ASYMPTOTIC STABILITY OF LINEARIZATIONS OF A VECTOR FIELD IN $\mathbb{R}^3$ WITH A SINGULAR POINT DOES NOT IMPLY GLOBAL STABILITY

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In the paper we present a negative solution to the higher-dimensional Markus-Yamabe global stability conjecture. Namely, we prove the following

Theorem 1. *There exists a C^1- vector field in $\mathbb{R}^3$ with a singular point that possesses the following properties:*

1) all the eigenvalues of the Jacobian matrix of the field have negative real parts everywhere;

2) the singular point is not globally attractive.

Remark 0. Theorem 1 is not valid in $\mathbb{R}^2$ ([4, 5, 6]).

The global stability problem for a vector field in $\mathbb{R}^n$ that satisfies condition 1) of Theorem 1 was stated by Markus and Yamabe [7] in 1960.

Earlier the global stability problem under some additional assumptions was investigated by G.Meisters [1], C.Olech ([1, 2]), P.Hartman [3] and others. In 1988 Barabanov [8] had made an attempt to show that Theorem 1 is valid in $\mathbb{R}^n$ for $n \geq 4$. There were some gaps found in his paper. Using the ideas presented in the Barabanov's paper J.Bernat and J.Llibre [9] have proved the $n-$ dimensional version of Theorem 1 with $n \geq 4$. A positive solution of the two-dimensional problem was obtained by C.Gutierrez [4] and independently by R.Fessler [5] in 1993. A little later in the same year it was independently obtained by the author [6]* . Thus the Markus-Yamabe global stability problem is solved completely now.

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The plan of the proof of Theorem 1

Definition 1. Say that a real $n \times n$-matrix has *stable type*, if all its eigenvalues have negative real parts.

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*The author's solution of the three-dimensional problem was announced in his talk at the Workshop on Dynamical Systems, Trieste, May 22 - June 2, 1995. In October, 1995 this result was discussed with A.Cima. After this at the end of 1995 a polynomial counterexample to the three-dimensional Markus-Yamabe conjecture was independently obtained in the joint work by A.van den Essen, E.Hubbers, A.Cima, F.Mañosas, A.Gasull. The method they used is quite different from the one of the author.

In the proof of Theorem 1 we use the following characterization of a three-dimensional stable type matrix.

Proposition 1. *Let A be a real 3×3-matrix, χ be its characteristic polynomial. A has stable type, if and only if the following inequalities hold:*
1) $\operatorname{tr} A < 0$*; 2)* $\det A < 0$*; 3)* $\chi(\operatorname{tr} A) > 0$.

For the completeness of presentation let us prove Proposition 1. Firstly suppose A is a stable type 3×3-matrix with the characteristic polynomial χ. Let us prove that inequalities 1) - 3) hold. Inequalities 1), 2) are obvious. Let us prove inequality 3). By assumption, the characteristic polynomial χ has at least one real negative root. Let λ_1 be the minimal one of such roots. Then $\operatorname{tr} A < \lambda_1$, since other roots have negative real parts. The polynomial χ is positive in the interval $(-\infty, \lambda_1$. Indeed by construstion, χ has no roots in the latter, and $\chi(x) \to +\infty$, as $x \to -\infty$. Hence $\chi(\operatorname{tr} A) > 0$.

Now suppose a real 3×3-matrix with the characteristic polynomial χ satisfies the inequalities from Proposition 3. Let us prove that all its eigenvalues have negative real parts. Suppose the contrary. By inequality 2), this means that χ has a unique real negative root λ_1, and the two other roots have nonnegative real parts. Hence $\lambda_1 < \operatorname{tr} A < 0$ (inequality 1)). Therefore $\chi(\operatorname{tr} A) < 0$, since $\chi(0) = \det A < 0$, and λ_1 is a unique real negative root of χ. This contradicts inequality 3). Proposition 1 is proved.

The idea of proof of Theorem 1. The proof of Theorem 1 is based on the following

Remark 1. Let (x, y, z) be a fixed positively oriented orthonormal coordinate system in $\mathbb{R}^3$, $r = (x^2 + y^2)^{\frac{1}{2}}$. There exists a vector field with a stable type Jacobian matrix in a neighbourhood of the circle $z = r = 1$ such that the latter is its hyperbolic closed trajectory with the stable manifold $r = 1$ and the unstable manifold $r = z > 0$.

Example 1. Let v be a vector field in $\mathbb{R}^3 \setminus Oz$ with the following coordinate representation:

$$\begin{aligned} \dot{x} &= (r-1)x + 2(r-z+1)y \\ \dot{y} &= -2(r-z+1)x + (r-1)y \\ \dot{z} &= 3r^2 - z(2r+1). \end{aligned}$$

The field v is rotation invariant with respect to the Oz-axis. The circle $z = r = 1$ is its hyperbolic closed trajectory with the stable manifold $r = 1$ and the unstable manifold $r = z > 0$. The origin is a unique singular point of the field. It is attractive, and its attraction basin is the set $r < 1$ (fig.1). The Jacobian matrix of the field v has stable type in the circle. Indeed by rotation invariancy of the field, for the test of the last property it suffices to check it only at the point $(1, 0, 1)$. The Jacobian matrix of v at this point is equal to

$$\begin{pmatrix} 1 & 2 & 0 \\ -4 & 0 & 2 \\ 4 & 0 & -3 \end{pmatrix}.$$

This is a stable type matrix. One can check this by using Proposition 1.

For the proof of Theorem 1 we show that there exists a vector field with the following properties:

Theorem 2. *There exists a vector field in $\mathbb{R}^3$ that satisfies condition 1) of Theorem 1 and possesses the following properties:*

1) 0 is its unique singular point;

2) the field is rotation invariant with respect to the Oz-axis;

3) the circle $z = r = 1$ is a hyperbolic closed trajectory; its stable manifold is the cylinder $r = 1$, and the intersection of its unstable manifold with the set $\{r \leq 2\}$ is the surface $\{0 < r = z \leq 2\}$;

4) the attractive basin of the point 0 is the set $\{r < 1\}$ (fig. 1').

Theorem 2 is proved in Section 2.

Proof of Theorem 2

Definition. Let v be an Oz- rotation invariant vector field in $\mathbb{R}^3$. The *factorization* of the field v is the vector field in the half-plane $\{y = 0\} \cap \{x \geq 0\}$ that is the orthogonal projection of the field $v|_{y=0}$ to the plane $y = 0$. The *rotation part* of the field v is its orthogonal projection to the lines tangent to the Oz- rotation circles.

We are looking for an Oz - rotation invariant vector field in $\mathbb{R}^3$ that satisfies condition 1) of Theorem 1 and has a factorization with phase portrait in a neighbourhood of the set $x \leq 1$ depicted at fig. 1. Condition 1) of Theorem 1 means that the Jacobian matrix of the field satisfies the inequalities from Proposition 1.

Example 2. Let a vector field $\tilde{v}$ in $\mathbb{R} \times \mathbb{R}_+ \cup \{0\}$ (with the coordinates (z, x)) have singular points only at $(0, 0)$, $(1, 1)$ and possess the first integral $F_l(z, x) = (x-1)^l(\frac{z}{x} - 1)$ in the set $\{x > 0\}$, $l > 0$. Then its phase portrait[1] is of the type depicted at fig. 1. The vector field from Example 1 possesses the first integral F_3.

The idea of the proof of Theorem 2. For the proof of Theorem 2 we show that for any $l \in \mathbb{N}$ large enough there exists an Oz- rotation invariant vector field in $\mathbb{R}^3$ with a stable type Jacobian matrix such that the correspondent factorization possesses the following properties:

1) its singular points are $(0, 0)$, $(1, 1)$;

2) the function F_l from Example 2 is its first integral in the set $\{0 < x \leq 2\}$;

3) all its trajectories in the set $x > 1$ approach the line $z = \frac{l}{l-1}x - \frac{2l}{(l-1)^2}$ at the infinity in the positive direction (fig. 1').

Remark 2. Let an Oz-rotation invariant vector field in $\mathbb{R}^3$ have the Jacobian matrix A. Then A has stable type everywhere, if and only if this is valid in the half-plane $\{y = 0\} \cap \{x \geq 0\}$. The function $\operatorname{tr} A$ is determined by the factorization of the field, i.e, does not depend on its rotation part.

In the proof of Theorem 2 we use the following formulas for an Oz - rotation invariant vector field and a one with the first integral F_l.

Remark 3. A vector field v in $\mathbb{R}^3$ is Oz - rotation invariant, if and only if its coordinate representation has the following form:

$$\begin{aligned}
\dot{x} &= f(z, r)x + h(z, r)y \\
\dot{y} &= f(z, r)y - h(z, r)x \\
\dot{z} &= q(z, r).
\end{aligned}$$

[1] In this place by the *phase portrait* of a vector field we mean the one without a fixed orientation of the phase curves.

If the functions f, h, q are C^1 in $\mathbb{R}\times(\mathbb{R}_+\cup\{0\})$, and $\frac{\partial q}{\partial r}(z,0)\equiv 0$, then the correspondent vector field v is C^1. Its factorization is the vector field

$$\begin{aligned}\dot{x} &= f(z,x)x\\ \dot{z} &= q(z,x).\end{aligned}$$

Its rotation part is the sum of the terms from its expression that contain the function h. Its divergence (i.e., the trace of its Jacobian matrix) is equal to

$$r\frac{\partial f}{\partial r}(z,r)+2f(z,r)+\frac{\partial q}{\partial z}(z,r)$$

Remark 4. A vector field in $\mathbb{R}\times\mathbb{R}_+\cup\{0\}$ with the only singular points $(0,0)$, $(1,1)$ possesses the first integral F_l from Example 2, $l>0$, if and only if its coordinate representation has the following form:

$$\begin{aligned}\dot{x} &= x(x-1)f(z,x)\\ \dot{z} &= (lx^2-z(x(l-1)+1))f(z,x).\end{aligned}\tag{0}$$

Indeed

$$\frac{\partial F_l}{\partial x}=\frac{(x-1)^{l-1}}{x^2}(l(zx-x^2)-z(x-1));$$

$$\frac{\partial F_l}{\partial z}=\frac{(x-1)^l}{x}.$$

Therefore the field possesses the first integral F_l in the set $\{x>0\}$, if and only if the ratio of its z- and x- components is equal to

$$-\frac{\frac{\partial F_l}{\partial x}}{\frac{\partial F_l}{\partial z}}=\frac{lx^2-z(x(l-1)+1)}{x(x-1)}.$$

The vector field (0) is directed as at fig.1 and $(1,1)$ is its nondegenerate (hyperbolic) singular point, if and only if $f|_{x>0}>0$. The divergence of an Oz- rotation invariant vector field in $\mathbb{R}^3$ with the factorization (0) at the circle $z=r=1$ is equal to $(1-l)f$. Hence in the case when $f>0$ it is negative, if and only if $l>1$.

Theorem 2 is implied by the following

Main Lemma 1. *There exist an $L>1$ and families $f_1(z,r)(l)$, $f_2(z,r)(l)$, $f_1,f_2:\mathbb{R}\times(\mathbb{R}_+\cup\{0\})\to\mathbb{R}_+$ of positive C^∞- functions*[2] *that depend on the parameter $l\in\mathbb{N}$ with the following properties:*

1) $\frac{\partial((r(l-1)+1)f_2)}{\partial r}|_{r=0}\equiv 0$, $f_1|_{r\leq 2}\equiv f_2|_{r\leq 2}$;

2) for any natural $l>L$ there exists a positive C^∞- function $h_l:\mathbb{R}\times\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+$ such that the field

$$\begin{aligned}\dot{x} &= (r-1)f_1(z,r)x+h_l(z,r)y\\ \dot{y} &= (r-1)f_1(z,r)y-h_l(z,r)x\\ \dot{z} &= (lr^2-z(r(l-1)+1))f_2(z,r)\end{aligned}$$

[2] From now on we omit the sign l in the expressions for f_1, f_2 for simplicity

has a stable type Jacobian matrix everywhere in $\{y=0\}\cap\{x\geq 0\}$.

Lemma 1 is proved in 2-7.

The vector field from Lemma 1 with l large enough is a one we are looking for. Indeed it is C^1 (Remark 3 and statement 1) of Lemma 1) and satisfies condition 1) of Theorem 1 (Remark 2). It has a unique singular point at 0 and possesses the first integral F_l from Example 2 in the set $r\leq 2$ (Remark 4). Its closed orbit $z=r=1$ is hyperbolic as in Theorem 2: the singular point $(1,1)$ of its factorization is hyperbolic with the stable manifold $x=1$; the intersection of its unstable manifold with the set $x\leq 2$ is $\{0<x=z\leq 2\}$.

Let us present families f_1, f_2, h_l that satisfy the statements of Lemma 1. Let $\Psi:\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+$ be a positive C^∞- function such that

$$\Psi\leq 1;\quad \Psi|_{r\leq 2}\equiv 1,\quad \Psi(r)|_{r\geq 3}=\frac{1}{r+1},$$

$\phi:\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+\cup\{0\}$ be a C^∞- function such that $\phi(0)=0$, $\phi'|_{[0,1)}>0$, $\phi|_{[1,+\infty)}\equiv 1$ (fig.2). Below we show that there exists an $L>1$ such that for any $l>L$ and m large enough dependently on l the families

$$f_2(z,r)=\frac{1}{r(l-1)+1}\,\frac{1}{(z^2+r^2+1)^{\frac{1}{3}}},\quad f_1(z,r)=\Psi(r)f_2(z,r),$$

$$h_l(z,r)=h_{l,m}(z,r)=\frac{m}{r}\phi(r(l-1))\sqrt{\int_{z-54r}^{+\infty}\frac{dt}{(t^2+1)^{\frac{2}{3}}}}$$

satisfy[3] the statements of Lemma 1.

Remark 5. Let f_1, f_2 be as above, $l\geq 2$. The factorization $\tilde{v}$ of a correspondent vector field from Lemma 1, which does not depend on h_l, possesses the following property: all its trajectories in the set $x>1$ have the asymptotic line $z=\frac{l}{l-1}x-\frac{2l}{(l-1)^2}$ at the infinity in the positive direction. One can prove this by straightforward calculation of the derivative of the function $\Lambda(z,x)=z-\frac{l}{l-1}x$ along $\tilde{v}$ in the lines $\Lambda=const$.

2. Pre-motivations and the scheme of the proof of Lemma 1. We are looking for a vector field of the type as in Lemma 1 with a stable type Jacobian matrix. This means that the latter satisfies the inequalities from Proposition 1.

Firstly let us motivate the construction of a vector field as in Lemma 1. Let v be an Oz- rotation invariant vector field in $\mathbb{R}^3$ with the rotation part v_r and the Jacobian matrix A. The function $\det A$ is the sum of a function that does not depend on v_r and a function $\det'$ quadratic in v_r. The same is valid for $\chi(\operatorname{tr}A_{l,m})$. Let the circle $z=r=1$ be v- invariant. Then $\det A=\det'$ at the point $(1,0,1)$. Indeed it suffices to check this in the case when $v_r(1,0,1)=0$. Then the whole circle $z=r=1$ consists of the singular points of the field v, and hence, $\det A(1,0,1)=0=\det'(1,0,1)$.

For the proof of Lemma 1 we construct a vector field as in the latter with the Jacobian matrix A such that the main contribution to $\det A$, $\chi(\operatorname{tr}A)$ is brought

[3] From now on a function $\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ and its continuation to the set $\mathbb{R}\times\{0\}$ are denoted by the same symbol.

by the terms of their expressions that contain the coordinate components of the rotation part of the field. To do this we construct families f_1, f_2 of the same type as in Lemma 1 and a family $h_{l,m} = \frac{m}{r} g_l(z,r)$ of nonnegative functions, $g_l|_{r=0} \equiv 0$ that depends on the natural parameter l and $m > 0$ with the following property: there exists an $L > 1$ such that for any natural $l > L$ and $m > 0$ large enough with respect to l the triple $(f_1, f_2, h_l = h_{l,m})$ satisfies the statements of Lemma 1.

Definition 3. Let $f_1, f_2 : \mathbb{R}\times\mathbb{R}_+\cup\{0\} \to \mathbb{R}_+$ be families of positive C^∞- functions that satisfy condition 1) of Lemma 1 (see footnote 2), $g_l : \mathbb{R}\times\mathbb{R}_+\cup\{0\} \to \mathbb{R}$ be a family of C^∞- functions that depend on the natural parameter $l \geq 2$ such that $g_l(z,0) \equiv 0$. The *family $v_{l,m}$ of vector fields in $\mathbb{R}^3$ associated to the triple* $(f_1,\ \ f_2,\ \ g_l)$ is the one of the fields constructed from the functions f_1, f_2, $h_l(z,r) = \frac{m}{r} g_l(z,r)$, $m > 0$ by the formula from Lemma 1.

Lemma 1 is implied by the following

Lemma 2. *There exist an $L > 1$, families f_1, f_2 and g_l as in Definition 3, $g_l|_{r>0} > 0$ such that for any natural $l > L$ there exists an $M > 0$ with the following property: for any $m > M$ the correspondent vector field $v_{l,m}$ from Definition 3 has a stable type Jacobian matrix everywhere in $\{y = 0\} \cap \{x \geq 0\}$.*

Lemma 2 is proved in 3-7.

3. The plan of the proof of Lemma 2.

Definition 4. Let f_1, f_2, g_l, $v_{l,m}$ be as in Definition 3. The *family $A_{l,m}(z,r)$ of matrix functions associated* to the triple $(f_1,\ \ f_2,\ \ g_l)$ defined for $z \in \mathbb{R}$, $r \geq 0$ is the one of the Jacobian matrices of the fields $v_{l,m}$ at the point $(r, 0, z)$.

In the proof of Lemma 2 we use the following properties of the matrix family $A_{l,m}$:

Preliminary Remark 6. For a triple $(f_1,\ \ f_2,\ \ g_l)$ as in Definition 3 the correspondent matrix family $A_{l,m} = (a_{ij}(z,r)(l,m))$ has the following type:

$$\begin{pmatrix} a_{11}(z,r)(l) & \frac{m}{r} g_l(z,r) & a_{13}(z,r)(l) \\ -m\frac{\partial g_l}{\partial r}(z,r) & a_{22}(z,r)(l) & -m\frac{\partial g_l}{\partial z}(z,r) \\ a_{31}(z,r)(l) & 0 & a_{33}(z,r)(l) \end{pmatrix}. \tag{1}$$

Its only elements[4] that depend either on m or on g_l are a_{12}, a_{21}, a_{23}.

To sketch the proof of Lemma 2 we introduce some definitions.

Definition 5. Let f_1, f_2 be as in Definition 3. The *families tr_l, $a_{ij} = a_{ij}(l)$* (see footnote 4) *of functions* in $(z,r) \in \mathbb{R}\times\mathbb{R}_+\cup\{0\}$ *associated to* f_1, f_2, $(i,j) \neq (1,2)$, $(2,1)$ $(2,3)$ are respectively the trace and the correspondent elements of the matrix family $A_{l,m}$ associated to a triple $(f_1,\ \ f_2,\ \ g_l)$, where g_l is a family as in Definition 3.

Remark 7. The families a_{ij}, tr_l associated to f_1, f_2 as above depend on the parameter l and are well-defined (Remark 6). The families a_{31}, a_{33} are determined by the choice of f_2. The families a_{11}, a_{22}, a_{13} are determined by the choice of f_1; they are linear in f_1 and its partial derivatives.

[4] From now on for simplicity we omit the sign l in the expressions for the families $a_{ij}(l)$ of the matrix $A_{l,m}$ elements

Definition 6. Let $(f_1,\ \ f_2,\ \ g_l)$ be a triple as in Definition 3, $A_{l,m}$ be the correspondent matrix family, χ be its characteristic polynomial. Define $det'_{l,m}$ $(\chi(tr)'_{l,m})$ to be the sum of the products of matrix elements in the expression for $\det A_{l,m}$ (respectively $\chi(\operatorname{tr} A_{l,m})$) that contain m.

Remark 8. Let f_1, f_2, g_l, $A_{l,m}$ be as above. The differences $\det A_{l,m} - det'_{l,m}$, $\chi(\operatorname{tr} A_{l,m}) - \chi(tr)'_{l,m}$ do not depend neither on m nor on g_l. Both $det'_{l,m}$, $\chi(tr)'_{l,m}$ are products of m^2 and a function that does not depend on m.

Lemma 2 is implied by the following

Lemma 3. *There exist an $L > 1$ and a triple $(f_1,\ \ f_2,\ \ g_l)$ as in Definition 3, $g_l|_{r>0} > 0$ such that for any natural $l > L$, $m \neq 0$ the correspondent families tr_l, $det'_{l,m}$, $\chi(tr)'_{l,m}$ from Definitions 5, 6 satisfy the following inequalities everywhere:*

1) $tr_l < 0$;

2) there exists a $c_l > 0$ such that

$$-det'_{l,m} > m^2 c_l |\det A_{l,m} - det'_{l,m}|, \tag{2}$$

$$\chi(tr)'_{l,m} > m^2 c_l |\chi(\operatorname{tr} A_{l,m}) - \chi(tr)'_{l,m}|. \tag{3}$$

Lemma 3 is proved in 4-7.

Proof of Lemma 2. Let L, f_1, f_2, g_l be as in Lemma 3. Let us prove that they satisfy the statement of Lemma 2. To do this we check that for any natural $l > L$ and m large enough dependently on l the matrix $A_{l,m}$ satisfies the inequalities 2), 3) from Proposition 1. This together with the latter and inequality 1) from Lemma 3 will prove Lemma 2. For a fixed natural $l > L$ let c_l be a correspondent constant from Lemma 3. For $m > \sqrt{\frac{2}{c_l}}$ $\det A_{l,m} < 0$, $\chi(\operatorname{tr} A_{l,m}) > 0$ (inequalities (2), (3)). Lemma 2 is proved.

Preliminary Remark 9. In the proof of Lemma 3 we use the following formulas:

$$\begin{cases} det'_{l,m}(z,r) = \frac{m^2}{2r}\left(\frac{\partial g_l^2}{\partial r}a_{33} - \frac{\partial g_l^2}{\partial z}a_{31}\right)(z,r); \\ \chi(tr)'_{l,m}(z,r) = -\frac{m^2}{2r}\left(\frac{\partial g_l^2}{\partial r}(a_{11}+a_{22}) + \frac{\partial g_l^2}{\partial z}a_{31}\right)(z,r); \\ a_{31}(z,r) = 2lrf_2(z,r) - z(l-1)f_2(z,r) + (lr^2 - z(r(l-1)+1))\frac{\partial f_2}{\partial r}(z,r); \\ a_{33}(z,r) = -(r(l-1)+1)f_2(z,r) + (lr^2 - z(r(l-1)+1))\frac{\partial f_2}{\partial z}(z,r). \end{cases} \tag{4}$$

4. Motivation of the construction of $(f_1,\ \ f_2,\ \ g_l)$ and the sketch of the proof of Lemma 3.

We are looking for $(f_1,\ \ f_2,\ \ g_l)$ as in Definition 3 such that the estimates from Lemma 3 hold. This implies that for any l large enough, $m \neq 0$ $tr_l = a_{11} + a_{22} + a_{33} < 0$, $det'_{l,m} < 0$, $\chi(tr)'_{l,m} > 0$.

To motivate the construction of a triple $(f_1,\ \ f_2,\ \ g_l)$ as in Lemma 3 we make the two following Remarks:

Preliminary Remark 10. Let f_1, f_2 be as in Definition 3, a_{ij} be the correspondent families from Definition 5. For any $l \geq 3$ the following inequalities hold:

$$a_{31}(1,1) > 0, \quad a_{31}(0,0) = 0,$$

$$tr_l(1,1)<0, \quad a_{33}(1,1)<0, \quad a_{33}(0,0)<0.$$

$$a_{22}|_{r=1}=0, \ \ a_{11}|_{r=1}>0, \ \ a_{11}|_{r=0}=a_{22}|_{r=0}<0,$$

Therefore *the main contribution to $tr_l(1,1)$ is brought by a_{33}*. The terms in the expressions (4) for a_{31}, a_{33} that contain the partial derivatives of the family f_2 vanish at $(1,1)$.

Remark 11. The vector field from Example 1 coincides with the vector field $v_{3,1}$ from a family $v_{l,m}$ as in Definition 3. The correspondent matrix $A_{3,1}$ has stable type at $(1,1)$. The families $A_{l,m}$ and g_l possess the following properties:

$$a_{21}(1,1)|_{l=3}=-4=-\frac{\partial g_3}{\partial r}(1,1)<0,$$

$$a_{23}(1,1)|_{l=3}=2=-\frac{\partial g_3}{\partial z}(1,1)>0,$$

$$g_3(1,1)>0, \ \ det'_{3,1}(1,1)<0, \ \ \chi(tr)'_{3,1}(1,1)>0$$

(Example 1). Let $l=3$, $m=1$. *The term $\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}a_{33}$* is a unique negative term at $(1,1)$ in the expression (4) for $det'_{l,m}$ and hence *brings the main contribution to $det'_{l,m}(1,1)$*. *The term $-\frac{m^2}{2r}\frac{\partial g_l^2}{\partial z}a_{31}$* is a unique positive term at $(1,1)$ in the expression (4) for $\chi(tr)'_{l,m}$ and hence *brings the main contribution to $\chi(tr)'_{l,m}(1,1)$*. On the other hand, this term vanishes in the set $r=0$. Hence in this set $\chi(tr)'_{l,m}=-\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}(a_{11}+a_{22})$.

The idea of the proof of Lemma 3. For the proof of Lemma 3 we construct a triple as in the latter with the following properties: 1) for any $l\geq 2$

$$g_l|_{r>0}>0, \quad \frac{1}{r}\frac{\partial g_l^2}{\partial r}(z,r)>0, \quad \frac{\partial g_l^2}{\partial z}(z,r)\leq 0; \tag{5}$$

2) for any l large enough a) $a_{33}<0$, $a_{31}|_{r>0}>0$, $a_{22}|_{r\leq\frac{1}{l-1}}<0$, b) for any (z,r) each of $a_{11}(z,r)$, $a_{22}(z,r)$ is either negative or "small enough" with respect to $a_{33}(z,r)$ (in the sense of (6)), c) the main contribution to $det'_{l,m}$ is brought by the (negative) term $\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}a_{33}$ from (4), d) the contribution to $\chi(tr)'_{l,m}$ of the (positive) term $-\frac{m^2}{2r}\frac{\partial g_l^2}{\partial z}a_{31}$ is greater than the one of the negative terms in its expression (4) in the set $r\geq\frac{1}{l-1}$, the term $-\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}a_{22}$ from (4) possesses the same property in the set $r<\frac{1}{l-1}$, e) the main contributions to a_{31}, a_{33} are brought by the terms in their expressions (4) that do not contain neither z nor the partial derivatives of f_2, f) the (negative) ratios $\frac{a_{31}}{a_{33}}$, $\frac{a_{33}}{a_{31}}$ are bounded in the set $r\geq\frac{1}{l-1}$, the bounds are independent on l;

3) the restriction of the (negative) ratio $\frac{\frac{\partial g_l^2}{\partial r}}{\frac{\partial g_l^2}{\partial z}}$ to the set $r\geq\frac{1}{l-1}$ concidered as a function in (l,z,r) is constant.

The precise statements of the above properties 1), 2)a)-d), 2)f), 3) are presented below (inequalities (6)-(8) and properties (ii)-(iv)).

Let us motivate the construction of a triple $(f_1, \ \ f_2, \ \ g_l)$ as above. We are looking for a family f_2 of positive functions such that the main contribution to the correspondent family a_{33} is brought by the terms in its expression (4) that do not contain $\frac{\partial f_2}{\partial z}$. Firstly let $f_2(z,r) = f_2(r)$ be independent on z. Then $a_{33}(z,r) = -(r(l-1)+1)f_2(r) < 0$. The correspondent family a_{31} is the sum of a family of functions independent on z and a family linear in z. If $a_{31} \geq 0$ everywhere then the latter should vanish identically. This means that

$$\frac{\partial((r(l-1)+1)f_2)}{\partial r} = 0, \quad \text{i.e.,}$$

$$f_2(r) = \frac{c}{r(l-1)+1}, \quad c \in \mathbb{R}.$$

Example 3. Let $f_1 \equiv f_2 \equiv \frac{1}{r(l-1)+1}$, tr_l, a_{ij} be the correspondent families from Definition 5. Then

$$a_{31}(z,r) = \frac{lr}{r(l-1)+1}\left(2 - \frac{(l-1)r}{r(l-1)+1}\right) \geq 0, \quad a_{33} = -1 < 0.$$

There exist $L > 1,\ c,\ c',\ c_1,\ c_2 > 0$ such that for any $l > L$

1) the following inequalities hold everywhere:

$$\begin{cases} a_{11} < -\frac{ca_{33}}{l} \\ a_{22} < -\frac{ca_{33}}{l} \end{cases}, \tag{6}$$

$$-c_1\frac{lr}{r(l-1)+1}a_{33}(z,r) \leq a_{31}(z,r) \leq -c_2\frac{lr}{r(l-1)+1}a_{33}(z,r); \tag{7}$$

2) for any $z \in \mathbb{R}$, $r \leq \frac{1}{l-1}$

$$a_{22}(z,r) < c'a_{33}(z,r). \tag{8}$$

One can show this by a straightforward calculation. System (6) implies that $tr_l < 0$ for any l large enough.

Definition 7. Let f_1, f_2 be as in Definition 3. Say that the pair $(f_1,\ f_2)$ is *good*, if there exist $L > 1,\ c,\ c',\ c_1,\ c_2 > 0$ such that for any $l > L$ $a_{33} < 0$ and inequalities (6)-(8) hold in the correspondent sets from Example 3.

The plan of the proof of Lemma 3. For the proof of Lemma 3 we construct a triple $(f_1,\ f_2,\ g_l)$ as in the latter with the following properties: (i) the pair $(f_1,\ f_2)$ is good; (ii) for any $l \geq 2$ inequalities (5) hold everywhere; (iii) the restriction of the family g_l^2 to the set $r \geq \frac{1}{l-1}$ is of the type $\theta(z - 2c_2r)$, where θ is a positive function (independent on l) with $\frac{d\theta}{dt} < 0$, c_2 is a constant from (7); (iv) the inequality

$$\frac{\partial g_l^2}{\partial r} \geq -2c_2\frac{\partial g_l^2}{\partial z} \tag{9}$$

holds everywhere. Property (i) implies that $tr_l < 0$ for any l large enough. As it is (implicitly) shown in 5, properties (i)-(iv) imply the above properties 2c),d) (and hence positivity of the left-hand sides in (2), (3)). We prove the existence of a triple with properties (i)-(iv) such that for any l large enough there exists a $c_l > 0$ such that (2), (3) hold everywhere. This will prove Lemma 3.

The proof of Lemma 3 is split into 4 subsections. In 5 for each good pair $(f_1,\ f_2)$ we construct a class of "appropriate" families g_l as in Definition 3 with the properties (ii)-(iv). This class is determined by the constant c_2 and presented in Remark 12. Each g_l from the class is determined by the choice of θ and c_2. Then in 6-7 we construct a good pair $(f_1,\ f_2)$ with the following property: there exists a representative g_l from the correspondent class such that estimates (2), (3) hold. In the proof of the latters we use estimates (10), (11) of their left-hand sides. These estimates are proved in 5.

At the end of 5 we reduce Lemma 3 to Basic Technical Lemma, which is stated at the same place. The latter implies the existence of $(f_1,\ f_2)$, g_l as at the end of the last item. This Lemma is proved in 6-7.

5. The construction of the class of appropriate families g_l and Basic Technical Lemma.

Let $(f_1,\ f_2)$ be a fixed good pair. We are looking for families g_l such that (5) hold and the correspondent left-hand sides in (2), (3) are positive. As it is shown below, this is satisfied provided that g_l possesses the following properties.

Proposition 2. *Let $(f_1,\ f_2)$ be a good pair, c_2 be a correspondent constant from (7). Let g_l be a family as in Definition 3 with properties (ii)-(iv) from the end of 4. Then there exist $L > 1$, $c'' > 0$ such that for any $l > L$, $m \neq 0$ the following inequalities hold:*

$$det'_{l,m}(z,r) \leq \frac{m^2}{6r}(\frac{\partial g_l^2}{\partial r}a_{33})(z,r) < 0; \tag{10}$$

$$\chi(tr)'_{l,m}(z,r) \geq -\frac{m^2 c''}{4r}(\frac{\partial g_l^2}{\partial r}a_{33})(z,r) > 0. \tag{11}$$

Proof. The right estimates in (10), (11) are obvious. Let us prove the left estimate in (10). By (4)-(7), (9), for any l large enough

$$det'_{l,m}(z,r) \leq \frac{m^2}{2r}(\frac{\partial g_l^2}{\partial r}a_{33} + \frac{l}{l-1}c_2\frac{\partial g_l^2}{\partial z}a_{33})(z,r) \leq \frac{m^2}{6r}(\frac{\partial g_l^2}{\partial r}a_{33})(z,r).$$

Now let us prove the left estimate in (11). Firstly let us estimate $\chi(tr)'_{l,m}$ in the set $r \geq \frac{1}{l-1}$. By (4)-(7), for any l large enough, $r \geq \frac{1}{l-1}$

$$\begin{aligned}\chi(tr)'_{l,m}(z,r) \ &\geq \ \frac{m^2}{2r}(-\ \frac{\partial g_l^2}{\partial r}(a_{11}\ +\ a_{22}) + \frac{c_1}{2}\frac{\partial g_l^2}{\partial z}a_{33})(z,r) \\ &\geq \frac{m^2}{2r}(\frac{\partial g_l^2}{\partial r}\frac{2ca_{33}}{l} - \frac{c_1}{4c_2}\frac{\partial g_l^2}{\partial r}a_{33})(z,r) = -(\frac{c_1}{4c_2} - \frac{2c}{l})\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}a_{33}(z,r).\end{aligned}$$

The constant factor in the brackets in the right-hand side of the above inequality is greater than $\frac{c_1}{8c_2}$ whenever l is large enough.

Now let us estimate $\chi(tr)'_{l,m}$ in the set $r < \frac{1}{l-1}$. By (4)-(8), for any l large enough, $r < \frac{1}{l-1}$

$$\chi(tr)'_{l,m}(z,r) \geq -\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}(a_{11} + a_{22})(z,r) \geq -\frac{m^2}{2r}\frac{\partial g_l^2}{\partial r}(-\frac{c}{l} + c')a_{33}(z,r).$$

This is the place we use estimate (8). The right-hand side of the above inequality is greater than $-\frac{m^2c'}{4r}\frac{\partial g_l^2}{\partial r}a_{33}(z,r)$ when l is large enough. This together with the estimate from the last item proves the existence of $c''>0$ such that (11) holds whenever l is large enough. Proposition 2 is proved.

Below we construct a class of families g_l that satisfy the conditions of Proposition 2 for the fixed f_1, f_2.

Firstly let us motivate the construction of g_l. Let c_2, f_1, f_2 be as above, $\theta:\mathbb{R}\to\mathbb{R}_+$ be a positive C^∞-function with $\frac{d\theta}{dt}<0$. Put

$$g_l(z,r)=\sqrt{\theta(z-2c_2r)}.$$

The family g_l possesses properties (ii)-(iv). But it does not satisfy the conditions of Definition 3, since $g_l(z,0)\not\equiv 0$.

Now let us modify the above g_l in order to obtain a family that satisfies the conditions of Definition 3 as well. To do this we introduce the following

Definition 8. Let $b>0$, $\theta:\mathbb{R}\to\mathbb{R}_+$ be a positive C^∞- function with $\frac{d\theta}{dt}<0$, $\phi:\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+\cup\{0\}$ be a fixed C^∞-function such that $\phi(t)=1$ for $t\geq 1$, $\phi(0)=0$, $\phi'|_{[0,1)}>0$ (fig. 2). Define

$$g_{l,\theta,b}(z,r)=\sqrt{\theta(z-2br)}\phi(r(l-1)).$$

Remark 12. Let f_1, f_2, c_2 be as in Proposition 2. For any θ as above the family $g_l=g_{l,\theta,c_2}$ satisfies the conditions of Proposition 2.

The class of families from Remark 12 is the one we are looking for.

Below we show that there exist f_1, f_2, c_2, θ as above such that the triple $(f_1,\ f_2,\ g_l=g_{l,\theta,c_2})$ satisfies the statements of Lemma 3.

Lemma 3 is implied by Proposition 2 and the following

Basic Technical Lemma 4. *There exist a good pair $(f_1,\ f_2)$ and a function θ as in Definition 8 such that for any l large enough, $b>0$ there exists a $c_l>0$ such that the families from Definitions 5, 6 correspondent to the triple $(f_1,\ f_2,\ g_l=g_{l,\theta,b})$ satisfy the following inequalities:*

$$-\frac{1}{r}\frac{\partial g_{l,\theta,b}^2}{\partial r}(z,r)a_{33}(z,r)>c_l|\det A_{l,m}-det'_{l,m}|(z,r), \tag{12}$$

$$-\frac{1}{r}\frac{\partial g_{l,\theta,b}^2}{\partial r}(z,r)a_{33}(z,r)>c_l|\chi(\operatorname{tr}A_{l,m})-\chi(tr)'_{l,m}|(z,r). \tag{13}$$

Lemma 4 is proved in 6-7.

Proof of Lemma 3. Let θ and a good pair $(f_1,\ f_2)$ be as in Lemma 4, c_2 be a correspondent constant from (7). Let us prove that f_1, f_2, $g_l=g_{l,\theta,c_2}$ satisfy the statements of Lemma 3. For any l large enough $tr_l<0$ ($a_{33}<0$ and inequalities (6)), and there exists a $c_l>0$ such that (2), (3) hold. This follows from (12), (13) and Proposition 2. Lemma 3 is proved.

6. Motivations of the construction of f_1, f_2, θ and the sketch of the proof of Lemma 4.

We show that one can choose θ as in Lemma 3 from the following class of functions.

Example 4. Let $s > \frac{1}{2}$. Define

$$\theta_s(t) = \int_t^{+\infty} \frac{d\tau}{(\tau^2+1)^s}.$$

The function $\theta = \theta_s$ satisfies the conditions of Definition 8.

Below we prove that there exists a triple $(f_1,\ f_2,\ \theta = \theta_s)$ that satisfies the statements of Lemma 4. We construct such a triple with $s = \frac{2}{3}$.

Remark 13. We show that the families f_1, f_2 from the end of 1 and $\theta = \theta_{\frac{2}{3}}$ satisfy the statements of Lemma 4 and the correspondent constant c_2 from (7) can be chosen to be equal to 27. (The last statement is proved in 7.) Hence the triple $(f_1,\ \ f_2,\ \ g_l)$ with $g_l = g_{l,\theta,27}$ satisfies the statements of Lemma 3 (and hence Lemma 2 too). Therefore the triple $(f_1,\ f_2,\ h_l)$ from the end of 1 satisfies the statements of Lemma 1.

In the proof of Lemma 4 we use the two following estimates of $\frac{d\theta}{dt}$ and $\frac{1}{r}\frac{\partial g^2_{l,\theta,b}}{\partial r}$ valid for $\theta = \theta_s$.

Remark 14. For any $s > 0$, $\theta = \theta_s$ there exists a $p > 0$ such that

$$\theta > -p\frac{d\theta}{dt}. \tag{14}$$

This estimate is proved in 7 for $s = \frac{2}{3}$.

Proposition 3. *Let θ, $g_{l,\theta,b}$ be as in Definition 8, and there exist a $p > 0$ such that (14) holds. Then for any $l \geq 2$, $b > 0$ there exists a $q > 0$ such that*

$$\frac{1}{r}\frac{\partial g^2_{l,\theta,b}}{\partial r}(z,r) > -\frac{q}{r+1}\frac{d\theta}{dt}(z-2br). \tag{15}$$

Proof. Let us estimate the left-hand side of (15).

$$\frac{1}{r}\frac{\partial g^2_{l,\theta,b}}{\partial r}(z,r) = -2b\frac{\phi^2(r(l-1))}{r}\frac{d\theta}{dt}(z-2br) + \frac{2(l-1)\phi(r(l-1))}{r}\phi'(r(l-1))\times$$

$$\theta(z-2br) \geq -\frac{d\theta}{dt}(z-2br)(2b\frac{\phi^2(r(l-1))}{r} + 2(l-1)p\phi'(r(l-1))\frac{\phi(r(l-1))}{r}).$$

The second factor in the right-hand side of the above inequality does not depend on z. It is positive for any $r \geq 0$ and equals $\frac{2b}{r}$ when $r \geq \frac{1}{l-1}$. Indeed its first term is nonnegative everywhere and equals $\frac{2b}{r}$ when $r \geq \frac{1}{l-1}$. Its second term is positive when $0 \leq r < \frac{1}{l-1}$ and vanishes when $r \geq \frac{1}{l-1}$. Therefore there exists a $q > 0$ such that this factor is greater than $\frac{q}{r+1}$. Proposition 3 is proved

The plan of the proof of Lemma 4. For the proof of Lemma 4 we construct a good pair $(f_1,\ f_2)$ and a $\theta = \theta_s$ such that for any l large enough and $b > 0$ there exists a $c_l > 0$ such that the correspondent families a_{ij} satisfy the following inequalities:

$$\begin{cases} \frac{1}{r+1}\frac{d\theta}{dt}(z-2br)a_{33}(z,r) > c_l|\det A_{l,m} - det'_{l,m}|(z,r) \\ \frac{1}{r+1}\frac{d\theta}{dt}(z-2br)a_{33}(z,r) > c_l|\chi(\operatorname{tr} A_{l,m}) - \chi(tr)'_{l,m}|(z,r). \end{cases} \tag{16}$$

This together with (14), (15) will prove Lemma 4.

Preliminary Remark 15. In the proof of Lemma 4 we use the following formulas:

$$\det A_{l,m} - det'_{l,m} = a_{22}(a_{11}a_{33} - a_{13}a_{31}), \tag{17}$$

$$\chi(\operatorname{tr} A_{l,m}) - \chi(tr)'_{l,m} = -(a_{11}+a_{22})(a_{11}+a_{33})(a_{22}+a_{33}) + a_{13}(a_{11}+a_{33})a_{31}. \tag{18}$$

Preliminary Remark 16. The right-hand sides in (17), (18) are homogeneous polynomials of degree 3 in a_{ij}; each of their monomials contains at least one of the following terms: a_{11}, a_{22}, a_{13}. Therefore they are polynomials of degree 3 in f_1, f_2 and their partial derivatives such that each of their monomials contains either f_1 or its partial derivative (Remark 7).

To motivate the construction of f_1, f_2, θ we make the following

Remark 17. Let $(f_1,\ f_2)$ be a good pair, θ be as in Definition 8, and inequalities (16) hold for the correspondent families from Definition 5 and Remark 8. Then for any $r \geq 0$

$$\int_0^{+\infty} \frac{|\det A_{l,m} - det'_{l,m}|}{|a_{33}|}(\zeta,r)d\zeta, \qquad \int_0^{+\infty} \frac{|\chi(\operatorname{tr} A_{l,m}) - \chi(tr)'_{l,m}|}{|a_{33}|}(\zeta,r)d\zeta < +\infty.$$

The motivation of the construction of f_1, f_2, θ. We are looking for a good pair $(f_1,\ f_2)$ such that there exists a $\theta = \theta_s$ that satisfies estimates (16). This implies that the correspondent integrals from Remark 17 should be finite. We show that one can choose f_2 as in Lemma 4 of the following type:

Example 5. Put

$$f(z,r) = \frac{1}{(z^2+r^2+1)^{\frac{s}{2}}}, \quad s > 0, \quad f_2(z,r) = \frac{f(z,r)}{r(l-1)+1}.$$

It appears that for $s < \frac{4}{5}$, $f_1 \equiv f_2$ the pair $(f_1,\ f_2)$ is *good*. For any l large enough the ratios $\frac{|a_{ij}|}{f}$ $((i,j) \neq (1,2),(2,1),(2,3))$, $\frac{f}{|a_{33}|}$ are bounded from above uniformly in (z,r), and therefore there exists a $p_l > 0$ such that the expressions in the integrals from Remark 17 are less than $p_l f^2$ (Remark 16). Hence *if* $\frac{1}{2} < s < \frac{4}{5}$ (e.g., $s = \frac{2}{3}$) then *the integrals from Remark 17 are finite* whenever l is large enough.

Now let us motivate the construction of f_1, f_2, θ. Firstly put

$$\theta = \theta_{\frac{2}{3}}, \quad f_1(z,r) \equiv f_2(z,r) \equiv \frac{1}{r(l-1)+1}\frac{1}{(z^2+r^2+1)^{\frac{1}{3}}}.$$

The pair $(f_1,\ f_2)$ is good. For any $l\geq 2$, $b>0$ there exists a $q>0$ such that

$$-\frac{d\theta}{dt}(z-2br)\geq q|a_{33}|^2(z,r). \tag{19}$$

On the other hand for any l large enough there exists a $q_l>0$ such that the following inequality holds:

$$|\det A_{l,m}-det'_{l,m}|+|\chi(\operatorname{tr}A_{l,m})-\chi(tr)'_{l,m}|<q_l|a_{33}|^3. \tag{20}$$

If this were valid with the change of the right-hand side to $q_l\frac{|a_{33}|^3(z,r)}{r+1}$ then for any l large enough *estimates (16) would hold* (inequality (19)). But for a fixed $l\geq 2$ there is no q_l such that this modified estimate (20) holds.

Let Ψ be a function as at the end of 1. For the proof of Lemma 4 we show that if the above f_1 is changed to $\Psi(r)f_2(z,r)$, then the modified estimate (20) from the last item will hold. To do this we use Remark 16 and prove that the correspondent families a_{ij} satisfy the following stronger system of simple estimates.

Remark 18. Let $(f_1,\ f_2)$ be a good pair, a_{ij} be the correspondent families. Let there exist an $L>1$ with the following property: for any $l>L$ there exists a $p_l>0$ such that

$$|a_{11}|(z,r),\quad |a_{22}|(z,r),\quad |a_{13}|(z,r)<\frac{p_l|a_{33}|(z,r)}{r+1}.$$

Then for any l large enough there exists a $q_l>0$ such that the left-hand side of (20) is less than $\frac{q_l|a_{33}|^3(z,r)}{r+1}$. This follows from Remark 16 and (7).

We show that the new triple $(f_1(z,r)=\Psi(r)f_2(z,r),\ f_2,\ \theta=\theta_{\frac{2}{3}})$ satisfies the statements of Lemma 4.

As it is shown below, the following Lemma implies Lemma 4. Then in 7 we prove that the above Ψ, f_2, θ possess all the properties from this Lemma.

Lemma 5. *Let Ψ be a function as at the end of 1. There exist C^∞-functions $f(z,r)$, $f:\mathbb{R}\times\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+$, $\frac{\partial f}{\partial r}|_{r=0}=0$, θ as in Definition 8 and a $p>0$ with the following properties:*

1) inequality (14) holds;

2) for any $b>0$ there exists a $q>0$ such that

$$-\frac{d\theta}{dt}(z-2br)>qf^2(z,r).$$

3) Let $f_2(z,r)=\frac{1}{r(l-1)+1}f(z,r)$, $f_1(z,r)=\Psi(r)f_2(z,r)$, a_{ij} be the families from Definition 5 correspondent to the tuple $(f_1,\ f_2)$. For any $l\geq 2$ there exists a $p_l>0$ such that

$$|a_{11}|(z,r),\quad |a_{22}|(z,r),\quad |a_{13}|(z,r)<\frac{p_l}{r+1}f(z,r).$$

4) There exist $L\geq 3$, c, c', c_1, c_2, c_3, $c_4>0$ such that for any $l>L$

a) $c_3f<-a_{33}<c_4f$;

b) inequalities (6)-(8) hold in the correspondent sets from Example 3.

Lemma 5 is proved in 7.

Proof of Lemma 4. Let L, f_1, f_2, θ be as in Lemma 5. Let us prove that the triple $(f_1,\ f_2,\ \theta)$ satisfies the statements of Lemma 4. By construction, the pair $(f_1,\ f_2)$

satisfies the conditions of Definition 3. It is good by statement 4b) of Lemma 5. Let us prove estimates (12), (13). By Proposition 2 and statement 1) of Lemma 5, to do this it suffices to prove (16). Let us estimate the left-hand sides of the latters. For any $l > L$, $b > 0$ there exists a $q > 0$ such that inequality (19) holds (statements 2), 4a) of Lemma 5). Now let us estimate the right-hand sides of (16). For any $l > L$ there exists a $q_l > 0$ such that the moduli in the right-hand sides of (16) are less than $\frac{q_l |a_{33}|^3(z,r)}{r+1}$. This follows from Remark 18 and statements 3), 4a), 4b) of Lemma 5. This together with (19) proves that for any $l > L$ there exists a $c_l > 0$ such that (16) hold. Lemma 4 is proved.

7. Proof of Lemma 5.

Let $\Psi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ be as at the end of 1,

$$f(z,r) = \frac{1}{(z^2 + r^2 + 1)^{\frac{1}{3}}}, \quad \theta(t) = \theta_{\frac{2}{3}}(t) = \int_t^{+\infty} \frac{d\tau}{(\tau^2+1)^{\frac{2}{3}}},$$

f_1, f_2 be the correspondent families from Lemma 5. Below we show that f, θ, $c_2 = 27$ satisfy the statements of Lemma 5.

Remark 19. The function Ψ satisfies the inequality

$$\Psi(r) \leq \frac{4}{r+1}.$$

The function f is C^∞, positive, and $\frac{\partial f}{\partial r}|_{r=0} = 0$.
Statement 2) of Lemma 5 is obvious.

Proof of statement 1). Let us estimate $\theta(t)$.

$$\theta(t) \geq \theta(|t|) = \int_{|t|}^{+\infty} \frac{d\tau}{(\tau^2+1)^{\frac{2}{3}}} \geq \int_{|t|}^{+\infty} \frac{d\tau}{(\tau+1)^{\frac{4}{3}}} = \frac{3}{(|t|+1)^{\frac{1}{3}}} \geq \frac{3}{2(t^2+1)^{\frac{1}{3}}}$$

$$= -\frac{3}{2}(t^2+1)^{\frac{1}{3}} \frac{d\theta}{dt}(t) \geq -\frac{3}{2}\frac{d\theta}{dt}(t).$$

Statement 1) is proved.

Proof of statement 4a). We use the following formula for the family a_{33} correspondent to the tuple (f_1, f_2):

$$a_{33}(z,r) = -f(z,r)\Big(1 + \frac{2}{3}\frac{\left(\frac{lr^2}{r(l-1)+1} - z\right)z}{z^2+r^2+1}\Big). \tag{21}$$

Let us prove that for any l large enough

$$\left|\frac{\left(\frac{lr^2}{r(l-1)+1} - z\right)z}{z^2+r^2+1}\right| < \frac{4}{3}. \tag{22}$$

This together with (21) will prove inequality 4a) with $c_3 = \frac{1}{9}$, $c_4 = \frac{17}{9}$.

We use the following

Inequality 1. *For any $t,\ z\in\mathbb{R}$*

$$|\frac{tz+z^2}{t^2+z^2+1}|<\frac{5}{4}.$$

Proof. It suffices to prove Inequality 1 for $t,\ z\geq 0$. In this case the former is implied by the following inequality:

$$5(z^2+t^2+1)-4(tz+z^2)=z^2-4tz+4t^2+5+t^2=(z-2t)^2+t^2+5>0.$$

Inequality 1 is proved.

Now let $l\geq 2$. Let us estimate the left-hand side of (22). It is less or equal to

$$\frac{l}{l-1}\frac{r|z|+z^2}{z^2+r^2+1}<\frac{5}{4}\frac{l}{l-1}$$

(Inequality 1). For any l large enough the right-hand side of the above inequality is less than $\frac{4}{3}$. Therefore there exists an $L>1$ such that for any $l>L$, $c_3=\frac{1}{9}$, $c_4=\frac{17}{9}$ inequality 4a) holds. Statement 4a) of Lemma 5 is proved.

Proof of estimate (7). We use the following formula for the family a_{31} correspondent to $f_1,\ f_2$:

$$a_{31}(z,r)=(\frac{lr}{r(l-1)+1}(2-\frac{(l-1)r}{r(l-1)+1})-\frac{2}{3}\frac{(\frac{lr^2}{r(l-1)+1}-z)r}{z^2+r^2+1})f(z,r). \tag{23}$$

Let us prove that there exist $0<c_1<c_2$ such that for any l large enough (7) holds with the change of a_{33} to $-f$. This together with statement 4a) proved above will prove (7).

It suffices to show that for any l large enough

$$|\frac{(\frac{lr^2}{r(l-1)+1}-z)r}{z^2+r^2+1}|\leq\frac{4}{3}\frac{lr}{r(l-1)+1}. \tag{24}$$

This together with the inequality

$$2\frac{lr}{r(l-1)+1}\geq\frac{lr}{r(l-1)+1}(2-\frac{(l-1)r}{r(l-1)+1})\geq\frac{lr}{r(l-1)+1}$$

and (23) will prove the modified estimate (7) from the last item with $c_1=\frac{1}{9}$, $c_2=3-\frac{1}{9}$. This means that the estimate (7) from Example 3 is valid with $c_2=27$ (statement 4a) with $c_3=\frac{1}{9}$ proved above).

Below we consider that $l\geq 2$. Let us estimate the left-hand side of (24). It is less or equal to

$$\frac{lr}{r(l-1)+1}\frac{r^2+\frac{|z|(r(l-1)+1)}{l}}{z^2+r^2+1}.$$

The second ratio in the above expression is less or equal to

$$\frac{r^2+|z|r}{z^2+r^2+1}+\frac{1}{l}\frac{|z|}{z^2+r^2+1}<\frac{5}{4}+\frac{1}{l}$$

(Inequality 1). Therefore for any l large enough this ratio is less than $\frac{4}{3}$. This proves estimate (24). Estimate (7) is proved.

Proof of estimates (6), (8). The first estimate in (6) follows from statement 4a), which is proved above. Let us prove its third inequality. Then we prove (8) and the second inequality of (6).

We use the formula

$$a_{22}(z,r) = \frac{r-1}{r(l-1)+1}\Psi(r)f(z,r). \tag{25}$$

Let $L \geq 3$, c_3, c_4 be as in 4a). By the latter and (25), for any $l > L$

$$a_{22} < \frac{f}{l-1} \leq -\frac{a_{33}}{c_3(l-1)}.$$

Therefore for $c = \frac{2}{c_3}$, $l > L$ the third inequality from (6) holds.

Now let us prove (8). By (25), for any $l \geq 3$, $z \in \mathbb{R}$, $r \leq \frac{1}{l-1}$

$$a_{22} \leq -\frac{1}{4}\Psi f = -\frac{f}{4}.$$

By 4a) and the last inequality, for $c' = \frac{1}{4c_4}$, $l > L$ (8) holds in the set $r \leq \frac{1}{l-1}$. Estimate (8) is proved.

Let us prove the second estimate of (6). To do this we show that there exists a $q > 0$ such that for any $l \geq 2$ $a_{11} < \frac{qf}{l-1}$. This together with 4a) will prove the second estimate of (6). We use the formula

$$\begin{aligned} a_{11}(z,r) &= \frac{\partial}{\partial r}\left(\frac{r(r-1)}{r(l-1)+1}\Psi(r)f(z,r)\right) = \left(\frac{r(r-1)}{r(l-1)+1}\left(\frac{\partial \ln f(z,r)}{\partial r} + \frac{d\ln\Psi(r)}{dr}\right)\right. \\ &+ \left.\frac{2r-1}{r(l-1)+1} - \frac{(l-1)r(r-1)}{(r(l-1)+1)^2}\right)\Psi(r)f(z,r) = \left(\frac{r(r-1)}{r(l-1)+1}\left(-\frac{2}{3}\frac{r}{z^2+r^2+1}\right.\right. \\ &+ \left.\left.\frac{d\ln\Psi(r)}{dr}\right) + \frac{(l-1)r^2}{(r(l-1)+1)^2} + \frac{2r-1}{(r(l-1)+1)^2}\right)\Psi(r)f(z,r). \end{aligned} \tag{26}$$

The first factor in the right-hand side of (26) is bounded from above by

$$\frac{1}{l-1}\left((r+1)\left(\frac{2}{3}\frac{r}{z^2+r^2+1} + \left|\frac{\partial \ln\Psi(r)}{\partial r}\right|\right) + 1 + 2\right).$$

The second factor in the last expression does not depend on l and is bounded from above uniformly in (z,r). This proves the estimate from the beginning of the item. Estimate (6) is proved. The proof of estimates (6)-(8) is completed. Statement 4b) of Lemma 5 is proved.

Proof of statement 3). Let $l \geq 2$. We prove the estimates from statement 3) of Lemma 5 with the change of their right-hand side to $p_l\Psi(r)f(z,r)$. This together with Remark 19 will prove statement 3).

Firstly let us estimate $|a_{22}|$. By (25),

$$|a_{22}|(z,r) \leq \frac{r+1}{r(l-1)+1}\Psi(r)f(z,r) \leq \Psi(r)f(z,r).$$

This proves the estimate for a_{22} from the last item with $p_l = 1$.

Now let us estimate a_{11}. For any fixed $l \geq 2$ there exists a $p_l > 0$ such that the module of the first factor in the right-hand side of (26) is less than p_l for all $r \geq 0$, $z \in \mathbb{R}$. Indeed for a fixed $l \geq 2$ each its term has a bounded module. This proves the estimate for $|a_{11}|$.

Let us estimate $|a_{13}|$. We use the formula

$$a_{13}(z,r) = \frac{r(r-1)}{r(l-1)+1}\Psi(r)\frac{\partial f}{\partial z}(z,r) = -\frac{2}{3}\frac{r(r-1)z}{(r(l-1)+1)(z^2+r^2+1)}\Psi(r)f(z,r). \tag{27}$$

The module of the second ratio in the right-hand side of (27) is bounded from above uniformly in $l \geq 2$, (z,r). This proves the estimate for $|a_{13}|$. Statement 3) of Lemma 5 is proved. The proof of Lemma 5 is completed.

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