Density of compositions of thin film planar billiard reflections in symplectomorphism pseudogroup.

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1 Main result: density of compositions of billiard reflections

It is well-known that billiard reflections acting in the space of oriented geodesics preserve the canonical symplectic form [1, 2, 3, 4, 5, 8]. However only a tiny part of symplectomorphisms are realized by reflections. There is an important open question stated in [7]: which symplectomorphisms can be realized by compositions of reflections?

In the present paper we consider the case of planar billiards. We show that the above compositions are dense in some domain in the space of local symplectomorphisms of appropriate region.

Let $\gamma \subset \mathbb{R}^2$ be germ of planar $C^\infty$-smooth convex curve at a point $O$ equipped with natural length parameter $s$, $s(O) = 0$. We consider that the orientation induced by the parameter $s$ is counterclockwise. Consider the (germ of) space $\Pi$ of oriented lines intersecting $\gamma$ transversally at one point and directed outside the convex domain bounded by $\gamma$. It is identified with the space of pairs $(s, \phi)$, where $s$ is the point of intersection of an oriented line $L$ with $\gamma$ and $\phi$ is its angle with the tangent line $T_s \gamma$. More precisely, $\phi$ is the oriented angle between the negatively, clockwise oriented tangent line $T_s \gamma$ and the oriented line $L$, $0 < \psi_1(s) < \phi < \psi_2(s) < \pi$. The domain $\Pi$ is a topological disk.

The billiard reflection from the curve $\gamma$, which will be denoted by $T_\gamma$, is defined on $\Pi$ as follows. Namely, take an oriented line $L$ as above, let $A$ denote its intersection point with $\gamma$. The image $T_\gamma(L)$ is the line symmetric to $L$ with respect to the tangent line $T_A \gamma$ and directed at $A$ inside the convex domain bounded by $\gamma$.

It is well-known that billiard reflections with respect to all the curves preserve a canonical symplectic (area) form on the space $\Pi$ of oriented lines
In the above coordinates $(s, \phi)$ the invariant symplectic form is
\[ \omega = \sin \phi ds \wedge d\phi. \]

Consider a small deformation $\gamma_\varepsilon$ of the curve $\gamma$ depending on a small parameter $\varepsilon$. In what follows we study the compositions
\[ \Delta T_\varepsilon := T_{\gamma_\varepsilon}^{-1} \circ T_\gamma. \]

The derivative in $\varepsilon$ of the composition $\Delta T_\varepsilon$ at $\varepsilon = 0$ is a vector field on $\Pi$ that is divergence free with respect to the area form $\omega$. There vector fields were introduced and studied by Ron Perline in [6].

We study the Lie algebra generated by the vector fields coming from all the deformations $\gamma_\varepsilon$. We prove that the latter Lie algebra is dense in the space of all the divergence free vector fields. As a corollary, this implies that the compositions of reflections from curves and their inverses are dense in the space of area-preserving mappings defined on $\Pi$.

Let us pass to the statements of main results.

We consider deformations $\gamma_\varepsilon$ of the curve $\gamma$ defined by $C^\infty$-smooth functions $h(s)$ as follows. Consider the exterior unit normal field $\vec{n}(s)$ on $\gamma$. By definition, $\gamma_\varepsilon$ is the parametrized curve
\[ \gamma_\varepsilon(s) := \gamma(s) + \varepsilon h(s) \vec{n}(s). \]

The vector field
\[ v_h := \frac{d\Delta T_\varepsilon}{d\varepsilon}|_{\varepsilon=0} \]
on $\Pi$ preserves the canonical area form. The fields $v_h$ and similar fields in any dimensions were calculated by Ron Perline [6]. In our planar case in the coordinates $(s, \phi)$ on $\Pi$ one has
\[ v_h := (-2h(s) \cot \phi, 2h'(s)). \]  

(1.1)

In what follows we denote
\[ y := \cot \phi. \]

The vector fields $v_h$ corresponding to all the possible $C^\infty$-smooth functions $h(s)$ form the vector subspace
\[ L_0 := \{ (-h(s)y, h'(s)) \mid h \text{ is a function of the parameter } s \} \]
in the space of vector fields on $\Pi$. It is not a Lie algebra.
Proposition 1.1 The commutator

\[ [L_0, L_0] = L_1 \]

is a Lie algebra. It coincides with the vector space of fields of type \((h(s), yh'(s))\) where \(h(s)\) runs through all the \(C^\infty\)-smooth functions of one variable.

Theorem 1.2 The Lie algebra of vector fields generated by the vector space \(L_0\) is dense in the space of area-preserving vector fields on \(\Pi\) with respect to the area form \(\omega\) in the topology of uniform convergence with derivatives on compact subsets in \(\Pi\).

Fix arbitrary small \(\sigma > 0\) and \(\delta > 0\) such that the line intersecting the arc \(|s| \leq \sigma\) of the curve \(\gamma\) by angle between \(\delta\) and \(\pi - \delta\) (including these values) intersects this arc once. Reflections from the curves \(\gamma\) and \(\gamma_\varepsilon\) are well-defined for \(\varepsilon\) small enough on the following compact subset in \(\Pi\):

\[ \Pi_{O,\sigma,\delta} := \{(s, \phi) \mid |s| \leq \sigma, \delta \leq \phi \leq \pi - \delta\}. \]

Corollary 1.3 For every \(\sigma, \delta > 0\) as above those compositions of billiard reflections from curves close to \(\gamma\) and of their inverses that are well-defined on \(\Pi_{O,\sigma,\delta}\) are dense in the space of symplectomorphisms from \(\Pi_{O,\sigma,\delta}\) onto a subset in \(\Pi\) in the topology of uniform convergence.

Remark 1.4 Consider the case of a closed convex planar curve \(\gamma\). Recall that the phase cylinder associated to such a curve \(\gamma\) is the space of oriented lines intersecting \(\gamma\) transversally. It would be interesting to study extendability of the above theorem and corollary to the Lie algebra of Hamiltonian vector fields and Hamiltonian symplectomorphisms of the phase cylinder. It would be interesting to study extensions of these results to higher dimensions. This is a work in progress.

2 Proof of Theorem 1.2

Let \(\mathfrak{g}\) denote the Lie algebra generated by the vector space \(L_0\), see Theorem 1.2. Each divergence free vector field on \(\Pi\) is Hamiltonian with respect to the form \(\omega\), by simple connectivity of the domain \(\Pi\). Lie bracket of two Hamiltonian vector fields is the Hamiltonian vector field for the Poisson bracket of their Hamiltonians. Therefore, for the proof of Theorem 1.2 it suffices to prove density of the Hamiltonians of the vector fields from the algebra \(\mathfrak{g}\) in the space of functions of two variables \((s, \phi)\).
In the proof of Theorem 1.2 we use the following formulas for a Hamiltonian vector field $\chi_H$ with Hamiltonian $H(s, \phi)$ and for Poisson bracket in the coordinates $(s, \phi)$:

$$\chi_H = \frac{1}{\sin \phi} \left( \frac{\partial H}{\partial \phi}, - \frac{\partial H}{\partial s} \right); \quad \{F, G\} = \frac{1}{\sin \phi} \left( \frac{\partial F}{\partial s} \frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi} \frac{\partial G}{\partial s} \right). \quad (2.1)$$

We also use the following trigonometric formulas:

$$|\sin \phi| = \frac{1}{\sqrt{1 + \cot^2 \phi}} = \frac{1}{\sqrt{1 + y^2}}; \quad y' = \cot'(\phi) = -(1 + y^2). \quad (2.2)$$

Note that our angles $\phi$ always lie in the interval $(0, \pi)$. Thus, $\sin \phi > 0$, and we can omit the above module sign.

**Proposition 2.1** Each vector field $(-h(s)y, h'(s)) \in L_0$, $y = \cot \phi$, is Hamiltonian with the Hamiltonian

$$H(s, \phi) = -h(s)\sin \phi = -\frac{1}{\sqrt{1 + y^2}}h(s).$$

The proposition follows from the first formula in (2.1).

For every $d \in \mathbb{Z}_{\geq 0}$ and every function $f(s)$ set

$$H_{d,f}(s, \phi) := \frac{y^d}{\sqrt{1 + y^2}}f(s), \quad y := \cot \phi.$$

**Proposition 2.2** For every $d, k \in \mathbb{Z}_{\geq 0}$ and every two functions $f(s), g(s)$ one has

$$\{H_{d,f}, H_{k,g}\} = H_{d+k-1,dfg' - kg'f} + H_{d+k+1, (d-1)f'g' - (k-1)f'g}. \quad (2.3)$$

**Proof** For every $m \in \mathbb{Z}_{\geq 0}$ one has

$$\left( \frac{g^m}{\sqrt{1 + y^2}} \right)' = -(m - 1) \frac{g^{m+1}}{\sqrt{1 + y^2}} + m \frac{g^{m-1}}{\sqrt{1 + y^2}},$$

which follows from (2.2). Therefore, for every function $q(s)$ one has

$$\frac{\partial H_{m,q}}{\partial \phi} = -((m - 1)H_{m+1,q} + mH_{m-1,q}).$$

The latter formula together with (2.1) imply (2.3).
Corollary 2.3 For every $d \in \mathbb{Z}_{\geq 0}$ let $\Lambda_d$ denote the vector space of functions of the type $H_{d,f}$, where $f(s)$ runs through all the $C^\infty$-smooth functions in one variable. One has

$$\{\Lambda_0, \Lambda_0\} = \Lambda_1, \quad \{\Lambda_1, \Lambda_1\} \subset \Lambda_1,$$  

(2.4)

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} + \Lambda_{d+k+1} \text{ whenever } (d, k) \neq (1, 1), (0, 0),$$

(2.5)

$$\{\Lambda_0, \Lambda_k\}/\Lambda_{\leq k} = \Lambda_{k+1}(\text{mod } \Lambda_{\leq k}), \quad \Lambda_{\leq k} := \sum_{d=0}^{k} \Lambda_d.$$  

(2.6)

Therefore, the Lie algebra generated by $\Lambda_0$ under the Poisson bracket coincides with

$$\mathfrak{g} = \oplus_{d \geq 0} \Lambda_d.$$

Proof The corollary follows from formula (2.3). In more detail, let us prove the left formula in (2.4). One has

$$\{H_{0,f}, H_{0,g}\} = H_{1,f'g' - g'f},$$

by (2.3). It is clear that each function $\eta(s)$ can be represented by expression $f'g - g'f$, since the functions under question are defined on an interval. This proves the left formula in (2.4). The proof of statement (2.6) is analogous. The other statements of the corollary follow immediately from (2.3).  

Proposition 2.4 The above Lie algebra $\mathfrak{g} = \oplus_{d \geq 0} \Lambda_d$ is dense in the space of smooth functions of two variables $(s, \phi)$ in $C^\infty$-topology: in the topology of uniform convergence with derivatives of all orders on compact subsets.

Proof The polynomials in two variables $(s, y)$ divided by $\sqrt{1 + y^2}$ are dense (Weierstrass Theorem). Therefore, finite sums

$$\sum_{d=0}^{m} \frac{y^d}{\sqrt{1 + y^2}} f_d(s) = \sum_{d=0}^{m} H_{d,f_d},$$

where $f_d$ are arbitrary smooth functions, are also dense. This implies the statement of the proposition.  

Corollary 2.3 together with Proposition 2.4 imply Theorem 1.2, which, in its turn, implies Corollary 1.3.
References


