is a domain on the Riemann sphere, V_ε is a one-parameter family (depending on the parameter $\varepsilon \geq 0$) of domains on the sphere. We say that the family V_ε *converges* to V as $\varepsilon \to 0$, if it converges to V in the Hausdorff sense, i.e., if the maximal distance from a point of the boundary ∂V_ε to the boundary ∂V tends to zero, and the same is true for the boundaries ∂V and ∂V_ε interchanged. By convergence of a family of functions holomorphic in V_ε depending on the same parameter ε we mean uniform convergence of these functions on compact subsets of V.

Theorem 6.4.17 *Suppose that* (6.4.7) *is a generic saddle-node family of vector fields* (*see Definition* 6.4.14), $\alpha = \alpha_i(\varepsilon)$ *is a continuous family of* t*-coordinates of their singularities,* $S = S_i$ *is a sector associated to it* (*see Definition* 6.2.4). *There exist an* $r > 0$, *a neighborhood* U_p *of the origin on the* p*-axis, and a family* Ω_ε *of simply connected domains on the* t*-axis that contain* $\alpha(\varepsilon)$ *and do not contain* $-\alpha(\varepsilon)$ (*this family depends on the same parameter* ε *and is defined for all small values* $\varepsilon \neq 0$) *such that the following statements hold :*

(1) *The connected component containing* $\alpha(\varepsilon)$ *of* $\Omega_\varepsilon \cap (S^r \setminus [0, -\alpha(\varepsilon)])$ *converges to* S^r *as* $\varepsilon \to 0$ (*see Definition* 6.4.16).

(2) *Let* $\Omega'_\varepsilon = \Omega_\varepsilon \setminus [\alpha_0(\varepsilon), \alpha_1(\varepsilon)]$. *The canonical integral* I_ε *of the perturbed field* (6.4.7) *at the singular point* $(0, \alpha(\varepsilon))$ (*see Definition* 6.4.11) *is a multivalued holomorphic function on* $\widetilde{\Omega_\varepsilon} = U_p \times \Omega_\varepsilon$ *branched along the line* $t = \alpha(\varepsilon)$. *This function has a single-valued branch on* $\widetilde{\Omega_\varepsilon}' = U_p \times \Omega'_\varepsilon$. *This branch, when appropriately normalized* (*see Remark* 6.4.12), *converges to the sectorial canonical integral* (6.4.4) *of the nonperturbed field on* $\widetilde{S}^r = U_p \times S^r$.

This theorem is proved in [39] for saddle-nodes of arbitrary multiplicity (but for a less general class of deformations (6.4.7) in the case of multiplicity two). In fact, its version from [39] in the latter case is equivalent to Theorem 6.4.17.

Corollary 6.4.18 *Let* (6.4.7) *be a generic saddle-node family of vector fields,* $(0, \alpha_i(\varepsilon))$ *its singularities,* $i = 0, 1$, *and* S_i *the corresponding sectors* (*see Definition* 6.2.4). *Accordingly, suppose that* $r > 0$, U_p, *and* $\Omega_\varepsilon(i)$ *are the constant, the neighborhood, and the domains* Ω_ε *corresponding to* $\alpha = \alpha_i$ *from the preceding theorem,* $\Omega'_\varepsilon(i) = \Omega_\varepsilon(i) \setminus [\alpha_0(\varepsilon), \alpha_1(\varepsilon)]$, $I_{i,\varepsilon}(t)$ *is the canonical integral of the perturbed field at the singular point* $(0, \alpha_i(\varepsilon))$ (*see Definition* 6.4.11). *More precisely, we take its single-valued branch in the domain* $\widetilde{\Omega_\varepsilon}'(i) = U_p \times \Omega'_\varepsilon(i)$ *; we set* $I_{2,\varepsilon} = I_{0,\varepsilon}$. *Let* C^j *be the connected component of* $S_0^r \cap S_1^r$, $j = 0, 1$ (*we assume that* $C^0 \supset \mathbb{R}_-$, $C^1 \supset \mathbb{R}_+$), *and let* ϕ_j *be the corresponding component* (6.4.5) *of the Martinet-Ramis invariant of the nonperturbed field. There exists a family* $C(\varepsilon)$ *of connected components of* $S_0^r \cap S_1^r \cap \Omega'_\varepsilon(0) \cap \Omega'_\varepsilon(1)$ *that converges to* C^j *as* $\varepsilon \to 0$ *and possesses the following property : the transition function* Φ_ε *between appropriately normalized integrals* $I_{l,\varepsilon}$ *in* $\widetilde{C}(\varepsilon) = U_p \times C(\varepsilon)$, $I_{j+1,\varepsilon} = \Phi_\varepsilon \circ I_{j,\varepsilon}$, *is holomorphic in a domain* (*depending on* ε) *that converges to the domain of* ϕ_j, *and* $\Phi_\varepsilon \to \phi_j$ *as* $\varepsilon \to 0$.

Corollary 6.4.18 and its extension to higher multiplicities are contained in [39].

Now we formulate another corollary of Theorem 6.4.17, on convergence of appropriate separatrices of the perturbed field to the sectorial central manifolds of the saddle-node. We use this corollary further in the proof of Theorem 6.2.5.

Definition 6.4.19 The *sectorial separatrix* of a saddle-node (6.4.1) over a good sector S is the zero curve of the corresponding canonical sectorial integral (or equivalently, the image of the central manifold of the formal normal form under the inverse of the normalizing change of variables). The *horizontal separatrix* of a typical singular point of a two-dimensional holomorphic vector field is the zero curve of the corresponding canonical integral (see Definition 6.4.11).

Remark 6.4.20 Let (6.4.1) be a saddle-node vector field, S a good sector (see Definition 6.1.2), and Γ the corresponding sectorial separatrix (see the preceding definition). There exists an $r > 0$ such that Γ contains the graph

$$p = q(t) \tag{6.4.8}$$

of a function $q(t)$ with the following properties :

(i) q is holomorphic in S^r and continuous in its closure ;

(ii) this is the unique function satisfying (i) whose graph is tangent to the field.

Remark 6.4.21 Consider a two-dimensional holomorphic vector field in coordinates (p, t) with a typical singularity. Let the eigenline of its linearization operator with the largest eigenvalue be parallel to the p-axis. Then the corresponding horizontal separatrix (see Definition 6.4.19) contains the graph

$$p = q(t) \tag{6.4.9}$$

of a holomorphic function, the graph contains the singularity. This is the unique graph of a holomorphic function tangent to the field and passing through the singularity.

Corollary 6.4.22 *Let* (6.4.7) *be a generic saddle-node family. Then the horizontal separatrices at the singularities of the perturbed field converge to the sectorial separatrices of the saddle-node over the corresponding sectors* (*see the previous Definition*). *More precisely, let* $b(\varepsilon) = (0, \alpha(\varepsilon))$ *be a singularity family,* S *be the sector associated to* α (*see Definition* 6.2.4). *Let* $q(t)$ *be the function whose graph* (6.4.8) *is contained in the sectorial separatrix over* S, *and* $q_\varepsilon(t)$ *the function with graph* (6.4.9) *contained in the horizontal separatrix of the perturbed field at* $b(\varepsilon)$. *There exist an* $r > 0$ *and a family* Ω_ε *of domains in the* t*-line,* $\alpha(\varepsilon) \in \Omega_\varepsilon$, $-\alpha(\varepsilon) \notin \Omega_\varepsilon$, *satisfying the following statements :*

(1) *the connected component containing* $\alpha(\varepsilon)$ *of the intersection* $(S^r \setminus [0, -\alpha(\varepsilon)]) \cap \Omega_\varepsilon$ *converges to* S^r, *as* $\varepsilon \to 0$ (*see Definition* 6.4.16) ;

(2) *the function* $q(t)$ *is holomorphic in* S^r, q_ε *is holomorphic in* Ω_ε, *and* $q_\varepsilon \to q$.

The generalization of the corollary to arbitrary dimension and multiplicity is stated and proved in [39].

6.4.3 Projectivization. Proof of Theorem 6.2.5

For the proof of Theorem 6.2.5 we projectivize all the linear equations involved. The projectivization of a linear equation is a tangent line field on the product $\mathbb{P}^1 \times \{|t| < 1\}$ that is the pushforward of the linear equation under the tautological projection $\mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$ (or a holomorphic vector field on the latter product contained in the tangent line field).

The projectivization of a two-dimensional irregular equation (6.1.1) is a holomorphic vector field on $\mathbb{P}^1 \times \{|t| < 1\}$ having a pair of singularities on the fiber $\mathbb{P}^1 \times 0$ (which correspond to the eigenlines of the matrix $A(0)$, the coordinate lines in our case). These singularities are saddle-nodes of the same order k as the Poincaré rank of the equation under consideration ($k = 1$). The projectivization transforms the graphs of the canonical sectorial solutions of (6.1.1) to the sectorial separatrices of the corresponding saddle-node singularities of the projectivization (and hence, the solutions themselves to the corresponding functions (6.4.8)). Indeed, the images of the canonical solutions under the tautological projection are functions holomorphic in the corresponding sectors and continuous in their closures (by construction), and their graphs are tangent to the projectivization. By uniqueness (see Remark 6.4.20), they coincide with those defining the corresponding sectorial separatrices.

The projectivization of a perturbed equation from a generic family (6.2.1) is a holomorphic vector field on the same space $\mathbb{P}^1 \times \{|t| < 1\}$ with four typical singularities : a pair of singularities in each fiber $\mathbb{P}^1 \times \alpha_i(\varepsilon)$, $i = 0, 1$. Analogously, the projectivization transforms the graphs of the monodromy eigenfunctions at the singularities of the perturbed linear equation to the horizontal separatrices of the projectivization.

The projectivization of a generic family of linear equations becomes a generic saddle-node family (locally near each saddle-node singularity of the projectivization of the nonperturbed equation) after applying an appropriate family of changes of the space variable. Now the preceding corollary applied to the family of projectivizations says that the horizontal separatrices converge to the sectorial separatrices of the projectivized nonperturbed equation. This means that the branches in S_i' of the monodromy eigenfunctions (taken up to multiplication by constants) converge to the canonical basic solutions of the nonperturbed equation (also taken up to multiplication by constants). Therefore, appropriately normalized monodromy eigenfunctions converge to appropriately normalized canonical basic solutions. This proves Theorem 6.2.5 modulo Corollary 6.4.22.

6.4.4 Convergence of the horizontal separatrices. A brief proof of Corollary 6.4.22

We give a brief proof of Corollary 6.4.22 independent on Theorem 6.4.17 (the complete text of the proof may be found in [39]).

Let us prove the statements of Corollary 6.4.22, say, for

$$\alpha = \alpha_0, \ S = S_0 : \quad \text{let us show that} \quad q_\varepsilon \to q.$$

To do this, we show that the functions q_ε are holomorphic in domains Ω_ε large enough (satisfying statement (1) of Corollary 6.4.22) and form a normal family (i.e., precompact in the topology of uniform convergence on compact sets in S^r) : more precisely,

$$|q_\varepsilon'(t)| < 1, \ |q_\varepsilon(t)| \leq |t - \alpha_0(\varepsilon)| \quad \text{for any} \quad t \in \Omega_\varepsilon. \tag{6.4.10}$$

(Recall that by definition, $q_\varepsilon(\alpha_0(\varepsilon)) = 0$.) Then the limit of any convergent sequence q_{ε_n}, $\varepsilon_n \to 0$, is a function holomorphic in the sector S^r and continuous in its closure that vanishes at 0. Its graph is tangent to the saddle-node field. Therefore, by the uniqueness statement of Remark 6.4.20, the limit coincides with q. This together with normality proves the convergence $q_\varepsilon \to q$.

For the proof of the bounds (6.4.10) we consider the following family K of tangent cones at the points of the phase plane and the corresponding cone K' :

$$K = \{|\dot p| < |\dot t|\}, \qquad K' = \{|p| < |t - \alpha_0(\varepsilon)|\}.$$

The inequalities (6.4.10) are equivalent to the inclusions

$$T\Gamma q_\varepsilon \subset K, \qquad \Gamma q_\varepsilon \subset K', \tag{6.4.11}$$

where Γq_ε is the graph of the function q_ε.

For the proof of the inclusions (6.4.11), we consider an appropriate constant multiple

$$v_\theta(\varepsilon) = e^{i\theta}(6.4.7), \quad \theta \in \mathbb{R} \text{ is independent on } \varepsilon,$$

of the vector field family (6.4.7). We choose the number θ so that the singular point $b_0(\varepsilon) = (0, \alpha_0(\varepsilon))$ of the perturbed field $v_\theta(\varepsilon)$ from the new family is hyperbolic with the stable manifold $W^s = \{t = \alpha_0(\varepsilon)\}$ and the unstable manifold being locally the horizontal separatrix $W^u = \Gamma q_\varepsilon$. More precisely,

(1) the eigenvalue of the linearization operator of $v_\theta(\varepsilon)$ at $b_0(\varepsilon)$ at the eigenline tangent to the line $\{t = \alpha_0(\varepsilon)\}$ has a negative real part ;

(2) the other eigenvalue has a positive real part ;

(3) the previous conditions hold "uniformly" : the real part of the former eigenvalue is bounded away from zero ; the argument of the latter eigenvalue is bounded away from $\pi/2 + \pi\mathbb{Z}$.

The above conditions will be satisfied if, e.g., $\theta < -\pi/2$ and θ is close enough to $-\pi/2$.

In the proof of (6.4.11) we use the fact that for any θ satisfying (1)–(3)) there exists a bidisc U in the phase space (independent of ε) such that for any ε small enough the tangent cone field K is invariant under the real flow of the perturbed field $v_\theta(\varepsilon)$: each cone of K is mapped under a positive time flow map strictly inside the cone of K at the image of the point under consideration. This implies that the cone K' is also $v_\theta(\varepsilon)$-invariant.

The inclusions (6.4.11) hold a priori at the singular point $b_0(\varepsilon)$, and hence in its neighborhood (whose size depends on the parameter). By invariance of K, they remain valid in all the trajectories of the field $v_\theta(\varepsilon)$ in the unstable manifold Γq_ε that go out from the singular point. These trajectories saturate a domain in Γq_ε bijectively projected onto some domain in the t-line (denoted by Ω_ε). If the bidisc U is chosen in an appropriate way (say, centered at 0 and so that its height in the coordinate p is at least two times greater than its width in the coordinate t ; denote by V its projection to the t-line), then the previous domain Ω_ε is saturated by the real trajectories of the quadratic vector field

$$\dot{t} = e^{i\theta}(t - \alpha_0(\varepsilon))(t - \alpha_1(\varepsilon))$$

in the disc V that go out from its repelling singular point $\alpha_0(\varepsilon)$ (see Fig. 6.8(a)). The family of the domains Ω_ε thus constructed satisfies statement (1) of Corollary 6.4.22, at least for some sector S associated to α_0. (In fact one can achieve this for an arbitrary given sector S associated to α_0 by appropriate choice of θ.) This proves Corollary 6.4.22.

In fact, the domain Ω_ε converges to a domain (denoted Ω, see Fig. 6.8(b)) bounded by a cardioid-like curve having a "cusp" at 0 with tangency to the ray $\arg t = \pi - \theta$. Recall that the closure of the sector S_0 is disjoint from $i\mathbb{R}_-$. One can achieve that the latter cusp ray be arbitrarily close to $i\mathbb{R}_-$ (so that the limit domain Ω contains the sector S_0^r for appropriate $r > 0$) by choosing a $\theta < -\pi/2$ close enough to $-\pi/2$.

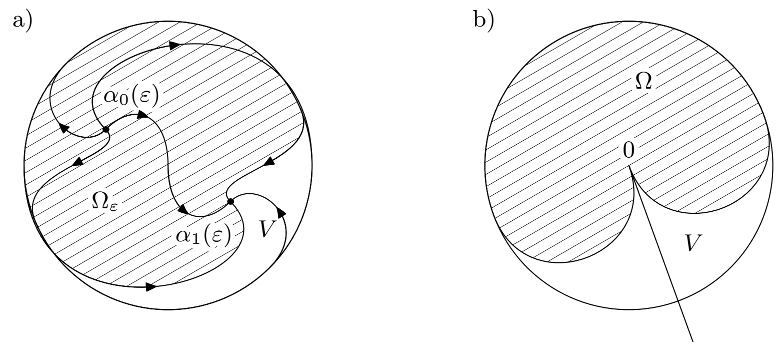

FIG. 6.8 – The domains Ω_ε and Ω, where the separatrices have bounded slopes

Bibliographie

[1] Abels, H. ; Margulis, G.A. ; Soifer, G.A. Semigroups containing proximal linear maps. - Israel J. Math., 91 (1995), 1-30.

[2] Abikoff, W. Real analytic theory of Teichmüller space. - Lect. Notes in Math., 820, Springer-Verlag (1980).

[3] Agol, I. Tameness of hyperbolic $3-$manifolds. - Preprint http ://arxiv.org/abs/math/0405568.

[4] Ahlfors, L. Lectures on quasiconformal mappings. - Wadsworth (1987).

[5] Ahlfors, L. ; Bers, L. Riemann's mapping theorem for variable metrics, Ann. of Math. (2) 72 (1960), 385–404.

[6] Apanasov, B.N. ; Gusevskii N.A. ; Krushkal, S.L., Kleinian groups and uniformization in examples and problems (in Russian). - Novosibirsk : Nauka, 1981.

[7] Arnold, V.I. Dopolnitelnye glavy teorii obyknovennykh differentsialnykh uravnenii. (Russian) [Supplementary chapters to the theory of ordinary differential equations] - "Nauka", Moscow, 1978.

[8] Arnold, V.I. ; Ilyashenko, Yu.S. Ordinary differential equations, in : *Dynamical Systems–1*, Itogi Nauki i Tekhniki, VINITI, Moscow, 1985 ; English transl. in : *Dynamical Systems I*, Encyclopaedia Math. Sci., Springer, Berlin, 1988, 1–148.

[9] Arnold V.I., Varchenko A.N., Gussein-Zade S.M. Singularities of differentiable mappings. - Nauka publ., Moscow, 1982.

[10] Balser, W. ; Jurkat, W.B. ; Lutz, D.A. Birkhoff invariants and Stokes' multipliers for meromorphic linear differential equations. - J. Math. Anal. Appl. 71 (1979), No.1, 48-94.

[11] Bănică, C. ; Forster, O. Complete intersections in Stein Manifolds. - Manuscripta Math. 37 (1982), no. 3, 343-356.

[12] Berndtsson, Bo ; Ransford, T.J. Analytic multifunctions, the $\overline{\partial}$- equation, and a proof of the corona theorem.- Pacific J.Math. 124 (1986) no 1, 57-72.

[13] Bers, L. Simultaneous uniformization. - Bull. Amer. Math. Society. 66 (1960) 94-97.

[14] Breuillard, E. ; Gelander, T. On dense free subgroups of Lie groups. - J.Algebra, 261 (2003), No. 2, 448-467.

[15] Brown, G. On commutators in a simple Lie algebra. - Proc. Amer. Math. Soc. 14 1963 763–767.

[16] Buff, X. ; Cheritat, A. Ensembles de Julia quadratiques de mesure de Lebesgue strictement positive. - Comptes Rendus Mathématique 2005, Volume 341, no 11, 669-674.

[17] Calegari, D. ; Gabai, D. Shrinkwrapping and the taming of hyperbolic 3-manifolds. - J. Amer. Math. Soc. 19 (2006), no. 2, 385–446.

[18] Candel, A., Uniformization of surface laminations. - Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 4, 489–516.

[19] Candel, A. ; Gómez-Mont, X. Uniformization of the leaves of a rational vector field. - Ann. Inst. Fourier (Grenoble) 45 (1995), no. 4, 1123–1133.

[20] Chern, S.-S., An elementary proof of the existence of isothermal parameters on a surface. - Proc. Amer. Math. Soc. 6 (1955), 771–782.

[21] Choquet-Bruhat, Y. ; de Witt-Morette, C. ; Dillard-Bleick, M. Analysis, Manifolds and Physics. - North-Holland, 1977.

[22] Dawson, C.M. ; Nielsen, M.A. The Solovay-Kitaev algorithm. - Quantum Inf. Comput. 6 (2006), no. 1, 81–95.

[23] Douady, A. ; Buff, X. Le théorème d'intégrabilité des structures prèsque complexes. (French) [Integrability theorem for almost complex structures] - The Mandelbrot set, theme and variations, 307–324, London Math. Soc. Lecture Note Ser., 274, Cambridge Univ. Press, Cambridge, 2000.

[24] Douady, A., Estrada, F., Sentenac, P. Champs de vecteurs polynômiaux sur $\mathbb{C}$. - To appear in the Proceedings of Bodilfest.

[25] Duval, A. Biconfluence et groupe de Galois. - J. Fac. Sci. Univ. Tokyo Sect. IA Math. 38 (2) (1991), 211–223.

[26] J. Écalle, Invariants holomorphes simples des transformations de multiplicateur 1. - C. R. Acad. Sci. Paris Sér. A–B, 276 (1973), A375–A378.

[27] Écalle, J. Les fonctions résurgentes. Tome III. L'Équation du pont et la classification analytique des objects locaux. - Publications Mathématiques d'Orsay, 85-5. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. 587 pp

[28] Écalle, J. Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac. (French) [Introduction to analyzable functions and constructive proof of the Dulac conjecture] Actualités Mathématiques. [Current Mathematical Topics] - Hermann, Paris, 1992. ii+340 pp.

[29] Epstein, D. B. A. Almost all subgroups of a Lie group are free. - J. Algebra 19, 1971, 261–262.

[30] Gamburd, A. ; Jakobson, D. ; Sarnak, P. Spectra of elements in the group ring of $SU(2)$. - J. Eur. Math. Soc. (JEMS) 1 (1999), no. 1, 51–85.

[31] Garnier, R. Sur les singularités irregulières des équations différentielles linéaires. - J. Math. Pures Appl. 8, no 2 (1919), 99–198.

[32] Gelander, T. On deformations of free subgroups in compact Lie groups. - To appear in Israel J. Math.

[33] Ghys, É. ; Carrière, Y. Relations d'équivalence moyennables sur les groupes de Lie. - C. R. Acad. Sci. Paris Sér. I Math, vol. 300 (1985), No.19, 677-680.

[34] Ghys, É. Sur l'uniformisation des laminations paraboliques. - In Integrable systems and foliations, ed. C.Albert, R.Brouzet, J.-P. Dufour (Montpellier, 1995), Progress in Math. 145 (1996), 73-91.

[35] Glutsyuk, A., Uniformization of leaves of one-dimensional holomorphic foliations. - Thèse Ph.D., Département de Mathématiques, Université d'État de Moscou, 1996 (en russe).

[36] Glutsyuk, A., The hyperbolicity of phase curves of a generic polynomial vector field in $\mathbb{C}^n$. - Functsionalnyi Analiz i iego Prilozheniia, 2(1994), 1-11 (version anglaise en Functional analysis and its Applications, 2(1994), 77-84).

[37] Glutsyuk, A., Hyperbolicity of leaves of a generic one-dimensional holomorphic foliation on a nonsingular projective algebraic manifold. - Trudy Matematicheskogo Instituta im. V.A.Steklova, v.213 (1996), 90-111 (version anglaise en Proceedings of Steklov Mathematical Institute, v.213 (1996), 83-103).

[38] Glutsyuk, A., Stokes operators via limit monodromy of a generic deformation. - Journal of Dynamical and Control Systems, v.5 (1999) no 1, 101-135.

[39] Glutsyuk, A., Confluence of singular points and the nonlinear Stokes Phenomena. - Trudy Moskovskogo Matematicheskogo Obshchestva, v.62 (2000), p.54-104 (en russe, la version anglaise de ce journal est "Proceedings of Moscow Mathematical Society").

[40] Glutsyuk, A., Resonant confluence of singular points and Stokes phenomena. - Journal of Dynamical and Control Systems, vol. 10 (2004), No. 2 (April), pp. 253–302.

[41] Glutsyuk, A., Confluence of singular points and Stokes phenomena. - Proceedings of NATO Advanced Study Institute "Normal Forms, Bifurcations and Finiteness Problems in Differential Equations", Montreal, July 6-19 2002 (C. Rousseau and Yu. Ilyashenko, eds.), NATO Science Series II Math. Phys. Chem., (2004), vol.137, pp. 267-294. Kluwer Academic Publishers, Dordrecht.

[42] Glutsyuk, A., On the monodromy group of confluenting linear equations. - Moscow Math. J., 5 (2005), no. 1, 67-90.

[43] Glutsyuk, A., Nonuniformizable skew cylinders : a counterexample to the simultaneous uniformization problem. - C.R.Acad.Sci.Paris, Série 1 Math., t.332 (2001), p.209-214.

[44] Glutsyuk, A., On simultaneous uniformization and local nonuniformizability. - C.R.Acad.Sci.Paris, Série 1 Math., t.334 (2002), p.489-494.

[45] Glutsyuk, A., Simultaneous metric uniformization of foliations by Riemann surfaces. - Commentarii Mathematici Helvetici, vol. 79, Issue 4 (2004), pp.704-752.

[46] Glutsyuk, A., Upper bounds of topology of complex polynomials in two variables. - Mosc. Math. J. 5 (2005), no. 4, 781–828.

[47] Glutsyuk, A., An explicit formula for period determinant. - Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 887–917.

[48] Glutsyuk, A., A survey on minimality of horospheric laminations associated to rational functions. - Dans Fields Institute Communications 2007, Vol : 51, pp. 269-287. Proceedings of the Partially hyperbolic dynamics, laminations, and Teichmuller flow Workshop, January 5-9, 2006.

[49] Glutsyuk, A., On density of horospheres in dynamical laminations. -
http ://xxx.lanl.gov/abs/math.DS/0605644

[50] Glutsyuk, A., Instability of free nondiscrete subgroups in Lie groups. - Soumis au journal "Transformation Groups". Disponible sur l'archive :
http ://arxiv.org/abs/math/0409556

[51] Glutsyuk, A., Simple proofs of uniformization theorems. - Soumis aux Fields Institute Communications. Disponible sur l'archive :
http ://xxx.lanl.gov/abs/math/0510071

[52] Glutsyuk, A. ; Ilyashenko, Yu., Restricted infinitesimal Hilbert sixteenths problem. - Doklady Academii Nauk, 2006, v. 407, no 2, 154-159 (in Russian). English translation in Doklady Mathematics, 2006, vol. 73, no 2, 185-189.

[53] Glutsyuk, A. ; Ilyashenko, Yu., Restricted version of the insinitesimal Hilbert 16-th problem. - Moscow Math. J. 7 (2007), no. 2, 281-325.

[54] Goto, M. A theorem on compact semi-simple groups. - J. Math. Soc. Japan 1, (1949). 270–272.

[55] Grigoriev, A. Uniform asymptotic bound on the number of zeros of Abelian integrals. - Preprint arXiv :math.DS/0305248 v1, 17 May 2003.

[56] Grigoriev, A. - Ph. D. thesis, the Weizmann Institute of Sciences, December 2001.

[57] Gromov, M. Pseudoholomorphic curves in symplectic manifolds. - Invent. Math. 82 (1985), no. 2, 307–347.

[58] Haefliger, A., Some remarks on foliations with minimal leaves. - J. Differential Geometry, 15 (1980), no 2, 269284.

[59] Hirschbühl, R. Commutators in classical Lie algebras. - Linear Algebra Appl. 142 (1990), 91–111.

[60] Ilyashenko, Yu.S. The density of an individual solution and the ergodicity of the family of solutions of the equation $d\eta/d\xi = P(\xi,\eta)/Q(\xi,\eta)$. (Russian) - Mat. Zametki 4 1968 741–750.

[61] Ilyashenko, Yu.S. Generation of limit cycles under the perturbation of the equation $dw/dz = -R_z/R_w$, where $R(z,w)$ is a polynomial. - Math. Sbornik, vol. 78 (1969) no. 3 360–373.

[62] Ilyashenko. An example of equations $dw/dz = P_n(z, w)/Q_n(z, w)$ having a countable number of limit cycles and arbitrarily high Petrovskiĭ-Landis genus. - Mat. Sb. (N.S.) 80 (122) 1969 388–404.

[63] Ilyashenko, Yu.S. Foliations by analytic curves. (Russian) - Mat. Sb. (N.S.) 88(130) (1972), 558–577.

[64] Ilyashenko, Yu.S., Nondegenerate B-groups. (Russian.) - Dokl. Akad. Nauk SSSR 208 (1973), 1020–1022.

[65] Ilyashenko, Yu.S. Topology of phase portraits of analytic differential equations on a complex projective plane. (Russian.) - Trudy Sem. Petrovsk. No. 4 (1978), 83–136.

[66] Ilyashenko, Yu.S. Finiteness theorems for limit cycles. - Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, 94. American Mathematical Society, Providence, RI, 1991.

[67] Yu. S. Ilyashenko, ed., Nonlinear Stokes Phenomena. - Adv. Soviet Math. **14**, Amer. Math. Soc., Providence, RI, 1993.

[68] Ilyashenko, Yu.S. Covering manifolds for analytic families of leaves of foliations by analytic curves. - Topol. Methods Nonlinear Anal. 11 (1998), no. 2, 361–373.

[69] Ilyashenko, Yu.S. Centennial history of Hilbert's 16th problem. - Bull AMS, v 39 (2002), no 3, 301–354

[70] Ilyashenko, Yu.S., Variation of argument and Bernstein index for holomorphic functions on Riemann surfaces. - Math. Res. Lett. 14 (2007), no. 3, 433–442.

[71] Ilyashenko, Yu.S. ; Khovanskii, A.G., Galois groups, Stokes operators and a theorem of Ramis. - Funct. Anal. Appl., 24, no 4 (1991), 286–296.

[72] Ilyashenko, Yu.S. ; Pyartli, A.S. The monodromy group at infinity of a generic polynomial vector field on the complex projective plane. - Russian J. Math. Phys. 2 (1994), no. 3, 275–315.

[73] Ilyashenko, Yu.S. ; Shcherbakov, A.A. Skew Cylinders and Simultaneous Uniformization. - Proc. of Steklov Math. Inst., vol.213 (1996), 112-123

[74] Ilyashenko Yu.S. ; Yakovenko S.Yu. Counting real zeros of analytic functions satisfying linear ordinary differential equations. - Journal of Differential equations 126 (1996), No. 1, 87-105.

[75] Jurkat, W.B. ; Lutz, D.A. ; Peyerimhoff, A. Birkhoff invariants and effective calculations for meromorphic linear differential equations. - J. Math. Anal. Appl. 53 (1976), no.2, 438–470.

[76] Jurkat, W.B. ; Lutz, D.A. ; Peyerimhoff, A. Birkhoff invariants and effective calculations for meromorphic linear differential equations. - Houston J. Math. 2, no 2 (1976), 207–238.

[77] Kahn, J. ; Lyubich, M.Yu. ; Rempe, L. A note on hyperbolic leaves and wild laminations of rational functions. - Preprint.

[78] Kaimanovich, V.A. ; Lyubich, M.Yu. Conformal and harmonic measures on laminations associated with rational maps. - Mem. Amer. Math. Soc. 173 (2005), No. 820, vi+119 pp.

[79] Kaloshin, V. ; Rodnianski, I. Diophantine properties of elements of $SO(3)$. - Geom. Funct. Anal. 11 (2001), No. 5, 953–970.

[80] Kitaev, A. Yu. Quantum computations : algorithms and error correction. (Russian) - Uspekhi Mat. Nauk 52 (1997), no. 6(318), 53–112 ; translation in Russian Math. Surveys 52 (1997), no. 6, 1191–1249.

[81] Khovanskii, A.G., Real analytic manifolds with the property of finiteness, and complex Abelian integrals. - Funct. Anal. Appl. vol. 18 no. 2 (1984), 40–50.

[82] Khovanskii, A. G. ; Yakovenko, S.Yu., Generalized Rolle theorem in $\mathbb{R}^n$ and $\mathbb{C}$. - Journal of Dynamical and Control Systems, vol. 2, N 1, 1996, 103—-123.

[83] Korn, A., Zwei Anwendungen der Methode der sukzessiven Annäherungen. - Schwarz Festschrift, Berlin (1919), pp. 215-229.

[84] Landis, E. M. ; Petrovskiĭ, I. G. On the number of limit cycles of the equation $dy/dx = P(x,y)/Q(x,y)$, where P and Q are polynomials. (Russian) - Mat. Sb. N.S. 43(85) (1957), 149–168.

[85] Lavrentiev, M.A., Sur une classe des représentations continues. - Mat. Sb., 42 (1935), 407-434.

[86] Lichtenstein, L., Zur Theorie der konformen Abbildungen ; Konforme Abbildungen nicht-analytischer singularitätenfreier Flächenstücke auf ebene Gebiete. - Bull. Acad. Sci. Cracovie, (1916), 192-217.

[87] Lins Neto, A. Simultaneous uniformization for the leaves of projective foliations by curves. - Bol. Soc. Brasil. Mat. (N.S.) 25 (1994), no. 2, 181–206

[88] Lyubich, M. Yu. Dynamics of rational transformations : topological picture. (Russian) - Uspekhi Mat. Nauk 41 (1986), no. 4(250), 35–95, 239.

[89] Lyubich, M. ; Minsky, Y. Laminations in holomorphic dynamics. - J. Differential Geom. 47 (1997), no. 1, 17–94.

[90] Mardesić, P. An explicit bound of the multiplicity of zeros of generic Abelian integrals. - Nonlinearity 4 (1991), 845-852.

[91] P. Mardesic, R. Roussarie, and C. Rousseau, Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms. - Mosc. Math. J., 4(2) :455–502, 2004.

[92] Martinet, J. Remarques sur la bifurcation nœud-col dans le domaine complexe, - Astérisque no 150-151, 1987, p. 131-149.

[93] J. Martinet and J.-P. Ramis, Problèmes de modules pour des équations différentielles non linéaires du premier ordre. - Inst. Hautes Études Sci. Publ. Math. 55 (1982), 63–164.

[94] Morrey, C. B., Jr. On the solutions of quasi-linear elliptic partial differential equations. - Trans. Amer. Math. Soc. 43 (1938), no. 1, 126–166.

[95] Nielsen, M.A. ; Chuang, I.L. Quantum computation and quantum information. - Cambridge University Press, Cambridge, 2000. xxvi+676 pp.

[96] Nishino, T. Nouvelles recherches sur les fonctions entières de plusieurs variables complexes (II). Fonctions entières qui se reduisent à celles d'une variable. - J. Math. Kyoto Univ. 9-2 (1969), 221-274.

[97] Novikov, D. ; Yakovenko, S.Yu., A complex analog of Rolle theorem and polynomial envelops of irreducible differential equations in the complex domain. - J. London Math. Soc. (2), 56 (1997), no.2, 305-319.

[98] Novikov, D. ; Yakovenko, S.Yu., La borne simplement exponentielle pour le nombre de zéros réels isolés des intégrales complètes abéliennes. - Comptes Rendus Acad. Sci. Paris, série I, 320 (1995), 853-858 (brief announcement)

[99] Novikov, D. ; Yakovenko, S.Yu. Simple exponential estimate for the number of zeros of complete Abelian integrals. - Ann. Inst. Fourier (Grenoble), vol. 45 (1995), 897–927.

[100] Novikov, D. ; Yakovenko, S.Yu. Redundant Picard-Fuchs systems for Abelian integrals. - J. of Differential Equations, 177 (2001), 267-306.

[101] Petrov, G. S. The number of zeros of complete elliptic integrals. (Russian) - Funktsional. Anal. i Prilozhen. 18 (1984), no. 2, 73–74.

[102] Pushkar', I.A. A multidimensional generalization of Ilyashenko's theorem on abelian integrals. (Russian) - Funktsional. Anal. i Prilozhen. 31 (1997) no. 2, 34–44

[103] J.-P. Ramis, Phénomène de Stokes et filtration Gevrey sur le groupe de Picard-Vessiot. - C. R. Acad. Sci. Paris Sér. I Math. 301 (5) (1985), 165–167.

[104] Ramis, J.-P. Confluence et résurgence. - J. Fac. Sci. Tokyo, Sect. IA, Math. 36 (1989), No.3, p.703-716.

[105] Roitman, M. ; Yakovenko, S.Yu. On the number of zeros of analytic functions in a neighborhood of a Fuchsian singular point with real spectrum. - Math. Res. Letters, 3 (1996), no 3 , 359–371.

[106] Shcherbakov, A.A. The exhaustion method for skew cylinders. - Algebra i Analiz, 12(5) (2000). 178-206.

[107] Sibuya, Y., Stokes phenomena. - Bull. Amer. Math. Soc., 83 (1977), pp. 1075–1077.

[108] Slodkowski, Z. Holomorphic motions and polynomial hulls. - Proc. Amer. Math. Soc. 111 (1991), no. 2, 347–355.

[109] Stein, K. Überlagerungen holomorph-vollständiger komplexer Räume. - Arch. Math. 1956, 7 no 5, 354-361.

[110] Sullivan, D. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. - Ann. of Math. (2) 122 (1985), no. 3, 401–418.

[111] Sullivan, D.P. ; Thurston, W.P. Extending holomorphic motions. - Acta Math. 157 (1986), no. 3-4, 243–257.

[112] Tits, J. Free subgroups in linear groups. - J.Algebra, vol. 20 (1972), 250-270.

[113] Varchenko, A. N. Estimation of the number of zeros of an Abelian integral depending on a parameter, and limit cycles. - Funct. Anal. Appl. vol. 18 no. 2 (1984), 14–25.

[114] Varchenko, A.N. Critical values and the determinant of periods. (Russian) - Uspekhi Mat. Nauk 44 (1989), no. 4(268), 235–236 ; translation in Russian Math. Surveys 44 (1989), no. 4, 209–210

[115] Verjovsky, A., A uniformization theorem for holomorphic foliations. - The Lefschetz centennial conference, Part III (Mexico City, 1984), 233–253, Contemp. Math., 58, III, Amer. Math. Soc., Providence, RI, 1987.

[116] Vinberg, E. B. ; Onishchik, A. L. Seminar po gruppam Li i algebraicheskim gruppam. (Russian) [A seminar on Lie groups and algebraic groups] - Second edition. URSS, Moscow, 1995.

[117] Voronin, S.M. Analytic classification of germs of conformal mappings $(C, 0) \to (C, 0)$. (Russian) - Funktsional. Anal. i Prilozhen. 15 (1981), no. 1, 1–17.

[118] Voronin, S.M. ; Meshcheryakova, Yu.I. Analytic classification of generic degenerate elementary singular points of germs of holomorphic vector fields on the complex plane (in Russian). - *Izv. Vyssh. Uchebn. Zaved. Mat.* (1) (2002), 13–16.

[119] C. Zhang, Quelques études en théorie des équations fonctionnelles et en analyse combinatoire. - Thèse, Institut de Recherche Mathématique Avancée, Univ. Louis Pasteur et CNRS (URA 01), 1994.