The uses of Dyson-Schwinger equations

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Contents

CHAPTER 1

Introduction

1.0.1. Some historical references. These lecture notes concern the study of the asymptotics of large systems of particles in very strong mean field interaction and in particular the study of their fluctuations. Examples are given by the distributions of eigenvalues of Gaussian random matrices, β-ensembles, random tilings and discrete β-ensembles, or several random matrices. These models display a much stronger interaction between the particles than the underlying randomness so that classical tools from probability theory fail. Fortunately, these model have in common that their correlators (basically moments of a large class of test functions) obey an infinite system of equations that we will call the Dyson-Schwinger equations. They are also called loop equations, Master equations or Ward identities. Dyson-Schwinger equations are usually derived from some invariance or some symmetry of the model, for instance by some integration by parts formula. We shall argue in these notes that even though these equations are not closed, they are often asymptotically closed (in the limit where the dimension goes to infinity) so that we can asymptotically solve them and deduce asymptotic expansions for the correlators. This in turn allows to retrieve the global fluctuations of the system, and eventually even more local information such as rigidity.

This strategy has been developed at the formal level in physics [**2**] for a long time. In particular in the work of Eynard and collaborators [**48, 47, 46, 15**], it was shown that if one assumes that correlators expand formally in the dimension N, then the coefficients of these expansions obey the so-called topological recursion. For instance, in [**26, 27**], it was shown that assuming a formal expansion holds, Dyson-Schwinger equations induce recurrence relations on the terms in the expansion which can be solved by algebraic geometry means. These recurrence relations can even be interpreted as topological recursion, so that the coefficients of these expansions can be given combinatorial interpretations. In fact, it was realized in the seminal works of t'Hooft [**86**] and Brézin-Parisi-Itzykson-Zuber [**40**] that moments of Gaussian matrices and matrix models can be interpreted as the generating functions for maps. One way to retrieve this result is by using Dyson-Schwinger equations and checking that asymptotically they are similar to the topological recursion formulas obeyed by the enumeration of maps, as found by Tutte [**90**]. In this case, one first need to analyze the limiting behavior of the system, given by the so-called equilibrium measure or spectral curve, and then the Dyson-Schwinger equations, that is the topological recursion, will provide the large dimension expansion of the observables.

The study of the asymptotics of our large system of particles also starts with the analysis of its limiting behaviour. I usually derive this limiting behaviour as the minimizer of an energy functional appearing as a large deviation rate functional [**7**], or in concentration of measure estimates [**71**], but, according to fields, people can prefer to see it as the optimizer of Fekete points [**79**], or as the solution of a Riemann-Hilbert problem [**34**]. This study often amounts to the analysis of some equation. The same type of analysis appears in combinatorics when one counts for example triangulations of the sphere. Indeed, it can be seen, thanks to Tutte surgery [**90**], that the generating function for this enumeration satisfies some equation. Sometimes, one can solve explicitly this equation, for instance thanks to the quadratic method and catalytic variables [**21, 24**] or [**53**, Section 2.9]. In our models, we will also be able to derive equations for our equilibrium measure thanks to Dyson-Schwinger equations. But sometimes, these equations may have several solutions, for instance in the setting of a double well potential in β-models. The absence of uniqueness of solutions to these equations prevents the analysis of many interesting models, such as several matrix models at low temperature. In good cases such as the β-models, we may still get uniqueness for instance if we add the information that the equilibrium measure minimizes a strictly convex energy. Dyson-Schwinger equation can then be regarded as the equations satisfied by the critical points of this energy.

The Dyson-Schwinger equations will be our key to get precise informations on the convergence to equilibrium, such as large dimension expansion of the free energy or fluctuations. These types of questions were attacked also in the Riemann Hilbert problems community based on a fine study of the asymptotics of orthogonal polynomials [**51, 32, 41, 10, 23, 33**]. It seems to me however that such an approach is more rigid as it requires more technical steps and assumptions and can not apply in such a great generality than loop equations. Yet, when it can be used, it provides eventually more detailed information. Moreover, in certain cases, such as the case of potentials with Fisher Hartwig singularities, Riemann Hilbert techniques could be used but not yet loop equations [**62, 35**].

To study the asymptotic properties of our models we need to get one step further than the formal approach developped in the physics litterature. In other words, we need to show that indeed correlators can expand in the dimension up to some error which is quantified in the large N limit and shown to go to zero. To do so, one needs in general a priori concentration bounds in order to expand the equations around their limits. For β-models, such a priori concentration of measure estimates can be derived thanks to a result of Boutet de Monvel, Pastur and Shcherbina [**22**] or Maida and Maurel-Segala [**71**]. It is roughly based on the fact that the logarithm of the density of such models is very close to a distance of the empirical measure to its equilibrium measure, hence implying a priori estimates on this distance. In more general situations where densities are unknown, for instance when one considers the distributions of the traces of polynomials in several matrices, one can rely on abstract concentration of measures estimates for instance in the case where the density is strictly log-concave or the underlying space has a positive Ricci curvature (e.g $SU(N)$) [**57**]. Dyson-Schwinger equations are then crucial to obtain optimal concentration bounds and asymptotics.

This strategy was introduced by Johansson [**63**] to derive central limit theorems for β-ensembles with convex potentials. It was further developped by Shcherbina and collaborators [**1, 82**] and myself, together with Borot [**52**], to study global fluctuations for β-ensembles when the potential is off-critical in the sense that the equilibrium has a connected support and its density vanishes like a square root at its boundary. These assumptions allow to linearize the Dyson-Schwinger equations around their limit and solve these linearizations by inverting the so-called Master operator. The case where the support of the density has finitely many connected component but the potential is off-critical was adressed in [**80, 14**]. It displays the additional tunneling effect where eigenvalues may jump from one connected support to the other, inducing discrete fluctuations. However, it can also be solved asymptotically after a detailed analysis of the case where the number of particles in each connected components is fixed, in which case Dyson-Schwinger equations can be asymptotically solved. These articles assumed that the potentials are real analytic in order to use Dyson-Schwinger equations for the Stieltjes functions. We will see that these techniques generalize to sufficiently smooth potentials by using more general Dyson-Schwinger equations. Global fluctuations, together with estimates of the Wasserstein distance, were obtained in [**67**] for off-critical, one-cut smooth potentials. One can obtain by such considerations much more precise estimates such as the expansion of the partition function up to any order for general off-critical cases with fixed filling fractions, see [**14**]. Such expansion can also be derived by using Riemann-Hilbert techniques, see [**42**] in a perturbative setting and [**28**] in two cut cases and polynomial potential.

But β-models on the real line serve also as toy models for many other models. Borot, Kozlowski and myself considered more general potentials depending on the empirical measure in [**16**]. We studied also the case of more complicated interactions (in particular sinh interactions) in [**17**] : the main problems are then due to the non-linearity of the interaction which induces multi-scale phenomenon. The case of critical potentials was tackled recently in [**37**]. Also Dyson-Schwinger (often called Ward identities) equations are instrumental to study Coulomb gas systems in higher dimension. One however has to deal with the fact that Ward identities are not nice functions of the empirical measure anymore, so that an additional term, the anisotropic term, has to be controlled. This could very nicely be done by Leblé and Serfaty [**68**] by using local large deviations estimates. Recently we also generalized this approach to study discrete β-ensembles and random tilings [**12**] by analyzing the so-called Nekrasov's equations in the spirit of Dyson-Schwinger equations.

The same approach can be developed to study multi-matrix questions. Originally, I developed this approach to study fluctuations and large dimension expansion of the free energy with E. Maurel Segala [**54, 55**] in the context of several random matrices. In this case we restrict ourselves to perturbations of the quadratic potential to insure convergence and stability of our equations. We could extend this study to the case of unitary or orthogonal matrices following the Haar measure (or perturbation of this case) in [**29, 56**]. This strategy was then applied in a closely related setting by Chatterjee [**25**], see also [**31**].

Dyson-Schwinger equations are also central to derive more local results such as rigidity and universality, showing that the eigenvalues are very close to their deterministic locus and that their local fluctuations does not depend much on the model. For instance, in the case of Wigner matrices with non Gaussian entries, a key tool to prove rigidity is to show that the Stieltjes transform approximately satisfies the same quadratic equation than in the Gaussian case up to the optimal scale [**44, 43, 4**]. Recently, it was also realized that closely connected ideas could lead to universality of local fluctuations, on one hand by using the local relaxation flow [**44, 69, 19**], by using Lindenberg strategy [**84, 85**] or by constructing approximate transport maps [**82, 5, 50**]. Such ideas could be generalized in the discrete Beta ensembles [**61**] where universality could be derived thanks to optimal rigidity (based on the study of Nekrasov's equations) and comparisons to the continuous setting.

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1.0.2. A toy model. Let us give some heuristics for the type of analysis we will do in these lectures thanks to a toy model. We will consider the distribution of N real-valued variables $\lambda_1, \ldots, \lambda_N$ and denote by

$$\hat{\mu}^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

their empirical measure : for a test function f, $\hat{\mu}^N(f) = \frac{1}{N}\sum f(\lambda_i)$. Then, the correlators are moments of the type

$$M(f_1, \ldots, f_p) = \mathbb{E}[\prod_{i=1}^{p} \hat{\mu}^N(f_i)]$$

where f_i are test functions, which can be chosen to be polynomials, Stieltjes functionals or some more general set of smooth test functions. Dyson-Schwinger equations are usually retrieved from some underlying invariance or symmetries of the model. Let us consider the continuous case where the law of the λ_i's is absolutely continuous with respect to $\prod d\lambda_i$ and given by

$$dP_N^V(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N^V} \exp\{-\sum_{i_1=1}^{N}\sum_{i_2=1}^{N} V(\lambda_{i_1}, \lambda_{i_2})\} \prod d\lambda_i$$

where V is some symmetric smooth function. Then a way to get equations for the correlators is simply by integration by parts (which is a consequence of the invariance of Lebesgue measure under translation) : Let $f_0, f_1, \ldots, f_\ell$ be continuously differentiable functions. Then

$$\begin{aligned}
\mathbb{E}[\hat{\mu}^N(f_0')\prod_{i=1}^{\ell}\hat{\mu}^N(f_i)] &= \mathbb{E}[\left(\frac{1}{N}\sum_k \partial_{\lambda_k} f_0(\lambda_k))\right)\prod_{i=1}^{\ell}\hat{\mu}^N(f_i)] \\
&= -\frac{1}{N}\mathbb{E}\left[(\frac{dP_N^V}{d\lambda})^{-1}\sum_k f_0(\lambda_k)\partial_{\lambda_k}\left(\prod_{i=1}^{\ell}\hat{\mu}^N(f_i)(\frac{dP_N^V}{d\lambda})\right)\right] \\
&= 2N\mathbb{E}[\left(\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\hat{\mu}^N(x_1)d\hat{\mu}^N(x_2)\right)\prod_{i=1}^{\ell}\hat{\mu}^N(f_i)] \\
&\quad -\frac{1}{N}\sum_{j=1}^{\ell}\mathbb{E}[\left(\hat{\mu}^N(f_0 f_j')\right)\prod_{i\neq j}\hat{\mu}^N(f_i)]
\end{aligned}$$

where we noticed that since V is symmetric $\partial_x V(x,x) = 2\partial_x V(x,y)|_{y=x}$. The case $\ell = 0$ refers to the case $f_1 = \cdots = f_\ell = 1$. We will call the above equations Dyson -Schwinger equations. One would like to analyze the asymptotics of the correlators. The idea is that if we can prove that $\hat{\mu}^N$ converges, then we can linearize the above equations around this limit, and hopefully solve them asymptotically by showing that only few terms are relevant on some scale, solving these simplified equations and then considering the equations at the next order of correction. Typically in the case above, we see that if $\hat\mu^N$ converges towards μ^* almost surely (or in L^p) then by the previous equation (with $\ell=0$) we must have

$$\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\mu^*(x_1)d\mu^*(x_2)=0\,. \tag{1.1}$$

We can then linearize the equations around μ^* and we find that if we set $\Delta_N=\hat\mu^N-\mu^*$, we can rewrite the above equation with $\ell=0$ as

$$\mathbb{E}[\Delta_N(\Xi f_0)]=\frac{1}{N}\mathbb{E}[\hat\mu^N(f_0')]-2\mathbb{E}[\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\Delta_N(x_1)d\Delta_N(x_2)] \tag{1.2}$$

where Ξ is the Master operator given by

$$\Xi f_0(x)=2f_0(x)\int\partial_{x_1}V(x,x_1)d\mu^*(x_1)+2\int f_0(x_1)\partial_{x_1}V(x_1,x)d\mu^*(x_1)\,.$$

Let us show heuristically how such an equation can be solved asymptotically. Let us assume that we have some a priori estimates that tell us that Δ_N is of order δ_N almost surely (or in all L^k's)[that is that for sufficiently smooth functions g, $\Delta_N(g)=(\hat\mu^N-\mu^*)(g)$ is with high probability (i.e with probability greater than $1-N^{-D}$ for all D and N large enough) at most of order $\delta_N C_g$ for some finite constant C_g]. Then, the right hand side of (1.2) should be smaller than $\max\{\delta_N^2,N^{-1}\}$ for sufficiently smooth test functions. Hence, if we can invert the master operator Ξ, we see that the expectation of Δ_N is of order at most $\max\{\delta_N^2,N^{-1}\}$. We would like to bootstrap this estimate to show that δ_N is at most of order N^{-1}. This requires to estimate higher moments of Δ_N. Let us do a similar derivation from the Dyson-Schwinger equations when $\ell=1$ to find that if $\bar\Delta_N(f)=\Delta_N(f)-\mathbb{E}[\Delta_N(f)]$,

$$\begin{aligned}\mathbb{E}[\Delta_N(\Xi f_0)\bar\Delta_N(f_1)] &= -2\mathbb{E}[\int f_0(x_1)\partial_{x_1}V(x_1,x_2)d\Delta_N(x_1)d\Delta_N(x_2)\bar\Delta_N(f_{0)}] \\ &\quad +\frac{1}{N}\mathbb{E}[\Delta_N(f_0')\bar\Delta_N(f_1)]+\frac{1}{N^2}\mathbb{E}[\hat\mu^N(f_0f_1')]\,.\end{aligned} \tag{1.3}$$

Again if Ξ is invertible, this allows to bound the covariance $\mathbb{E}[\Delta_N(f_0)\bar\Delta_N(f_1)]$ by $\max\{\delta_N^3,\delta_N^2/N,N^{-2}\}$, which is a priori better than δ_N^2 unless δ_N is of order $1/N$. Since $\Delta_N(f)-\bar\Delta_N(f)$ is at most of order δ_N^2 by (1.2), we deduce that also $\mathbb{E}[\Delta_N(f_0)\Delta_N(f_1)]$ is at most of order δ_N^3. We can plug back this estimate into the previous bound and show recursively (by considering higher moments) that δ_N can be taken to be of order $1/N$ up to small corrections. We then deduce that

$$C(f_0,f_1)=\lim_{N\to\infty}N^2\mathbb{E}[(\Delta_N-\mathbb{E}[\Delta_N])(f_0)(\Delta_N-\mathbb{E}[\Delta_N])(f_1)]=\mu^*(\Xi^{-1}f_0f_1')$$

and

$$m(f_0)=\lim_{N\to\infty}N\mathbb{E}[\Delta_N(f_0)]=\mu^*((\Xi^{-1}f_0)')\,.$$

We can consider higher order equations (with $\ell\ge1$) to deduce higher orders of corrections, and the convergence of higher moments.

1.0.3. Rough plan of the lecture notes. We will apply these ideas in several cases where V has a logarithmic singularity in which case the self-interaction term in the potential has to be treated with more care. More precisely we will examine the following models.

(1) *The law of the GUE.* We consider the case where the λ_i are the eigenvalues of the GUE and we take polynomial test functions. In this case the operator Ξ is triangular and easy to invert. Convergence towards μ^* and a priori estimates on Δ_N can also be proven from the Dyson-Schwinger equations.
(2) *The Beta ensembles.* We take smooth test functions. Convergence of $\hat{\mu}^N$ is proven by large deviation principle and quantitative estimates on δ_N are obtained by concentration of measure. The operator Ξ is invertible if μ^* has a single cut, with a smooth density which vanishes like a square root at the boundary of the support. We then obtain full expansion of the correlators. In the case where the equilibrium measure has p connected components in its support, we can still follow the previous strategy if we fix the number of eigenvalues in a small neighborhood of each connected pieces (the so-called filing fractions). Summing over all possible choices of filing fractions allows to estimate the partition functions as well as prove a form of central limit theorem depending on the fluctuations of the filling fractions.
(3) *Discrete Beta ensembles.* These distributions include the law of random tilings and the λ_i's are now discrete. Integration by parts does not give nice equations a priori but Nekrasov found a way to write new equations by showing that some observables are analytic. These equations can in turn be analyzed in a spirit very similar to continuous Beta-ensembles.
(4) *Several matrix models.* In this case, large deviations results are not yet known despite candidates for the rate function were proposed by Voiculescu [**94**] and a large deviation upper bound was derived [**9**]. However, we can still write the Dyson-Schwinger equations and prove that limits exist provided we are in a perturbative setting (with respect to independent GUE matrices). Again in perturbative settings we can derive the expansion of the correlators by showing that the Master operator is invertible.

We will discuss also one idea related with our approach based on Dyson-Schwinger to study more local questions, in particular universality of local fluctuations. The first is based on the construction of approximate transport maps as introduced in [**5**]. The point is that the construction of this transport maps goes through solving a Poisson equation $Lf = g$ where L is the generator of the Langevin dynamics associated with our invariant measure. It is symmetric with respect to this invariant measure and therefore closely related with integration by parts. In fact, solving this Poisson equation is at the large N limit closely related with inverting the master operator Ξ above, and in general follows the strategy we developed to analyze Dyson-Schwinger equations. Another strategy to show universality of local fluctuations is by analyzing the Dyson-Schwinger equations but for less smooth test functions, that is prove local laws. We will not developp this approach here. These ideas were developed in [**61**] for discrete beta-ensembles, based on a strategy initiated in [**20**]. The argument is to show that optimal bounds on Stieltjes functionals can be derived from Dyson-Schwinger equation away from the support of the equilibrium measure, but at some distance. It is easy to get it at distance of order $1/\sqrt{N}$, by straightforward concentration inequalities. To get estimates up to distance of order $1/N$, the idea is to localize the measure. Rigidity follows from this approach, as well as universality eventually.

CHAPTER 2

The example of the GUE

In this section, we show how to derive topological expansions from Dyson-Schwinger equations for the simplest model : the GUE. The Gaussian Unitary Ensemble is the sequence of $N \times N$ hermitian matrices $X_N, N \geq 0$ such that $(X_N(ij))_{i\leq j}$ are independent centered Gaussian variables with variance $1/N$ that are complex outside of the diagonal (with independent real and imaginary parts). Then, we shall discuss the following expansion, true for all integer k

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}(X_N^k)] = \sum_{g\geq 0} \frac{1}{N^{2g}} M_g(k)\,.$$

This expansion is called a topological expansion because $M_g(k)$ is the number of maps of genus g which can be build by matching the edges of a vertex with k labelled half-edges. We remind here that a map is a connected graph properly embedded into a surface (i.e so that edges do not cross). Its genus is the smallest genus of a surface so that this can be done. This identity is well known [**96**] and was the basis of several breakthroughs in enumerative geometry [**60, 64**]. It can be proven by expanding the trace into products of Gaussian entries and using Wick calculus to compute these moments. In this section, we show how to derive it by using Dyson-Schwinger equations.

2.0.1. Combinatorics versus analysis. In order to calculate the electromagnetic momentum of an electron, Feynman used diagrams and Schwinger used Green's functions. Dyson unified these two approaches thanks to Dyson-Schwinger equations. On one hand they can be thought as equations for the generating functions of the graphs that are enumerated, on the other they can be seen as equations for the invariance of the underlying measure. A baby version of this idea is the combinatorial versus the analytical characterization of the Gaussian law $\mathcal{N}(0,1)$. Let X be a random variable with law $\mathcal{N}(0,1)$. On one hand it is the unique law with moments given by the number of matchings :

$$\mathbb{E}\left[X^n\right] = \#\left\{\text{pair partitions of } n \text{ points}\right\} =: P_n\,. \tag{2.1}$$

On the other hand, it is also defined uniquely by the integration by parts formula

$$\mathbb{E}\left[Xf(X)\right] \quad = \quad \mathbb{E}\left[f'(X)\right] \tag{2.2}$$

(2.3)

for all smooth functions f going to infinity at most polynomially. If one applies the latter to $f(x) = x^n$ one gets

$$m_{n+1} := \mathbb{E}\left[X^{n+1}\right] \quad = \quad \mathbb{E}\left[nX^{n-1}\right] = nm_{n-1}\,.$$

This last equality is the induction relation for the number P_{n+1} of pair partitions of $n+1$ points by thinking of the n ways to pair the first point. Since $P_0 = m_0 = 1$ and $P_1 = m_1 = 0$, we conclude that $P_n = m_n$ for all n. Hence, the integration by parts formula and the combinatorial interpretation of moments are equivalent.

2.0.2. GUE : combinatorics versus analysis. When instead of considering a Gaussian variable we consider a matrix with Gaussian entries, namely the GUE, it turns out that moments are as well described both by integration by parts equations and combinatorics. In fact moments of GUE matrices can be seen as generating functions for the enumeration of interesting graphs, namely maps, which are sorted by their genus. We shall describe the full expansion, the so-called topological expansion, at the end of this section and consider more general colored cases in section 7. In this section, we discuss the large dimension expansion of moments of the GUE up to order $1/N^2$ as well as central limit theorems for these moments, and characterize these asymptotics both in terms of equations similar to the previous integration by parts, and by the enumeration of combinatorial objects.

Let us be more precise. A matrix $X = (X_{ij})_{1\le i,j\le N}$ from the GUE is the random $N \times N$ Hermitian matrix so that for $k < j$, $X_{kj} = X^{\mathbb{R}}_{kj} + iX^{i\mathbb{R}}_{kj}$, with two independent real centered Gaussian variables with covariance $1/2N$ (denoted later $\mathcal{N}\left(0, \frac{1}{2N}\right)$) variables $X^{\mathbb{R}}_{kj}, X^{i\mathbb{R}}_{kj}$) and for $k \in \{1, \dots, N\}$, $X_{kk} \sim \mathcal{N}\left(0, \frac{1}{N}\right)$. then, we shall prove that

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}(X^k)] = M_0(k) + \frac{1}{N^2}M_1(k) + o(\frac{1}{N^2}) \tag{2.4}$$

where

- $M_0(k) = C_{k/2}$ denotes the Catalan number : it vanishes if k is odd and is the number of non-crossing pair partitions of $2k$ (ordered) points, that is pair partitions so that any two blocks (a, b) and (c, d) is such that $a < b < c < d$ or $a < c < d < b$. C_k can also be seen to be the number of rooted trees embedded into the plane and k edges, that is trees with a distinguished edge and equipped with an exploration path of the vertices $v_1 \to v_2 \to \cdots \to v_{2k}$ of length $2k$ so that (v_1, v_2) is the root and each edge is visited twice (once in each direction). C_k can also be seen as the number of planar maps build over one vertex with valence k : namely take a vertex with valence k, draw it on the plane as a point with k half-edges. Choose a root, that is one of these half-edges. Then the set of half-edges is in bijection with k ordered points (as we drew them on the plane which is oriented). A matching of the half-edges is equivalent to a pairing of these points. Hence, we have a bijection between the graphs build over one vertex of valence k by matching the end-points of the half-edges and the pair partitions of k ordered points. The pairing is non-crossing iff the matching gives a planar graph, that is a graph that is properly embedded into the plane (recall that an embedding of a graph in a surface is proper iff the edges of the graph do not cross on the surface). Hence, $M_0(k)$ can also be interpreted as the number of planar graphs build over a rooted vertex with valence k. Recall that the genus g of a graph (that is the minimal genus of a surface in which it can be properly embedded) is given by Euler formula :

$$2 - 2g = \#Vertices + \#Faces - \#Edges\,,$$

where the faces are defined as the pieces of the surface in which the graph is embedded which are separated by the edges of the graph. If the surface as minimal genus, these faces are homeomorphic to discs.

- $M_1(k)$ is the number of graphs of genus one build over a rooted vertex with valence k. Equivalently, it is the number of rooted trees with $k/2$ edges and exactly one cycle.

Moreover, we shall prove that for any $k_1, \ldots, k_p$ $(\mathrm{Tr}(X^{k_j}) - \mathbb{E}[\mathrm{Tr}(X^{k_j})])_{1 \le j \le p}$ converges in moments towards a centered Gaussian vector with covariance

$$M_0(k,\ell) = \lim_{N\to\infty} \mathbb{E}\left[(\mathrm{Tr}(X^k) - \mathbb{E}[\mathrm{Tr}(X^k)])(\mathrm{Tr}(X^\ell) - \mathbb{E}[\mathrm{Tr}(X^\ell)])\right]\,.$$

$M_0(k,\ell)$ is the number of connected planar rooted graphs build over a vertex with valence k and one with valence ℓ. Here, both vertices have labelled half-edges and two graphs are counted as equal only if they correspond to matching half-edges with the same labels (and this despite of symmetries). Equivalently $M_0(k,\ell)$ is the number of rooted trees with $(k+\ell)/2$ edges and an exploration path with $k+\ell$ steps such that k consecutive steps are colored and at least an edge is explored both by a colored and a non-colored step of the exploration path.

Recall here that convergence in moments means that all mixed moments converge to the same mixed moments of the Gaussian vector with covariance M. We shall use that the moments of a centered Gaussian vector are given by Wick formula :

$$m(k_1,\ldots,k_p) = \mathbb{E}[\prod_{i=1}^p X_{k_i}] = \sum_\pi \prod_{\text{blocks } (a,b) \text{ of } \pi} M(k_a,k_b)$$

which is in fact equivalent to the induction formula we will rely on :

$$m(k_1,\ldots,k_p) = \sum_{i=2}^p M(k_1,k_i)m(k_2,\ldots,k_{i-1},k_{i+1},\ldots,k_p)\,.$$

Convergence in moments towards a Gaussian vector implies of course the standard weak convergence as convergence in moments implies that the second moments of $Z_N := (\mathrm{Tr}(X^{k_j}) - \mathbb{E}[\mathrm{Tr}(X^{k_j})])_{1\le j\le p}$ are uniformly bounded, hence the law of Z_N is tight. Moreover, any limit point has the same moments than the Gaussian vector. Since these moments do not blow too fast, there is a unique such limit point, and hence the law of Z_N converges towards the law of the Gaussian vector with covariance M. We will discuss at the end of this section how to generalize the central limit theorem to differentiable test functions, that is show that $Z_N(f) = \mathrm{Tr} f(X) - \mathbb{E}[\mathrm{Tr} f(X)]$ converges towards a centered Gaussian variable for any bounded differentiable function. This requires more subtle uniform estimates on the covariance of $Z_N(f)$ for which we will use Poincaré's inequality.

The asymptotic expansion (2.4) as well as the central limit theorem can be derived using combinatorial arguments and Wick calculus to compute Gaussian moments. This can also be obtained from the Dyson -Schwinger (DS) equation, which we do below.

2.0.2.1. *Dyson-Schwinger Equations.* Let :

$$Y_k := \mathrm{Tr} X^k - \mathbb{E}\mathrm{Tr} X^k$$

We wish to compute for all integer numbers $k_1, \dots, k_p$ the correlators :

$$\mathbb{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^p Y_{k_i}\right].$$

By integration by parts, one gets the following Dyson-Schwinger equations

LEMMA 2.1. *For any integer numbers $k_1, \dots, k_p$, we have*

$$\mathbb{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^p Y_{k_i}\right] = \mathbb{E}\left[\frac{1}{N}\sum_{\ell=0}^{k_1-2} \mathrm{Tr} X^{\ell} \mathrm{Tr} X^{k_1-2-p} \prod_{i=2}^p Y_{k_i}\right]$$

$$+\mathbb{E}\left[\sum_{i=2}^p \frac{k_i}{N} \mathrm{Tr} X^{k_1+k_i-2} \prod_{j=2, j\neq i}^p Y_{k_j}\right] \tag{2.5}$$

PROOF. Indeed, we have

$$\begin{aligned} \mathbb{E}\left[\mathrm{Tr} X^{k_1} \prod_{i=2}^p Y_{k_i}\right] &= \sum_{i,j=1}^N \mathbb{E}\left[X_{ij} (X^{k_1-1})_{ji} \prod_{i=2}^p Y_{k_i}\right] \\ &= \frac{1}{N}\sum_{i,j=1}^N \mathbb{E}\left[\partial_{X_{ji}}\left((X^{k_1-1})_{ji} \prod_{i=2}^p Y_{k_i}\right)\right] \end{aligned}$$

where we noticed that since the entries are Gaussian independent complex variables, for any smooth test function f,

$$\mathbb{E}[X_{ij} f(X_{k\ell}, k \le \ell)] = \frac{1}{N}\mathbb{E}[\partial_{X_{ji}} f(X_{k\ell}, k \le \ell)]\,. \tag{2.6}$$

But, for any $i, j, k, \ell \in \{1, \dots, N\}$ and $r \in \mathbb{N}$

$$\partial_{X_{ji}} (X^r)_{k\ell} = \sum_{s=0}^{r-1} (X^s)_{kj} (X^{r-s-1})_{i\ell}$$

where $(X^0)_{ij} = 1_{i=j}$. As a consequence

$$\partial_{X_{ji}}(Y_r) = r X_{ij}^{r-1}\,.$$

The Dyson-Schwinger equations follow readily. $\diamond$

EXERCISE 2.2. Show that

(1) If X is a GUE matrix, (2.6) holds. Deduce (2.1).
(2) take X to be a GOE matrix, that is a symmetric matrix with real independent Gaussian entries $N_{\mathbb{R}}(0, \frac{1}{N})$ above the diagonal, and $N_{\mathbb{R}}(0, \frac{2}{N})$ on the diagonal. Show that

$$\mathbb{E}[X_{ij} f(X_{k\ell}, k \le \ell)] = \frac{1}{N}\mathbb{E}[\partial_{X_{ji}} f(X_{k\ell}, k \le \ell)] + \frac{1}{N}\mathbb{E}[\partial_{X_{ij}} f(X_{k\ell}, k \le \ell)]\,.$$

Deduce that a formula analogous to (2.1) holds provided we have an additional term $N^{-1}\mathbb{E}\left[k_1 \mathrm{Tr} X^{k_1} \prod_{i=2}^p Y_{k_i}\right]$.

2.0.2.2. *Dyson-Schwinger equation implies genus expansion.* We will show that the DS equation (2.1) can be used to show that :

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right] = M_0(k) + \frac{1}{N^2}M_1(k) + o(\frac{1}{N^2})$$

Next orders can be derived similarly. Let :

$$m_k^N := \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right]$$

By the DS equation (with no Y terms), we have that :

$$m_k^N = \mathbb{E}\left[\sum_{\ell=0}^{k-2}\frac{1}{N}\mathrm{Tr}X^\ell\frac{1}{N}\mathrm{Tr}X^{k-\ell-2}\right]. \tag{2.7}$$

We now assume that we have the self-averaging property that for all $\ell \in \mathbb{N}$:

$$\mathbb{E}\left[\left(\frac{1}{N}\mathrm{Tr}X^\ell - \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^\ell\right]\right)^2\right] = o(1)$$

as $N \to \infty$ as well as the boundedness property

$$\sup_N m_\ell^N < \infty\,.$$

We will show both properties are true in Lemma 2.3. If this is true, then the above expansion (2.7) gives us :

$$m_k^N = \sum_{\ell=0}^{k-2} m_\ell^N m_{k-\ell-2}^N + o(1)$$

As $\{m_\ell^N, \ell \leq k\}$ are uniformly bounded, they are tight and so any limit point $\{m_\ell, \ell \leq k\}$ satisfies

$$m_k = \sum_{\ell=0}^{k-2} m_\ell m_{k-\ell-2}, m_0 = 1, m_1 = 0\,.$$

This equation has clearly a unique solution.

On the other hand, let $M_0(k)$ be the number of maps of genus 0 with one vertex with valence k. These satisfy the Catalan recurrence :

$$M_0(k) = \sum_{\ell=0}^{k-2} M_0(\ell)M_0(k-\ell-2)$$

This recurrence is shown by a Catalan-like recursion argument, which can be seen as a toy model for topological recursion, which goes by considering the matching of the first half edge with the ℓth half-edge, dividing each map of genus 0 into two sub-maps (both still of genus 0) of size ℓ and $k-\ell-2$, for $\ell \in \{0, \ldots, k-2\}$.

Since m and M_0 both satisfy the same recurrence (and $M_0(0) = m_0^N = 1, M_0(1) = m_1^N = 0$), we deduce that $m = M_0$ and therefore we proved by induction (assuming the self-averaging works) that :

$$m_k^N = M_0(k) + o(1) \text{ as } N \to \infty$$

It remains to prove the self-averaging and boundedness properties.

LEMMA 2.3. *There exists finite constants D_k and E_k, $k \in \mathbb{N}$, independent of N, so that for integer number ℓ, every integer numbers $k_1, \ldots, k_\ell$ then :*

$$\text{a)} \qquad c^N(k_1, \ldots, k_p) := \mathbb{E}\left[\prod_{i=1}^{\ell} Y_{k_i}\right] \ \textit{satisfies}\ |c^N(k_1, \ldots, k_p)| \le D_{\sum k_i}$$

and

$$\text{b)} \qquad m_{k_1}^N := \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^{k_1}\right] \ \textit{satisfies}\ |m_{k_1}^N| \le E_{k_1}\,.$$

PROOF. The proof is by induction on $k = \sum k_i$. It is clearly true for $k = 0, 1$ where $E_0 = 1, E_1 = 0$ and $D_k = 0$. Suppose the induction hypothesis holds for $k-1$. To see that $b)$ holds, by the DS equation, we first observe that :

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right] &= \mathbb{E}\left[\sum_{\ell=0}^{k-2}\frac{1}{N}\mathrm{Tr}X^\ell \frac{1}{N}\mathrm{Tr}X^{k-\ell-2}\right] \\
&= \sum_{\ell=0}^{k-2}(m_\ell^N m_{k-\ell-2}^N + \frac{1}{N^2}c^N(\ell, k-\ell-2))
\end{aligned}$$

Hence, by the induction hypothesis we deduce that

$$\left|\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^k\right]\right| \le \sum_{\ell=0}^{k-2}(E_\ell E_{k-2-\ell} + D_{k-2}) := E_k\,.$$

To see that $a)$ holds, we use the DS equation as follows

$$\begin{aligned}
\mathbb{E}\left[Y_{k_1}\prod_{j=2}^{p} Y_{k_j}\right] &= \mathbb{E}\left[\mathrm{Tr}X_{k_1}\prod_{j=2}^{p} Y_{k_j}\right] - \mathbb{E}\left[\mathrm{Tr}X_{k_1}\right]\mathbb{E}\left[\prod_{j=2}^{p} Y_{k_j}\right] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{\ell=0}^{k-2}\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2}\prod_{j=2}^{p} Y_{k_j}\right] \\
&\quad + \mathbb{E}\left[\sum_{i=2}^{p}\frac{k_i}{N}\mathrm{Tr}X^{k_1+k_i-2}\prod_{j=2, j\neq i}^{p} Y_{k_j}\right] \\
&\quad - \mathbb{E}\left[\frac{1}{N}\sum_{\ell=0}^{k-2}\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2}\right]\mathbb{E}\left[\prod_{j=2}^{p} Y_{k_j}\right]\,.
\end{aligned}$$

We next substract the last term to the first and observe that

$$\begin{aligned}
&\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2} - \mathbb{E}[\mathrm{Tr}X^\ell \mathrm{Tr}X^{k_1-\ell-2}] \\
&\qquad = NY_\ell m_{k_1-2-\ell}^N + NY_{k_1-2-\ell}m_\ell^N + Y_\ell Y_{k_1-2-\ell} - c^N(\ell, k_1-2-\ell)
\end{aligned}$$

to deduce

$$\mathbb{E}\left[Y_{k_1}\prod_{j=2}^{p}Y_{k_j}\right] = 2\sum_{\ell=0}^{k_1-2} m_\ell^N c^N(k_1-2-\ell, k_2, \dots, k_p)$$
$$+\sum_{i=2}^{p} k_i m^N_{k_1+k_i-2} c^N(k_2, .., k_{i-1}, k_{i+1}, ., k_p)$$
$$-\frac{1}{N}\sum_{\ell=0}^{k_1-2}[c^N(\ell, k_1-2-\ell)c^N(k_2,\dots,k_p) - c^N(\ell, k_1-2-\ell, k_2, \dots, k_p)]$$
$$+\frac{1}{N}\sum_{i=2}^{p} k_i c^N(k_1+k_i-2, k_2, .., k_{i-1}, k_{i+1}, ., k_p) \tag{2.8}$$

which is bounded uniformly by our induction hypothesis. ◇

As a consequence, we deduce

COROLLARY 2.4. For all $k \in \mathbb{N}$, $\frac{1}{N}\mathrm{Tr}(X^k)$ converges almost surely towards $M_0(k)$.

PROOF. Indeed by Borel Cantelli Lemma it is enough to notice that it follows from the summability of

$$P\left(|\mathrm{Tr}(X^k - \mathbb{E}\left(\mathrm{Tr}(X^k)\right)| \geq N\varepsilon\right) \leq \frac{c^N(k,k)}{\varepsilon^2 N^2} \leq \frac{D_{2k}}{\varepsilon^2 N^2}.$$

◇

2.0.3. Central limit theorem. The above self averaging properties prove that $m_k^N = M_0(k) + o(1)$. To get the next order correction we analyze the **limiting covariance** $c^N(k,\ell)$. We will show that

LEMMA 2.5. *For all $k, \ell \in \mathbb{N}$, $c^N(k,\ell)$ converges as N goes to infinity towards the unique solution $M_0(k,\ell)$ of the equation*

$$M_0(k,\ell) \quad = \quad 2\sum_{p=0}^{\ell-2} M_0(p)M_0(k-2-p,\ell) + \ell M_0(k+\ell-2)$$

so that $M_0(k,\ell) = 0$ if $k+\ell \leq 1$.

As a consequence we will show that

COROLLARY 2.6. $N^2(m_k^N - M_0(k)) = m_k^1 + o(1)$ where the numbers $(m_k^1)_{k\geq 0}$ are defined recursively by :

$$m_k^1 = 2\sum_{\ell=0}^{k-2} m_\ell^1 M_0\left(k-\ell-2\right) + \sum_{\ell=0}^{k-2} M_0(\ell, k-\ell-2)$$

PROOF. (Of Lemma 2.5) Observe that $c^N(k,\ell)$ converges for $K = k+\ell \leq 1$ (as it vanishes uniformly). Assume you have proven convergence towards $M_0(k,\ell)$ up to K. Take $k_1 + k_2 = K+1$ and use (2.8) with $p = 1$ to deduce that $c^N(k_1,k_2)$ satisfies

$$c^N(k_1,k_2) = 2\sum_{\ell=0}^{k_1-2} m_\ell^N c^N(k_1-\ell-2, k_2) + k_2 m^N_{k_1+k_2-2} + \frac{1}{N}\sum c^N(\ell, k_1-\ell-2, k_2)\,.$$

Lemma 2.3 implies that the last term is at most of order $1/N$ and hence we deduce by our induction hypothesis that $c(k_1, k_2)$ converges towards $M_0(k_1, k_2)$ which is given by the induction relation

$$M_0(k_1,k_2) \quad = \quad 2\sum_{\ell=0}^{k_1} M_0(\ell)M_0(k_1-2-\ell,k_2)+k_2M_0(k_1+k_2-2)\,.$$

Moreover clearly $M_0(k_1, k_2) = 0$ if $k_1 + k_2 \leq 1$. There is a unique solution to this equation. $\diamond$

EXERCISE 2.7. Show by induction that

$$M_0(k,\ell) = \#\{\text{planar maps with 1 vertex of degree } \ell \text{ and one vertex of degree } k\}$$

PROOF. (of Corollary 2.6) Again we prove the result by induction over k. It is fine for $k = 0, 1$ where $c^1_k = 0$. By (2.8) with $p = 0$ we have :

$$\begin{aligned} N^2(m_k^N - M_0(k)) \quad &= \quad 2\sum M_0(\ell)N^2\left(m_{k-\ell-2}^N - M_0(k-2-\ell)\right) \\ &\quad +\sum N^2\left(m_\ell^N - M_0(\ell)\right)\left(m_{k-\ell-2} - M_0(k-2-\ell)\right) \\ &\quad +\sum c^N(\ell, k-\ell-2) \end{aligned}$$

from which the result follows by taking the large N limit on the right hand side. $\diamond$

EXERCISE 2.8. Show that $c^1_k = m_1(k)$ is the number of planar maps with genus 1 build on a vertex of valence k.(The proof goes again by showing that $m_1(k)$ satisfies the same type of recurrence relations as c^1_k by considering the matching of the root : either it cuts the map of genus 1 into a map of genus 1 and a map of genus 0, or there remains a (connected) planar maps.)

THEOREM 2.9. *For any polynomial function* $P = \sum \lambda_k x^k$, $Z_N(P) = \mathrm{Tr}P - \mathbb{E}[\mathrm{Tr}P]$ *converges in moments towards a centered Gaussian variable* $Z(P)$ *with covariance given by*

$$\mathbb{E}[Z(P)\bar{Z}(P)] = \sum \lambda_k \bar{\lambda}_{k'} M_0(k,k')\,.$$

PROOF. It is enough to prove the convergence of the moments of the Y_k's. Let

$$c^N(k_1,\ldots,k_p) = \mathbb{E}\left[Y_{k_1}\cdots Y_{k_p}\right]\,.$$

Then we claim that, as $N \to \infty$, $c^N(k_1, \ldots, k_p)$ converges to $G(k_1, \ldots k_p)$ given by :

$$G(k_1,\ldots,k_p) = \sum_{i=2}^{k} M_0(k_1,k_i)G(k_2,\ldots,\hat{k}_i,\ldots,k_p) \tag{2.9}$$

where $\hat{}$ is the absentee hat.

This type of moment convergence is equivalent to a Wick formula and is enough to prove (by the moment method) that $Y_{k_1}, \ldots, Y_{k_p}$ are jointly Gaussian. Again, we will prove this by induction by using the DS equations. Now assume that (2.9) holds for any $k_1, \ldots, k_p$ such that $\sum_{i=1}^p k_i \leq k$. (induction hypothesis) We use

(2.8). Notice by the a priori bound on correlators of Lemma 2.3(a) that the terms with a $1/N$ are negligible in the right hand side and m_k^N is close to $M_0(k)$, yielding

$$\mathbb{E}\left[Y_{k_1}\prod_{j=2}^{p} Y_{k_j}\right] = 2\sum_{\ell=0}^{k_1-2} M_0(\ell)c^N(k_1-2-\ell,k_2,\ldots,k_p)$$
$$+\sum_{i=2}^{p} k_i M_0(k_1+k_i-2)c^N(k_2,..,k_{i-1},k_{i+1},.,k_p)+O(\frac{1}{N})$$

By using the induction hypothesis, this gives rise to :

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{p} Y_{k_i}\right] &= 2\sum M_0(\ell)G(k_1-\ell-2,k_2,\ldots,k_p)\\ &\quad+\sum k_i M_0(k_i+k_j-2)G(k_2,\ldots,\hat{k}_i,\ldots k_p)+o(1)\end{aligned}$$

It follows that

$$\begin{aligned}G(k_1,\ldots,k_p) &= 2\sum M_0(\ell)G(k_1-\ell-2,k_2,\ldots,k_p)\\ &\quad+\sum k_i M_0(k_i+k_j-2)G(k_2,\ldots,\hat{k}_i,\ldots k_p)\,.\end{aligned}$$

But using the induction hypothesis, we get

$$G(k_1,\ldots,k_p)=\sum_{i=2}^{p}(2\sum M_0(\ell)M(k_1-\ell-2,k_i)+k_iM_0(k_i+k_j-2))G(k_2,\ldots,\hat{k}_i,\ldots k_p)$$

which yields the claim since

$$M_0(k_1,k_i)=2\sum M_0(\ell)M(k_1-\ell-2,k_i)+k_iM_0(k_1+k_i-2)\,.$$

$\diamond$

2.0.4. Generalization. One can generalize the previous results to smooth test functions rather than polynomials. We have

LEMMA 2.10. *Let σ be the semi-circle law given by*

$$d\sigma(x)=\frac{1}{2\pi}\sqrt{4-x^2}dx\,.$$

(1) *For any bounded continuous function f with polynomial growth at infinity*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)=\int f(x)d\sigma(x)\qquad a.s.$$

(2) *For any C^2 function f with polynomial growth at infinity*

$$Z(f)=\sum f(\lambda_i)-\mathbb{E}(\sum f(\lambda_i))$$

converges in law towards a centered Gaussian variable.

Our proof will only show convergence : the covariance is well known and can be found for instance in [**77**, (3.2.2)].

EXERCISE 2.11. Show that for all $n\in\mathbb{N}$, $\int x^n d\sigma(x)=M_0(n)$.

PROOF. The convergence of $\frac{1}{N}\sum_{i=1}^N f(\lambda_i)$ follows since polynomials are dense in the set of continuous functions on compact sets by Weierstrass theorem. Indeed, our bounds on moments imply that we can restrict ourselves to a neighborhood of $[-2,2]$:

$$\frac{1}{N}\sum_{i=1}^N \lambda_i^{2p} 1_{|\lambda_i|\geq M} \leq \frac{1}{M^{2k}}\frac{1}{N}\sum \lambda_i^{2k+2p}$$

has moments asymptotically bounded by $\sigma(x^{2k+2p})/M^{2k} \leq 2^{2p}(2/M)^{2k}$. This allows to approximate moments by truncated moments and then use Weierstrass theorem.

To derive the central limit theorem, one can use concentration of measure inequalities such as Poincaré inequality. Indeed, Poincaré inequalities for Gaussian variables read : for any C^1 real valued function F on $\mathbb{C}^{N(N-1)/2}\times\mathbb{R}^N$

$$\mathbb{E}\left[(F(X_{k\ell},k,l)-\mathbb{E}[F(X_{k\ell},k,l)])^2\right] \leq \frac{2}{N}\mathbb{E}\left[\sum_{i,j}\left|\partial_{X_{ij}}F(X_{k\ell},k,l)\right|^2\right] .$$

Taking $F = \mathrm{Tr}f(X)$ we find that $\partial_{X_{ij}}F(X_{k\ell},k,l) = f'(X)_{ji}$. Indeed, we proved this point for polynomial functions f so that we deduce

$$\mathbb{E}\left[(\mathrm{Tr}(f(X))-\mathbb{E}[\mathrm{Tr}(f(X))])^2\right] \leq \frac{2}{N}\mathbb{E}\left[\mathrm{Tr}(f'(X)^2)\right] .$$

Hence, if we take a C^1 function f, whose derivative is approximated by a polynomial P_ε on $[-M,M]$ (with $M>2$) up to an error $\varepsilon>0$, and whose derivative grows at most like x^{2K} for $|x|\geq M$, we find

$$\begin{aligned}\mathbb{E}&\left[(\mathrm{Tr}f(X)-\mathbb{E}[\mathrm{Tr}f(X)]-(\mathrm{Tr}P_\varepsilon(X)-\mathbb{E}[\mathrm{Tr}P_\varepsilon(X)]))^2\right]\\ &\leq 4\mathbb{E}\left[(\varepsilon^2+\frac{1}{N}\sum(P_\varepsilon^2(\lambda_i)+\lambda_i^{2K})1_{|\lambda_i|\geq M})\right]\end{aligned}$$

where the right hand side goes to zero as N goes to infinity and then ε goes to zero. This shows the convergence of the covariance of $Z(f)$. We then proceed similarly to show that the approximation is good in any L^p, hence deriving the convergence in moments.

◇

2.0.5. GUE topological expansion . The "topological expansion" reads

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left[X^k\right]\right] = \sum_{g\geq 0}\frac{1}{N^{2g}}M_g(k)$$

where $M_g(k)$ is the number of rooted maps of genus g build over a vertex of degree k. Here, a "map" is a connected graph properly embedded in a surface and a "root" is a distinguished oriented edge. A map is assigned a genus, given by the smallest genus of a surface in which it can be properly embedded. This complete expansion (not that the above series is in fact finite) can be derived as well either by Wick calculus or by Dyson-Schwinger equations : we leave it as an exercise to the reader. We will see later that cumulants of traces of moments of the GUE are related with the enumeration of maps with several vertices.

CHAPTER 3

Wigner random matrices

In this section, we investigate random matrices with non-Gaussian entries. It turns out that the study of such matrices requires often additional tools from linear algebra, such as Schur complement formula, which provide other types of equations. These equations are interesting in their own right but quite different from Dyson-Schwinger equations and therefore we shall not investigate them in details in these notes. However, because we believe that heavy tails matrices still cary a lot of exciting open problems, we take a few pages to describe a few techniques which were developed to study them, and in particular the new type of equations to which they are related. We restrict ourselves to symmetric matrices for simplicity.

3.1. Law of large numbers : light tails

We first consider the case where the entries $(x_{ij})_{i\leq j}$ are independent, centered, with variance one and all moments finite. Then, the spectral measure of the symmetric $N\times N$ matrix X with entries $X_{i,j}=x_{i,j}/\sqrt{N}$ converges towards the semi-circle law as proved by Wigner [**95**]. We first present a proof of the convergence of moments very similar to the one we gave in the Gaussian case and then turn to the weaker convergence of the Stieltjes transforms.

3.1.1. Convergence in moments.

Theorem 3.1. *For all integer numbers k*

$$\lim_{N\to\infty}\mathbb{E}[\frac{1}{N}\mathrm{Tr}(X^k)]=M_0(k)\,.$$

Proof. We shall first generalize the previous proof to this setting thanks to the generalized integration by parts formula, see e.g [**70**, Proposition 3.1] :

Lemma 3.2. *Let ξ be a random variable such that $\mathbb{E}[|\xi|^{p+2}]<\infty$ for some integer number p. Then for any C^{p+1} function ϕ we have*

$$\mathbb{E}[\xi\phi(\xi)]=\sum_{\ell=1}^{p}\frac{\kappa_{\ell+1}}{\ell!}\mathbb{E}[\partial^\ell\phi(\xi)]+\varepsilon_p$$

where the κ's are the cumulant of ξ and there exists a finite constant C_p only depending on p such that

$$|\varepsilon_p|\leq C_p\|\partial^{p+1}\phi\|_\infty\,.$$

We deduce the approximate Dyson-Schwinger equation

$$m_\ell^N:=\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}X^\ell\right]=\mathbb{E}[\sum_{k=0}^{\ell-2}\frac{1}{N}\mathrm{Tr}X^k\frac{1}{N}\mathrm{Tr}X^{\ell-k-2}]+\eta_\ell^N$$

where since we assumed the entries have all moments finite

$$\eta_\ell^N = \frac{1}{N^2}\sum_{r\ge 2}\frac{\kappa_{r+1}}{r!N^{\frac{r-1}{2}}}\sum_{i,j=1}^N \mathbb{E}[\partial_{X_{j,i}}^r (X^{\ell-1})_{i,j}]\,.$$

Note here that since we deal with polynomials the above sum is finite. To check that η_ℓ^N goes to zero as N goes to infinity, it is enough to show that $\mathbb{E}[\partial_{X_{j,i}}^r X_{i,j}^{\ell-1}]$ is bounded uniformly in i,j for all ℓ. To prove this we bound uniformly by induction over $n=\sum_{s=1}^r n_s$ the more general quantity

$$\mathbb{E}[\prod_{s=1}^r (X^{n_s})_{i_s j_s}] \quad = \quad \mathbb{E}[\frac{1}{N}\sum_{k=1}^N\sum_{\ell\ge 1}\frac{\kappa_{\ell+1}}{\ell! N^{\frac{\ell-1}{2}}}\partial_{X_{i_1 k}}^\ell (X_{kj_1}^{n_1-1}\prod_{s=2}^r X_{i_s j_s}^{n_s})]$$

Indeed, this term is clearly bounded when $n=1$, and the above relation gives moments for a given n in terms of moments with lower total degree, so that the induction hypothesis holds. We conclude that $\eta_\ell^N = O(N^{-1/2})$. To conclude following the previous arguments we need to show that we have self averaging :

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(X^{\ell_0})\prod_{j=1}^p\left(\frac{1}{N}\mathrm{Tr}(X^{\ell_j}) - \mathbb{E}[\frac{1}{N}\mathrm{Tr}(X^{\ell_j})]\right)\right]$$

goes to zero when $p\ge 1$ for all integer numbers $\ell_0,\ell_1,\dots,\ell_p$. This can be done again by induction over $K=\sum_{j=0}^p \ell_j$ by simultaneously showing that m_ℓ^N is bounded for $\ell\le K$. Indeed, we observe that

$$\begin{aligned} c^N(\ell_0,\ell_1,\dots,\ell_p) \quad &= \quad 2\sum_{r=0}^{\ell_0-2} m_{\ell_0-r-2}^N c^N(r,\ell_1,\dots,\ell_p) \\ &\quad + \sum_{j=1}^p \ell_j c(\ell_0+\ell_j-2,\ell_1\dots,\ell_{j-1},\ell_{j+1},\dots\ell_p) + \zeta_{\ell_0,\dots,\ell_p} \end{aligned}$$

where ζ is the term containing derivatives of order greater or equal to two. But, as for η, it can be written as a sum of bounded terms of the form $\mathbb{E}[\prod_{s=1}^r X_{i_s j_s}^{n_s}]$ with such a renormalization that it is at most of order $N^{-1/2}$. We can then proceed as in the previous proof. $\diamond$

Following [**70**], the central limit theorem can be deduced in the same spirit by Lindenberg strategy as soon as the entries have the same fourth moment than the Gaussian variable. To remove the hypothesis that all moments are finite, we must however consider other test functions than polynomials. We next introduce another approach to the convergence of the spectral measure based on Stieltjes transform and a different type of equation based on linear algebra, more precisely the Schur complement formula. This approach generalizes to heavy tails matrices.

3.1.2. Convergence of the Stieltjes transform. We begin by recalling some classical results concerning the Stieltjes transform of a probability measure.

DEFINITION 3.3. Let μ be a positive, finite measure on the real line. The *Stieltjes transform* of μ is the function

$$G_\mu(z) := \int_{\mathbb{R}}\frac{\mu(dx)}{z-x}\,,\ z\in\mathbb{C}\setminus\mathbb{R}\,.$$

Note that for $z \in \mathbb{C} \setminus \mathbb{R}$, both the real and imaginary parts of $1/(x-z)$ are continuous bounded functions of $x \in \mathbb{R}$, and further $|G_\mu(z)| \le \mu(\mathbb{R})/|\Im z|$. These crucial observations are used repeatedly in what follows.

REMARK 3.4. The generating function $\hat{\eta}(z)$ of moments is closely related to the Stieltjes transform of the semicircle distribution σ : for $|z| < 1/4$,

$$\begin{aligned}
\hat{\eta}(z) &= \sum_{k=0}^{\infty} z^k \int x^{2k}\sigma(x)dx = \int \left(\sum_{k=0}^{\infty} \left(zx^2\right)^k\right)\sigma(x)dx \\
&= \int \frac{1}{1-zx^2}\sigma(x)dx \\
&= \int \frac{1}{1-\sqrt{z}x}\sigma(x)dx = \frac{-1}{\sqrt{z}}G_\sigma(1/\sqrt{z})\,.
\end{aligned}$$

where the third equality uses that the support of σ is the interval $[-2,2]$, and the fourth uses the symmetry of σ. Using the fact that $\int x^{2k}d\sigma(x) = M_0(k)$ is the Catalan number (see Exercise 2.11) which satisfies the induction relation

$$M_0(k) = \sum_{\ell=0}^{k-1} M_0(\ell)M_0(k-\ell-1)$$

it is not hard to deduce from the above that

$$G_\sigma(z) - \frac{1}{z} = \frac{1}{z}G_\sigma(z)^2 \tag{3.1}$$

Stieltjes transforms can be inverted. In particular, one has

THEOREM 3.5. *For any open interval I with neither endpoint on an atom of μ,*

$$\begin{aligned}
\mu(I) &= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_I \frac{G_\mu(\lambda+i\epsilon)-G_\mu(\lambda-i\epsilon)}{2i}d\lambda \\
&= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_I \Im G_\mu(\lambda+i\epsilon)d\lambda\,. \qquad (3.2)
\end{aligned}$$

PROOF. Note first that because

$$\Im G_\mu(i) = \int \frac{1}{1+x^2}\mu(dx)\,,$$

we have that $G_\mu \equiv 0$ implies $\mu = 0$. So assume next that G_μ does not vanish identically. Then, since

$$\lim_{y\uparrow+\infty} y\Im G_\mu(iy) = \lim_{y\uparrow+\infty}\int \frac{y^2}{x^2+y^2}\mu(dx) = \mu(\mathbb{R})$$

by bounded convergence, we may and will assume that $\mu(\mathbb{R}) = 1$, i.e. that μ is a probability measure.

Let x be distributed according to μ, and denote by c_ϵ a random variable, independent of x, Cauchy distributed with parameter ϵ, i.e. the law of c_ϵ has density

$$\frac{\epsilon dx}{\pi(x^2+\epsilon^2)}\,. \tag{3.3}$$

Then, $\Im G_\mu(\lambda+i\epsilon)/\pi$ is nothing but the density (with respect to Lebesgue measure) of the law of $x+c_\epsilon$ evaluated at $\lambda \in \mathbb{R}$. The convergence in (3.2) is then just a rewriting of the weak convergence of the law of $x+c_\epsilon$ to that of x, as $\epsilon \to 0$. ◇

We will study the Stieltjes transform of Wigner matrices thanks to the following Schur complement formula.

LEMMA 3.6. *Let X be a symmetric matrix, and let X_i denote the i-th column of X with the entry $X(i,i)$ removed (i.e., X_i is an $N-1$-dimensional vector). Let $X^{(i)}$ denote the matrix obtained by erasing the i-th column and row from X. Then, for every $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$(X - zI)^{-1}(i,i) = \frac{1}{X(i,i) - z - X_i^*(X^{(i)} - zI_{N-1})^{-1}X_i}\,. \tag{3.4}$$

Proof of Lemma 3.6 Note first that from Cramer's rule,

$$(X - zI_N)^{-1}(i,i) = \frac{\det(X^{(i)} - zI_{N-1})}{\det(X - zI)}\,. \tag{3.5}$$

Write next

$$X - zI_N = \begin{pmatrix} X^{(N)} - zI_{N-1} & X_N \\ X_N^* & X(N,N) - z \end{pmatrix},$$

and use the matrix identity

$$\begin{aligned} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det\left(\begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}\right) \\ &= \det A \det(D - CA^{-1}B) \end{aligned} \tag{3.6}$$

with $A = X^{(N)} - zI_{N-1}$, $B = X_N$, $C = X_N^*$ and $D = X(N,N) - z$ to conclude that

$$\begin{aligned} &\det(X - zI_N) = \\ &\det(X^{(N)} - zI_{N-1}) \det\left[X(N,N) - z - X_N^*(X^{(N)} - zI_{N-1})^{-1}X_N\right]. \end{aligned}$$

The last formula holds in the same manner with $X^{(i)}$, X_i and $X(i,i)$ replacing $X^{(N)}, X_N$ and $X(N,N)$ respectively. Substituting in (3.5) completes the proof of Lemma 3.6. $\diamond$

We shall deduce from Schur formula the convergence of the Stieltjes transform

THEOREM 3.7. *Assume that X_{ij} are centered, with variance $1/N$ and so that $\sup_N \sup_{ij} \mathbb{E}[(\sqrt{N}|X_{ij}|)^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Let X be the $N \times N$ Hermitian matrix with independent entries $(X_{ij})_{i \geq j}$ above the diagonal. Then*

$$G_N(z) = \frac{1}{N}\mathrm{Tr}(z - X)^{-1}$$

converges almost surely when N goes to infinity towards the unique solution of

$$G(z) = \frac{1}{z - G(z)}$$

going to zero at infinity

PROOF. The first point in the proof is that $G_N(z)$, if it converges in expectation, will converge almost surely by concentration of measure. We give a concentration result due to C. Bordenave, P. Caputo and D. Chafai **[11]** which holds for the eigenvalues $\lambda_1, \ldots, \lambda_N$ of any self-adjoint random matrix X provided the column vectors $\{(X_{ij})_{i \leq j}, 1 \leq j \leq N\}$ are independent. It is based on Azuma's-Hoeffding inequality.

LEMMA 3.8. *Let* $\|f\|_{TV}$ *be the total variation norm,*

$$\|f\|_{TV} = \sup_{x_1 < \cdots < x_n} \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

Then, for any $\delta > 0$ *and any function* f *with finite total variation norm so that* $E[|\frac{1}{N}\sum_{i=1}^N f(\lambda_i)|] < \infty$,

$$P\left(\left|\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) - \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)]\right| \geq \delta\|f\|_{TV}\right) \leq 2e^{-\frac{N\delta^2}{8}}$$

REMARK 3.9. Note that the above speed is not optimal for laws μ, ν which have sufficiently fast decaying tails, in which case $\sum_{i=1}^N f(\lambda_i) - \mathbb{E}[\sum_{i=1}^N f(\lambda_i)]$ is of order one. However it is the optimal rate for instance for heavy tails matrices where the central limit theorem holds for $N^{-1/2}(\sum_{i=1}^N f(\lambda_i) - \mathbb{E}[\sum_{i=1}^N f(\lambda_i)])$.

REMARK 3.10. Note that we only required independence of the vectors, rather than the entries.

Proof of Lemma 3.8. Let us first recall Azuma-Hoeffding's inequality.

LEMMA 3.11. *(Azuma-Hoeffding's inequality) Suppose* $M_k, k \geq 0$ *is a martingale for the filtration* $\mathcal{F}_k$ *and* $|M_k - M_{k-1}| \leq c_k$. *Then for all* $t \geq 0$

$$P(M_n - M_0 \geq t) \leq \exp\{-\frac{t^2}{2\sum_{k=1}^n c_k^2}\}\,.$$

We finally prove Lemma 3.8 for a continuously differentiable function f, the generalization to all functions with finite variation norm then holds by density. We then have $\|f\|_{TV} = \int |f'(x)|dx$. We apply Azuma-Hoeffding 's inequality to

$$M_k = \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)|\mathcal{F}_k]$$

where $\mathcal{F}_k$ is the filtration generated by $\{X_N(i,j), 1 \leq i \leq j \leq k\}$ for Wigner matrices. M_k is a martingale obviously and

$$M_N - M_0 = \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) - \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i)]\,.$$

Therefore we need to bound for each $k \in \{1, \cdots, N\}$

$$M_k - M_{k-1} \quad = \quad \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) - \frac{1}{N}\sum_{i=1}^{N} f(\tilde{\lambda}_i)|\mathcal{F}_k]\,.$$

where in the above expectation λ_i and $\tilde{\lambda}_i$ are the eigenvalues of the $N \times N$ matrix X_N and Z_N respectively, where Z_N has the same entries than X_N except for the kth vector where we take independent copies. Hence the eigenvalues λ and $\tilde{\lambda}$ are the eigenvalues of two operators which differ at most by a rank one perturbation. This implies that their spectral measures are close by the following lemma :

LEMMA 3.12. *let X, Y be two $N \times N$ Hermitian matrices so that $Y - X$ has rank one. Let $\lambda_1 \le \lambda_2 \cdots \le \lambda_N$ (resp. $\tilde\lambda_1 \le \tilde\lambda_2 \cdots \le \tilde\lambda_N$) be the ordered eigenvalues of X and Y respectively. Then, for any C^1 function g on the real line*

$$\left|\sum_{i=1}^N g(\lambda_i) - \sum_{i=1}^N g(\tilde\lambda_i)\right| \le 2\|g\|_{TV}\,.$$

PROOF. Since the two matrices differ only by a rank one matrix, the eigenvalues λ_i and $\tilde\lambda_i$ are interlaced by Weyl interlacing property, see e.g [**3**, Theorem A.7] :

$$\tilde\lambda_{i-1} \le \lambda_i \le \tilde\lambda_{i+1}\,.$$

If g is increasing we deduce that

$$\sum_{i=1}^{N-2} g(\tilde\lambda_i) \le \sum_{i=2}^{N-1} g(\lambda_i) \le \sum_{i=3}^{N} g(\tilde\lambda_i)$$

which implies

$$|\sum_{i=1}^N g(\lambda_i) - \sum_{i=1}^N g(\tilde\lambda_i)| \le 2\|g\|_\infty \tag{3.7}$$

Decomposing $f(x) - f(0)$ as the difference of two increasing functions

$$f(x) - f(0) = \int_0^x f'(y)1_{f'(y)\ge 0}dy - \int_0^x (-f')(y)1_{f'(y)<0}dy$$

proves the claim since

$$\|f\|_{TV} \ge \|\int_0^x f'(y)1_{f'(y)\ge 0}dy\|_\infty + \|\int_0^x (-f')(y)1_{f'(y)<0}dy\|_\infty$$

$\diamond$

To complete the proof of Lemma 3.8 notice that a consequence of the previous lemma gives that

$$|M_k - M_{k-1}| \le \frac{2}{N}\|f\|_{TV}$$

which allows to conclude by Azuma-Hoeffding's inequality that for all $\delta > 0$

$$P\left(\frac{1}{N}\sum_{i=1}^N f(\lambda_i) - \mathbb{E}[\frac{1}{N}\sum_{i=1}^N f(\lambda_i)] \ge \delta\right) \le e^{-\frac{\delta^2 N}{8\|f\|_{TV}^2}}\,.$$

The other bound is obtained by changing f into $-f$. $\diamond$

We therefore can prove the convergence of $\mathbb{E}[G_N(z)]$ instead of $G_N(z)$.

The main idea in the proof is that the convergence and fluctuations of the term $X_i^T(X^{(i)} - zI_{N-1})^{-1}X_i$ in terms of G_N will provide the convergence and fluctuations of $G_N(z)$. To this end let us write

$$\begin{aligned} X_i^T(X_N^{(i)} - zI)^{-1}X_i &= \sum_{j\ne k} X_{ij}X_{ik}(X_N^{(i)} - zI)^{-1}_{jk} + \sum_j |X_{ij}|^2(X^{(i)} - zI)^{-1}_{jj} \\ &=: O(z) + D(z) \end{aligned}$$

We first observe that the off diagonal terms $O(z)$ will always be negligible

LEMMA 3.13. *Under the assumptions of Theorem 3.7, for all $\varepsilon > 0$, for any matrix C such that $N^{-1}\mathrm{Tr}(CC^T)$ is bounded independently of N*

$$\lim_{N\to\infty} \mathbb{P}(|\sum_{j\neq k} X_{ij}X_{ik}C_{jk}| \geq \delta) = 0\,.$$

PROOF. Chebyshev's inequality and independence yields

$$\mathbb{P}(|\sum_{j\neq k} X_{ij}X_{ik}C_{jk}| \geq \delta) \leq \frac{1}{\delta^2}\sum_{j,k}\mathbb{E}[|X_{ij}|^2]^2 C_{jk}^2 = \frac{1}{\delta^2 N^2}\mathrm{Tr}(CC^T) \leq \frac{C}{\delta^2 N}$$

which proves the claim. $\diamond$

We hence need only to focus on the diagonal term $D(z)$ and consider the equation

$$(z-X)^{-1}_{ii} = \frac{1}{z - \sum_{j=1}^N |X_{ij}|^2 (z - X^{(i)})^{-1}_{jj} + \varepsilon_i^N(z)} \tag{3.8}$$

with some $\varepsilon_i^N(z)$ going to zero in probability with N going to infinity. We treat the case of Wigner matrices so that $\sqrt{N}X_{ij}$ belongs to $L^{2+\varepsilon}$ (recall $X^{(i)}$ is independent from $X_{ij}, 1 \leq j \leq N$. In this case, the law of large numbers (conditionally to $X^{(i)}$) insures that

$$\lim_{N\to\infty}\left(\sum_j |X_{ij}|^2 (X_N^{(i)} - zI)^{-1}_{jj} - \frac{1}{N}\sum_j (X_N^{(i)} - zI)^{-1}_{jj}\right) = 0 \qquad a.s.$$

Hence, we see that

$$(z-X)^{-1}_{ii} = \frac{1}{z - \frac{1}{N}\mathrm{Tr}(z-X^{(i)})^{-1} + \varepsilon_i^N(z)'}$$

where $\varepsilon_i^n(z)'$ goes to zero in probability. Finally, Lemma 3.12 we get

$$|\frac{1}{N}\mathrm{Tr}(z-X^{(i)})^{-1} - \frac{1}{N}\mathrm{Tr}(z-X)^{-1}| \leq \frac{2}{N\Im z}\,.$$

Therefore, we conclude that for all $z \in \mathbb{C}\backslash\mathbb{R}$, all $i \in \{1, \cdots, N\}$, there exists $\varepsilon_i^N(z)$ going to zero in probability such that

$$(z-X)^{-1}_{ii} = \frac{1}{z - G_N(z) + \varepsilon_i(z)}\,.$$

We deduce that

$$\Delta_i^N(z) = (z-X)^{-1}_{ii} - \frac{1}{z - G_N(z)} = (z-X)^{-1}_{ii}\frac{1}{z-G_N(z)}\varepsilon_i^N(z)$$

goes to zero in probability since $(z-X)^{-1}_{ii}$ and $\frac{1}{z-G_N(z)}$ are bounded by $1/|\Im z|$. Since it is also uniformly bounded for the same reason, we deduce that $\Delta_i^N(z)$ goes to zero in L^1. As a consequence

$$\mathbb{E}[G_N(z)] = \mathbb{E}\left[\frac{1}{z-G_N(z)}\right] + o(1) = \frac{1}{z - \mathbb{E}[G_N(z)]} + o(1)$$

where we used Lemma 3.8 as well as the fact that $|z - G_N(z)| \geq |\Im z|$. Solving this quadratic equation, since we know that $\mathbb{E}[G_N(z)]$ goes to zero as $\Im z$ goes to infinity, implies that

$$\mathbb{E}[G_N(z)] = \frac{z - \sqrt{z^2-4}}{2} + o(1)\,.$$

◇

3.2. Law of large numbers : heavy tails

For heavy tails matrices, the spectral measure still converges but we have a different limit than the semi-circle law. Here are the models we would like to study :

Definition 3.14 (Models of symmetric heavy tailed matrices with i.i.d. sub–diagonal entries).
Let $A = (a_{i,j})_{i,j=1\leq N}$ be a random symmetric matrix with i.i.d. sub-diagonal entries $(a_{i,j})_{i\leq j}$.

1. We say that A is a **Lévy matrix** of parameter α in $]0,2[$ when $A = X/a_N$ where the entries x_{ij} of X have absolute values in the domain of attraction of an α-stable distribution, more precisely for all $u \geq 0$

$$\mathbf{P}\left(|x_{ij}| \geq u\right) = \frac{L(u)}{u^\alpha} \tag{3.9}$$

with a slowly varying function L, and

$$a_N = \inf\{u : P\left(|x_{ij}| \geq u\right) \leq \frac{1}{N}\}$$

($a_N = \tilde{L}(N)N^{1/\alpha}$, with $\tilde{L}(\cdot)$ a slowly varying function).

2. We say that A is a **Wigner matrix with exploding moments** with parameter $(C_k)_{k\geq 1}$ whenever the entries of A are centered, and for any $k \geq 1$

$$\lim_{N\to\infty} N\mathbb{E}\big[(a_{ij})^{2k}\big] = C_k, \tag{3.10}$$

with $(C_{k+1})_{k\geq 0}$ the sequence of moments of a unique measure m.

A particular case of matrices with exploding moments is the case of the the adjacency matrix of an Erdös-Rényi graph, i.e. of a matrix A such that $A_{ij} = 1$ with probability p/N and 0 with probability $1 - p/N$. It is an exploding moments Wigner matrix, with $C_k = p$ for all $k \geq 1$ (and $m = p\delta_1$).

The main assumption we will make on the matrices we shall consider is a bit more general than these two types of models and reads as follows.

Assumption 3.15. Let μ_N be the law of $a_{ij}, i \leq j$. Assume that uniformly on t in compacts of $\mathbb{C}^-$

$$\lim_{N\to\infty} N\int (e^{-itx^2} - 1)d\mu_N(x) = \Phi(t)$$

with Φ such that there exists g on $\mathbb{R}^+$ bounded by Cy^κ for some $\kappa > -1$ such that for $t \in \mathbb{C}^-$,

$$\Phi(t) = \int_0^\infty g(y)e^{\frac{iy}{t}}\,dy\,. \tag{3.11}$$

Furthermore assume that X with law μ_N can be decomposed into the law of $A+B$ where

$$P(A \neq 0) \ll N^{-1} \qquad \mathbb{E}[B^2] \ll N^{-1/2}$$

Note that these hypotheses are fulfiled by our examples.

-In the case of Lévy matrices, $\Phi(\lambda)=-\sigma(i\lambda)^{\alpha/2}$ and the expression

$$-\sigma(i\lambda)^{\alpha/2}=\int_{y=0}^{+\infty}C_\alpha y^{\frac{\alpha}{2}-1}e^{i\frac{y}{\lambda}}\mathrm{d}y$$

shows the existence of g satisfying (3.11) : $g(y)=C_\alpha y^{\frac{\alpha}{2}-1}$. The last point is satisfied with for some $a\in(0,1/2(2-\alpha))$

$$A=1_{|x_{ij}|>N^a a_N}\frac{x_{ij}}{a_N},\quad B=1_{|x_{ij}|\le N^a a_N}\frac{x_{ij}}{a_N}\,.$$

- In the case of Wigner matrices with exploding moment, one first needs to use the following formula, for $\xi\in\mathbb{C}$ with positive real part :

$$1-e^{-\xi}=\int_0^{+\infty}\frac{J_1(2\sqrt{t})}{\sqrt{t}}e^{-t/\xi}\mathrm{d}t, \tag{3.12}$$

where J_1 the Bessel function of the first kind defined by $J_1(s)=\frac{s}{2}\sum_{k\ge0}\frac{(-s^2/4)^k}{k!(k+1)!}$. It follows that

$$\begin{aligned}N(\phi_N(\lambda)-1)&=N\mathbb{E}(e^{-i\lambda a^2}-1)\\&=-N\mathbb{E}\int_0^{+\infty}\frac{J_1(2\sqrt{t})}{\sqrt{t}}e^{-\frac{t}{i\lambda a^2}}\mathrm{d}t=\int_0^{+\infty}g_N(y)e^{i\frac{y}{\lambda}}\mathrm{d}y\end{aligned}$$

with

$$g_N(y):=-N\frac{\mathbb{E}[|a|J_1(2\sqrt{y}|a|)]}{\sqrt{y}}=-N\mathbb{E}[a^2\frac{J_1(2\sqrt{ya^2})}{\sqrt{ya^2}}]=\int f_y(x)\mathrm{d}m_N(x)$$

for $f_y(x):=-\frac{J_1(2\sqrt{xy})}{\sqrt{xy}}$ and m_N the measure with kth moment given by $N\mathbb{E}[(a_{ij})^{2k}]$. As m_N converges weakly to m and f_y is continuous and bounded, we have

$$g_N(y)\to-\int\frac{J_1(2\sqrt{xy})}{\sqrt{xy}}\mathrm{d}m(x)=:g(y)\,.$$

Theorem 3.16. *Under Assumption 3.15, G_N converges almost surely towards G given , for $z\in\mathbb{C}^+$, by*

$$G(z)=i\int_0^\infty e^{itz}e^{\rho_z(t)}dt$$

where $\rho_z:\mathbb{R}^+\to\{x+iy;x\le0\}$ is the unique solution analytic in $z\in\mathbb{C}^+$ of the fixed point equation

$$\rho_z(t)=\int_0^\infty g(y)e^{\frac{iy}{t}z+\rho_z(\frac{y}{t})}dy$$

Proof. Again, because of Lemma 3.8, it is enough to prove that G_N converges in L^1. We wish to use again the Schur complement formula (3.8). The new difficulty is that the diagonal elements of the resolvent are not approximately deterministic anymore, but rather behave like independent random variables. Indeed $\sum_j|X_{ij}|^2(zI-X^{(i)})^{-1}_{jj}$ remains random in the limit $N\to\infty$ as can be seen if we compute for instance its Fourier transform for $t \in \mathbb{R}^+$:

$$\begin{aligned}
\mathbb{E}[e^{-it\sum_j |X_{ij}|^2 (zI - X^{(i)})^{-1}_{jj}}] &= \mathbb{E}\left[\prod_{j=1}^N \mathbb{E}_{X_i}[e^{-it|X_{ij}|^2(-X^{(i)}+zI)^{-1}_{jj}}]\right] \\
&= \mathbb{E}\left[\prod_{j=1}^N (1 + \frac{1}{N}\Phi(t(z - X_N^{(i)})^{-1}_{jj}) + o(\frac{1}{N}))\right] \\
&= \mathbb{E}\left[e^{\frac{1}{N}\sum_{j=1}^N \Phi(t(z-X_N^{(i)})^{-1}_{jj}) + o(1)}\right]
\end{aligned}$$

where in the second line we used Assumption 3.15. To prove the convergence of G_N it is thus natural to study the order parameter

$$\rho_z^N(t) := \frac{1}{N}\sum_{j=1}^N \Phi(t(z - X_N)^{-1}_{jj})\,.$$

First, we can show that $\rho_z^N(t)$ self-averages as in Lemma 3.8 thanks to the fact that Φ is smooth on $\mathbb{C}^-$. Indeed, we can use also Azuma-Hoefding inequality and the same martingale decomposition to reduce the problem to bound uniformly

$$\sum_{i=1}^N \Phi((z - X^{(j)})^{-1}_{ii}) - \sum_{i=1}^N \Phi((z - X^{(j+1)})^{-1}_{ii})$$

where $X^{(j)} - X^{(j-1)}$ has rank one. But, following Lemma C.3 in **[52]**, we notice that if X, Y are two Hermitian matrices so that $X - Y$ has rank one, then for any function f with finite total variation norm, we have

$$\left|\sum_{i=1}^N f((z-X)^{-1}_{ii}) - \sum_{i=1}^N f((z-Y)^{-1}_{ii})\right| \le \|f\|_{TV} \sum_{i=1}^N \left|\left((z-X)^{-1} - (z-Y)^{-1}\right)_{ii}\right|$$

But $M := (z-X)^{-1} - (z-Y)^{-1}$ has rank one and is uniformly bounded by $2/|\Im z|$, hence $M = \pm\|M\| ee^*$ for some unit vector v. It follows that

$$\left|\sum_{i=1}^N f((z-X)^{-1}_{ii}) - \sum_{i=1}^N f((z-Y)^{-1}_{ii})\right| \le \|f\|_{TV}\frac{2}{|\Im z|}\sum_{i=1}^N |v_i|^2 = \|f\|_{TV}\frac{2}{|\Im z|}\,,$$

which is the desired bound.

To get an equation for the order parameter ρ_z^N, notice that by Schur complement formula and symmetry that

$$\mathbb{E}[\rho_z^N(t)] = \mathbb{E}\left[\Phi\left(t\left(z - X(1,1) + X_1^T(zI - X^{(1)})^{-1}X_1\right)^{-1}\right)\right]\,.$$

Again, we may neglect the off diagonal terms as in Lemma 3.13 (note that the A_i's may be assumed to vanish with high probability and then the L^2 norm argument holds), and therefore deduce that since Φ is smooth by continuous by assumption

$$\mathbb{E}[\rho_z^N(t)] = \mathbb{E}\left[\Phi\left(t\left(z - \sum_{k=2}^N |X_{ik}|^2(zI - X^{(1)})^{-1}_{kk}\right)^{-1}\right)\right] + o(1)\,.$$

Therefore we deduce from Assumption (3.11) that

$$\begin{aligned}
\mathbb{E}[\rho_z^N(t)] &= \int_0^\infty g(y)\mathbb{E}[e^{i\frac{y}{t}(z-X_{ii}-\sum |X_{ij}|^2(z-X^i)^{-1}_{jj})}]dy + o(1)\\
&= \int_0^\infty g(y)e^{i\frac{y}{t}z}\prod_{j=1}^N(1+\frac{1}{N}\Phi(\frac{y}{t}(z-X^i)^{-1}_{jj}))]dy + o(1)\\
&= \int_0^\infty g(y)e^{i\frac{y}{t}z}e^{\rho_z^N(y/t)}dy + o(1) \qquad (3.13)
\end{aligned}$$

It is not hard to see that ρ_z^N is sequentially tight as a continuous function on $\mathbb{R}^+$, for instance by Arzela-Ascoli theorem. By (3.13), any limit point ρ_z satisfies

$$\rho_z(t) = \int_0^\infty g(y)e^{i\frac{y}{t}z}e^{\rho_z(y/t)}dy\,.$$

We also note that by definition, ρ_z^N takes its values in $\{x+iy, x\le 0\}$. We claim that there exists at most one solution with values with non positive real part for $\Im z$ big enough. Indeed if we had two such solutions ρ and $\tilde\rho$, and we denote $\Delta(t)=|\rho_z(t)-\tilde\rho_z(t)|$ their difference, then as $|g(y)|\le Cy^\kappa, \kappa>-1$

$$\Delta(t)\le C\int_0^\infty y^\kappa\Delta(y/t)\wedge 1e^{-\Im zy/t}dy = Ct^{\kappa+1}\int y^\kappa\Delta(y)\wedge 1e^{-\Im zy}dy$$

where we used that $\rho,\tilde\rho$ have non positive real parts. Integrating under $t^\kappa e^{-\Im zt}dt$ on both sides yields

$$I := \int y^\kappa\Delta(y)e^{-\Im zy}dy \le C\int t^{2\kappa+1}e^{-\Im zt}dt\times\int y^\kappa\Delta(y)\wedge 1e^{-\Im zy}dy$$

Since $\int y^\kappa\Delta(y)\wedge 1e^{-\Im zy}dy$ is finite and smaller than I, we deduce for $\Im z$ large enough so that

$$C\int t^{2\kappa+1}e^{-\Im zt}dt<1$$

that $I=0$. But we claim that for each given t, $N\to\rho_z^N(t)$ is analytic away from the real axis. Indeed, Φ is analytic on $\{x+iy, y<0\}$ by (3.11) and $(z-X)^{-1}_{ii}$ is analytic on $\Im z>0$, with image in $\{x+iy, y<0\}$. We have also seen it is uniformly bounded. Hence any limit point must be analytic on $\{\Im z>0\}$ by Montel's theorem. We conclude that $\rho_z(t)$ is uniquely determined by its values for $\Im z$ large and therefore uniquely defined by our equation. To conclude, $\rho_z^N(t)$ converges almost surely and in L^1 towards $\rho_z(t)$.

This characterizes also the limit of G_N. Indeed, by concentration inequalities we have almost surely that

$$\begin{aligned}
G_N(z) &= \mathbb{E}[G_N(z)] + o(1)\\
&= \mathbb{E}[\frac{1}{z-\sum|X_{ji}|^2(z-X^{(1)})^{-1}_{jj}}] + o(1)\\
&= i\int_0^\infty dt\mathbb{E}\left[e^{it(z-\sum|X_{ji}|^2(z-X^{(1)})^{-1}_{jj})}\right] + o(1)\\
&= i\int_0^\infty dte^{itz+\rho_z(t)} + o(1)\,.
\end{aligned}$$

As G_N is tight on $\Omega_\varepsilon=\{\Im z\ge\varepsilon\}$ for all $\varepsilon>0$ by Arzela-Ascoli theorem, and its limit points are analytic by Montel's theorem (as G_N is uniformly bounded on Ω_ε), this implies the convergence of G_N on $\mathbb{C}^+$ to the unique analytic function on $\mathbb{C}^+$ given by the above formula for $\Im z$ large enough. $\diamond$

3.3. CLT

The goal of this section is to prove the CLT for heavy tail random matrices. We shall use martingale technology. This strategy was used by Bai and Silverstein (see [**83**]) in the context of Wigner matrices. We shall prove

THEOREM 3.17. *Under the assumptions of Theorem 3.16 and if we assume additionally that we can write*

$$\Phi(x+y) = \int_{\mathbb{R}^+\times\mathbb{R}^+} e^{\frac{it}{x}+\frac{is}{y}}\, d\tau(t,s)$$

with a measure $d\tau(t,s) = \delta_{t=0}d\mu(s) + \delta_{s=0}d\mu(t) + f(t,s)dsdt$ *with* $|f(t,s)| \le Ct^\kappa + Cs^\kappa$, $\kappa > -2$, *then for all* $z_1, \ldots, z_p \in \mathbb{C}\backslash\mathbb{R}$,

$$\left(\mathbf{Z}_N(z_i) := \frac{1}{\sqrt{N}}\left(\mathrm{Tr}((z_i - X)^{-1}) - \mathbb{E}[\mathrm{Tr}((z_i - X)^{-1})]\right), 1 \le i \le p\right)$$

converges in law towards a centered Gaussian vector.

As an exercise, you can check that the new assumption is verified for Erdös-Renyi matrices and for Lévy except for the bound on f (some argument then needs to be adapted). Notice that the scaling of the fluctuations is $N^{-\frac{1}{2}}$ as for independent variables, but differently from light tails matrices.

We let $\mathbb{E}_k$ be the conditional expectation with respect to the $k \times k$ left upper corner of X (this means we integrate over X_{ij}, i or j being greater than k, and keep the other variables fixed), and we write

$$M_k = \frac{1}{\sqrt{N}}\sum_{\ell=0}^{k}(\mathbb{E}_{\ell+1} - \mathbb{E}_\ell)[\mathrm{Tr}(z - X)^{-1}]$$

M_k is a martingale and $M_{N-1} = Z_N(z)$. We leave the following as an exercise :

EXERCISE 3.18. Let $\varepsilon_\ell^N(z) := (\mathbb{E}_\ell - \mathbb{E}_{\ell+1})(\mathrm{Tr}(z-X)^{-1})$ and assume that for any $z \in \mathbb{C}\backslash\mathbb{R}$, there exists a deterministic constant $\epsilon_N(z)$ going to zero with N such that

$$|\varepsilon_\ell^N(z)| \le \epsilon_N(z) \quad a.s.$$

whereas there exists a deterministic function $C(z,z')$ such that

$$\lim_{N\to\infty}\sum_{\ell=1}^{N}\mathbb{E}_\ell[\varepsilon_\ell^N(z)\varepsilon_\ell^N(z')] = C(z,z') \quad a.s. \tag{3.14}$$

Then show that for any $z_1, \ldots, z_p \in \mathbb{C}\backslash\mathbb{R}$,

$$\left(Z_N(z_i) := \frac{1}{\sqrt{N}}(\mathrm{Tr}((z_i - X)^{-1}) - \mathbb{E}[\mathrm{Tr}((z_i - X)^{-1})]), 1 \le i \le p\right)$$

converges in law towards a centered Gaussian vector with covariance $C(z,z')$.

Let us show that the hypotheses of the exercise are fulfilled. First we need another formula from linear algebra, namely, see [**83**, Lemma A.5] :

$$\varepsilon_k^N(z) := \frac{1}{\sqrt{N}}(\mathbb{E}_k - \mathbb{E}_{k-1})\left[\frac{1 + X_k^T G_k(z)^2 X_k}{z - X_{kk} - X_k^T G_k(z) X_k}\right]$$

where $G_k(z) = (z - X^{(k)})^{-1}$ is the resolvent of the matrix X where the kth line and column were removed. Now

$$|X_k^T G_k(z)^2 X_k| \leq \sum \frac{\langle X_k, u_i \rangle^2}{|\lambda_i - z|^2} = \frac{1}{\Im z} \Im(X_k^T G_k(z) X_k)$$

from which it follows that

$$|\varepsilon_k^N(z)| \leq \frac{2}{\Im z \sqrt{N}}$$

so that our first hypothesis is fulfilled. Hence the main point is to check the convergence of the covariance (3.14).

We first observe we can again remove the off-diagonal terms in $\varepsilon_k^N(z)$ as there are small with high probability. Hence we consider

$$\delta_k^N(z) := \frac{1}{\sqrt{N}} (\mathbb{E}_k - \mathbb{E}_{k-1}) \left[\frac{1 + \sum_i |X_{ki}|^2 [G_k(z)^2]_{ii}}{z - \sum_i |X_{ki}|^2 G_k(z)_{ii}} \right]$$

and observe that it is also bounded by $C(z)/\sqrt{N}$ as $|\Im z||G^2(z)_{jj}| \leq \Im G(z)_{jj}$.

We are going to show that as k/N goes to u

$$\lim_{N \to \infty} N \mathbb{E}_k[\delta_k^N(z) \delta_k^N(z')] = C_u(z, z') \quad a.s. \tag{3.15}$$

which will give (3.14) with

$$C(z, z') = \int_0^1 C_u(z, z') du \, .$$

Let us denote by $\psi_z(X)$ the function on $N \times N$ self-adjoint matrices X given by :

$$\psi_z(X) = \frac{1 + \sum_{i \neq k} |X_{ki}|^2 (z - X^k)_{ii}^{-2}}{z - \sum_{i \neq k} |X_{ki}|^2 (z - X^k)_{ii}^{-1}}$$

Then, we can write

$$\begin{aligned} E_k &:= N \mathbb{E}_k[\delta_k^N(z) \delta_k^N(z')] \\ &= \mathbb{E}_{Z'} \mathbb{E}_Z \mathbb{E}_u[\psi_z(X(W,u,Z)) \psi_z(X(W,u,Z'))] \\ &\quad - \mathbb{E}_{Z,u}[\psi_z(X(W,u,Z))] \mathbb{E}_{Z,u}[\psi_{z'}(X(W,u,Z))] \end{aligned}$$

where the matrix $X(W,u,Z)$ has $k-1 \times k-1$ first square upper left corner given by W(kept fixed here), the kth vector given by $u \in \mathbb{C}^k$, and the last $N-k$ columns $(X_{ij})_{i \leq j}$, $j \geq k+1$, given by Z. $\mathbb{E}_Y$ means that we integrate over Y. To estimate this expectation we pass to Fourier transform again, and use the fact that

$$1 + \sum_{i \neq k} |X_{ki}|^2 (z - X^k)_{ii}^{-2} = \partial_z (z - \sum_i |X_{ki}|^2 (z - X^k)_{ii}^{-1})$$

so that

$$\begin{aligned} \psi_z(X) &= \partial_z \ln(z - \sum_i |X_{ki}|^2 (z - X^k)_{ii}^{-1}) \\ &= \int_0^\infty \frac{dt}{t} \partial_z e^{itz - it \sum_i |X_{ki}|^2 (z - X^{(k)})_{ii}^{-1}} dt \end{aligned}$$

where we used the representation $\ln z = \int_0^\infty t^{-1}(e^{itz} - 1) dt$. Here the integral is not singular at $t = 0$ as the derivative brings a term linear in t. Also, we assumed that the imaginary part of z is positive so that the integral converges. Otherwise we would have chosen t in $(-\infty, 0)$. To simplify the notations hereafter, we take z, z' with positive imaginary part, the general case simply requires to change the integration over t, s below from $\mathbb{R}^+$ to $\mathbb{R}^-$) Up to cut the integrals to make everything well defined we can use the above representation to compute E_k, and permute integration and derivatives. Then, we have

$$E_k = E_{k,1} - E_{k,2}(z)E_{k,2}(z')$$

with

$$E_{k,1} = \int_0^\infty \frac{dt}{t}\int_0^\infty \frac{ds}{s}\partial_z\partial_{z'}e^{itz+isz'}$$
$$\times\mathbb{E}[e^{-it\sum_i X(W,u,Z)^2_{ki}(z-X(W,u,Z)^{(k)})^{-1}_{ii}-is\sum_i X(W,u,Z')^2_{ki}(z'-X(W,u,Z')^{(k)})^{-1}_{ii}}]$$
$$E_{k,2}(z) = \int_0^\infty \frac{dt}{t}\partial_z e^{itz}\mathbb{E}[e^{-it\sum_i X(W,u,Z)^2_{ki}(z-X(W,u,Z)^{(k)})^{-1}_{ii}}]$$

where we first expectation holds on Z, Z', u and the second on Z, u. For the second term we recognize

$$\mathbb{E}[e^{-it\sum_i X(W,u,Z)^2_{ki}(z-X(W,u,Z)^{(k)})^{-1}_{ii}}] = e^{\rho_z^N(t)} = e^{\rho_z(t)} + o(1)$$

so that we get

$$E_{k,2}(z) \simeq \int_0^\infty \frac{dt}{t}\partial_z e^{itz+\rho_z(t)}$$

The first term is not so easy. Indeed, $X(W,u,Z)_{ki} = X(W,u,Z')_{ki} = u_i$ for $i \le k$ but are independent $X(W,u,Z)_{ki} = Z_{ki}, X(W,u,Z')_{ki} = Z'_{ki}$ for $i \ge k+1$. Hence, taking the expectation we now get :

$$E_{k,1} \simeq \int_0^\infty \frac{dt}{t}\int_0^\infty \frac{ds}{s}\partial_z\partial_{z'}e^{itz+isz'}e^{u\rho^{N,\frac{k}{N}}_{z,z'}(t,s)+(1-u)(\rho_z(t)+\rho_{z'}(s))} \tag{3.16}$$

where

$$\rho^{N,\frac{k}{N}}_{z,z'}(t,s) = \frac{1}{k}\sum_{\ell=1}^{k}\Phi\left(t(z-X(W,u,Z)^{(k)})^{-1}_{\ell,\ell} + s(z'-X(W,u,Z')^{(k)})^{-1}_{\ell,\ell}\right)$$

and we noticed that

$$\frac{1}{N-k}\sum_{\ell=k+1}^{N}\left(\Phi(t(z-X(W,u,Z)^{(k)})^{-1}_{\ell,\ell}) + \Phi(s(z'-X(W,u,Z')^{(k)})^{-1}_{\ell,\ell})\right) \simeq \rho_z(t)+\rho_{z'}(s)\,.$$

Hence, we have a new parameter coming in, namely $\rho^{N,\frac{k}{N}}_{z,z'}(t,s)$. Again we can show that this order parameter self-averages. We can derive an equation for this parameter thanks to our additional assumption on Φ as well as Schur complement formula. For the latter we notice that for $\ell \le k$ we have

$$\frac{1}{(z-X(W,u,Z)^{(k)})^{-1}_{\ell,\ell}} \simeq z - \sum |X(W,u,Z)^{(k)}_{\ell i}|^2(z-X(W,u,Z)^{(k,\ell)})^{-1}_{ii}$$

where

$$X(W,u,Z)^{(k)}_{\ell i} = \begin{cases} W_{\ell i}, i \le k-1 \\ Z_{\ell,i+1}, i \ge k \end{cases}$$

We can write (up to admit that reduction by one dimension does not change the parameter, which can be verified by Weyl interlacing, and that we can remove again off diagonal terms) thanks to our hypothesis on Φ,

$$\begin{aligned}
&\bar\rho^{N,\frac{k}{N}}_{z,z'}(t,s)=\mathbb{E}\big[\rho^{N,\frac{k}{N}}_{z,z'}(t,s)\big]\\
&=\int_0^\infty\int_0^\infty e^{i\frac{v}{t}z+i\frac{v'}{s}z'}\mathbb{E}[\exp\big\{-i\frac{v}{t}\sum_\ell|X(W,u,Z)^{(k)}_{\ell 1}|^2(z-X(W,u,Z)^{(k,1)})^{-1}_{\ell\ell}\\
&\quad -i\frac{v'}{s}\sum_\ell|X(W,u,Z')^{(k)}_{\ell 1}|^2(z-X(W,u,Z')^{(k,1)})^{-1}_{\ell\ell}\big\}]\mathrm{d}\tau(v,v')\\
&=\int_0^\infty\int_0^\infty e^{i\frac{v}{t}z+i\frac{v'}{s}z'}e^{u\bar\rho^{N,\frac{k}{N}}_{z,z'}(\frac{v}{t},\frac{v'}{s})+(1-u)(\rho_z(\frac{v}{t})+\rho_{z'}(\frac{v'}{s}))}\mathrm{d}\tau(v,v')+o(1)
\end{aligned}$$

It is not hard to see as before that this order parameter is tight as a continuous function of two variables. We can then verify that our assumption guarantees that there exists a unique solution for $\Im z,\Im z'$ large enough to the equation

$$\rho^u_{z,z'}(t,s)=\int_0^\infty\int_0^\infty e^{i\frac{v}{t}z+i\frac{v'}{s}z'}e^{u\rho^u_{z,z'}(\frac{v}{t},\frac{v'}{s})+(1-u)(\rho_z(\frac{v}{t})+\rho_{z'}(\frac{v'}{s}))}\mathrm{d}\tau(v,v')$$

such that when z' goes to infinity or s goes to zero (resp. $z\to\infty,t\to 0$), $\rho^u_{z,z'}(t,s)$ goes to $\rho_z(t)$ (resp. $\rho_{z'}(s)$). Note here that $\rho^u_{z,z'}$ has also a non positive real part (as ρ_z). Indeed, if we have two solutions $\rho^u_{z,z'},\tilde\rho^u_{z,z'}$ then

$$\Delta(t,s)=|\rho^u_{z,z'}(t,s)-\tilde\rho^u_{z,z'}(t,s)|\le C\int_0^\infty\int_0^\infty e^{-\frac{v}{t}\Im z-\frac{v'}{s}\Im z'}\Delta(\frac{v}{t},\frac{v'}{s})|f(v,v')|dvdv'\,.$$

Hence, integrating both sides with respect to $(t^\kappa+s^\kappa)tse^{-\Im zt-\Im z's}dtds$ yields

$$I:=\int\Delta(t,s)(t^\kappa+s^\kappa)e^{-\Im zt-\Im z's}dtds\le C\int(t^\kappa+s^\kappa)^2tse^{\Im zt-\Im z's}dtdsI$$

Since I is obviously finit, we deduce that it vanishes as soon as

$$C\int(t^\kappa+s^\kappa)^2tse^{\Im zt-\Im z's}dtds<1\,.$$

Hence, $\rho^u_{z,z'}$ is uniquely defined for $\Im z$ or $\Im z'$ large enough. Its analytic extension is thus also uniquely defined. And hence $\bar\rho^{N,\frac{k}{N}}_{z,z'}$ converges towards $\rho^u_{z,z'}$ for all z,z' in $(\mathbb{C}^+)^2$. This guarantees the announced convergence (3.15) with

$$C_u(z,z')=\int_0^\infty\frac{dt}{t}\int_0^\infty\frac{ds}{s}\partial_z\partial_{z'}e^{itz+isz'}(e^{u(\rho^u_{z,z'}(t,s)-\rho_z(t)-\rho_z(s))}-1)e^{\rho_z(t)+\rho_{z'}(s)}\,.$$

CHAPTER 4

Beta-ensembles

Closely related to random matrices are the so-called Beta-ensembles. Their distribution is the probability measure on $\mathbb{R}^N$ given by

$$\mathrm{d}P_N^{\beta,V}(\lambda_1,\ldots,\lambda_N)=\frac{1}{Z_N^{\beta,V}}\Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)}\prod_{i=1}^N \mathrm{d}\lambda_i$$

where $\Delta(\lambda)=\prod_{i<j}|\lambda_i-\lambda_j|$.

Remark 4.1. In the case $V(X)=\frac{1}{2}x^2$ and $\beta=2$, $P_N^{2,x^2/4}$ is exactly the law of the eigenvalues for a matrix taken in the GUE as we were considering in the previous chapter (the case $\beta=1$ corresponds to GOE and $\beta=4$ to GSE). This is left as a (complicated) exercise, see e.g. **[3]**.

β ensembles also represent strongly interacting particle systems. It turns out that both global and local statistics could be analyzed in some details. In these lectures, we will discuss global asymptotics in the spirit of the previous chapter. This section is strongly inspired from **[13]**. However, in that paper, only Stieltjes functions were considered, so that closed equations for correlators were only retrieved under the assumption that V is analytic. In this section, we consider more general correlators, allowing sufficiently smooth (but not analytic) potentials. We did not try to optimize the smoothness assumption.

4.1. Law of large numbers and large deviation principles

Notice that we can rewrite the density of β-ensembles as :

$$\begin{aligned}\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} &= \frac{1}{Z_N^{\beta,V}}\exp\left\{\frac{1}{2}\beta\sum_{i\neq j}\ln|\lambda_i-\lambda_j|-\beta N\sum V(\lambda_i)\right\}\\ \text{"}=\text{"} &\quad \frac{1}{Z_N^{\beta,V}}\exp\left\{-\beta N^2\mathcal{E}\left(\hat{\mu}_N\right)\right\}\end{aligned}$$

where $\hat{\mu}_N$ is the empirical measure (total mass 1), and for any probability measure μ on the real line, we denote by $\mathcal{E}$ the energy

$$\mathcal{E}(\mu)=\int\int[\frac{1}{2}V(x)+\frac{1}{2}V(y)-\frac{1}{2}\ln|x-y|]\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

(the "$=$" is in quotes because we have thrown out the fact that $\ln|x-y|$ is not well defined for a Dirac mass on the "self-interaction" diagonal terms)

Assumption 4.2. Assume that $\liminf_{|x|\to\infty}\frac{V(x)}{\ln(|x|)}>1$ (i.e. $V(x)$ goes to infinity fast enough to dominate the log term at infinity) and V is continuous.

THEOREM 4.3. *If Assumption 4.2 holds, the empirical measure converges almost surely for the weak topology*

$$\hat{\mu}_N \Rightarrow \mu_V^{\mathrm{eq}}\,, a.s$$

where μ_V^{eq} is the equilibrium measure for V, namely the minimizer of $\mathcal{E}(\mu)$.

One can derive this convergence from a related large deviation principle [**8**] that we now state. Below, the set $\mathcal{P}(\mathbb{R})$ of probability measures on the real line is endowed with its weak topology.

THEOREM 4.4. *If Assumption 4.2 holds, the law of $\hat{\mu}_N$ under $P_N^{\beta,V}$ satisfies a large deviation principle with speed N^2 and good rate function*

$$I(\mu) = \beta\mathcal{E}(\mu) - \beta \inf_{\nu\in\mathcal{P}(\mathbb{R})} \mathcal{E}(\nu)\,.$$

In other words, I has compact level sets and for any closed set F of $\mathcal{P}(\mathbb{R})$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \ln P_N^{\beta,V}(\hat{\mu}_N \in F) \le -\inf_F I$$

whereas for any open set O of $\mathcal{P}(\mathbb{R})$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \ln P_N^{\beta,V}(\hat{\mu}_N \in O) \ge -\inf_O I$$

To deduce the convergence of the empirical measure, we first prove the existence and uniqueness of the minimizers of $\mathcal{E}$.

LEMMA 4.5. *Suppose Assumption 4.2 holds, then :*

- *There exists a unique minimizer μ_V^{eq} to $\mathcal{E}$. It is characterized by the fact that there exists a finite constant C_V such that the effective potential*

$$V_{\mathrm{eff}}(x) := V(x) - \int \ln|x-y|\, d\mu_V^{\mathrm{eq}}(y) - C_V$$

  vanishes on the support of μ_V^{eq} and is non negative everywhere.
- *For any probability measure μ, we have the decomposition*

$$\mathcal{E}(\mu) = \mathcal{E}(\mu_V^{\mathrm{eq}}) + \int_0^\infty \frac{ds}{s} \left| \int e^{isx} d(\mu - \mu_V^{\mathrm{eq}})(x) \right|^2 + \int V_{\mathrm{eff}}(x) d\mu(x)\,. \tag{4.1}$$

PROOF. We notice that with $f(x,y) = \frac{1}{2}V(x) + \frac{1}{2}V(y) - \frac{1}{2}\ln|x-y|$,

$$\mathcal{E}(\mu) = \int f(x,y) d\mu(x) d\mu(y) = \sup_{M\ge 0} \int f(x,y) \wedge M d\mu(x) d\mu(y)$$

by monotone convergence theorem. Observe also that the growth assumption we made on V insures that there exists $\gamma > 0$ and $C > -\infty$ such that

$$f(x,y) \ge \gamma(\ln(|x|+1) + \ln(|y|+1)) + C\,, \tag{4.2}$$

so that $f \wedge M$ is a bounded continuous function. Hence, $\mathcal{E}$ is the supremum of the bounded continuous functions $\mathcal{E}_M(\mu) := \int\int f(x,y) \wedge M d\mu(x) d\mu(y)$, defined on the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$, equipped with the weak topology. Hence $\mathcal{E}$ is lower semi-continuous. Moreover, the lower bound (4.2) on f yields

$$L_M := \{\mu \in \mathcal{P}(\mathbb{R}) : \mathcal{E}(\mu) \le M\} \subset \left\{ \int \ln(|x|+1) d\mu(x) \le \frac{M-C}{2\gamma} \right\} =: K_M \tag{4.3}$$

where K_M is compact. Hence, since L_M is closed by lower semi-continuity of $\mathcal{E}$ we conclude that L_M is compact for any real number M. This implies that $\mathcal{E}$ achieves its minimal value. Let $\mu_V^{\rm eq}$ be a minimizer. Writing that $\mathcal{E}(\mu_V^{\rm eq}+\epsilon\nu)\ge\mathcal{E}(\mu_V^{\rm eq})$ for any measure ν with zero mass so that $\mu_V^{\rm eq}+\epsilon\nu$ is positive for ϵ small enough gives the announced characterization in terms of the effective potential $V_{\rm eff}$.

For the second point, take μ with $\mathcal{E}(\mu)<\infty$ and write

$$V = V_{\rm eff} + \int \ln|.-y|\,\mathrm{d}\mu_V^{\rm eq}(y) + C_V$$

so that

$$\mathcal{E}(\mu) = \mathcal{E}(\mu_V^{\rm eq}) - \frac{1}{2}\int\int \ln|x-y|\,d(\mu-\mu_V^{\rm eq})(x)d(\mu-\mu_V^{\rm eq})(y) + \int V_{\rm eff}(x)d\mu(x)\,.$$

On the other hand, we have the following equality for all $x,y\in\mathbb{R}$

$$\ln|x-y| = \int_0^\infty \frac{1}{2t}\left(e^{-\frac{1}{2t}} - e^{-\frac{|x-y|^2}{2t}}\right)dt\,.$$

One can then argue [**7**] that for all probability measure μ with $\mathcal{E}(\mu)<\infty$(in particular with no atoms), we can apply Fubini's theorem and the fact that $\mu-\mu_V^{\rm eq}$ is massless, to show that

$$\begin{aligned}
\Sigma(\mu) &:= \int\int \ln|x-y|\,d(\mu-\mu_V^{\rm eq})(x)d(\mu-\mu_V^{\rm eq})(y)\\
&= -\int_0^\infty \frac{1}{2t}\int\int e^{-\frac{|x-y|^2}{2t}}d(\mu-\mu_V^{\rm eq})(x)d(\mu-\mu_V^{\rm eq})(y)dt\\
&= -\int_0^\infty \frac{1}{2\sqrt{2\pi t}}\int e^{-\frac{1}{2}t\lambda^2}\left|\int e^{i\lambda x}d(\mu-\mu_V^{\rm eq})(x)\right|^2 d\lambda dt\\
&= -\int_0^\infty \left|\int e^{iyx}d(\mu-\mu_V^{\rm eq})(x)\right|^2\frac{dy}{y}
\end{aligned}$$

This term is concave non-positive in the measure μ as it is quadratic in μ, and in fact non degenerate as it vanishes only when all Fourier transforms of μ equal those of $\mu_V^{\rm eq}$, implying that $\mu=\mu_V^{\rm eq}$. Therefore $\mathcal{E}$ is as well strictly convex as it differs from this function only by a linear term. Its minimizer is thus unique. $\diamond$

Remark 4.6. Note that the characterization of $\mu_V^{\rm eq}$ implies that it is compactly supported as $V_{\rm eff}$ goes to infinity at infinity.

Remark 4.7. It can be shown that the equilibrium measure has a bounded density with respect to Lebesgue measure if V is C^2. Indeed, if f is C^1 from $\mathbb{R}\to\mathbb{R}$ and ε small enough so that $\varphi_\varepsilon(x)=x+\varepsilon f(x)$ is a bijection, we know that

$$I(\varphi_\varepsilon\#\mu_V^{\rm eq})\ge I(\mu_V^{\rm eq})\,,$$

where we denoted by $\varphi\#\mu$ the pushforward of μ by φ given, for any test function g, by :

$$\int g(y)d\varphi\#\mu(y) = \int g(\varphi(x))d\mu(x)\,.$$

As a consequence, we deduce by arguing that the term linear in ε must vanish that

$$\frac{1}{2}\int\int\frac{f(x)-f(y)}{x-y}d\mu_V^{\rm eq}(x)d\mu_V^{\rm eq}(y) = \int V'(x)f(x)d\mu_V^{\rm eq}(x)\,.$$

By linearity, we may now take f to be complex valued and given by $f(x) = (z-x)^{-1}$. We deduce that the Stieltjes transform $S_{\rm eq}(z) = \int (z-x)^{-1} d\mu_V^{\rm eq}(x)$ satisfies

$$\frac{1}{2} S_{\rm eq}(z)^2 = \int \frac{V'(x)}{z-x} d\mu_V^{\rm eq}(x) = S_{\rm eq}(z) V'(\Re(z)) + f(z)$$

with

$$f(z) = \int \frac{V'(x) - V'(\Re(z))}{z-x} d\mu_V^{\rm eq}(x) \,.$$

f is bounded on compacts if V is C^2. Moreover, we deduce that

$$S(z) = V'(\Re(z)) - \sqrt{V'(\Re(z))^2 + 2f(z)} \,.$$

But we can now let z going to the real axis and we deduce from Theorem 3.5 that $\mu_V^{\rm eq}$ has bounded density $\sqrt{V'(x)^2 - 4f(x)}$.

Note also that it follows, since $V'(x)^2 - 4f(x)$ is smooth that when the density of $\mu_V^{\rm eq}$ vanishes at a it vanishes like $|x-a|^{q/2}$ for some integer number $q \geq 1$.

Because the proof of the large deviation principle will be roughly the same in the discrete case, we detail it here.

Proof of Theorem 4.4 We first consider the non-normalized measure

$$\frac{\mathrm{d}Q_N^{\beta,V}}{\mathrm{d}\lambda} = \exp\left\{ \frac{1}{2}\beta \sum_{i \neq j} \ln|\lambda_i - \lambda_j| - \beta N \sum V(\lambda_i) \right\}$$

and prove that it satisfies a weak large deviation principle, that is that for any probability measure μ,

$$\begin{aligned} -\beta \mathcal{E}(\mu) &= \limsup_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \ln Q_N^{\beta,V}(d(\hat\mu_N, \mu) < \delta) \\ &= \liminf_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{N^2} \ln Q_N^{\beta,V}(d(\hat\mu_N, \mu) < \delta) \end{aligned}$$

where d is a distance compatible with the weak topology, such as the Vasershtein distance.

To prove the upper bound observe that for any $M > 0$

$$\begin{aligned} Q_N^{\beta,V}(d(\hat\mu_N, \mu) < \delta) &\leq \int_{d(\hat\mu^N, \mu) < \delta} e^{-\beta N^2 \int_{x \neq y} f(x,y) \wedge M d\hat\mu^N(x) d\hat\mu^N(y)} \prod e^{-\beta V(\lambda_i)} d\lambda_i \\ &= e^{\beta N M} \int_{d(\hat\mu^N, \mu) < \delta} e^{-\beta N^2 \int f(x,y) \wedge M d\hat\mu^N(x) d\hat\mu^N(y)} \prod e^{-\beta V(\lambda_i)} d\lambda_i \end{aligned}$$

where in the first line we used that the λ_i are almost surely distinct. Now, using that for any finite M, $\mathcal{E}_M$ is continuous, we get

$$Q_N^{\beta,V}(d(\hat\mu_N, \mu) < \delta) \leq e^{\beta N M} e^{-\beta N^2 \mathcal{E}_M(\mu) + N^2 o(\delta)} \left(\int e^{-\beta V(\lambda)} d\lambda\right)^N$$

Taking first the limit N going to infinity, then δ going to zero and finally M going to infinity yields

$$\limsup_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^2} \ln Q_N^{\beta,V}(d(\hat\mu_N, \mu) < \delta) \leq -\beta \mathcal{E}(\mu) \,.$$

To get the lower bound, we may choose μ with no atoms as otherwise $\mathcal{E}(\mu) = +\infty$. We can also assume μ compactly supported, as we can approximate it by

$\mu_M(dx) = 1_{|x|\le M} d\mu/\mu([-M,M])$ and it is not hard to see that $\mathcal{E}(\mu_M)$ goes to $\mathcal{E}(\mu)$ as M goes to infinity. Let x_i be the i^{th} classical location of the particles given by $\mu((-\infty, x_i]) = i/N$. $x_i < x_{i+1}$ and we have for N large enough and $p > 0$, if $u_i = \lambda_i - x_i$,

$$\Omega = \cap_i \{|u_i| \le N^{-p}, u_i \le u_{i+1}\} \subset \{d(\hat{\mu}_N, \mu) < \delta\}$$

so that we get the lower bound

$$Q_N^{\beta,V}(d(\hat{\mu}_N,\mu) < \delta) \quad \ge \quad \int_\Omega \prod_{i>j} |x_i - x_j + u_i - u_j|^\beta \prod_{i=1}^N \exp\left(-N\beta V\left(x_i + u_i\right)\right) du_i$$

Observe that by our ordering of x and u, we have $|x_i - x_j + u_i - u_j| \ge \max\{|x_i - x_j|, |u_i - u_j|\}$ and therefore

$$\prod_{i>j} |x_i - x_j + u_i - u_j|^\beta \ge \prod_{i>j+1} |x_i - x_j|^\beta \prod_i |x_{i+1} - x_i|^{\beta/2} \prod_i |u_{i+1} - u_i|^{\beta/2}$$

where for $i > j+1$

$$\ln|x_i - x_j| \ge \int_{x_{i-1}}^{x_i} \int_{x_j}^{x_{j+1}} \ln|x-y| d\mu(x) d\mu(y)$$

whereas

$$\ln|x_i - x_{i-1}| \ge 2\int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} 1_{x>y} \ln|x-y| d\mu_V^{\mathrm{eq}}(x) d\mu_V^{\mathrm{eq}}(y)\,.$$

We deduce that

$$\sum_{i>j+1} \ln|x_i - x_j| + \frac{1}{2}\sum_i \ln|x_{i+1} - x_i| \ge \frac{N^2}{2} \int\int \ln|x-y| d\mu_V^{\mathrm{eq}}(x) d\mu_V^{\mathrm{eq}}(y)\,.$$

Moreover, V is continuous and μ compactly supported, so that

$$\frac{1}{N}\sum_{i=1}^N V(x_i + u_i) = \frac{1}{N}\sum_{i=1}^N V(x_i) + o(1)\,. \tag{4.4}$$

Hence, we conclude that

$$\begin{aligned} Q_N^{\beta,V}(d(\hat{\mu}_N,\mu) < \delta) \quad &\ge \quad \exp\{-\beta N^2 \mathcal{E}(\mu)\} \int_\Omega \prod_i |u_{i+1} - u_i|^{\beta/2} \prod du_i \\ &\ge \quad \exp\{-\beta N^2 \mathcal{E}(\mu) + o(N^2)\} \qquad (4.5) \end{aligned}$$

which gives the lower bound. To conclude, it is enough to prove exponential tightness. But with K_M as in (4.3) we have by (4.2)

$$Q_N^{\beta,V}(K_M^c) \le \int_{K_M^c} e^{-2\gamma N(N-1)\int \ln(|x|+1) d\hat{\mu}^N(x) - CN^2} \prod d\lambda_i \le e^{N^2(C' - 2\gamma M)}$$

with some finite constant C' independent of M. Hence, exponential tightness follows :

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \ln Q_N^{\beta,V}(K_M^c) = -\infty$$

from which we deduce a full large deviation principle for $Q_N^{\beta,V}$ and taking $F = O$ be the whole set of probability measures, we get in particular that

$$\lim_{N\to\infty} \frac{1}{N^2} \ln Z_N^{\beta,V} = -\beta \inf \mathcal{E}\,.$$

$\diamond$

We also have large deviations from the support : the probability that some eigenvalue is away from the support of the equilibrium measure decays exponentially fast if V_{eff} is positive there. This was proven for the quadratic potential in [**6**], then in [**3**] but with the implicit assumption that there is convergence of the support of the eigenvalues towards the support of the limiting equilibrium measure. In [**52, 16**], it was proved that large deviations estimate for the support hold in great generality. Hence, if the effective potential is positive outside of the support S of the equilibrium measure, there is no eigenvalue at positive distance of the support with exponentially large probability. It was shown in [**49**] that if the effective potential is not strictly positive outside of the support of the limiting measure, eigenvalues may deviate towards the points where it vanishes. For completeness, we summarize the proof of this large deviation principle below.

Theorem 4.8. *Let S be the support of μ_V^{eq}. Assume Assumption 4.2 and that V is C^2. Then, V_{eff} is a good rate function and we have the following large deviations from the support estimates. Then, for any closed set F in S^c*

$$\limsup_{N\to\infty} \frac{1}{N} \ln P_N^{\beta,V}(\exists i \in \{1,N\} : \lambda_i \in F) \le -\inf_F V_{\text{eff}}\,,$$

whereas for any open set $O \subset S^c$

$$\liminf_{N\to\infty} \frac{1}{N} \ln P_N^{\beta,V}(\exists i \in \{1,N\} : \lambda_i \in O) \ge -\inf_O V_{\text{eff}}\,.$$

Proof. Observe first that V_{eff} is continuous as V is and $x \to \int \ln|x-y| d\mu_V^{\text{eq}}(y)$ is continuous by Remark 4.7. Hence, as V_{eff} goes to infinity at infinity, it is a good rate function.

We shall use the representation

$$\frac{\Upsilon_N^{\beta,V}(\mathsf{F})}{\Upsilon_N^{\beta,V}(\mathbb{R})} \le P_N^{\beta,V}\left[\exists i \quad \lambda_i \in \mathsf{F}\right] \le N\,\frac{\Upsilon_N^{\beta,V}(\mathsf{F})}{\Upsilon_N^{\beta,V}(\mathbb{R})} \tag{4.6}$$

where, for any measurable set X :

$$\Upsilon_N^{\beta,V}(\mathsf{X}) = P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{\mathsf{X}} d\xi\, e^{\left\{-N\beta\, V(\xi) + (N-1)\beta \int d\hat\mu_{N-1}(\lambda) \ln|\xi-\lambda|\right\}}\right]. \tag{4.7}$$

We shall hereafter estimate $\frac{1}{N}\ln \Upsilon_N^{\beta,V}(\mathsf{X})$. We first prove a lower bound for $\Upsilon_N^{\beta,V}(\mathsf{X})$ with X open. For any $x \in \mathsf{X}$ we can find $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset X$. Let $\delta_\varepsilon(V) = \sup\{|V(x)-V(y)|, |x-y| \le \varepsilon\}$. Using twice Jensen inequality, we lower bound $\Upsilon_N^{\beta,V}(\mathsf{X})$ by

$$\begin{aligned}
&\geq P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{x-\varepsilon}^{x+\varepsilon} d\xi e^{\left\{-N\beta V(\xi)+(N-1)\beta\int d\hat{\mu}_{N-1}(\eta)\ln|\xi-\eta|\right\}}\right] \\
&\geq e^{-N\beta\left(V(x)+\delta_\varepsilon(V)\right)} P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{x-\varepsilon}^{x+\varepsilon} d\xi e^{\left\{(N-1)\beta\int d\hat{\mu}_{N-1}(\lambda)\ln|\xi-\lambda|\right\}}\right] \\
&\geq 2\varepsilon\, e^{-N\beta\left(V(x)+\delta_\varepsilon(V)\right)} e^{\left\{(N-1)\beta P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int d\hat{\mu}_{N-1}(\lambda)\, H_{x,\varepsilon}(\lambda)\right]\right\}}
\end{aligned}$$

$$\geq 2\varepsilon\, e^{-N\beta\left(V(x)+\delta_\varepsilon(V)\right)} e^{\left\{(N-1)\beta P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int d\hat{\mu}_{N-1}(\lambda)\,\phi_{x,K}(\lambda)H_{x,\varepsilon}(\lambda)\right]\right\}} \tag{4.8}$$

where we have set :

$$H_{x,\varepsilon}(\lambda)=\int_{x-\varepsilon}^{x+\varepsilon}\frac{d\xi}{2\varepsilon}\ln|\xi-\lambda| \tag{4.9}$$

and $\phi_{x,K}$ is a continuous function which vanishes outside of a large compact K including the support of μ_V^{eq}, is equal to one on a ball around x with radius $1+\varepsilon$ (note that H is non-negative outside $[x-(1+\varepsilon),x+1+\varepsilon]$ resulting with the lower bound (4.8)) and on the support of μ_V^{eq}, and takes values in $[0,1]$. For any fixed $\varepsilon>0$, $\phi_{x,K}H_{x,\varepsilon}$ is bounded continuous, so we have by Theorem 4.4 (note that it applies as well when the potential depends on N as soon as it converges uniformly on compacts) that :

$$\Upsilon_N^{\beta,V}(\mathsf{X})\geq 2\varepsilon\, e^{-\frac{N\beta}{2}\left(V(x)+\delta_\varepsilon(V)\right)} e^{\left\{(N-1)\beta\int d\mu_V^{\mathrm{eq}}(\lambda)\,\phi_{x,K}(\lambda)\,H_{x,\varepsilon}(\lambda)+NR(\varepsilon,N)\right\}} \tag{4.10}$$

with $\lim_{N\to\infty}R(\varepsilon,N)=0$ for all $\varepsilon>0$. Letting $N\to\infty$, we deduce since $\int d\mu_V^{\mathrm{eq}}(\lambda)\,\phi_{x,K}(\lambda)\,H_{x,\varepsilon}(\lambda)=\int d\mu_V^{\mathrm{eq}}(\lambda)\,H_{x,\varepsilon}(\lambda)$ that :

$$\liminf_{N\to\infty}\frac{1}{N}\ln\Upsilon_N^{\beta,V}(\mathsf{X})\geq-\beta\,\delta_\varepsilon(V)-\beta\Big(V(x)-\int d\mu_V^{\mathrm{eq}}(\lambda)\,H_{x,\varepsilon}(\lambda)\Big) \tag{4.11}$$

Exchanging the integration over ξ and x, observing that $\xi\to\int d\mu_V^{\mathrm{eq}}(\lambda)\ln|\xi-\lambda|$ is continuous by Remark 4.7 and then letting $\varepsilon\to 0$, we conclude that for all $x\in\mathsf{X}$,

$$\liminf_{N\to\infty}\frac{1}{N}\ln\Upsilon_N^{\beta,V}(\mathsf{X})\geq-\beta\,V_{\mathrm{eff}}(x)\,. \tag{4.12}$$

We finally optimize over $x\in\mathsf{X}$ to get the desired lower bound. To prove the upper bound, we note that for any $M>0$,

$$\Upsilon_N^{\beta,V}(\mathsf{X})\leq P_{N-1}^{\beta,\frac{NV}{N-1}}\left[\int_{\mathsf{X}} d\xi\, e^{\left\{-N\beta V(\xi)+(N-1)\beta\int d\hat{\mu}_{N-1}(\lambda)\ln\max\left(|\xi-\lambda|,M^{-1}\right)\right\}}\right].$$

Observe that there exists C_0 and $c>0$ and d finite such that for $|\xi|$ larger than C_0 :

$$W_\mu(\xi)=V(\xi)-\int d\mu(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)\geq c\ln|\xi|+d$$

by the confinement Hypothesis (4.2), and this for all probability measures μ on $\mathbb{R}$. As a consequence, if $\mathsf{X}\subset[-C,C]^c$ for some C large enough, we deduce that :

$$\Upsilon_N^{\beta,V}(\mathsf{X})\leq\int_{\mathsf{X}} d\xi e^{-(N-1)\frac{\beta}{2}(c\ln|\xi|+d)}\leq e^{-N\frac{\beta}{4}c\ln C} \tag{4.13}$$

where the last bound holds for N large enough. Combining (4.12), (4.13) and (4.6) shows that

$$\limsup_{C\to\infty}\limsup_{N\to\infty}\frac{1}{N}\ln P_N^{\beta,V}\left[\exists i \quad |\lambda_i|\geq C\right]=-\infty\,.$$

Hence, we may restrict ourselves to X bounded. Moreover, the same bound extends to $P_{N-1}^{\beta,\frac{NV}{N-1}}$ so that we can restrict the expectation over $\hat\mu_{N-1}$ to probability measures supported on $[-C,C]$ up to an arbitrary small error $e^{-Ne(C)}$, provided C is large enough and where $e(C)$ goes to infinity with C. Recall also that $V(\xi)-2\int d\hat\mu_{N-1}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)$ is uniformly bounded from below by a constant D. As $\lambda\to\ln\max\big(|\xi-\lambda|,M^{-1}\big)$ is bounded continuous on compacts, we can use the large deviation principles of Theorem 4.4 to deduce that for any $\varepsilon>0$, any $C\geq C_0$,

$$\begin{aligned}(4.14)\Upsilon_N^{\beta,V}(\mathsf{X}) \quad&\leq\quad e^{N^2\tilde R(\varepsilon,N,C)}+e^{-N(e(C)-\frac{\beta}{2}D)}\\ &+\quad\int_{\mathsf{X}}d\xi\, e^{\left\{-N\beta V(\xi)+(N-1)\beta\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\left(|\xi-\lambda|,M^{-1}\right)+NM\varepsilon\right\}}\end{aligned}$$

with $\limsup_{N\to\infty}\tilde R(\varepsilon,N,C)$ equals to

$$\limsup_{N\to\infty}\frac{1}{N^2}\ln P_{N-1}^{\beta,\frac{NV}{N-1}}\left(\{\hat\mu_{N-1}([-C,C])=1\}\cap\{d(\hat\mu_{N-1},\mu_V^{\mathrm{eq}})>\varepsilon\}\right)\quad<\quad 0.$$

Moreover, $\xi\to V(\xi)-\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)$ is bounded continuous so that a standard Laplace method yields,

$$\limsup_{N\to\infty}\frac{1}{N}\,\ln\Upsilon_N^{\beta,V}(\mathsf{X})$$

$$\leq\max\left\{-\inf_{\xi\in\mathsf{X}}\left[\beta\Big(V(\xi)-\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\left(|\xi-\lambda|,M^{-1}\right)\Big)\right],-(e(C)-\frac{\beta}{2}D)\right\}.$$

We finally choose C large enough so that the first term is larger than the second, and conclude by monotone convergence theorem that $\int d\mu_V^{\mathrm{eq}}(\lambda)\ln\max\big(|\xi-\lambda|,M^{-1}\big)$ decreases as M goes to infinity towards $\int d\mu_V^{\mathrm{eq}}(\lambda)\ln|\xi-\lambda|$. This completes the proof of the large deviation.

$\diamond$

Hereafter we shall assume that

ASSUMPTION 4.9. V_{eff} is positive outside S.

REMARK 4.10. As a consequence of Theorem 4.8, we see that up to exponentially small probabilities, we can modify the potential at a distance ϵ of the support. Later on, we will assume we did so in order that V'_{eff} does not vanish outside S.

In these notes we will also use that particles stay smaller than M for some M large enough with exponentially large probability.

THEOREM 4.11. *Assume Assumption 4.2 holds. Then, there exists M finite so that*

$$\limsup_{N\to\infty}\frac{1}{N}\ln P_N^{\beta,V}\left(\exists i\in\{1,\ldots,N\}:|\lambda_i|\geq M\right)<0\,.$$

Here, we do not need to assume that the effective potential is positive everywhere, we only use it is large at infinity. The above shows that latter on, we can always change test functions outside of a large compact $[-M, M]$ and hence that L^2 norms are comparable to L^∞ norms.

4.2. Concentration of measure

We next define a distance on the set of probability measures on $\mathbb{R}$ which is well suited for our problem.

DEFINITION 4.12. For μ, μ' probability measures on $\mathbb{R}$, we set

$$D(\mu,\mu') = \left(\int_0^\infty \left| \int e^{iyx} d(\mu-\mu')(x) \right|^2 \frac{dy}{y} \right)^{\frac{1}{2}} .$$

It is easy to check that D defines a distance on $\mathcal{P}(\mathbb{R})$ (taking eventually the value $+\infty$, for instance on measure with Dirac masses). Moreover, we have the following property

PROPERTY 4.13. Let $f \in L^1(dx)$ such that $\hat{f}$ belongs to $L^1(dt)$, and set $\|f\|_{1/2} = \left(\int t|\hat{f}_t|^2 dt\right)^{1/2}$.

- Assume also f continuous. Then for any probability measures μ, μ'
$$\left| \int f(x) d(\mu-\mu')(x) \right| \leq 2\|f\|_{1/2} D(\mu,\mu') .$$
- Assume moreover $f, f' \in L^2$. Then

$$\|f\|_{1/2} \leq 2(\|f\|_{L^2} + \|f'\|_{L^2}) . \tag{4.15}$$

PROOF. For the first point we just use inverse Fourier transform and Fubini to write that

$$\begin{aligned} |\int f(x) d(\mu-\mu')(x)| &= |\int \hat{f}_t \widehat{\mu-\mu'}_t dt| \\ &\leq 2\int_0^\infty t^{1/2}|\hat{f}_t| t^{-1/2} |\widehat{\mu-\mu'}_t| dt \leq 2D(\mu,\mu')\|f\|_{1/2} \end{aligned}$$

where we finally used Cauchy-Schwarz inequality. For the second point, we observe that

$$\|f\|_{1/2}^2 = \int_0^\infty t|\hat{f}_t|^2 dt \leq \frac{1}{2}(\int |\hat{f}_t|^2 dt + \int |t\hat{f}_t|^2 dt) = \frac{\pi}{2}(\|f\|_{L^2}^2 + \|f'\|_{L^2})$$

from which the result follows. ◇

We are going to show that $\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ satisfies concentration inequalities for the D-distance. However, the distance between $\hat{\mu}_N$ and μ_V^{eq} is infinite as $\hat{\mu}_N$ has atoms. Hence, we are going to regularize $\hat{\mu}_N$ so that it has finite energy, following an idea of Maurel-Segala and Maida [**71**]. First define $\tilde{\lambda}$ by $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_i = \tilde{\lambda}_{i-1} + \max\{\sigma_N, \lambda_i - \lambda_{i-1}\}$ where σ_N will be chosen to be like N^{-p}. Remark that $\tilde{\lambda}_i - \tilde{\lambda}_{i-1} \geq \sigma_N$ whereas $|\lambda_i - \tilde{\lambda}_i| \leq N\sigma_N$. Define $\tilde{\mu}_N = \mathbb{E}_U\left[\frac{1}{N}\sum \delta_{\tilde{\lambda}_i + U_i}\right]$ where U_i are independent and equi-distributed random variables uniformly distributed on $[0, N^{-q}]$ (i.e. we smooth the measure by putting little rectangles instead of Dirac masses and make sure that the eigenvalues are at least distance N^{-p} apart). For further use, observe that we have uniformly $|\tilde{\lambda}_i + U_i - \lambda_i| \leq N^{1-p} + N^{-q}$. In the sequel we will take $q = p+1$ so that the first error term dominates. Then we claim that

LEMMA 4.14. *Assume V is C^1. For $3 < p+1 \le q$ there exists $C_{p,q}$ finite and $c > 0$ such that*

$$P_N^{\beta,V}\left(D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}}) \ge t\right) \le e^{C_{p,q} N \ln N - \beta N^2 t^2} + e^{-cN}$$

REMARK 4.15. Using that the logarithm is a Coulomb interaction, Serfaty et al could improve the above bounds to get the exact exponent in the term in $N \ln N$, as well as the term in N. This allows to prove central limit theorems under weaker conditions. Our approach seems however more robust and extends to more general interactions [**16**].

COROLLARY 4.16. Assume V is C^1. For all $q > 2$ there exists C finite and $c, c_0 > 0$ such that

$$\mathbf{P}\left(\sup_{\varphi} \frac{1}{N^{-q+2}\|\varphi\|_L + c_0 N^{-1/2}\sqrt{\ln N}\|\varphi\|_{\frac{1}{2}}} \left|\int \varphi \mathrm{d}\left(\hat{\mu}_N - \mu_V^{\mathrm{eq}}\right)\right| \ge 1\right) \le e^{-cN}$$

Moreover

$$\left|\int \frac{\varphi(x)-\varphi(y)}{x-y} d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(x) d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(y)\right| \le C\frac{1}{N}\ln N \|\varphi\|_{C^2} \tag{4.16}$$

with probability greater than $1 - e^{-cN}$. Here $\|\varphi\|_L = \sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|}$ and $\|\varphi\|_{C^k} = \sum_{\ell \le k}\|\varphi^{(\ell)}\|_\infty$. Note that we can modify φ outside a large set $[-M.M]$ up to modify the constant c.

PROOF. We take $q = p+1$. The triangle inequality yields :

$$\begin{aligned}
\left|\int \varphi \mathrm{d}\left(\hat{\mu}_N - \mu_V^{\mathrm{eq}}\right)\right| &= \left|\int \varphi \mathrm{d}\left(\hat{\mu}_N - \tilde{\mu}_N\right) + \int \varphi \mathrm{d}\left(\tilde{\mu}_N - \mu_V^{\mathrm{eq}}\right)\right| \\
&\le \left|\frac{1}{N}\sum_{i=1}^N \mathbb{E}_U[\varphi(\lambda_i) - \varphi(\tilde{\lambda}_i + U)]\right| \\
&\quad + \left|\int \hat{\varphi}(\lambda)(\widehat{\tilde{\mu}_N - \mu_V^{\mathrm{eq}}})(\lambda) d\lambda\right| \\
&\le \|\varphi\|_L N^{-q+2} + 2\|\varphi\|_{\frac{1}{2}} D\left(\tilde{\mu}_N, \mu_V^{\mathrm{eq}}\right)
\end{aligned}$$

where we noticed that $|\lambda_i - \tilde{\lambda}_i|$ is bounded by N^{-p+1} and U by N^{-q} and used Cauchy-Schwartz inequality. We finally use (4.15) to see that on $\{|\lambda_i| \le M\}$ we have by the previous lemma that for all φ

$$\left|\int \varphi \mathrm{d}\left(\hat{\mu}_N - \mu_V^{\mathrm{eq}}\right)\right| \le N^{-p+1}\|\varphi\|_L + t\|\varphi\|_{\frac{1}{2}}$$

with probability greater than $1 - e^{C_{p,q}N\ln N - \frac{\beta}{2}N^2t^2}$. We next choose $t = c_0\sqrt{\ln N/N}$ with $c_0^2 = 4|C_{p,q}|/\beta$ so that this probability is greater than $1 - e^{-c_0^2/2N\ln N}$. Theorem 4.11 completes the proof of the first point since it shows that the probability that one eigenvalue is greater than M decays exponentially fast.

We next consider

$$L_N(\phi) := \int \frac{\phi(x)-\phi(y)}{x-y} d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(x) d(\hat{\mu}^N - \mu_V^{\mathrm{eq}})(y)$$

on $\{\max|\lambda_i| \leq M\}$. Hence we can replace ϕ by $\phi\chi_M$ where χ_M is a smooth function, equal to one on $[-M, M]$ and vanishing outside $[-M-1, M+1]$. Hence assume that ϕ is compactly supported. If we denote by $\tilde{L}_N(\phi)$ the quantity defined as $L_N(\phi)$ but with $\tilde{\mu}_N$ instead of $\hat{\mu}^N$ we have that

$$\left|\tilde{L}_N(\phi) - L_N(\phi)\right| \leq 2\|\phi^{(2)}\|_\infty N^{-q+2}\,.$$

We can now replace ϕ by its Fourier representation to find that

$$\tilde{L}_N(\phi) = \int dtit\hat{\phi}(t)\int_0^1 d\alpha \int e^{i\alpha tx} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)\int e^{i(1-\alpha)tx} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)\,.$$

We can then use Cauchy-Schwartz inequality to deduce that

$$\begin{aligned}
|\tilde{L}_N(\phi)| &\leq \int dt|t\hat{\phi}(t)|\int_0^1 d\alpha|\int e^{i\alpha tx} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)|^2 \\
&= \int dt|t\hat{\phi}(t)|\int_0^1 \frac{td\alpha}{t\alpha}|\int e^{i\alpha tx} d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)|^2 \\
&\leq \int dt|t\hat{\phi}(t)|D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}})^2 \\
&\leq CD(\tilde{\mu}_N, \mu_V^{\mathrm{eq}})^2\|\phi\|_{C^2} \qquad (4.17)
\end{aligned}$$

where we noticed that

$$\begin{aligned}
\int dt|t\hat{\phi}(t)| &\leq \left(\int dt|t\hat{\phi}(t)|^2(1+t^2)\right)^{1/2}\left(\int dt(1+t^2)^{-1}\right)^{1/2} \\
&\leq C(\|\phi^{(2)}\|_{L^2} + \|\phi'\|_{L^2}) \leq C\|\phi\|_{C^2}
\end{aligned}$$

as we compactified ϕ. The conclusion follows from Theorem 4.11. $\diamond$

We next prove Lemma 4.14. We first show that :

$$Z_N^{\beta,V} \geq \exp\left(-N^2\beta\mathcal{E}(\mu_V^{\mathrm{eq}}) + CN\ln N\right)$$

The proof is exactly as in the proof of the large deviation lower bound of Theorem 4.4 except we take $\mu = \mu_V^{\mathrm{eq}}$ and V is C^1, so that

$$\frac{1}{N}\sum_{i=1}^N V(x_i + u_i) = \frac{1}{N}\sum_{i=1}^N V(x_i) + O(\frac{1}{N})\,.$$

This allows to improve the lower bound (4.5) into

$$\begin{aligned}
Z_N^{\beta,V} &\geq Q_N^{\beta,V}(d(\hat{\mu}_N, \mu_V^{\mathrm{eq}}) < \delta) \\
&\geq \exp\{-\beta N^2\mathcal{E}(\mu_V^{\mathrm{eq}}) + CN\ln N\} \qquad (4.18)
\end{aligned}$$

Now consider the unnormalized density of $Q_N^{\beta,V} = Z_N^{\beta,V}P_N^{\beta,V}$ on the set where $|\lambda_i| \leq M$ for all i

$$\begin{aligned}
\frac{dQ_N^{\beta,V}(\lambda)}{d\lambda} &= \prod_{i<j}|\lambda_i - \lambda_j|^\beta \exp\left(-N\beta\sum V(\lambda_i)\right) \\
&\leq \prod_{i<j}\left|\tilde{\lambda}_i - \tilde{\lambda}_j\right|^\beta \exp\left(-N\beta\sum V(\tilde{\lambda}_i)\right)
\end{aligned}$$

because the $\tilde{\lambda}$ only increased the differences. Observe that for $|\lambda_i| \le M$,

$$|V(\lambda_i) - V(\tilde{\lambda}_i + U_i)| \le \sup_{|x|\le M+1} |V'(x)|(N^{1-p} + N^{-q})\,.$$

Moreover for each $j > i$

$$\ln \left|\tilde{\lambda}_i - \tilde{\lambda}_j\right| = \mathbb{E} \ln \left|\tilde{\lambda}_i + u_i - \tilde{\lambda}_j - u_j\right| + O(N^{-q+p})\,.$$

Hence, we deduce that on $|\lambda_i| \le M$ for all i, there exists a finite constant C such that

$$\frac{\mathrm{d}P_N^{\beta,V}}{\mathrm{d}\lambda} \le \exp\left(-N^2\beta\,(\mathcal{E}(\tilde{\mu}_N) - \mathcal{E}(\mu_V^{\mathrm{eq}})) + CN\ln N + CN^{2-q+p} + CN^{3-p}\right)$$

As we chose $q = p+1$, $p > 2$, the error is at most of order $N \ln N$. We now use the fact that

$$\mathcal{E}(\tilde{\mu}_N) - \mathcal{E}(\mu_V^{\mathrm{eq}}) = D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}})^2 + \int (V_{\mathrm{eff}})(x) d(\tilde{\mu}_N - \mu_V^{\mathrm{eq}})(x)$$

where the last term is non-negative, and Theorem 4.11, to conclude

$$P_N^{\beta,V}\left(\{D(\tilde{\mu}_N, \mu_V^{\mathrm{eq}} \ge t\} \cap \{\max|\lambda_i| \le M\}\right) \le e^{CN\ln N - \beta N^2 t^2} \left(\int e^{-N\beta V_{\mathrm{eff}}(x)} dx\right)^N$$

where the last integral is bounded by a constant as V_{eff} is non-negative and goes to infinity at infinity faster than logarithmically. We finally remove the cutoff by M thanks to Theorem 4.11.

4.3. The Dyson-Schwinger equations

4.3.1. Goal and strategy. We want to show that for sufficiently smooth functions f that

- $$\mathbb{E}\left[\frac{1}{N}\sum f(\lambda_i)\right] = \mu_V^{\mathrm{eq}}(f) + \sum_{g=1}^{K} \frac{1}{N^g} c_g(f) + o(\frac{1}{N^K})$$
- $\sum f(\lambda_i) - \mathbb{E}[\sum f(\lambda_i)]$ converges to a centered Gaussian.

We will provide two approaches, one which deals with general functions and a second one, closer to what we will do for discrete β ensembles, where we will restrict ourselves to Stieltjes transform $f(x) = (z-x)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, which in fact gives these results for all analytic function f by Cauchy formula. The present approach allows to consider sufficiently smooth functions but we will not try to get the optimal smoothness. We will as well restrict ourselves to $K = 2$, but the strategy is similar to get higher order expansion. The strategy is similar to the case of the GUE :

- We derive a set of equations, the Dyson-Schwinger equations, for our observables (the correlation functions, that are moments of the empirical measure, or the moments of Stieltjes transform) : it is an infinite system of equations, a priori not closed. However, it will turn out that asymptotically it can be closed.

- We linearize the equations around the limit. It takes the form of a linearized operator acting on our observables being equal to observables of smaller magnitudes. Inverting this linear operator is then the key to improve the concentration bounds, starting from the already known concentration bounds of Corollary 4.16.
- Using optimal bounds on our observables and the inversion of the master operator, we recursively obtain their large N expansion.
- As a consequence, we derive the central limit theorem.

4.3.2. Dyson-Schwinger Equation. Hereafter we set $M_N = N(\hat{\mu}_N - \mu_V)$. We let Ξ be defined on the set of $C_b^1(\mathbb{R})$ functions by

$$\Xi f(x) = V'(x)f(x) - \int \frac{f(x)-f(y)}{x-y} d\mu_V(y)\,.$$

Ξ will be called the master operator. The Dyson -Schwinger equations are given in the following lemma.

LEMMA 4.17. *Let $f_i : \mathbb{R} \to \mathbb{R}$ be C_b^1 functions, $0 \le i \le K$. Then,*

$$\begin{aligned}
\mathbb{E}[M_N(\Xi f_0)\prod_{i=1}^K N\hat{\mu}_N(f_i)] &= (\frac{1}{\beta}-\frac{1}{2})\mathbb{E}[\hat{\mu}_N(f_0')\prod_{i=1}^K N\hat{\mu}_N(f_i)] \\
&+ \frac{1}{\beta}\sum_{\ell=1}^K \mathbb{E}[\hat{\mu}_N(f_0 f_\ell')\prod_{i\neq\ell} N\hat{\mu}_N(f_i)] \\
&+ \frac{1}{2N}\mathbb{E}[\int \frac{f_0(x)-f_0(y)}{x-y} dM_N(x)dM_N(y)\prod_{i=1}^p N\hat{\mu}_N(f_i)]
\end{aligned}$$

PROOF. This lemma is a direct consequence of integration by parts which implies that for all j

$$\begin{aligned}
\mathbb{E}[f_0'(\lambda_j)\prod_{i=1}^K N\hat{\mu}_N(f_i)] &= \beta\mathbb{E}\left[f_0(\lambda_j)\left(NV'(\lambda_j) - \sum_{k\neq j}\frac{1}{\lambda_j-\lambda_k}\right)\prod_{i=1}^K N\hat{\mu}_N(f_i)\right] \\
&- \sum_{\ell=1}^K \mathbb{E}[f_0(\lambda_j)f_\ell'(\lambda_j)\prod_{i\neq\ell} N\hat{\mu}_N(f_i)]
\end{aligned}$$

Summing over $j \in \{1,\ldots,N\}$ and dividing by N yields

$$\begin{aligned}
&\beta N\mathbb{E}\left[\left(\hat{\mu}_N(V'f_0) - \frac{1}{2}\int\int \frac{f_0(x)-f_0(y)}{x-y} d\hat{\mu}_N(x)d\hat{\mu}_N(y)\right)\prod_{i=1}^K N\hat{\mu}_N(f_i)\right] \\
= \;& (1-\frac{\beta}{2})\mathbb{E}[\hat{\mu}_N(f_0')\prod_{i=1}^K N\hat{\mu}_N(f_i)] + \sum_{\ell=1}^K \mathbb{E}[\hat{\mu}_N(f_0 f_\ell')\prod_{i\neq\ell} N\hat{\mu}_N(f_i)]
\end{aligned}$$

where we used that $(x-y)^{-1}(f(x)-f(y))$ goes to $f'(x)$ when y goes to x. We first take $f_\ell = 1$ for $\ell \in \{1,\ldots,K\}$ and f_0 with compact support and deduce that as $\hat{\mu}_N$ goes to μ_V almost surely as N goes to infinity, we have

$$\mu_V(f_0 V') - \frac{1}{2}\int \frac{f_0(x)-f_0(y)}{x-y} d\mu_V(x)d\mu_V(y) = 0\,. \tag{4.19}$$

This implies that μ_V has compact support and hence the formula is valid for all f_0. We then linearize around μ_V to get the announced lemma. $\diamond$

The central point is therefore to invert the master operator Ξ. We follow a lemma from [**5**]. For a function $h:\mathbb{R}\to\mathbb{R}$, we recall that $\|h\|_{C^j(\mathbb{R})}:=\sum_{r=0}^j\|h^{(r)}\|_{L^\infty(\mathbb{R})}$, where $h^{(r)}$ denotes the r-th derivative of h.

LEMMA 4.18. *Given $V:\mathbb{R}\to\mathbb{R}$, assume that μ_V^{eq} has support given by $[a,b]$ and that*

$$\frac{d\mu_V}{dx}(x)=S(x)\sqrt{(x-a)(b-x)}$$

with $S(x)\geq\bar{c}>0$ a.e. on $[a,b]$.

Let $g:\mathbb{R}\to\mathbb{R}$ be a C^k function and assume that V is of class C^p. Then there exists a unique constant c_g such that the equation

$$\Xi f(x)=g(x)+c_g$$

has a solution of class $C^{(k-2)\wedge(p-3)}$. More precisely, for $j\leq(k-2)\wedge(p-3)$ there is a finite constant C_j such that

$$\|f\|_{C^j(\mathbb{R})}\leq C_j\|g\|_{C^{j+2}(\mathbb{R})},\tag{4.20}$$

where, for a function h, $\|h\|_{C^j(\mathbb{R})}:=\sum_{r=0}^j\|h^{(r)}\|_{L^\infty(\mathbb{R})}$.

This solution will be denoted by $\Xi^{-1}g$. It is C^k if g is C^{k+2} and $p\geq k+1$. It decreases at infinity like $|V'(x)x|^{-1}$.

REMARK 4.19. The inverse of the operator Ξ can be computed, see [**5**]. For $x\in[a,b]$ we have that $\Xi^{-1}g(x)$ equals

$$\begin{aligned}Xi^{-1}g(x)=\frac{1}{\beta(x-a)(b-x)S(x)}\Bigg(&\int_a^b\sqrt{(y-a)(b-y)}\frac{g(y)-g(x)}{y-x}\,dy\\&-\pi\left(x-\frac{a+b}{2}\right)(g(x)+c_g)+c_2\Bigg),\end{aligned}$$

where c_g and c_2 are chosen so that $\Xi^{-1}g$ converges to finite constants at a and b. We find that for $x\in S$

$$\begin{aligned}\Xi^{-1}g(x)&=\frac{1}{\beta S(x)}PV\int_a^b g(y)\frac{1}{(y-x)\sqrt{(y-a)(b-y)}}dy\\&=\frac{1}{\beta S(x)}\int_a^b(g(y)-g(x))\frac{1}{(y-x)\sqrt{(y-a)(b-y)}}dy,\end{aligned}$$

and outside of S f is given by (see Remark 4.10).

$$f(x)=\left(V'(x)-\int(x-y)^{-1}d\mu_V^{\mathrm{eq}}(y)\right)^{-1}\int\frac{f(y)}{x-y}d\mu_V^{\mathrm{eq}}(y)\,.$$

REMARK 4.20. Observe that by Remark 4.7, the density of μ_V^{eq} has to vanish at the boundary like $|x-a|^{q/2}$ for some $q\in\mathbb{N}$. Hence the only case when we can invert this operator is when $q=1$. Moreover, by the same remark,

$$S(x)\sqrt{(x-a)(b-x)}=\sqrt{V'(x)^2-f(x)}=V'(x)-PV\int(x-y)^{-1}d\mu_V^{eq}(y)$$

so that S extends to the whole real line. Assuming that S is positive in $[a,b]$ we see that it is positive in a open neighborhood of $[a,b]$ since it is smooth. We can assume without loss of generality that it is smooth everywhere by the large deviation principle for the support.

We will therefore assume hereafter that V is off-critical in the following sense.

ASSUMPTION 4.21. $V:\mathbb{R}\to\mathbb{R}$ is of class C^p and $\mu_V^{\rm eq}$ has support given by $[a,b]$ and that

$$\frac{d\mu_V}{dx}(x)=S(x)\sqrt{(x-a)(b-x)}$$

with $S(x)\geq\bar c>0$ a.e. on $[a,b]$. Moreover, we assume that $(|V'(x)x|+1)^{-1}$ is integrable.

The first condition is necessary to invert Ξ on all test functions (in critical cases, Ξ is may not be surjective). The second implies that for $\Xi^{-1}f$ decays fast enough at infinity so that it belongs to L^1 (for f smooth enough) so that we can use the Fourier inversion theorem.

We then deduce from Lemma 4.17 the following :

COROLLARY 4.22. Assume that 4.21 with $p\geq 4$. Take f_0 C^k, $k\geq 3$ and f_i C^1. Let $g=\Xi^{-1}f_0$ be the C^{k-2} function such that there exists a constant c_g such that $\Xi f_0=g+c_g$. Then

$$\begin{aligned}&\mathbb{E}[\prod_{i=0}^K M_N(f_i)]=(\frac{1}{\beta}-\frac{1}{2})\mathbb{E}[\hat\mu_N((\Xi^{-1}f_0)')\prod_{i=1}^K M_N(f_i)]\\&+\frac{1}{\beta}\sum_{\ell=1}^K\mathbb{E}[\hat\mu_N(\Xi^{-1}f_0f_\ell')\prod_{i\neq\ell}M_N(f_i)]\\&+\frac{1}{2N}\mathbb{E}[\int\frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)\prod_{i=1}^K M_N(f_i)]\end{aligned}$$

4.3.3. Improving concentration inequalities. We are now ready to improve the concentration estimates we obtained in the previous section. We could do that by using the Dyson-Schwinger equations (this is what we will do in the discrete case) but in fact there is a quicker way to proceed by infinitesimal change of variables in the continuous case :

LEMMA 4.23. *Take $g\in C^4$ and assume $p\geq 4$. Then there exists universal finite constants C_V and $c>0$ such that for all $M>0$*

$$P_N^{\beta,V}\left(N|\int g(x)d(\hat\mu^N-\mu_V^{\rm eq})(x)|\geq C_V\|g\|_{C^4}\ln N+M\ln N\right)\leq e^{-cN}+N^{-M}\,.$$

PROOF. Take f compactly supported on a compact set K. Making the change of variable $\lambda_i=\lambda_i'+\frac{1}{N}f(\lambda_i')$, we see that $Z_N^{\beta,V}$ equals

$$\int\prod|\lambda_i-\lambda_j+\frac{1}{N}(f(\lambda_i)-f(\lambda_j))|^\beta\prod e^{-N\beta V(\lambda_i+\frac{1}{N}f(\lambda_i))}(1+\frac{1}{N}f'(\lambda_i))d\lambda_i \tag{4.21}$$

Observe that by Taylor's expansion there are $\theta_{ij}\in[0,1]$ such that

$$\begin{aligned}&\prod|\lambda_i-\lambda_j+\frac{1}{N}(f(\lambda_i)-f(\lambda_j))|\\&=\prod|\lambda_i-\lambda_j|\exp\{\frac{1}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}-\frac{1}{N^2}\sum_{i<j}\theta_{ij}\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)^2\}\end{aligned}$$

where the last term is bounded by $\|f'\|_\infty^2$. Similarly there exists $\theta_i \in [0,1]$ such that

$$V(\lambda_i + \frac{1}{N}f(\lambda_i)) = V(\lambda_i) + \frac{1}{N}f(\lambda_i)V'(\lambda_i) + \frac{1}{N^2}f(\lambda_i)^2 V''(\lambda_i + \frac{\theta_i}{N}f(\lambda_i))$$

where the last term is bounded for N large enough by $C_K(V)\|f\|_\infty^2$ with $C_K = \sup_{d(x,K)\le 1}|V''(x)|$. We deduce by expanding the right hand side of (4.21) that

$$\int \exp\{\frac{\beta}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} - \beta\sum_i V'(\lambda_i)f(\lambda_i)\}dP_N^{\beta,V}$$

$$\le \exp\{\beta C_K\|f\|_\infty^2 + \beta\|f'\|_\infty^2 + \|f'\|_\infty\}$$

Using Chebychev inequality we deduce that if f is C^1 and compactly supported
(4.22)

$$P_N^{\beta,V}\left(\left|\frac{1}{2N}\sum_{i,j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} - \sum V'(\lambda_i)f'(\lambda_i)\right| \ge M\ln N\right) \le N^{-M}e^{C(f)}$$

with $C(f) = C_K\|f\|_\infty^2 + (\beta+1)(1+\|f'\|_\infty)^2$. But

$$\frac{1}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} - \sum V'(\lambda_i)f'(\lambda_i)$$

$$= -N(\hat\mu^N - \mu)(\Xi f) + \frac{1}{2}N\int\int\frac{f(x)-f(y)}{x-y}d(\hat\mu^N - \mu_V^{\mathrm{eq}})(x)d(\hat\mu^N - \mu_V^{\mathrm{eq}})(y)$$

where if f is C^2 the last term is bounded by $C\|f\|_{C^2}\ln N$ with probability greater than $1-e^{-cN}$ by Corollary 4.16. Hence, we deduce from (4.23) that

$$P_N^{\beta,V}\left(N\left|(\hat\mu^N-\mu)(\Xi f)\right| \ge M\ln N\right) \le N^{C\|f\|_{C^2}-M}e^{C\|f\|_{C^1}^2} + e^{-cN}$$

and inverting f by putting $g = \Xi f$ concludes the proof for f with compact support. Again using Theorem 4.11 allows to extend the result for f with full support. $\diamond$

EXERCISE 4.24. Concentration estimates could as well be improved by using Dyson-Schwinger equations. However, using the Dyson-Schwinger equations necessitates to loose in regularity at each time, since it requires to invert the master operator. Hence, it requires stronger regularity conditions. Prove that if Assumption 4.21 holds with $p \ge 12$, for any f be C^k with $k \ge 11$. Then for $\ell = 1,2$, there exists C_ℓ such that

$$\left|\mathbb{E}[(N(\hat\mu_N - \mu_V^{\mathrm{eq}})(f))^\ell]\right| \le C_\ell\|f\|_{C^{3+4\ell}}\|f\|_{C^1}^{1_{\ell=2}}(\ln N)^{\frac{\ell+1}{2}}\,.$$

Hint : Use the DS equations, concentration, invert the master operator and bootstrap if you do not get the best estimates at once.

THEOREM 4.25. *Suppose that Assumption 4.21 holds with* $p \ge 10$. *Let* f *be* C^k *with* $k \ge 9$. *Then*

$$m_V(f) = \lim_{N\to\infty}\mathbb{E}[N(\hat\mu_N - \mu_V^{\mathrm{eq}})(f)] = (\frac{1}{\beta}-\frac{1}{2})\mu_V^{\mathrm{eq}}[(\Xi^{-1}f)']\,.$$

Let f_0, f_1 *be* C^k *with* $k \ge 9$ *and* $p \ge 12$. *Then*

$$C_V(f_0,f_1) = \lim_{N\to\infty}\mathbb{E}[M_N(f_0)M_N(f_1)] = m_V(f_0)m_V(f_1) + \frac{1}{\beta}\mu_V^{\mathrm{eq}}(f_1'\Xi^{-1}f_0)\,.$$

REMARK 4.26. Notice that as C is symmetric, we can deduce that for any f_0, f_1 in C^k with $k \geq 9$,

$$\mu_V^{\mathrm{eq}}(f_1'\Xi^{-1}f_0) = \mu_V^{\mathrm{eq}}(f_0'\Xi^{-1}f_1)\,.$$

PROOF. To prove the first convergence observe that

$$\mathbb{E}[M_N(f_0)] = (\frac{1}{\beta}-\frac{1}{2})\mathbb{E}[\hat{\mu}_N((\Xi^{-1}f_0)')] + \frac{1}{2N}\mathbb{E}[\int \frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)]\,. \tag{4.23}$$

The first term converges to the desired limit as soon as $(\Xi^{-1}f_0)'$ is continuous. For the second term we can use the previous Lemma and the basic concentration estimate 4.16 to show that it is neglectable. The arguments are very similar to those used in the proof of Corollary 4.16 but we detail them for the last time. First, not that if χ_M is the indicator function that all eigenvalues are bounded by M, we have by Theorem 4.11 that

$$|\mathbb{E}[(1-\chi_M)\int \frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)]| \leq \|\Xi^{-1}f_0\|_{C^1}N^2e^{-cN}\,.$$

We therefore concentrate on the other term, up to modify $\Xi^{-1}f_0$ outside $[-M,M]$ so that it decays to zero as fast as wished and is as smooth as the original function (it is enough to multiply it by a smooth cutoff function). In particular we may assume it belongs to L^2 and write its decomposition in terms of Fourier transform. With some abuse of notations, we still denote $(\widehat{\Xi^{-1}f_0})_t$ the Fourier transform of this eventually modified function. Then, we have

$$\begin{aligned}&|\mathbb{E}[\chi_M\int \frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)]|\\ \leq\ & \int |t(\widehat{\Xi^{-1}f_0})_t|\int_0^1 \mathbb{E}[\chi_M|M_N(e^{i\alpha t\cdot})|^2]d\alpha dt\end{aligned}$$

To bound the right hand side under the weakest possible hypothesis over f_0, observe that by Corollary 4.16 applied on only one of the M_N we have

$$\mathbb{E}[\chi_M|M_N(e^{i\alpha t\cdot})|^2] \leq C\sqrt{N\ln N}|t|\mathbb{E}[|M_N(e^{i\alpha t\cdot})|] + N^2e^{-cN} \tag{4.24}$$

where again we used that even though $e^{i\alpha t\cdot}$ has infinite 1/2 norm, we can modify this function outside $[-M,M]$ into a function with 1/2 norm of order $|t|$. We next use Lemma 4.23 to estimate the first term in (4.24) (with $\|e^{i\alpha t\cdot}\|_{C^4}$ of order $|\alpha t|^4+1$) and deduce that :

$$\begin{aligned}&|\mathbb{E}[\chi_M\int \frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)]|\\ \leq\ & C(\ln N)^{3/2}\sqrt{N}\int |t(\widehat{\Xi^{-1}f_0})_t||t|^5dt\\ \leq\ & C(\ln N)^{3/2}\sqrt{N}\|\Xi^{-1}f_0\|_{C^7}\\ \leq\ & C(\ln N)^{3/2}\sqrt{N}\|f_0\|_{C^9}\end{aligned}$$

Hence, we deduce that

$$\frac{1}{N}\left|\mathbb{E}[\int\frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)]\right|\le C(\ln N)^{3/2}\sqrt{N}^{-1}\|f_0\|_{C^9}$$

goes to zero if f_0 is C^9. This proves the first claim. Similarly, for the covariance, we use Corollary 4.22 with $p=1$ to find that for $f_0, f_1 C^k$,

$$\begin{aligned}
C_N(f_0,f_1) &= \mathbb{E}[(N(\hat\mu^N-\mu_V^{\mathrm{eq}}))(f_0)M_N(f_1)]\\
&= (\frac{1}{\beta}-\frac{1}{2})\mu_V^{\mathrm{eq}}((\Xi^{-1}f_0)')\mathbb{E}[M_N(f_1)]+\frac{1}{\beta}\mu_V^{eq}(\Xi^{-1}f_0f_1')\\
&\quad +(\frac{1}{\beta}-\frac{1}{2})\mathbb{E}[(\hat\mu_N-\mu_V^{\mathrm{eq}})((\Xi^{-1}f_0)')M_N(f_1)]\\
&\quad +\frac{1}{\beta}\mathbb{E}[(\hat\mu_N-\mu_V^{eq})(\Xi^{-1}f_0f_1')]\\
&\quad +\frac{1}{2N}\mathbb{E}[\int\frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)M_N(f_1)]
\end{aligned}\tag{4.25}$$

The first line converges towards the desired limit. The second goes to zero as soon as $(\Xi^{-1}f_0)'$ is $C^1$and f_1 is C^4, as well as the third line. Finally, we can bound the last term by using twice Lemma 4.23, Cauchy-Schwartz and the basic concentration estimate once

$$\begin{aligned}
&\left|\mathbb{E}[\int\frac{\Xi^{-1}f_0(x)-\Xi^{-1}f_0(y)}{x-y}dM_N(x)dM_N(y)M_N(f_1)]\right|\\
&\le\quad C(\ln N)^{5/2}\sqrt{N}\int dt|(\widehat{\Xi^{-1}f_0})_t|\|f_1\|_{C^4}|t|^6\\
&\le\quad C(\ln N)^{5/2}\sqrt{N}\|\Xi^{-1}f_0\|_{C^7}\|f_1|_{C^4}^{1/2}
\end{aligned}$$

which once plugged into (4.25) yields the result. ◇

4.3.4. Central limit theorem.

THEOREM 4.27. *Suppose that Assumption 4.21 holds with $p\ge 10$. Let f be C^k with $k\ge 9$. Then $M_N(f):=\sum_{i=1}^N f(\lambda_i)-N\mu_V^{\mathrm{eq}}(f)$ converges in law under $P_{\beta,V}^N$ towards a Gaussian variable with mean $m_V(f)$ and covariance $\sigma(f)=\mu_V^{eq}(f'\Xi^{-1}f)$.*

Observe that we have weaker assumptions on f than in Lemma 4.25. This is because when we use the Dyson-Schwinger equations, we have to invert the operator Ξ several times, hence requiring more and more smoothness of the test function f. Using the change of variable formula instead allows to invert it only once, hence lowering our requirements on the test function.

PROOF. We can take f compactly supported by Theorem 4.11. We come back to the proof of Lemma 4.23 but go one step further in Taylor expansion to see that the function

$$\Lambda_N(f):=\frac{\beta}{N}\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}-\frac{\beta}{2N^2}\sum_{i<j}\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)^2-\beta\sum V'(\lambda_i)f'(\lambda_i)$$
$$-\frac{\beta}{2N}\sum V''(\lambda_i)(f(\lambda_i))^2+\frac{1}{N}\sum_{i=1}^N f'(\lambda_i)$$

satisfies

$$\left|\ln \int e^{\Lambda_N(f)} dP_N^{\beta,V}\right| \le C\frac{1}{N}\|f\|_{C^1}^3$$

where the constant C may depend on the support of f. For any $\delta > 0$, with probability greater than $1 - e^{-C(\delta)N^2}$ for some $C(\delta) > 0$, the empirical measure $\hat{\mu}^N$ is at Vasershtein distance smaller than δ from μ_V^{eq}. On this set, for f C^1

$$\frac{1}{2N^2}\sum_{i,j}\left(\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)^2 + \frac{1}{N}\sum V''(\lambda_i)(f(\lambda_i))^2 = C(f) + O(\delta)$$

where

$$C(f) = \frac{1}{2}\int\left(\frac{f(x)-f(y)}{x-y}\right)^2 d\mu_V^{eq}(x)d\mu_V^{eq}(y) + \int V''(x)f(x)^2 d\mu_V^{eq}(x)$$

whereas

$$\frac{1}{N}\sum_{i=1}^N f'(\lambda_i) = M(f) + o(1), \quad \text{if } M(f) = \int f'(x)d\mu_V^{eq}(x)\,.$$

As $\Lambda_N(f)$ is at most of order N, we deduce by letting N and then δ going to zero that

$$Z_N(f) := \frac{\beta}{2N}\sum_{i,j=1}^N \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j} - \beta\sum V'(\lambda_i)f'(\lambda_i)$$

satisfies for any f C^1

$$\lim_{N\to\infty}\int e^{Z_N(f)}dP_N^{\beta,V} = e^{(\frac{\beta}{2}-1)M(f)+\frac{\beta}{2}C(f)}\,.$$

In the line above we took into account that we added a diagonal term to $Z_N(f)$ which contributed to the mean. We can now replace f by tf for real numbers f and conclude that $Z_N(f)$ converges in law towards a Gaussian variable with mean $(\frac{\beta}{2}-1)M(f)$ and covariance $\beta C(f)$. On the other hand we can rewrite $Z_N(f)$ as

$$Z_N(f) = \beta M_N(\Xi f) + \varepsilon_N(f)$$

where

$$\varepsilon_N(f) = \frac{\beta N}{2}\int \frac{f(x)-f(y)}{x-y}d(\hat{\mu}^N - \mu_V^{eq})(x)d(\hat{\mu}^N - \mu_V^{eq})(y)$$

Now, we can use Lemma 4.23 to bound the probability that $\varepsilon_N(f)$ is greater than some small δ. We again use the Fourier transform to write :

$$\varepsilon_N(f) = \frac{\beta N}{2}\int (it\hat{f}_t)\int_0^1 (\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{\alpha t}dt\,.$$

We can bound the L^1 norm of $\varepsilon_N(f)$ by Cauchy-Schwartz inequality by

$$\mathbb{E}[|\varepsilon_N(f)|] \le \frac{\beta N}{2}\int |t\hat{f}_t|\int_0^1 \mathbb{E}[|(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}|^2]^{1/2}\mathbb{E}[|(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}|^2]^{1/2}dtd\alpha\,.$$

Finally, Lemma 4.23 implies that

$$\mathbb{E}[|(\widehat{\hat{\mu}^N - \mu_V^{eq}})_{(1-\alpha)t}|^2]^{1/2} \le C|t|^4\frac{\ln N}{N} + N^{-C}$$

from which we deduce that there exists a finite constant C

$$\mathbb{E}[|\varepsilon_N(f)|] \le C\int |t|^5|\hat{f}_t|dt\frac{\ln N^2}{N}\,.$$

Thus, the convergence in law of $Z_N(f)$ implies the convergence in law of $M_N(\Xi f)$ towards a Gaussian variable with covariance $C(f)$ and mean $(\frac{1}{2}-\frac{1}{\beta})M(f)$. If f is C^9, we can invert Ξ and conclude that $M_N(f)$ converges towards a Gaussian variable with mean $m(f)=(\frac{1}{2}-\frac{1}{\beta})M(\Xi^{-1}(f))$ and covariance $C(\Xi^{-1}(f))$. To identify the covariance, it is enough to show that $C(f)=\mu_V^{eq}((\Xi f)'f)$. But on the support of μ_V^{eq}

$$(\Xi f)'(x) = V'' f(x) + PV\int \frac{f(x)-f(y)}{(x-y)^2} d\mu_V^{eq}(y)$$

from which the result follows.

$\diamond$

4.4. Expansion of the partition function

THEOREM 4.28. (1) *For f C^{17} and V C^{20},*

$$\mathbb{E}_{P^N_{\beta,V}}[\hat{\mu}^N(f)] = \mu_V^{\mathrm{eq}}(f) + \frac{1}{N} m_V(f) + \frac{1}{N^2} K_V(f) + o(\frac{1}{N^2}),$$

with $m_V(f)$ as in Theorem 4.25 and

$$K_V(f) = (\frac{1}{\beta}-\frac{1}{2}) m_V(((\Xi^{-1}f_0)') + \frac{1}{2}\int itdt \int \widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha C_V(e^{it\alpha\cdot}, e^{it(1-\alpha)\cdot}).$$

(2) *Assume V C^{20}, then*

$$\ln Z^N_{\beta,V} = C^0_\beta N\ln N + C^1_\beta \ln N + N^2 F_0(V) + N F_1(V) + F_2(V) + o(1)$$

with $C^0_\beta = \frac{\beta}{2}$, $C^1_\beta = \frac{3+\beta/2+2/\beta}{12}$ and

$$\begin{array}{rcl} F_0(V) &=& -\mathcal{E}(\mu_V^{\mathrm{eq}}) \\ F_1(V) &=& -(\frac{\beta}{2}-1)\int \ln \frac{d\mu_V^{\mathrm{eq}}}{dx} d\mu_V^{\mathrm{eq}} + f_1 \\ F_2(V) &=& -\beta\int_0^1 K_{V_\alpha}(V-V_0)d\alpha + f_2 \end{array} \tag{4.26}$$

where f_1, f_2 only depends on $b-a$, the width of the support of μ_V^{eq}.

PROOF. The first order estimate comes from Theorem 4.25. To get the next term, we notice that if $\Xi^{-1}f$ belongs to L^1 we can use the Fourier transform of $\Xi^{-1}f$ (which goes to infinity to zero faster than $(|t|+1)^{-3}$ as $\Xi^{-1}f$ is C^6) so that

$$\begin{aligned} &\mathbb{E}[\textstyle\int \frac{\Xi^{-1}f(x)-\Xi^{-1}f(y)}{x-y} dM_N(x)dM_N(y)] \\ &= \int dtit\widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha \mathbb{E}[\widehat{M_N}(e^{it\alpha\cdot})\widehat{M_N}(e^{it(1-\alpha)\cdot})] \\ &\simeq \int dtit\widehat{\Xi^{-1}f}(t)\int_0^1 d\alpha C_V(e^{it\alpha\cdot}, e^{it(1-\alpha)\cdot}) \end{aligned}$$

We can therefore use (4.23) to conclude that

$$
\begin{aligned}
N(\mathbb{E}[M_N(f)] - m(f)) &\simeq (\frac{1}{\beta} - \frac{1}{2}) m_V(((\Xi^{-1}f)') \\
&+ \frac{1}{2}\int dtit \int \widehat{\Xi^{-1}f}(t) \int_0^1 d\alpha C_V(e^{it\alpha\cdot}, e^{it(1-\alpha)\cdot})
\end{aligned}
$$

which proves the first claim. We used that f is C^{12} so that $(\Xi^{-1}f)'$ is C^9 and Theorem 4.25 for the convergence of the first term. For the second we notice that the covariance is uniformly bounded by $C(|t|^{12}+1)$, so we can apply monotone convergence theorem when $\int dt|\widehat{\Xi^{-1}f}(t)||t|^{13}$ is finite, so f C^{16+}.

To prove the second point, the idea is to proceed by interpolation from a case where the partition function can be explicitly computed, that is where V is quadratic. We interpolate V with a potential $V_0(x) = c(x-d)^2/4$ so that the limiting equilibrium measure $\mu_{c,d}$, which is a semi-circle law shifted by d and enlarged by a factor $\sqrt{c}$, has support $[a,b]$ (so $d = (a+b)/2$ and $c = (b-a)^2/16$). The advantage of keeping the same support is that the potential $V_\alpha = \alpha V + (1-\alpha)V_0$ has equilibrium measure $\mu_\alpha = \alpha\mu_V^{\mathrm{eq}} + (1-\alpha)\mu_{c,d}$ since it satisfies the characterization of Lemma 4.5. We then write

$$
\begin{aligned}
\ln \frac{Z^N_{\beta,V}}{Z^N_{\beta,V_0}} &= \int_0^1 \partial_\alpha \ln Z^N_{\beta,V_\alpha} d\alpha \\
&= -\beta N^2 \int_0^1 \mathbb{E}_{P^N_{\beta,V_\alpha}}[\hat{\mu}^N(V-V_0)] d\alpha
\end{aligned}
$$

It is not hard to see that if μ_V^{eq} satisfy hypotheses 4.2, so does μ_α and that the previous expansion can be shown to be uniform in α. Hence, we obtain the expansion from the first point if V is C^{20} with

$$
\begin{aligned}
F_0(V) &= -\beta \int_0^1 \mu_{V_\alpha}(V-V_0) d\alpha + f_0 \\
F_1(V) &= -\beta \int_0^1 m_{V_\alpha}(V-V_0) d\alpha + f_1 \\
F_2(V) &= -\beta \int_0^1 K_{V_\alpha}(V-V_0) d\alpha + f_2
\end{aligned}
$$

where f_0, f_1, f_2 are the coefficients in the expansion of Selberg integrals given in **[75]** :

$$
Z^N_{V_0,\beta} = N^{\frac{\beta N}{2}} N^{\frac{3+\beta/2+2/\beta}{12}} e^{N^2 f_0 + N f_1 + f_0 + o(1)}
$$

with f_0, f_1, f_2 only depending on $b-a$:

$$
\begin{aligned}
f_0 &= (\beta/2)\Big[-\frac{3}{4}+\ln\Big(\frac{b-a}{4}\Big)\Big] \\
f_1 &= (1-\beta/2)\ln\Big(\frac{b-a}{4}\Big) - 1/2 - \beta/4 + (\beta/2)\ln(\beta/2) + \ln(2\pi) - \ln\Gamma(1+\beta/2) \\
f_2 &= \chi'(0;2/\beta,1) + \frac{\ln(2\pi)}{2}
\end{aligned}
$$

The first formula of Theorem 4.28 is clear from the large deviation principle and the last is just what we proved in the first point. Let us show that the first order correction is given in terms of the relative entropy as stated in (4.34). Indeed, by integration by part and Remark 4.26 we have

$$
\begin{aligned}
(\frac{1}{\beta}-\frac{1}{2})^{-1} m_V(f) &= \mu_V^{\mathrm{eq}}[(\Xi^{-1}f)'] \\
&= -\int \Xi^{-1} f(\frac{d\mu_V^{\mathrm{eq}}}{dx})' dx \\
&= -\int \Xi^{-1} f(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx})' d\mu_V^{\mathrm{eq}} \\
&= -\int f'\Xi^{-1}(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx}) d\mu_V^{\mathrm{eq}}
\end{aligned}
$$

To complete our proof, we will first prove that if g is C^{10} ,

$$
\lim_{s\to 0} s^{-1}(\mu_{V-sf}^{\mathrm{eq}} - \mu_V^{\mathrm{eq}})(g) = \mu_V^{\mathrm{eq}}(\Xi^{-1} g f')\,. \tag{4.27}
$$

which implies the key estimate

$$
(\frac{1}{\beta}-\frac{1}{2})^{-1} m_V(f) = -\int f'\Xi^{-1}(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx}) d\mu_V^{\mathrm{eq}} = \partial_t \mu_{V+tf}(\ln\frac{d\mu_V^{\mathrm{eq}}}{dx})|_{t=0}\,. \tag{4.28}
$$

To prove (4.27), we first show that $m_V(f) = \int f(x) d\mu_V(x)$ is continuous in V in the sense that

$$
D(\mu_V,\mu_W) \le \sqrt{\|V-W\|_\infty}\,. \tag{4.29}
$$

Indeed, by Lemma 4.5 applied to $\mu=\mu_W$ and since $\int V_{\mathrm{eff}} d(\mu_W-\mu_V)\ge 0$, we have

$$
\begin{aligned}
D(\mu_W,\mu_V)^2 &\le \mathcal{E}(\mu_W) - \mathcal{E}(\mu_V) \\
&\le \inf\{\int W d\mu + \frac{1}{2}\Sigma(\mu)\} - \inf\{\int V d\mu + \frac{1}{2}\Sigma(\mu)\} \\
&\le \|W-V\|_\infty
\end{aligned}
$$

As a consequence $(\mu_{V-sf}^{\mathrm{eq}} - \mu_V^{\mathrm{eq}})(g)$ goes to zero like $\sqrt{s}$ for g Lipschitz and f bounded. We can in fact get a more accurate estimate by using the limiting Dyson-Schwinger equation (4.19) to μ_{V-sf}^{eq} and μ_V^{eq} and take their difference to get :

$$
\begin{aligned}
(\mu_{V-sf}^{\mathrm{eq}} &- \mu_V^{\mathrm{eq}})(\Xi_V g) = s\mu_{V-\frac{s}{\beta N}f}^{\mathrm{eq}}(gf') \\
&+ \frac{1}{2}\int \frac{g(x)-g(y)}{x-y} d(\mu_{V-sf}-\mu_V^{\mathrm{eq}})(x) d(\mu_{V-sf}-\mu_V^{\mathrm{eq}})(y)\,.
\end{aligned} \tag{4.30}
$$

The last term is at most of order s if g is C^2 by (4.29) (see a similar argument in (4.17)), and so is the first. Hence we deduce from (4.30) that $(\mu^{\rm eq}_{V-sf} - \mu^{\rm eq}_V)(g)$ is of order s if $g \in C^4$ and f is C^5. Plugging back this estimate into the last term in (4.30) together with (4.29), we get (4.27) for $g \in C^8$ and $f \in C^9$.

From (4.28), we deduce that

$$\begin{aligned} F_1(V) - f_1 &= -\beta \int_0^1 m_{V_\alpha}(V - V_0) d\alpha \\ &= (\frac{\beta}{2} - 1) \int_0^1 (\partial_\alpha \mu_{V_\alpha})(\ln \frac{d\mu^{\rm eq}_{V_\alpha}}{dx}) d\alpha - (\frac{\beta}{2} - 1) \int_0^1 \mu_{V_\alpha}(\partial_\alpha \ln \frac{d\mu^{\rm eq}_{V_\alpha}}{dx}) d\alpha \\ &= (\frac{\beta}{2} - 1) \int_0^1 \partial_\alpha [\mu_{V_\alpha}(\ln \frac{d\mu^{\rm eq}_{V_\alpha}}{dx})] d\alpha \end{aligned}$$

wich yields the result. Above in the second line the last term vanishes as $\mu^{\rm eq}_{V_\alpha}(1) = 1$.
$\diamond$

4.5. The Stieltjes transforms approach

Another common approach is to study the fluctuations of the Stieltjes transform, namely moments of :

$$Y_z = N(G_N(z) - \mathbb{E}[G_N(z)]) = \sum_{i=1}^N \frac{1}{z - \lambda_i} - \mathbb{E}\left[\sum_{i=1}^N \frac{1}{z - \lambda_i}\right]$$

for $z \in \mathbb{C} \backslash \mathbb{R}$. This requires that V is real analytic in order to get closed equations for correlators of this functional. Namely, we will assume that

ASSUMPTION 4.29. -V is real analytic,
-$\mu^{\rm eq}_V$ has a connected support $[a, b]$,
-$V_{\rm eff}$ is strictly positive outside the support of $\mu^{\rm eq}_V$.

Hereafter we will therefore assume that V is analytic in an open neighborhood $\mathcal{R}$ of the real line. All our contours and complex numbers will be taken in this neighborhood.

First notice that by integration by parts we have the so-called loop equations

LEMMA 4.30. *Let* $G_N(z) = \frac{1}{N} \sum \frac{1}{z - \lambda_i}$ *and* $G(z) = \int \frac{1}{z-x} d\mu^{\rm eq}_V(x)$ *for* $z \in \mathbb{C} \backslash \mathbb{R}$

$$\mathbb{E}\Bigg[\left(\frac{1}{N} \partial_z G_N(z) \left[-1 + \frac{\beta}{2}\right] + \frac{\beta}{2} G_N(z)^2 - \frac{\beta}{2\pi i} \oint \frac{V'(\xi)}{z - \xi} G_N(\xi) d\xi\right) \prod_k Y_{z_k}\Bigg]$$

$$= -\frac{1}{N} \mathbb{E}\Bigg[\sum_{j=1}^p \partial_{z_j} \frac{G_N(z) - G_N(z_j)}{z - z_j} \prod_{\ell \neq j} Y_{z_\ell}\Bigg] \tag{4.31}$$

PROOF. We start with

$$\int \mathrm{d}\lambda \sum_{i=1}^N \partial_{\lambda_i} \left[\frac{1}{z - \lambda_i} \frac{\mathrm{d}P^{\beta, V}_N}{\mathrm{d}\lambda} \prod_{j=1}^k Y_{z_j}\right] = 0$$

(This follows by integration by parts formula $\int_{-\infty}^{\infty} \partial_x f(x) \mathrm{d}x = 0$ for functions f vanishing at infinity) On the other hand, if we expand out this derivative we have :

$$\int \mathrm{d}P_N^{\beta,V} \left\{ \sum_{i=1}^N \frac{1}{(z-\lambda_i)^2} + \frac{1}{z-\lambda_i} \left\{ \beta \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} - \beta N V'(\lambda_i) \right\} \right.$$
$$\left. + \sum_j \sum_i \frac{1}{z-\lambda_i} \frac{1}{(z_j-\lambda_i)^2} \frac{1}{Y_{z_j}} \right\} \prod_{k=1}^p Y_{z_k} = 0$$

We now use the fact that V is real analytic so that Cauchy formula implies that

$$\sum \frac{V'(\lambda_i)}{z-\lambda_i} = -\frac{1}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} \sum \frac{1}{\xi - \lambda_i} \mathrm{d}\xi = -\frac{1}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} N G_N(\xi) \mathrm{d}\xi$$

where the contour encircles the λ_i's. We have seen in theorem 4.11 that when V_{eff} is positive outside the support of μ_V^{eq}, for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ so that

$$P_N^{\beta,V}\left(\exists i : \lambda_i \in [a-\varepsilon, b+\varepsilon]^c\right) \leq e^{-c(\varepsilon)N} .$$

This entitles us to change the probability measure to have support in $[a-\varepsilon, b+\varepsilon]$ up to exponentially small errors everywhere. We then can simply take a contour around $[a-\varepsilon, b+\varepsilon]$. $\diamond$

4.5.1. Analysis of the Dyson-Schwinger equation : heuristics. We know by Lemma 4.14 that G_N converges to G and hence (4.31) yields with $p = 0$ that

$$\frac{1}{2} G(z)^2 - \frac{1}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} G(\xi) \mathrm{d}\xi = 0 . \tag{4.32}$$

We next guess the corrections to this limit.

- *First order correction.* Setting $\Delta G_N := G_N - G$, (4.31) yields with $p = 0$ that

$$\mathbb{E}\left[\frac{1}{N} \partial_z G_N(z) \left[-1 + \frac{\beta}{2}\right] + K[\Delta G_N](z) + \frac{\beta}{2} (\Delta G_N(z))^2 \right] = 0 \tag{4.33}$$

  where

$$K f(z) = \beta G(z) f(z) - \frac{\beta}{2\pi i} \oint \frac{V'(\xi)}{z-\xi} f(\xi) \mathrm{d}\xi .$$

  By Lemma 4.14, we know that $\mathbb{E}[(\Delta G_N(z))^2]$ is smaller than $(\ln N)/N$. Assume that K is invertible with bounded inverse. Then, we deduce from (4.33) that $\mathbb{E}[\Delta G_N]$ is at most of order $\ln N/N$. If we can prove that $\mathbb{E}[(\Delta G_N(z))^2]$ is $o(N^{-1})$, then we deduce from (4.33) that

$$\lim_{N\to\infty} N \mathbb{E}[\Delta G_N(z)] = (1 - \frac{\beta}{2}) K^{-1}[\partial_z G](z) =: G_1(z) . \tag{4.34}$$

- *Limiting covariance.* To get the limiting covariance, let us take $p = 1$. Let $c(z, z') = \mathbb{E}[Y_z Y_{z'}]$. The Dyson -Schwinger equation then reads

$$\begin{aligned} K(c(., z'))(z) &= -\frac{\beta N}{2} \mathbb{E}[(\Delta G_N(z))^2 Y_{z'}] \\ &\quad -(\frac{\beta}{2} - 1) \partial_z \mathbb{E}[G_N(z) Y_{z'}] - \partial_{z'} \mathbb{E}[\frac{G_N(z) - G_N(z')}{z - z'}] \end{aligned} \tag{4.35}$$

The concentration estimates imply that

$$\mathbb{E}[(\Delta G_N(z))^2 Y_{z'}] = O((\ln N)^{3/2}/\sqrt{N})$$

and $\mathbb{E}[G_N(z)Y_{z'}]$ is of order $\ln N$, hence if K is reversible, we deduce that c is of order $O((\ln N)^{3/2}\sqrt{N})$. As we have shown in the previous point that $\mathbb{E}[\Delta G_N]$ is at most of order $\ln N/N$, we deduce that $\mathbb{E}[|\Delta G_N(z)|^2]$ is at most of order $O((\ln N/N)^{3/2})$ which completes the proof of (4.34).

To improve our estimate on $\mathbb{E}[(\Delta G_N(z))^2 Y_{z'}]$ we next use the concentration estimates on $Y_{z'}$ and our new bound on the covariance to obtain a bound of order $(\ln N/N)^{3/2} \times \sqrt{N \ln N} = (\ln N)^2/N$. This allows to improve our estimate thanks to (4.35) and a bound on c of order $(\ln N)^2$. Proceeding once more, we deduce that the last term in (4.35) is neglectable. Note also that $\mathbb{E}[G_N(z)Y_{z'}] = \mathbb{E}[(G_N(z) - G(z))Y_{z'}]$ goes also to zero. As this is an analytic function, its derivative goes as well to zero. Then, we deduce from (4.33) that

$$\lim_{N\to\infty} \mathbb{E}[N G_N(z) Y_{z'}] = -K^{-1}[\partial_{z'} \frac{G(.) - G(z')}{. - z'}](z) =: W(z, z') .$$

- *Second order correction.* Going back to (4.33) with $p = 0$ we have

$$K[\mathbb{E}[N(N\Delta G_N - G_1)]](z) = -\frac{\beta}{2}(\mathbb{E}[Y_z^2] + \mathbb{E}[N\Delta G_N(z)]^2) - (\frac{\beta}{2} - 1)\partial_z \mathbb{E}[N\Delta G_N(z)]$$

and we can go to the limit $N \to \infty$ to deduce

$$\lim_{N\to\infty} K[\mathbb{E}[N(N\Delta G_N - G_1)]](z) = -\frac{\beta}{2}(W(z,z) + G_1(z)^2) - (\frac{\beta}{2} - 1)\partial_z G_1(z)$$

so that taking the inverse of K yields the desired limit :

$$\lim_{N\to\infty} \mathbb{E}[N(N\Delta G_N - G_1)](z) = K^{-1}(-\frac{\beta}{2}(W(.,.) + G_1(.)^2) - (\frac{\beta}{2} - 1)\partial_z G_1)(z) .$$

The above heuristics can be made rigorous provided we invert the operator K (and show its inverse is continuous to neglect error terms after we inverted it). This is what we do next.

4.5.2. Inverting the master operator. Observe that we want to apply K to functions which are differences of Stieltjes transforms of probability measures and therefore going to infinity like $1/z^2$. We therefore search for f with such a decay satisfying $g(z) = Kf(z)$ for a given g. As a consequence g goes to infinity like $1/z$ at best. We can rewrite

$$Kf(z) = \beta(G(z) - V'(z))f(z) - \beta \oint \frac{V'(\xi) - V'(z)}{2\pi i(z - \xi)} f(\xi) d\xi .$$

We make the following crucial assumption of off-criticality :

Assumption 4.31. There exists a real analytic function S which does not vanish on a complex neighborhood of $[a - \varepsilon, b + \varepsilon]$ so that

$$\frac{d\mu_V^{\mathrm{eq}}}{dx} = S(x)\sqrt{(x-a)(b-x)} .$$

This implies that

$$G(z) - V'(z) = \pi\sqrt{(z-a)(z-b)}S(z)$$

with S real analytic and not vanishing on $[a-\varepsilon, b+\varepsilon]$. Indeed,(4.32) implies that

$$G(z)^2 - 2V'(z)G(z) + Q(z) = 0$$

with $Q(z) = 2\oint \frac{V'(\xi)-V'(z)}{2\pi i(z-\xi)} G(\xi)d\xi$. Solving this equation yields

$$G(z) = V'(z) - \sqrt{V'(z)^2 - Q(z)} \tag{4.36}$$

Remember that $G(z)$ is analytic outside of the support of μ_V^{eq} where its imaginary part jumps by $\pi d\mu_V^{\mathrm{eq}}/dx$. As V' is analytic, and $V'(z)^2 - Q(z)$ is analytic, we see that the discontinuity of G can only come from the square root term in (4.36) when it is complex, that is when $V'(z)^2 - Q(z)$ is negative on the real line. Hence, this square root becomes as z goes to the real line the density of μ_V^{eq}. The conclusion follows.

This behaviour is essential to invert K, in the spirit of Tricomi airfoil equation. Indeed, we write with $\sigma(z) = \sqrt{(z-a)(z-b)}$, for $g = Kf$, that is

$$g(z) = -\pi\beta S(z)\sigma(z)f(z) - \beta Q_f(z)$$

with $Q_f(z) = 2\oint \frac{V'(\xi)-V'(z)}{2\pi i(z-\xi)} f(\xi)d\xi$.

$$\begin{aligned}
\sigma(z)f(z) &= -\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\sigma(\xi)f(\xi)d\xi \\
&= -\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi) - Q_f(\xi))d\xi \\
&= -\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)} g(\xi)d\xi
\end{aligned}$$

where in the first line we took a contour around z and used Cauchy formula, in the second line we passed the contour around $[a,b]$ and used the definition of $Kf = g$, using that the residue at infinity vanishes because $\sigma(z)f(z)$ goes like $1/z$, and in the last line we used that Q_f/S is analytic. Hence, we deduce that

$$K^{-1}g(z) = -\frac{1}{\sigma(z)}\frac{1}{2\pi i}\oint \frac{1}{z-\xi}\frac{1}{\beta S(\xi)}(g(\xi))d\xi\,,$$

where the contour surrounds $[a-\varepsilon, b+\varepsilon]$. We note that away from $[a,b]$, K^{-1} is bounded. Also it maps holomorphic to holomorphic functions so that bounds on functions translate into bounds on its derivatives up to take slightly smaller imaginary part of the argument, thus giving continuity of the inverse.

4.5.3. Central limit theorem. To prove the central limit theorem we show by induction over p that

$$\lim_{N\to\infty} \mathbb{E}[\prod_{i=1}^{p} Y_{z_i}] = \sum_{i=2}^{p} W(z_1, z_i) \lim_{N\to\infty} \mathbb{E}[\prod_{\substack{\ell\neq i\\ \ell\geq 2}} Y_{z_\ell}]$$

We already showed that this is true for $p = 1, 2$ and assume we have proved it for $p \leq K$. This in particular imply that $\mathbb{E}[|Y_z|^{\frac{\lfloor 2K\rfloor}{2}}]$ is bounded uniformly in N for z away from the real axis (since we can take half of the $z_j = z$ and the other half to

be its conjugate). We then use the Dyson-Schwinger equation with $p = K$ in order to obtain the result for $K + 1$:
(4.37)

$$\mathbb{E}\left[(N\beta K(G_N - G)(z))\prod_{k=1}^{p} Y_{z_k}\right] = \mathbb{E}\left[\sum_{j=1}^{p}\partial_{z_j}\left[\frac{G_N(z) - G_N(z_j)}{z - z_j}\right]\prod_{\ell\neq j} Y_{z_\ell}\right] + \varepsilon_N(z)$$

where

$$\varepsilon_N(z) = -\mathbb{E}\left[\left(\partial_z G_N(z)\left[-1 + \frac{\beta}{2}\right] + \frac{\beta N}{2}(G_N(z) - G(z))^2\right)\prod_{k=1}^{p} Y_{z_k}\right].$$

We then use that by concentration inequality Lemma 4.14 and the induction bound

$$|\varepsilon_N(z)| \leq \frac{\ln N}{N}(\sqrt{N}\ln N)^n + \frac{1}{(\Im z)^2\prod|\Im z_{k|}}e^{-N\ln N}$$

where $n = 1$ if p is odd and 0 if p even. Indeed, we know by the induction bound that $\mathbb{E}[\prod_{k=1}^{p}|Y_{z_k}|]$ is of order one if p is even, but of order $\sqrt{N}\ln N$ if p is odd. Let us consider the case p odd which is slightly more complicated. Plugging back this estimate and inverting K yields that $|\mathbb{E}[N(G_N(z) - G(z))\prod_{k=1}^{p} Y_{z_k}]|$ is at most of order $\ln N\sqrt{N}$. Because we have already seen that $\mathbb{E}[N(G_N(z) - G(z))]$ is bounded, we deduce that $\mathbb{E}[\prod_{k=1}^{p+1} Y_{z_k}]$ is at most of order $\ln N\sqrt{N}$ and therefore we improve the previous bound (note $p+1$ is even) to

$$|\varepsilon_N(z)| = O(\frac{1}{N}(\frac{\ln N}{N})^{1/2}\ln N\sqrt{N}) = O\left(\frac{(\ln N)^{3/2}}{N}\right)$$

which in turns yields a better bound on $\mathbb{E}[\prod_{k=1}^{p+1} Y_{z_k}]$ of order $(\ln N)^{3/2}$. Bootstrapping this new bound once more shows that

$$|\varepsilon_N(z)| = O(\frac{1}{N}(\frac{\ln N}{N})^{1/2}\ln N^{3/2}) = O(\frac{(\ln N)^2}{N^{3/2}})$$

which now implies with the Dyson Schwinger equation and (4.34) (to take into account that we recenter with respect to the expectation instead of the limit) that

$$\mathbb{E}[\prod_{i=1}^{p} Y_{z_i} Y_z] = \sum_{j=1}^{p} K^{-1}\left[\partial_{z_j}[\frac{G(.) - G(z_j)}{. - z_j}]\right](z)\mathbb{E}\left[\prod_{\ell\neq j} Y_{z_\ell}\right] + o(1)$$

which provides the desired estimate.

CHAPTER 5

Discrete Beta-ensembles

We will consider discrete ensembles which are given by a parameter θ and a weight function w :

$$P_N^{\theta,w}(\vec{\ell}) = \frac{1}{Z_N^{\theta,\omega}} \prod_{i<j} I_\theta(\ell_j - \ell_i) \prod_i w(\ell_i, N)$$

where for $x \geq 0$ we have set

$$I_\theta(x) = \frac{\Gamma(x+1)\Gamma(x+\theta)}{\Gamma(x)\Gamma(x+1-\theta)}$$

where Γ is the usual Γ-function, $\Gamma(n+1) = n\Gamma(n)$. The coordinates $\ell_1, \ldots, \ell_N$ are discrete and belong to the set $\mathbb{W}_\theta$ such that

$$\ell_{i+1} - \ell_i \in \{\theta, \theta+1, \ldots\}$$

and $\ell_i \in (a(N), b(N))$ with $w(a(N), N) = w(b(N), N) = 0$ and $\ell_1 - a(N) \in \mathbb{N}, b(N) - \ell_N \in \mathbb{N}$.

Example 5.1. When $\theta = 1$ this probability measure arises in the setting of lozenge tilings of the hexagon. More specifically, if one looks at a "slice" of the hexagon with sides of size A, B, C, then the number of lozenges of a particular orientation is exactly N and the locations of these lozenges are distributed according the $P_N^{1,\omega}$. Along the vertical line at distance t of the vertical side of size A (see Figure 1), the distribution of horizontal lozenges corresponds to a potential of the form

$$w(\ell, N) = \left[(A+B+C+1-t-\ell)_{t-B}\,(\ell)_{t-C}\right],$$

where $(a)_n$ is the Pochhammer symbol, $(a)_n = a(a+1)\cdots(a+n-1)$.

More generally, as $x \to +\infty$ the interaction term scales like :

$$I_\theta(x) \approx |x|^{2\theta} \text{ as } x \to \infty$$

so the model for θ should be compared to the β ensemble model with $\beta \leftrightarrow 2\theta$. Note however that when $\theta \neq 1$, the particles configuration do not live on $\mathbb{Z}^{\mathbb{N}}$. These discrete β-ensembles were studied in [**12**]. Large deviation estimates can be generalized to the discrete setting but Dyson-Schwinger equations are not easy to establish. Indeed, discrete integration by parts does not give closed equations for our observables this time. A nice generalization was proposed by Nekrasov that allows an analysis similar to the analysis we developed for continuous β models. It amounts to show that some functions of the observables are analytic, in fact thanks

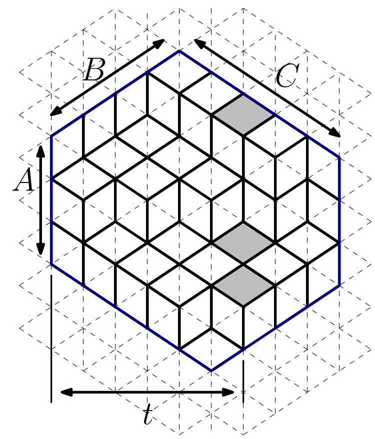

FIGURE 1. Lozenge tilings of a hexagon

to the fact that its possible poles cancel due to discrete integration by parts. We present this approach below.

5.1. Large deviations, law of large numbers

Let $\hat{\mu}_N$ be the empirical measure :

$$\hat{\mu}_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\ell_i/N}$$

ASSUMPTION 5.2. Assume that $a(N) = \hat{a}N + O(\ln N), b(N) = \hat{b}N + O(\ln N)$ for some finite $\hat{a}, \hat{b}$ and the weight $w(x, N)$ is given for $x \in (a(N), b(N))$ by :

$$w(x, N) = \exp\left(-NV_N\left(\frac{x}{N}\right)\right)$$

where $V_N(u) = 2\theta V_0(u) + \frac{1}{N}e_N(Nu)$. V_0 is continuous on $[\hat{a}, \hat{b}]$ and twice continuously differentiable in $(\hat{a}, \hat{b})$. It satisfies

$$|V_0''(u)| \le C(1 + \frac{1}{|u-\hat{a}|} + \frac{1}{|\hat{b}-u)|}).$$

e_N is uniformly bounded on $[a(N)+1, b(N)-1]/N$ by $C \ln N$ for some finite constant C independent of N.

For the sake of simplicity, we define V_0 to be constant outside of $[\hat{a}, \hat{b}]$ and continuous at the boundary.

EXAMPLE 5.3. In the setting of lozenge tilings of the hexagon of Example 5.1 we assume that for large N

$$A = \hat{A}N + O(1), B = \hat{B}N + O(1), C = \hat{C}N + O(1), t = \hat{t}N + O(1)$$

with $\hat{t} > \max\{\hat{B}, \hat{C}\}$. Then $a(N) = 0, b(N) = A + B + C + 1 - t$ obey $\hat{a} = 0, \hat{b} = \hat{A} + \hat{B} + \hat{C} - \hat{t}$. Moreover, the potential satisfies our hypothesis with

$$\begin{aligned} V_0(u) &= u \ln u + (\hat{A} + \hat{B} + \hat{C} - \hat{t} - u) \ln(\hat{A} + \hat{B} + \hat{C} - \hat{t} - u) \\ &\quad -(\hat{A} + \hat{C} - u) \ln(\hat{A} + \hat{C} - u) - (\hat{t} - \hat{C} + u) \ln(\hat{t} - \hat{C} + u) \end{aligned}$$

Notice that V_N is infinite at the boundary since w vanishes. However, particles stay at distance at least $1/N$ of the boundary and therefore up to an error of order $1/N$, we can approximate V_N by V_0.

THEOREM 5.4. *If Assumption 5.2 holds, the empirical measure converges almost surely :*

$$\hat{\mu}_N \to \mu_{V_0}$$

where μ_{V_0} is the equilibrium measure for V_0. It is the unique minimizer of the energy

$$\mathcal{E}(\mu) = \int \left(\frac{1}{2} V_0(x) + \frac{1}{2} V_0(y) - \frac{1}{2} \ln |x - y| \right) d\mu(x) d\mu(y)$$

subject to the constraint that μ is a probability measure on $[\hat{a}, \hat{b}]$ with density with respect to Lebesgue measure bounded by θ^{-1}.

REMARK 5.5. We have already seen that $\mathcal{E}$ is a strictly convex good rate function on the set of probability measures on $[\hat{a}, \hat{b}]$, see (4.1). To see that it achieves its minimal value at a unique minimizer, it is therefore enough to show that we are minimizing this function on a closed convex set. But the set of probability measures on $[\hat{a}, \hat{b}]$ with density bounded by $1/\theta$ is clearly convex. It can be seen to be closed as it is characterized as the countable intersection of closed sets given as the set of probability measures on $[\hat{a}, \hat{b}]$ so that

$$\left| \int f(x) d\mu(x) \right| \leq \frac{\|f\|_1}{\theta}$$

for bounded continuous function f on $[\hat{a}, \hat{b}]$ so that $\|f\|_1 = \int |f(x)| dx < \infty$.

The case where $\hat{a}, \hat{b}$ are infinite can also be considered **[12]**. This result can be deduced from a large deviation principle similar to the continuous case **[36]** :

THEOREM 5.6. *If Assumption 5.2 holds, the law of $\hat{\mu}^N$ under $P_N^{\theta,w}$ satisfies a large deviation principle in the scale N^2 with good rate function I which is infinite outside of the set $\mathcal{P}_\theta$ of probability measures on $[\hat{a}, \hat{b}]$ absolutely continuous with respect to the Lebesgue measure and with density bounded by $1/\theta$, and given on $\mathcal{P}_\theta$ by*

$$I(\mu) = 2\theta(\mathcal{E}(\mu) - \inf_{\mathcal{P}_\theta} \mathcal{E}) .$$

PROOF. The proofs are very similar to the continuous case, we only sketch the differences. In this discrete framework, because the particles have spacings bounded below by θ, we have, for all $x < y$,

$$\theta \# \{i : \ell_i \in N[x, y]\} \leq (y - x)N + \theta$$

so that

$$\hat{\mu}_N([x, y]) \leq \frac{|y - x|}{\theta} + \frac{1}{N} .$$

In particular, $\hat{\mu}^N$ can only deviate towards probability measures in $\mathcal{P}_\theta$. The proof of the large deviation upper bound is then exactly the same as in the continuous case. For the lower bound, the proof is similar and boils down to concentrate the particles very close to the quantiles of the measure towards which the empirical measure deviates : one just need to find such a configuration in $\mathbb{W}_\theta$. We refer the reader to [**36**].

In particular in the limit we will have :

$$\frac{\mathrm{d}\mu_V^{\mathrm{eq}}}{\mathrm{d}x} \leq \frac{1}{\theta}$$

The variational problem defining μ_V^{eq} in this case takes this bound into account. Noticing that $\mathcal{E}(\mu_V^{\mathrm{eq}} + t\nu) \geq \mathcal{E}(\mu_V^{\mathrm{eq}})$ for all ν with zero mass, non-negative outside the support of μ_V^{eq} and non-positive in the region where $d\mu_V^{\mathrm{eq}} = \theta^{-1}dx$, the characterization of the equilibrium measure is that $\exists C_V$ s.t. if we define :

$$V_{\mathrm{eff}}(x) = V_0(x) - \int \ln(|x-y|)\mathrm{d}\mu_V^{\mathrm{eq}}(y) - C_V$$

and V_{eff} satisfies :

$$\begin{cases} V_{\mathrm{eff}}(x) = 0 & \text{on } 0 < \frac{\mathrm{d}\mu_V^{\mathrm{eq}}}{\mathrm{d}x} < \frac{1}{\theta} \\ V_{\mathrm{eff}}(x) \geq 0 & \text{on } \frac{\mathrm{d}\mu}{\mathrm{d}x} = 0 \\ V_{\mathrm{eff}}(x) \leq 0 & \text{on } \frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{\theta} \end{cases}$$

The analysis of the large deviation principle and concentration are the same as in the continuous β ensemble case otherwise. ◇

5.2. Concentration of measure

As in the continuous case we consider the pseudo- distance D (4.12) and the regularization of the empirical measure $\tilde{\mu}_N$ given by the convolution of $\hat{\mu}^N$ with a uniform variable on $[0, \frac{\theta}{N}]$ (to keep measures with density bounded by $1/\theta$). We then have, as in the continuous case, the following concentration of measure estimates.

LEMMA 5.7. *Assume V_0 is C^1. There exists C finite such that for all $t \geq 0$*

$$P_N^{\theta,\omega}\left(D(\tilde{\mu}_N, \mu_{V_0}) \geq t\right) \leq e^{CN\ln N - N^2 t^2}$$

As a consequence, for any $N \in \mathbb{N}$, any $\varepsilon > 0$

$$P_N^{\theta,\omega}\left(\sup_{z:\Im z \geq \varepsilon}\left|\int \frac{1}{z-x} d(\hat{\mu}^N - \mu_{V_0})(x)\right| \geq \frac{t}{\varepsilon^2} + \frac{1}{\varepsilon^2 N}\right) \leq e^{CN\ln N - N^2 t^2}$$

PROOF. We set $Q_N^{\theta,\omega}(\ell) = N^{-\theta N^2} Z_N^{\theta,\omega} P_N^{\theta,\omega}(\ell)$ and set for a configuration ℓ, $\mathcal{E}(\ell) := \mathcal{E}(\hat{\mu}_N)$,

$$\mathcal{E}(\ell) = \frac{1}{N}\sum_{i=1}^N V_0(\frac{\ell_i}{N}) - \frac{2\theta}{N^2}\sum_{i<j}\ln|\frac{\ell_i}{N} - \frac{\ell_j}{N}|\,.$$

- *We first show that* $Q_N^{\theta,\omega}(\ell) = e^{-N^2 2\theta\mathcal{E}(\ell) + O(N\ln N)}$. Indeed, Stirling formula shows that $\ln\Gamma(x) = x\ln x - x - \ln\sqrt{2\pi x} + O(\frac{1}{x})$, which implies that

$$\prod_{i<j}\frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} = \prod_{i<j}|\ell_j - \ell_i|^{2\theta} e^{O(\sum_{i<j}\frac{1}{\ell_j - \ell_i})}$$

with $\sum_{i<j} \frac{1}{(\ell_j - \ell_i)} = O(N \ln N)$ as $\ell_j - \ell_i \geq \theta(j-i)$. Similarly, by our assumption on V_N, for all configuration ℓ so that $\ell_1 \neq a(N)$ and $\ell_N \neq a(N)$, we have :

$$\frac{1}{N}\sum_{i=1}^{N} V_N(\frac{\ell_i}{N}) = \frac{1}{N}\sum_{i=1}^{N} V_0(\frac{\ell_i}{N}) + O(\frac{\ln N}{N}).$$

Hence we deduce that for any configuration with positive probability :

$$Q_N^{\theta,\omega}(\ell) = e^{-N^2 2\theta \mathcal{E}(\ell) + O(N \ln N)} \tag{5.1}$$

- *We have the lower bound* $N^{-\theta N^2} Z_N^{\theta,\omega} \geq e^{-N^2 2\theta \mathcal{E}(\mu_{V_0}) + CN \ln N}$. To prove this bound we simply have to choose a configuration matching this lower bound. We let $(q_i)_{1\leq i\leq N}$ be the quantiles of μ_{V_0} so that

$$\mu_{V_0}([\hat{a}, q_i]) = \frac{i - 1/2}{N}.$$

Then we set

$$Q_i = a(N) + \theta(i-1) + \lfloor N q_i - a(N) - (i-1)\theta \rfloor$$

Because the density of μ_{V_0} is bounded by $1/\theta$, $q_{i+1} - q_i \geq \theta$ and therefore $Q_{i+1} - Q_i \geq \theta$. Moreover, $Q_1 - a(N)$ is an integer. Hence, Q is a configuration. We have by the previous point that

$$N^{-\theta N^2} Z_N^{\theta,\omega} \geq e^{-N^2 2\theta \mathcal{E}(Q) + O(N \ln N)} \tag{5.2}$$

We finally can compare $\mathcal{E}(Q)$ to $\mathcal{E}(\mu_{V_0})$. Indeed, by definition $Q_i \in [Nq_i, Nq_i + 1]$ and $Q_i - Q_j \geq \theta(i-j)$, so that

$$\begin{aligned}
\sum_{i<j} \ln |\frac{Q_i - Q_j}{N}| &\geq \sum_{i+[\frac{2}{\theta}]<j} \ln |\frac{Q_i - Q_j}{N}| + O(N \ln N) \\
&\geq \sum_{i+[\frac{2}{\theta}]<j} \ln |q_j - q_i - \frac{1}{N}| + O(N \ln N) \\
&= \sum_{i+[\frac{2}{\theta}]<j} \ln |q_j - q_i| + O(N \ln N) \\
&\geq N^2 \sum_{i+[\frac{2}{\theta}]<j} \int_{q_{j-1}}^{q_j} \int_{q_i}^{q_{i+1}} \ln |x-y| d\mu_{V_0}(x) d\mu_{V_0}(y) + O(N \ln N) \\
&\geq N^2 \int_{x<y} \ln |x-y| d\mu_{V_0}(x) d\mu_{V_0}(y) + O(N \ln N)
\end{aligned}$$

where we used that the logarithm is monotone and the density of μ_{V_0} uniformly bounded by $1/\theta$.

Moreover

$$|\sum_i \left(\frac{1}{N} V_0(\frac{Q_i}{N}) - \int_{q_i}^{q_{i+1}} V_0(x) d\mu_{V_0}(x) \right)| \leq C \sum_i \int_{q_i}^{q_{i+1}} (|\frac{Q_i}{N} - q_i| + |q_{i+1} - q_i|) d\mu_{V_0}$$

is bounded by C'/N.

We conclude that

$$\mathcal{E}(Q) \leq \mathcal{E}(\mu_{V_0}) + O(\frac{\ln N}{N})$$

so that we deduce the announced bound from (5.2).

- *We then show that* $Q_N^{\theta,\omega}(\ell) = e^{-N^2 2\theta\mathcal{E}(\tilde\mu_N)+O(N\ln N)}$. We start from (5.1) and need to show we can replace the empirical measure of $\hat\mu^N$ by $\tilde\mu_N$ and then add the diagonal term $i=j$ up to an error of order $N\ln N$. Indeed, if u,v are two independent uniform variables on $[0,\theta]$, independent of ℓ,

$$\sum_{i\neq j}\ln|\frac{\ell_i}{N}-\frac{\ell_j}{N}| - \sum_{i,j}\mathbb{E}[\ln|\frac{\ell_i}{N}-\frac{\ell_j}{N}+\frac{u-v}{N}|]$$
$$= -\sum_i \mathbb{E}[\ln|\frac{u-v}{N}|] + O(\sum_{i<j}\frac{1}{\ell_j-\ell_i}) = O(N\ln N)$$

whereas

$$\frac{1}{N}\sum_{i=1}^N (V_0(\frac{\ell_i}{N}) - \mathbb{E}[V_0(\frac{\ell_i}{N}+\frac{u}{N})]) = O(\frac{\ln N}{N})$$

- $P_N^{\theta,\omega}(\ell) \le e^{-N^2 2\theta D^2(\tilde\mu_N,\mu_{V_0})+O(N\ln N)}$.

We can now write

$$\mathcal{E}(\tilde\mu_N) = \mathcal{E}(\mu_{V_0}) + \int V_{eff}(x)d(\tilde\mu_N-\mu_{V_0})(x) + D^2(\tilde\mu_N,\mu_{V_0})$$

D^2 is indeed positive as $\tilde\mu_N$ and μ_{V_0} have the same mass. $V_{eff}(x)$ vanishes on the liquid regions of μ_{V_0}, is non-negative on the voids where $\tilde\mu_N-\mu_{V_0}$ is non-negative, and non positive on the frozen regions where $\tilde\mu_N-\mu_{V_0}$ is non-negative since $\tilde\mu_N$ has density bounded by $1/\theta$. Hence we conclude that

$$\int_{[\hat a,\hat b]} V_{eff}(x)d(\tilde\mu_N-\mu_{V_0})(x) \ge 0\,.$$

On the other hand the effective potential is bounded and so our assumption on $a(N)-N\hat a$ implies

$$N^2\int_{[\hat a,\hat b]^c} V_{eff}(x)d(\tilde\mu_N-\mu_{V_0})(x) = O(N\ln N)\,.$$

Hence, we can conclude by the previous two points.

$\diamond$

5.3. Nekrasov's equations

The analysis of the central limit theorem is a bit different than for the continuous β ensemble case. Introduce :

$$\begin{aligned} G_N(z) &= \frac{1}{N}\sum_{i=1}^N \frac{1}{z-\frac{\ell_i}{N}} \\ G(z) &= \int \frac{1}{z-x}\mathrm{d}\mu_V^{\mathrm{eq}}(x)\,. \end{aligned}$$

We want to study the fluctuations of $\{N(G_N(z)-G(z))\}$. To this end, we would like an analogue of Dyson-Schwinger equations in this discrete setting. The candidate given by discrete integration by parts is not suited to asymptotic analysis as it yields densities which depend on $\prod(1+(\ell_i-\ell_j)^{-1})$ which is not a function of $\hat\mu^N$. In this case the analysis goes by the **Nekrasov's equations** which Nekrasov calls "non-perturbative" Dyson-Schwinger equations. Assume that we can write :

ASSUMPTION 5.8.

$$\frac{w(x,N)}{w(x-1,N)} = \frac{\phi_N^+(x)}{\phi_N^-(x)}$$

where $\phi_N^\pm$ are analytic functions in some subset $\mathcal{M}$ of the complex plane which includes $[a(N), b(N)]$ and independent of N.

EXAMPLE 5.9. In the example of random lozenge tilings of Example 5.1 we can take

$$\phi_N^+(z) = \frac{1}{N^2}(t - C + z)(A + B + C - t - z), \quad \phi_N^-(z) = \frac{1}{N^2} z(A + C - z).$$

With these defined, Nekrasov's equation is the following statement.

THEOREM 5.10. *If Assumption 5.8 holds*

$$R_N(\xi) = \phi_N^-(\xi)\mathbb{E}_{P_N^{\theta,w}}\left[\prod_{i=1}^N \left(1 - \frac{\theta}{\xi - \ell_i}\right)\right] + \phi_N^+(\xi)\mathbb{E}_{P_N^{\theta,w}}\left[\prod_{i=1}^N \left(1 + \frac{\theta}{\xi - \ell_i - 1}\right)\right]$$

is analytic in $\mathcal{M}$.

PROOF. In fact this can be checked by looking at the poles of the right hand side and showing that the residues vanish. Noting that there is a residue when $\xi = \ell_i$ or $\ell_i - 1$ we find that the residue at $\xi = m$ is

$$-\theta\phi_N^-(m)\sum_i \sum_{\ell_i = m} P_N^{\theta,w}(\ell_1, .., \ell_{i-1}, m, \ell_{i+1}, \dots, \ell_N)\left[\prod_{j \neq i}^N \left(1 - \frac{\theta}{m - \ell_j}\right)\right]$$

$$+\theta\phi_N^+(m)\sum_i \sum_{\ell_i = m-1} P_N^{\theta,w}(\ell_1, .., \ell_{i-1}, m-1, \ell_{i+1}, \dots, \ell_N)\left[\prod_{j \neq i}^N \left(1 + \frac{\theta}{m - \ell_j - 1}\right)\right].$$

If $m = a(N) + 1$ the second term vanish since the configuration space is such that $\ell_i > a(N)$ for all i, whereas $\phi_N^-(a(N) + 1) = 0$. Hence both term vanish. The same holds at $b(N)$ and therefore we now consider $m \in (a(N) + 1, b(N))$. Similarly, a configuration where $\ell_i = m$ implies that $\ell_{i-1} \leq m - \theta$ whereas $\ell_i = m - 1$ implies $\ell_{i-1} \leq m - 1 - \theta$. However, the first term vanishes when $\ell_{i-1} = m - \theta$. Hence, in both sums we may consider only configurations where $\ell_{i-1} \leq m - 1 - \theta$. The same holds for $\ell_{i+1} \geq m + \theta$. Then notice that if ℓ is a configuration such that when we shift ℓ_i by one we still have a configuration, our specific choice of weight w and interaction with the function Γ imply that

$$\phi_N^-(m)P_N^{\theta,w}(\ell_1, .., m, \ell_{i+1}, \dots, \ell_N)\left[\prod_{j \neq i}^N \left(1 - \frac{\theta}{m - \ell_j}\right)\right]$$

$$= \phi_N^+(m)P_N^{\theta,w}(\ell_1, .., m - 1, \ell_{i+1}, \dots, \ell_N)\left[\prod_{j \neq i}^N \left(1 + \frac{\theta}{m - \ell_j - 1}\right)\right].$$

On the other hand a configuration such that when we shift the ith particle by one we do not get an admissible configuration has residue zero. Hence, we find that the residue at $\xi = \ell_i$ and $\ell_i - 1$ vanishes. $\diamond$

Nekrasov's equation a priori still contains the analytic function R_N as an unknown. However, we shall see that it can be asymptotically determined based on the sole fact that it is analytic, provided the equilibrium measure is off-critical.

ASSUMPTION 5.11. Uniformly in $\mathcal{M}$,

$$\phi_N^{\pm}(z) =: \phi^{\pm}(\frac{z}{N}) + \frac{1}{N}\phi_1^{\pm}(\frac{z}{N}) + O(\frac{1}{N^2})$$

Observe here that $\phi_1^{\pm}$ may depend on N and be oscillatory in the sense that it may depend on the boundary point. For instance, in the case of binomial weights, $\phi_N^+(x) = (\frac{M+1}{N} - x), \phi_N^-(x) = x$, we see that if M/N goes to $\mathbf{m}$, $\phi^-(x) = x$ and $\phi_1^-(x) = 0$, but

$$\phi^+(x) = m - x, \phi_1^+(x) = M + 1 - N\mathbf{m}$$

where the latter may oscillate, even if it is bounded. We will however hide this default of convergence in the notations. The main point is to assume the functions in the expansion are bounded uniformly in N and $z \in \mathcal{M}$.

EXAMPLE 5.12. With the example of lozenge tiling, we have

$$\phi^+(z) = (\hat{t} - \hat{C} + z)(\hat{A} + \hat{B} + \hat{C} - t - z), \quad \phi^-(z) = z(\hat{A} + \hat{C} - z).$$

whereas if $\Delta D = D - L\hat{D}$,

$$\phi_1^+(x) = x(\Delta t - \Delta C + \Delta A + \Delta B + \Delta C - \Delta t), \phi_1^-(x) = \frac{N}{L}x(\Delta A + \Delta C)\,.$$

To analyze the asymptotics of G_N, we expand the Nekrasov's equations around the equilibrium limit. We set $\xi = Nz$ for $z \in \mathbb{C}\backslash\mathbb{R}$. Since we know by Lemma 5.7 that $\Delta G_N(z) = G_N(z) - G(z)$ is small (away from $[a,b]$), we can expand the Nekrasov's equation of Lemma 5.10 to get :

$$R_N(\xi) = R_\mu(z) - \theta Q_\mu(z)\mathbb{E}\left[\Delta G_N(z)\right] + \frac{1}{N}E_\mu(z) + \Gamma_\mu(z) \tag{5.3}$$

where we have set :

$$\begin{aligned}
R_\mu(z) &:= \phi^-(z)e^{-\theta G(z)} + \phi^+(z)e^{\theta G(z)} \\
Q_\mu(z) &:= \phi^-(z)e^{-\theta G(z)} - \phi^+(z)e^{\theta G(z)} \\
E_\mu(z) &:= \phi^-(z)e^{-\theta G(z)}\frac{\theta^2}{2}\partial_z G(z) + \phi^+(z)e^{\theta G(z)}\left(\frac{\theta^2}{2} - \theta\right)\partial_z G(z) \\
&\quad +\phi_1^-(z)e^{-\theta G(z)} + \phi_1^+(z)e^{\theta G(z)}\,.
\end{aligned}$$

Γ_μ is the reminder term given by (5.3) which basically is bounded on $\{\Im z \geq \varepsilon\} \cap \mathcal{M}$ by

$$|\Gamma_\mu(z)| \leq C(\varepsilon)\left(\mathbb{E}[|\Delta G_N(z)|^2] + \frac{1}{N}|\partial_z\mathbb{E}[\Delta G_N(z)]| + o(\frac{1}{N})\right)$$

The a priori concentration inequalities of Lemma 5.7 show that $\Gamma_\mu(z) = O(\ln N/N)$. We deduce by taking the large N limit that R_μ is analytic in $\mathcal{M}$ and we set $\tilde{R}_\mu = R_N - R_\mu$.

Let us assume for a moment that we have the stronger control on Γ_μ

LEMMA 5.13. *For any $\varepsilon > 0$,*

$$\mathbb{E}[|\Delta G_N(z)|^2] + \frac{1}{N}|\partial_z\mathbb{E}[\Delta G_N(z)]| = o(\frac{1}{N})$$

uniformly on $\mathcal{M} \cap \{|\Im z| \geq \varepsilon\}$.

Let us deduce the asymptotics of $N\mathbb{E}[\Delta G_N(z)]$. To do that let us assume we are in a off-critical situation in the sense that

ASSUMPTION 5.14.

$$\theta Q_\mu(z) = \sqrt{(z-a)(b-z)}H(z) =: \sigma(z)H(z)$$

where H does not vanish in $\mathcal{M}$.

REMARK 5.15. Observe that if ρ is the density of the equilibrium measure,

$$e^{2i\pi\theta\rho(E)} = \frac{R_\mu(E) + Q_\mu(E-i0)}{R_\mu(E) + Q_\mu(E+i0)}\,.$$

Our assumption implies therefore that $\rho(E) = 0$ or $1/\theta$ outside $[a,b]$ and goes to these values as a square root. There is a unique liquid region, where the density takes values in $(0, 1/\theta)$, it is exactly $[a,b]$.

We now proceed with similar techniques as in the β ensemble case, to take advantage of equation (5.3) as we used the Dyson-Schwinger equation before.

LEMMA 5.16. *If Assumption 5.14 holds, for any* $z \in \mathcal{M}\backslash\mathbb{R}$,

$$\mathbb{E}[N\Delta G_N(z)] = m(z) + o(1) \tag{5.4}$$

with $m(z) = K^{-1}E_\mu(z)$ *where*

$$K^{-1}f(z) = \frac{1}{2i\pi\sigma(z)}\oint_{[a,b]}\frac{1}{\xi - z}\frac{1}{H(\xi)}f(\xi)d\xi\,.$$

REMARK 5.17. If we compare to the continuous setting, K is the operator of multiplication by $\theta Q_\mu(z)$ whereas in the continuous case it was multiplication by $\beta\frac{d\mu}{dx} = G(z) - V'(z)$. Choosing $\phi^+(z) = e^{-V'(z)/2}, \phi^-(z) = e^{+V'(z)/2}$ we see that $Q_\mu(z) = \sinh(\theta G_\mu - V'(z)/2)$ is the hyperbolic sinus of the density. Hence, the discrete and continuous master operators can be compared up to take a sinh.

PROOF. To get the next order correction we look at (5.3) :

$$\theta Q_\mu(z)\mathbb{E}\left[\Delta G_N(z)\right] = \frac{1}{N}E_\mu(z) - \tilde{R}_\mu(z) + \Gamma_\mu(z)$$

We can then rewrite as a contour integral for $z \in \mathcal{M}$:

$$\begin{aligned}
\sigma(z)\mathbb{E}\left[\Delta G_N(z)\right] &= \frac{1}{2i\pi}\oint_z \frac{1}{\xi - z}\left[\sigma(\xi)\mathbb{E}\left[\Delta G_N(\xi)\right]\right]\mathrm{d}\xi \\
&= \frac{1}{2i\pi}\oint_{[a,b]}\frac{1}{\xi - z}\frac{1}{H(\xi)}\left[\frac{1}{N}E_\mu(\xi) - \tilde{R}_\mu(\xi) + \Gamma_\mu(\xi)\right]\mathrm{d}\xi \\
&= \frac{1}{2i\pi}\oint_{[a,b]}\frac{1}{\xi - z}\frac{1}{H(\xi)}\left[\frac{1}{N}E_\mu(\xi)\right]\mathrm{d}\xi + o\left(\frac{1}{N}\right)
\end{aligned}$$

where we used that $\sigma\Delta G_N$ goes to zero like $1/z$ to deduce that there is no residue at infinity so that we can move the contour to a neighborhood of $[a,b]$, that $\tilde{R}_\mu/H$ is analytic in a neighborhood of $[a,b]$ to remove its contour integral, and assumed Lemma 5.13 holds to bound the reminder term, as the integral is bounded independently of N. $\diamond$

REMARK 5.18. The previous proof shows, without Lemma 5.13, that $\mathbb{E}[\Delta G_N(z)]$ is at most of order $\ln N/N$ sin Γ_μ is at most of this order by basic concentration estimates.

We finally prove Lemma 5.13. To do so, it is enough to bound $\mathbb{E}[|\Delta G_N(z)|^2]$ by $o(1/N)$ uniformly on $\mathcal{M}\cap\{|\Im z|\ge \epsilon/2\}$ by analyticity. Note that Lemma 5.7 already implies that this is of order $\ln N/N$. To improve this bound, we get an equation for the covariance. To get such an equation we replace the weight $w(x,N)$ by

$$w_t(x,N)=w(x,N)\left(1+\frac{t}{z'-x/N}\right)$$

for t very small. This changes the functions $\phi_N^{\pm}$ by

$$\phi_N^{+,t}(x)=\phi_N^{+}(x)\,(z'-x/N+t)\left(z'-x/N+\frac{1}{N}\right),$$

$$\phi_N^{-,t}(x)=\phi_N^{-}(x)\,(z'-x/N)\left(z'-x/N+t+\frac{1}{N}\right).$$

We can apply the Nekrasov's equations to this new measure for t small enough (so that the new weights w_t does not vanish for $z'\in\mathcal{M}$) to deduce that

$$R_N^t(\xi)=\phi_N^{-,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^N\left(1-\frac{\theta}{\xi-\ell_i}\right)\right]+\phi_N^{+,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^N\left(1+\frac{\theta}{\xi-\ell_i-1}\right)\right] \tag{5.5}$$

is analytic. We start expanding with respect to N by writing

$$\phi_N^{\pm,t}(Nx)/(z'-x)(t+z'-x)=(\phi^{\pm}(x)+\frac{1}{N}\phi_1^{\pm,t}(x)+o(\frac{1}{N}))$$

with

$$\phi_1^{+,t}(x)=\phi_1^{+}(x)+\frac{\phi^+(x)}{z'-x},\ \phi_1^{-,t}(x)=\phi_1^{-}(x)+\frac{\phi^-(x)}{t+z'-x}.$$

We set

$$\tilde{R}_N^t(x)=(R_N^t(Nx)-R_\mu(x))/(z'-x)(t+z'-x)$$

which is analytic up to a correction which is $o(1/N)$ and analytic away from z' in a neighborhood of which it has two simple poles. We divide both sides of Nekrasov equation by $(z'-x)(t+z'-x)$, and take $\xi=Nz$ and again using Lemma 5.7, we deduce that

$$\theta Q_\mu(z)\mathbb{E}_{P_N}^{\theta,w_t}\left[\Delta G_N(z)\right]=\tilde{R}_N^t(z)+\frac{1}{N}E_\mu^t(z)+\Gamma_\mu^t(z) \tag{5.6}$$

where

$$E_\mu^t(z)=E_\mu(z)+\frac{\phi^+(z)}{z'-z}e^{\theta G(z)}+\frac{\phi^-(z)}{t+z'-z}e^{-\theta G(z)}$$

and $\Gamma_\mu^t(z)$ is a reminder term. It is the sum of the reminder term coming from (5.3) and the error term coming from the expansion of $\phi^{\pm,t}$. The latter has single poles at z' and $z'+t$ and is bounded by $1/N^2$. We can invert the multiplication by Q_μ as before to conclude (taking a contour which does not include z' so that $\tilde{R}_N^t$ stays analytic inside) that

$$\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right]=K^{-1}[\frac{1}{N}E_\mu^t+\Gamma_\mu^t](z)+o(\frac{1}{N}),$$

where we noticed that the residues of ε_N are of order one.

We finally differentiate with respect to t and take $t=0$ (note therefore that we need no estimates under the tilted measure P_N^{θ,w_t}, but only those take at $t=0$

where we have an honest probability measure). Noticing that the operator K does not depend on t, we obtain, with $\bar{\Delta} G_N(z') = G_N(z') - \mathbb{E}[G_N(z')]$:

(5.7)
$$N^2 \mathbb{E}_{P_N^{\theta,w}}\left[\Delta G_N(z)\bar{\Delta} G_N(z')\right] = -K^{-1}[\frac{\phi^-(.)}{(z'-.)^2}e^{-\theta G(.)}](z) + NK^{-1}[\partial_t \Gamma_\mu^t|_{t=0}](z)$$

It is not difficult to see by a careful expansion in Nekrasov's equation (5.5) that

$$|\partial_t \Gamma_\mu^t(z)|_{t=0}| \leq C(\varepsilon)\Bigg(N\mathbb{E}[|\Delta G_N(z)|^2|\bar{\Delta} G_N(z')|] + |\partial_z \mathbb{E}[\Delta G_N(z)\bar{\Delta} G_N(z')]| + \frac{1}{N}\mathbb{E}[|\bar{\Delta} G_N(z')|]\Bigg) \tag{5.8}$$

By Lemma 5.7, it is at most of order $(\ln N)^3/\sqrt{N}$ so that we proved

$$N^2 \mathbb{E}_{P_N^{\theta,w}}\left[\Delta G_N(z)\bar{\Delta} G_N(z')\right] = -K^{-1}[\frac{\phi^-(.)}{(z'-.)^2}e^{-\theta G(.)}](z) + O((\ln N)^3\sqrt{N}) \tag{5.9}$$

This shows by taking $z' = \bar{z}$ that for $\Im z \geq \varepsilon$

$$\mathbb{E}[|N\Delta G_N(z)|^2] \leq (\ln N)^3\sqrt{N}\,. \tag{5.10}$$

We note here that $\Delta G_N(z)$ and $\bar{\Delta} G_N(z)$ only differ by $\ln N/N$ by Remark 5.18. This completes the proof of Lemma 5.13.

We derive the central limit theorem in the same spirit.

THEOREM 5.19. *If Assumption 5.14 holds, for any $z_1, \ldots, z_k \in \mathcal{M}\backslash\mathbb{R}$, $(N\Delta G_N(z_1) - m(z_1), \ldots, N\Delta G_N(z_k) - m(z_k))$ converges in distribution towards a centered Gaussian vector with covariance*

$$C(z, z') = -K^{-1}[\frac{\phi^-(.)}{(z'-.)^2}e^{-\theta G(.)}](z)$$

REMARK 5.20. It was shown in **[12]** that the above covariance is the same than for random matrices and is given by

$$C(z, w) = \frac{1}{(z-w)^2}\left(1 - \frac{zw - \frac{1}{2}(a+b)(z+w) + ab}{\sqrt{(z-a)(z-b)}\sqrt{(w-a)(w-b)}}\right).$$

It only depends on the end points and therefore is the same than for continuous β ensembles with equilibrium measure with same end points. However notice that the mean given in (5.12) is different.

PROOF. We first prove the convergence of the covariance by improving the estimates on the reminder term in (5.7) by a bootstrap procedure. It is enough to improve the estimate on $\partial_t \Gamma_\mu$ according to (5.7). But already, our new bound on the covariance (5.10) and Lemma 5.7 allow to bound the right hand side of (5.8) by $(\ln N)^4/N$. This allows to improve the estimate on the covariance as in the previous proof and we get :

$$\mathbb{E}[|N\Delta G_N(z)|^2] \leq C(\epsilon)(\ln N)^4\,. \tag{5.11}$$

In turn, we can again improve the estimate on $|\partial_t \Gamma_\mu(z)|$ since we now can bound the right hand side of (5.8) by $(\ln N)^5 N^{-1/2}$, which implies the desired convergence of $\mathbb{E}[\Delta G_N(z)\Delta G_N(z')]$ towards $C(z, z')$.

To derive the central limit theorem it is enough to show that the cumulants of degree higher than two vanish. To do so we replace the weight $w(x,N)$ by

$$w_t(x,N) = w(x,N)\prod_{i=1}^{p}\left(1+\frac{t_i}{z_i - x/N}\right).$$

The cumulants are then given by

$$N\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right]|_{t_1=t_2=\cdots=t_p=0}.$$

Indeed, recall that the cumulant of $N\bar{\Delta}G_N(z_1),\ldots N\bar{\Delta}G_N(z_p)$ is given by

$$\partial_{t_1}\cdots\partial_{t_p}\ln\mathbb{E}_{P_N^{\theta,w_t}}[\exp\{N\sum_{i=1}^{p}t_iG_N(z_i)\}]|_{t_1=t_2=\cdots=t_p=0}$$

which is also given by

$$\partial_{t_2}\cdots\partial_{t_p}\ln\mathbb{E}_{P_N^{\theta,w_t}}[N\bar{\Delta}G_N(z_1)]|_{t_1=t_2=\cdots=t_p=0}.$$

Noticing that $\mathbb{E}_{P_N^{\theta,w_t}}[\bar{\Delta}G_N(z)-\Delta G_N(z)]$ is independent of t, we conclude that it is enough to show that

$$N\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right]|_{t_1=t_2=\cdots=t_p=0}$$

goes to zero for $p\geq 2$. In fact, we can perform an analysis similar to the previous one. This changes the functions $\phi_N^{\pm}$ by

$$\phi_N^{+,t}(x) = \phi_N^{+}(z)\prod_{i=1}^{p}(z_i - x/N + t_i)\,,\phi_N^{-,t}(x) = \phi_N^{-}(z)\prod_{i=1}^{p}(z_i - x/N)\,.$$

We can apply the Nekrasov's equations to this new measure for t_i small enough (so that the new weights do not vanish) to deduce that

$$R_N^t(\xi) = \phi_N^{-,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^{N}\left(1-\frac{\theta}{\xi-\ell_i}\right)\right] + \phi_N^{+,t}(\xi)\mathbb{E}_{P_N^{\theta,w_t}}\left[\prod_{i=1}^{N}\left(1+\frac{\theta}{\xi-\ell_i-1}\right)\right]$$

is analytic. Expanding in N we deduce that

$$\mathbb{E}_{P_N^{\theta,w_t}}\left[\Delta G_N(z)\right] = K^{-1}[\frac{1}{N}E_\mu^t(z)+\Gamma_\mu^t(z)]$$

where

$$E_\mu^t(z) = E_\mu(z) + \sum_{i=1}^{p}\frac{\phi^+(x)}{z_i - x}e^{\theta G(z)} + \sum_{i=1}^{p}\frac{\phi^-(x)}{t_i+z_i-x}e^{-\theta G(z)}$$

and

$$\begin{aligned}|\partial_{t_1}\cdots\partial_{t_p}\Gamma_\mu^t|_{t_i=0}(z)| \leq C(\epsilon)\Bigg(&\mathbb{E}\left[(|\Delta G_N(z)|^2+\frac{1}{N^2})\prod|N\bar{\Delta}G_N(z_i)||\right]\\ &+\frac{1}{N}|\partial_z\mathbb{E}[\Delta G_N(z)\prod N\bar{\Delta}G_N(z_i)]|\Bigg).\end{aligned}$$

The contour in the definition of K^{-1} includes z and $[a,b]$ but not the z_i's. Taking the derivative with respect to $t_1,\ldots,t_p$ at zero we see that for $p\geq 1$

$$\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}\mathbb{E}_{P_N^{\theta,w_t}}\left[N\Delta G_N(z)\right] = K^{-1}[\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p}N\Gamma_\mu^t(z)]$$

where we used that the operator K is independent of t. We finally need to show that the right hand side goes to zero. It will, provided we show that for all $p\in\mathbb{N}$, all $z_1, \ldots, z_p \in \mathcal{M}\backslash[A,B]$ there exists C depending only on $\min d(z_i, [A,B])$ and p such that

$$\left|\mathbb{E}[\prod_{i=1}^{p} N\Delta G_N(z_i)]\right| \le C(\ln N)^{3p}.$$

This provides also bounds on $\mathbb{E}[|\Delta G_N(z)|^p]$ when p is even. Indeed $\partial_{t_2}\cdots\partial_{t_p} N\Gamma_\mu^t(z)$ can be bounded by a combination of such moments. We can prove this by induction over p. By our previous bound on the covariance, we have already proved this result for $p=2$ by (5.11). Let us assume we obtained this bound for all $\ell \le p$ for some $p \ge 2$. To get bounds on moments of correlators of order $p+1$, let us notice that $|\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p} N\Gamma_\mu^t|_{t=0}$ is at most of order $(\ln N)^{3p+2}$ if p is even by the induction hypothesis and Lemma 5.7(by bounding uniformly the Stieltjes functions depending on z). This is enough to conclude. If p is odd, we can only get bounds on moments of modulus of the Stieltjes transform of order $p-1$. We do that and bound also the Stieltjes transform depending on the argument z_1 by using Lemma 5.7. We then get a bound of order $(\ln N)^{3p+3}\sqrt{N}$ for $|\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p} N\Gamma_\mu^t|_{t=0}$. This provides a similar bound for the correlators of order $p+1$, which is now even. Using Hölder inequality back on the previous estimate and Lemma 5.7 on at most one term, we finally bound $|\partial_{t_1}\partial_{t_2}\cdots\partial_{t_p} N\Gamma_\mu^t|_{t=0}$ by $(\ln N)^{3(p+1)}$ which concludes the argument. $\diamond$

5.4. Second order expansion of linear statistics

In this section we show how to expand the expectation of linear statistics one step further. To this end we need to assume that $\phi_N^\pm$ expands to the next order.

ASSUMPTION 5.21. Uniformly in $\mathcal{M}$,

$$\phi_N^\pm(z) =: \phi^\pm(z) + \frac{1}{N}\phi_1^\pm(z) + \frac{1}{N^2}\phi_2^\pm(z) + O(\frac{1}{N^3})$$

LEMMA 5.22. *Suppose Assumption 5.21 holds. Then,*

$$\lim_{N\to\infty} \mathbb{E}[N^2\Delta G_N(z) - Nm(z)] - r(z) = 0 \tag{5.12}$$

with $r(z) = K^{-1}F_\mu(z)$ *where* $F_\mu(z)$ *is equal to*

$$\begin{aligned}
&= \phi^-(z)e^{-\theta G(z)}\left(\frac{\theta^2}{2}\partial_z m(z) - \frac{\theta^3}{3}\partial_z^2 G(z) + \frac{\theta^2}{2}[C(z,z) + (\frac{\theta}{2}\partial_z G(z) + m(z))^2]\right)\\
&\quad +\phi_1^-(z)e^{-\theta G(z)}\left(\frac{\theta^2}{2}\partial_z G(z) - \theta m(z)\right) + \phi_2^-(z)e^{-\theta G(z)}\\
&\quad +\phi^+(z)e^{\theta G(z)}\Bigg((\frac{\theta^2}{2} - \theta)\partial_z m(z) + (\frac{\theta^3}{3} + \theta - \frac{\theta^2}{2})\partial_z^2 G(z)\\
&\quad +\frac{\theta^2}{2}[(m(z) - \frac{2-\theta}{2}\partial_z G(z))^2 + C(z,z)]\Bigg)\\
&\quad +\phi_1^+(z)e^{\theta G(z)}[\theta m(z) + (\frac{\theta^2}{2} - \theta)\partial_z G(z)] + \phi_2^-(z)e^{\theta G(z)}
\end{aligned}$$

PROOF. The proof is as before to show that

$$\theta Q_\mu(z)\mathbb{E}[\Delta G_N(z)] = \frac{1}{N}E_\mu(z) + \frac{1}{N^2}F_\mu(z) + \tilde{R}_\mu^N(z) + o\left(\frac{1}{N^2}\right)$$

by using Nekrasov's equation of Theorem 5.10, expanding the exponentials and using Lemmas 5.19 and 5.16. We then apply K^{-1} on both sides to conclude.

◇

5.5. Expansion of the partition function

To expand the partition function in the spirit of what we did in the continuous case, we need to compare our partition function to one we know. In the continuous case, Selberg integrals were computed by Selberg. In the discrete case it turns out we can compute the partition function of binomial Jack measure **[12]** which corresponds to the choice of weight depending on two positive real parameters $\alpha, \beta > 0$ given by :

$$w_J(\ell) = (\alpha\beta\theta)^\ell \frac{\Gamma(M+\theta(N-1)+\frac{3}{2})}{\Gamma(\ell+1)\Gamma(M+\theta(N-1)+1-\ell)} \tag{5.13}$$

Then, the partition function can be computed explicitely and we find (see the work in progress with Borot and Gorin) :

THEOREM 5.23. *With summation going over $(\ell_1, \ldots, \ell_N)$ satisfying $\ell_1 \in \mathbb{Z}_{\geq 0}$ and $\ell_{i+1} - \ell_i \in \{\theta, \theta+1, \theta+2, \ldots\}$, $i \in \{1, \ldots, N-1\}$, we have*

$$\begin{aligned}
\mathbf{Z}_N^J &= \sum \prod_{1 \leq i < j \leq N} \frac{1}{N^{2\theta}} \frac{\Gamma(\ell_{i+1}-\ell_i+1)\Gamma(\ell_{i+1}-\ell_i+\theta)}{\Gamma(\ell_{i+1}-\ell_i)\Gamma(\ell_{i+1}-\ell_i+1-\theta)} \prod_{i=1}^N w_J(\ell_i) \\
&= (1+\alpha\beta\theta)^{MN} (\frac{\alpha\beta\theta}{N^2})^{\theta\frac{N(N-1)}{2}} \prod_{i=1}^N \frac{\Gamma(\theta(N+1-i))\Gamma(M+\theta(N-1)+\frac{3}{2})}{\Gamma(\theta)\Gamma(M+1+\theta(i-1))}.
\end{aligned}$$

On the other hand, the equilibrium measure μ_J for this model can be computed and we find that if $\frac{M}{N} \to (\mathbf{m}-\theta)$ and $q = \alpha\beta\theta$, there exists $\alpha, \beta \in (0, \mathbf{m})$ so that μ_J has density equal to 0 or $1/\theta$ outside (α, β), and in the liquid region (α, β) the density is given by :

$$\mu_J(x) = \frac{1}{\pi\theta} \operatorname{arccot} \left(\frac{x(1-q)+q\mathbf{m}-q\theta-\theta}{\sqrt{((x(1-q)+q\mathbf{m}-q\theta-\theta))^2+4xq(\mathbf{m}-x)}} \right),$$

where arccot is the reciprocal of the cotangent function. Therefore, depending on the choices of the parameters, the behavior of $\mu_J(x)$ as x varies from 0 to $\mathbf{m}$ is given by the following four scenarios (it is easy to see that all four do happen)

- Near zero $\mu_J(x) = 0$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = \theta^{-1}$ near $\mathbf{m}$;
- Near zero $\mu_J(x) = \theta^{-1}$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = \theta^{-1}$ near $\mathbf{m}$;
- Near zero $\mu_J(x) = 0$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = 0$ near $\mathbf{m}$;
- Near zero $\mu_J(x) = \theta^{-1}$, then $0 < \mu_J(x) < \theta^{-1}$, then $\mu_J(x) = 0$ near $\mathbf{m}$.

We want to interpolate our model with weight w with a Jack binomial model with weight w_J. To this end we would like to consider a model with the same liquid/frozen/void regions so that the model with weight $w^t w_J^{1-t}$, $t \in [0,1]$, corresponds to an equilibrium measure with the same liquid/frozen/void regions and an equilibrium measure given by the interpolation between both equilibrium measure. However, doing that we may have problems to satisfy the conditions of Nekrasov's equations if w/w_J may vanish or blow up. It is possible to circumvent this point by proving that the boundary points are frozen with overwhelming probability, hence allowing more freedom with the boundary point. In these lecture notes, we will not go to this technicality.

THEOREM 5.24. *Assume there exists M, q so that $\ln(w/w_J)$ is approximated, uniformly on $[\hat{a}, \hat{b}]$ by*

$$\ln \frac{w}{w_j}(Nx) = -N(V - V_J)(x) + \Delta_1 V(x) + \frac{1}{N}\Delta_2 V(x) + o(\frac{1}{N}).$$

where $V - V_J$ and $\Delta_1 V$ are analytic in $\mathcal{M}$, whereas $\Delta_2 V$ is bounded continuous on $[\hat{a}, \hat{b}]$. Assume moreover that $\phi_N^{\pm}$ satisfies Assumption 5.14. Then, we have

$$\ln \frac{Z_N^{\theta,w}}{Z_N^J} = -N^2 F_0(\theta, V) + N F_1(\theta, w) + F_0(\theta, w) + o(1)$$

with

$$\begin{aligned}
F_0(\theta, V) &= -2\theta\mathcal{E}(\mu) + 2\theta\mathcal{E}(\mu_J) \\
F_1(\theta, V) &= \frac{1}{2\pi i}\int_0^1 \int_C (V_J - V)(z) m_t(z) dt + \frac{1}{2\pi i}\int_0^1 \int_C \Delta_1 V(z) G_t(z) dt \\
F_2(\theta, V) &= \frac{1}{2\pi i}\int_0^1 \int_C ((V_J - V)(z) r_t(z) + \Delta_1 V(z) m_t(z) + \Delta_2 V(z) G_t(z))\, dz dt
\end{aligned}$$

PROOF. We consider P_N^{θ,w_t} the discrete β model with weight $w^t w_J^{1-t}$. We have

$$\begin{aligned}
\ln \frac{Z_N^{\theta,w}}{Z_N^J} &= \int_0^1 P_N^{\theta,w_t}\left(\sum_i \ln \frac{w}{w_J}(\ell_i)\right) dt \\
&= \int_0^1 P_N^{\theta,w_t}\left(\hat{\mu}^N\left(N^2(V_J - V) + N\Delta_1 V + \Delta_2 V\right)\right) dt + o(1).
\end{aligned}$$

Denote μ_t the equilibrium measure for $w^t w_J^{1-t}$. Clearly

$$\lim_{N\to\infty} \int_0^1 P_N^{\theta,w_t}(\hat{\mu}^N(\Delta_2 V)) dt = \int_0^1 \mu_t(\Delta_2 V)\, dt.$$

For the first two terms we use the analyticity of the potentials and Cauchy formula to express everything in terms of Stieltjes functions

$$\begin{aligned}
&\int_0^1 P_N^{\theta,w_t}(\hat{\mu}^N\left(N^2(V_J - V) + N\Delta_1 V\right)) dt \\
&= \frac{1}{2\pi i}\int_0^1 \int_C \left(N^2(V_J - V) + N\Delta_1 V\right)(z) P_N^{\theta,w_t}(G_N(z)) dz dt.
\end{aligned}$$

We then use Lemma 5.22 since all our assumptions are verified. This provides an expansion :

$$\ln \frac{Z_N^{\theta,w}}{Z_N^J} = -N^2 F_0(\theta, V) + N F_1(\theta, w) + F_0(\theta, w) + o(1).$$

Again by taking the large N limit we can identify $F_0(\theta, V) = -\mathcal{E}(\mu_V)$. For F_1 we find

$$F_1(\theta, w) = \frac{1}{2\pi i}\int_0^1 \int_C (V_J - V)(z) m_t(z) dt + \frac{1}{2\pi i}\int_0^1 \int_C \Delta_1 V(z) G_t(z) dt$$

and

$$F_2(\theta, w) = \frac{1}{2\pi i} \int_0^1 \int_C \left((V_J - V)(z) r_t(z) + \Delta_1 V(z) m_t(z) + \Delta_2 V(z) G_t(z) \right) dz dt$$

$\diamond$

CHAPTER 6

Continuous Beta-models : the several cut case

In this section we consider again the continuous β-ensembles, but in the case where the equilibrium measure has a disconnected support. The strategy has to be modified since in this case the master operator Ξ is not invertible. In fact, the central limit theorem is not true as if we consider a smooth function f which equals one on one connected piece of the support but vanishes otherwise, and if we expect that the eigenvalues stay in the vicinity of the support of the equilibrium measure, the linear statistic $\sum f(\lambda_i)$ should be an integer and therefore can not fluctuate like a Gaussian variable. It turns out however that the previous strategy works as soon as we fix the filling fractions, the number of eigenvalues in a neighborhood of each connected piece of the support. The idea will therefore be to obtain central limit theorems conditionally to filling fractions. We will as well expand the partition functions for such fixed filling fractions. The latter expansion will allow to estimate the distribution of the filling fractions and to derive their limiting distribution, giving a complete picture of the fluctuations. These ideas were developed in [**14, 16**]. [**14**] also includes the case of hard edges. After this work, a very special case (two connected components and a polynomial potential) could be treated in [**28**] by using Riemann Hilbert. I will here follow the strategy of [**14**], but will use general test functions instead of Stieltjes functionals as in Section 4. So as in Section 4, we consider the probability measure

$$\mathrm{d}P_N^{\beta,V}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_N^{\beta,V}}\Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)}\prod_{i=1}^N \mathrm{d}\lambda_i\,.$$

By Theorems 4.4 and 4.3, if V satisfies Assumption 4.2, we know that the empirical measure of the λ's converges towards the equilibrium measure μ_V^{eq}. We shall hereafter assume that $\mu_V^{\mathrm{eq}} = \mu_V$ has a disconnected support but a off-critical density in the following assumption.

Assumption 6.1. $V : \mathbb{R} \to \mathbb{R}$ is of class C^p and μ_V^{eq} has support given by $S = \cup_{i=1}^K [a_i, b_i]$ with $b_i < a_{i+1} < b_{i+1} < a_{i+2}$ and

$$\frac{d\mu_V}{dx}(x) = H(x)\sqrt{\prod_{i=1}^K (x-a_i)(b_i-x)}$$

where H is a continuous function such that $H(x) \geq \bar{c} > 0$ a.e. on S.

We discuss this assumption in Lemma 6.5. Let us notice that the fact that the support μ_V has a finite number of connected components is guaranteed when V is analytic. Also, the fact that the density vanishes as a square root at the boundary of the support is generic, cf [**66**]. Remember, see Lemma 4.5, that μ_V is described by the fact that the effective potential V_{eff} is non-negative outside of the support of

μ_V.We will also assume hereafter that Assumption 4.2 holds and that $V_{\rm eff}$ is strictly positive outside S. By Theorem 4.8, we therefore know that the eigenvalues will remain in $S_\varepsilon = \cup_{i=1}^p S_\varepsilon^i, S_\varepsilon^i := [a_i-\varepsilon, b_i+\varepsilon]$ with probability greater than $1-e^{-C(\varepsilon)N}$ with some $C(\varepsilon) > 0$ for all $\varepsilon > 0$. We take ε small enough so that S_ε is still the union of p disjoint connected components $S_i, 1 \le i \le p$. Moreover, we will assume that V is C^1 so that the conclusions of Theorem 4.14 and Corollary 4.16 still hold. In particular

COROLLARY 6.2. Assume V is C^1. There exists $c > 0$ and C finite such that

$$P_N^{\beta,V}\left(\max_{1\le i\le p} |\#\{j : \lambda_j \in [a_i-\varepsilon, b_i+\varepsilon]\} - N\mu_V([a_i,b_i])| \ge C\sqrt{N}\ln N\right) \le e^{-cN}$$

We can therefore restrict our study to the probability measure given, if we denote by $N_i = \#\{j : \lambda_j \in [a_i-\varepsilon, b_i+\varepsilon]\}$, $\hat{n}_i = N_i/N$ and $\hat{n} = (\hat{n}_1, \dots, \hat{n}_K)$, by

$$dP_{N,n}^{\beta,V}(\lambda_1,\dots,\lambda_N) = 1_{\max_i |N_i - N\mu([a_i,b_i])| \le C\sqrt{N}\ln N} \frac{1_{S_\varepsilon}}{Z_{N,\varepsilon}^{\beta,V}} \Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)} \prod_{i=1}^N d\lambda_i$$

since exponentially small corrections do not affect our polynomial expansions. As $\varepsilon > 0$ is kept fixed we forget it in the notations and denote

$$dP_{N,\hat{n}}^{\beta,V}(\lambda_1,\dots,\lambda_N) = \frac{1_{N_i=\hat{n}_i N} 1_{S_\varepsilon}}{Z_{N,\hat{n}}^{\beta,V}} \Delta(\lambda)^\beta e^{-N\beta\sum V(\lambda_i)} \prod_{i=1}^N d\lambda_i$$

the probability measure obtained by conditioning the filling fractions to be equal to $\hat{n} = (n_1, \dots, n_p)$. Clearly, we have

$$Z_N^{\beta,V} = \sum_{|N_i - N\mu([a_i,b_i])| \le C\sqrt{N}\ln N} \frac{N!}{N_1!\cdots N_p!} Z_{N,\hat{n}}^{\beta,V} \tag{6.1}$$

$$P_N^{\beta,V} = \sum_{|N_i - N\mu([a_i,b_i])| \le C\sqrt{N}\ln N} \frac{N!}{N_1!\cdots N_K!} \frac{Z_{N,\hat{n}}^{\beta,V}}{Z_N^{\beta,V}} P_{N,\hat{n}}^{\beta,V} \tag{6.2}$$

where the combinatorial term $\frac{N!}{N_1!\cdots N_K!}$ comes from the ordering of the eigenvalues to be distributed among the cuts. Hence, we will retrieve large N expansions of the partition functions and linear statistics of the full model from those of the fixed filling fraction models.

6.1. The fixed filling fractions model

To derive central limit theorems and expansion of the partition function for fixed filling fractions we first need to check that we have the same type of results that before we fix the filling fractions. We leave the following Theorem as an exercise, its proof is similar to the proof of Theorem 4.4. Recall the notation :

$$\mathcal{E}(\mu) = \int\int [\frac{1}{2}V(x) + \frac{1}{2}V(y) - \frac{1}{2}\ln|x-y|] d\mu(x) d\mu(y) .$$

THEOREM 6.3. *Fix $n_i \in (0,1)$ so that $\sum n_i = 1$. Under the above assumptions*

- *Assume that $(\hat{n}_i)_{1\le i\le K}$ converges towards $(n_i)_{1\le i\le K}$. The law of the vector of p empirical measures $\hat{\mu}_i^N = \frac{1}{N_i}\sum_{j=N_1+\cdots+N_{i-1}+1}^{N_1+\cdots+N_i} \delta_{\lambda_j}$ under $P_{N,\hat{n}}^{\beta,V}$ satisfies a large deviation principle on the space of p tuples of probability* *measures on $S_i = [a_i - \varepsilon, b_i + \varepsilon]$, $1 \le i \le p$, in the scale N^2 with good rate function $I_n = J_n - \inf J_n$ where*

$$J_n(\mu_1, \ldots, \mu_p) = \beta \mathcal{E}(\sum_{i=1}^{K} n_i \mu_i)\,.$$

- *J_n achieves its minimal value uniquely at $(\mu_i^n)_{1 \le i \le p}$. Besides there exists p constants C_i^n such that the effective potential*

$$V_{eff}^n(x) = V(x) - \int \ln|x-y| d(\sum n_i \mu_i^n(y)) - C_i^n \tag{6.3}$$

is greater or equal to 0 on S_i and equal to 0 on the support of μ_i^n.

- *The conclusions of Lemma 4.14 and Corollary 4.16 hold in the fixed filling fraction case in the sense that for $\hat{n} = N_i/N, \sum N_i = N$ we can smooth $\sum \hat{n}_i \hat{\mu}_i^N = \hat{\mu}^N$ into $\tilde{\mu}_N$ (by pulling appart eigenvalues and taking the convolution by a small uniform variable), so that there exists $c > 0, C_{p,q} < \infty$ such that for $t > 0$*

$$P_{N,\hat{n}}^{\beta,V}\left(D(\tilde{\mu}_N, \sum \hat{n}_i \mu_i^{\hat{n}}) \ge t\right) \le e^{C_{p,q} N \ln N - \beta N^2 t^2} + e^{-cN}$$

Note above that the filling fractions N_i/N may vary when N grows : the first two statements hold if we take the limit, and the last with $\hat{n}_i = N_i/N$ exactly equal to the filling fractions (the measures μ_i^n are defined for any given n_i such that $\sum n_i = 1$). The last result does not hold if $\hat{n}$ is replaced by its limit n, unless $\hat{n}$ is close enough to n. To get the expansion for the fixed filling fraction model it is essential to check that they are off critical if the $\hat{n}_i$ are close to $\mu(S_i)$:

LEMMA 6.4. *Assume V is analytic. Fix $\varepsilon > 0$. There exists $\delta > 0$ so that if $\max_i |n_i - \mu_V(S_i)| \le \delta$, $(\mu_i^n)_{1 \le i \le p}$ are off-critical in the sense that there exists $a_i^n < b_i^n$ in S_ε^i and H_i^n uniformly bounded below by a positive constant on S_ε^i such that*

$$d\mu_i^n(x) = H_i^n(x) \sqrt{(x - a_i^n)(b_i^n - x)} dx\,.$$

PROOF. We first observe that $n \to \int f d\mu_i^n$ is smooth for all smooth functions f. Indeed, take two filling fractions n, m and denote in short by $\mu^n = \sum n_i \mu_i^n$. Recall that μ^n minimizes $\mathcal{E}$ on the set of probability measures with filling fractions n. We decompose $\mathcal{E}$ as

$$\mathcal{E}(\nu) = \beta \int V_{\text{eff}}^n(x) d(\nu - \mu^n)(x) + \frac{\beta}{2} D^2(\nu, \mu^n) - \beta \sum C_h^n(\nu([\hat{a}_h, \hat{b}_h]) - n_h) \tag{6.4}$$

where V_{eff}^n is the effective potential for the measure μ^n. Note here that we used that as $\nu - \mu^n$ has zero mass to write

$$\int \ln|x-y| d(\nu - \mu^n)(x)(\nu - \mu^n)(y) = -D^2(\nu, \mu^n)\,.$$

We then take ν a measure with filling fractions m and since μ^m minimizes $\mathcal{E}$ among such measures,

$$\mathcal{E}(\mu^m) \le \mathcal{E}(\nu)\,. \tag{6.5}$$

We choose ν to have the same support than μ^n so that $\int V_{\text{eff}}^n(x) d(\nu - \mu^n)(x) = 0$ and notice that $\int V_{\text{eff}}^n(x) d(\mu^m - \mu^n)(x) \ge 0$. Hence, we deduce from (6.4) and (6.5) that

$$D^2(\mu^m, \mu^n) \le D^2(\nu, \mu^n)\,.$$

Finally we choose $\nu = \mu^n + \sum_i (m_i - n_i) \frac{1_{B_i}}{|B_i|} dx$ with B_i is an interval in the support of μ_i^n where its density is bounded below by some fixed value. For $\max |m_i - n_i|$ small enough it is a probability measure. Then, it is easy to check that

$$D^2(\mu^m, \mu^n) \le D^2(\mu, \mu^n) \le C\|m - n\|_\infty^2$$

from which the conclusion follows from (4.15).

Next, we use the Dyson-Schwinger equation with the test function $f(x) = (z-x)^{-1}$ to deduce that $G_i^n(z) = \int (z-x)^{-1} d\mu_i^n(x)$ satisfies the equation

$$G_i^n(z) \left(\sum n_j G_j^n(z) \right) = \int \frac{V'(x)}{z-x} d\mu_i^n(x) = V'(z) G_i^n(z) + f_i^n(z)$$

where $f_i^n(z) = -\int (V'(y) - V'(z))(y-z)^{-1} d\mu_i^n(y)$. Hence we deduce that

$$G_i^n(z) = \frac{1}{2n_i} \left(V'(z) - \sum_{j \ne i} n_j G_j^n(z) - \sqrt{(V'(z) - \sum_{j \ne i} n_j G_j^n(z))^2 - 4 n_i f_i^n(z)} \right) .$$

The imaginary part of G_i^n gives the density of μ_i^n in the limit where z goes to the real axis. Since the first term in the above right hand side is obviously real, the latter is given by the square root term and therefore we want to show that

$$F(z, n) = \left(V'(z) - \sum_{j \ne i} n_j G_j^n(z) \right)^2 - 4 n_i f_i^n(z)$$

vanishes only at two points a_i^n, b_i^n for $z \in S_i$. The previous point shows that F is Lipschitz in the filling fraction n as V is C^3 (since then f_n^i is the integral of a C^1 function under μ_i^n) whereas Assumption 6.1 implies that at $n_i^* = \mu_V(S_i)$, F vanishes at only two points and has non-vanishing derivative at these points. This implies that the points where $F(z, n)$ vanishes in S_i are at distance of order at most $\max |n_i - m_i|$ of a_i, b_i. However, to guarantee that there are exactly two such points, we use the analyticity of V which guarantees that $F(., n)$ is analytic for all n so that we can apply Rouché theorem. As $F(z, n^*)$ does not vanish on the boundary of some compact neighborhood K of a_i, for n close enough to n^*, we have $|F(z, n) - F(z, n^*)| \le |F(z, n^*)|$ for $z \in \partial K$. This guarantees by Rouché's theorem, since $F(., n)$ is analytic in neighborhood of S_i as V is, that $F(., n)$ and $F(., n^*)$ have the same number of zeroes inside K. $\diamond$

To apply the method of Section 4, we can again use the Dyson-Schwinger equations and in fact Lemma 4.17 still holds true : Let $f_i : \mathbb{R} \to \mathbb{R}$ be C_b^1 functions, $0 \le i \le p$. Then, taking the expectation under $P_{N,\hat{n}}^{\beta,V}$, we deduce

$$\begin{aligned}
\mathbb{E}[M_N(\Xi f_0) \prod_{i=1}^p N\hat{\mu}_N(f_i)] &= (\frac{1}{\beta} - \frac{1}{2}) \mathbb{E}[\hat{\mu}_N(f_0') \prod_{i=1}^p N\hat{\mu}_N(f_i)] \\
&+ \frac{1}{\beta} \sum_{\ell=1}^p \mathbb{E}[\hat{\mu}_N(f_0 f_\ell') \prod_{i \ne \ell} N\hat{\mu}_N(f_i)] \\
&+ \frac{1}{2} \mathbb{E}[\int \frac{f_0(x) - f_0(y)}{x-y} dM_N(x) dM_N(y) \prod_{i=1}^p N\hat{\mu}_N(f_i)] \\
&+ O(e^{-cN})
\end{aligned}$$

where the last term comes from the boundary terms which are exponentially small by the large deviations estimates of Theorem 4.8. We still denoted $M_N(f) = \sum f(\lambda_i) - N\sum \hat{n}_i \mu_i^{\hat{n}}(f)$ but this time the mass in each S_i is fixed so this quantity is unchanged if we change f by adding a piecewise constant function on the S_i's. We therefore have this time to find for any sufficiently smooth function g a function f such that there are constants C_j so that

$$\Xi^{\hat{n}} f(x) = V'(x)f(x) - \sum_{i=1}^{p} \hat{n}_i \int \frac{f(x)-f(y)}{x-y} d\mu_i^{\hat{n}}(y) = g(x) + C_j,\, x \in S_j\,.$$

By the characterization of $\mu^{\hat{n}}$, if $S_j^{\hat{n}} = [a_j^{\hat{n}}, b_j^{\hat{n}}]$ denotes the support of $\mu^{\hat{n}}$ inside S_ε^i, this question is equivalent to find f so that on every $[a_j^{\hat{n}}, b_j^{\hat{n}}]$,

$$\Xi^{\hat{n}} f(x) := PV \int \frac{f(y)}{x-y} H_j^{\hat{n}}(y)\sqrt{(y-a_j^{\hat{n}})(b_j^{\hat{n}}-y)}dy = g(x) + C_j$$

This question was solved in [**76**] under the condition that g, f are Hölder with some positive exponent. Once one gets existence of these functions, the property of the inverse are the same as before since inverting the operator on one S_i will correspond to the same inversion. For later use, we prove a slightly stronger statement :

LEMMA 6.5. *Let $\theta \in [0,1]$ and set for $n_i \in (0,1), \sum n_i = 1$. Let S_i^n denote the support of μ_i^n. We set, for $i \in \{1,\dots,K\}$, all $x \in S_i^n$*

$$\Xi_\theta^n f(x) := V'(x)f(x) - n_i \int \frac{f(x)-f(y)}{x-y} d\mu_i^n(y) + \theta \sum_{j\neq i} n_j \int \frac{f(x)-f(y)}{x-y} d\mu_j^n(y)\,.$$

Then for all $g \in C^k$, $k > 2$, there exist constants $C_j, 1 \le j \le p$, so that the equation

$$\Xi_\theta^n f(x) = g(x) + C_j, x \in S_j^n$$

has a unique solution which is Hölder for some exponent $\alpha > 0$. We denote by $(\Xi_\theta^n)^{-1}g$ this solution. There exists finite constant D_j such that

$$\|(\Xi_\theta^n)^{-1} g\|_{C^j} \le D_j \|g\|_{C^{j-2}}\,.$$

PROOF. Let us first recall the result from [**76**, section 90] which solves the case $\theta = 1$. Let $S_k^n = [a_k^n, b_k^n]$. Because of the characterization of the equilibrium measure, inverting Ξ_1 is equivalent to seek for f Hölder such that there are K constants $(C_k)_{1\le k\le K}$ such that on S_k

$$K_1 f(t) = PV \int_{\cup S_k} \frac{f(x)}{t-x} dx = g(t) + C_k$$

for all $k \in \{1,\dots,K\}$. Then, by [**76**, section 90], if g is Hölder, there exists a unique solution and it is given by

$$K_1^{-1} g(x) := f(x) = \frac{\sigma(x)}{\pi} \sum_k PV \int_{S_k} \frac{dy}{\sigma(y)} \frac{1}{y-x}(g(y) + C_k)$$

where $\sigma(x) = \sqrt{\prod (x-a_i^n)(x-b_i^n)}$. The proof shows uniqueness and then exhibits a solution. To prove uniqueness we must show that $K_1 f = C_k$ has a unique solution, namely zero. To do so one remarks that

$$\Phi(z) = \int_{\cup S_k} \frac{f(x)}{x-z} dx$$

is such that $\Psi(z) = (\Phi(z) - C_k/2)\sqrt{(x-a_k^n)(x-b_k^n)}$ is holomorphic in a neighborhood of S_k^n and vanishes at a_k^n, b_k^n. Indeed, $K_1 f = C_k$ is equivalent to

$\Phi^+(x) + \Phi^-(x) = C_k$ implies that $\Psi^+(x) = \Psi^-(x)$ on the cuts. Hence

$$\Phi(z) - C_k/2 = [(z-a_k^n)(z-b_k^n)]^{1/2}\Omega(z) \tag{6.6}$$

with Ω holomorphic in a neighborhood of S_k^n, and so $\Phi'(z)\sigma(z)$ is holomorphic everywhere. Hence, since Φ' goes to zero at infinity like $1/z^2$, $P(z) = \Phi'(z)\sigma(z)$ is a polynomial of degree at most $K-2$. We claim that this is a contradiction with the fact that then the periods of Φ vanish, see [**38**, Section II.1] for details. Let us roughly sketch the idea. Indeed, because $\Phi = u + iv$ is analytic outside the cuts, if $\Lambda = \cup\Lambda_k$ is a set of contours surrounding the cuts and Λ^c the part of the imaginary plan outside Λ, we have by Stockes theorem

$$J = \int_{\Lambda^c} \left((\partial_x u)^2 + (\partial_y u)^2\right) dxdy = \int_{\Lambda} u d\bar{v}$$

Letting Λ going to S we find

$$\int_{\Lambda} u d\bar{v} = \int_S u^+ dv^+ - \int_S u^- dv^-$$

But by the condition $\Phi^+ + \Phi^- = C_k$ we see that $u^+ + u^- = \Re(C_k), d(v^+ + v^-) = 0$ and hence

$$J = \sum_k \Re(C_k)\int_{S_k} dv^+ = \sum_k \Re(C_k)(v^+(b_k^n) - v^+(a_k^n))\,.$$

On the other hand $\Phi(z) = \int_{-\infty}^z P(\xi)/\sigma(\xi)d\xi$ for any path avoiding the cuts and hence converges towards finite values on the cuts. But since $\Phi'(\xi) = \frac{P(\xi)}{\sigma(x)}$ is analytic outside the cuts, going to zero like $1/z^2$ at infinity,

$$0 = \int_{\Lambda_k} \Phi'(\xi)d\xi = 2\int_{S_k^n} \Phi'(x)dx = 2(\Phi(b_k^n) - \Phi(a_k^n))$$

Thus, $v(b_k^n) - v(a_k^n) = 0$ and we conclude that $J = 0$. Therefore Φ vanishes, and so does f.

Next, we consider the general case $\theta \in (0,1)$. We show that Ξ_θ^n is injective on the space of Hölder functions. Again, it is sufficient to consider the homogeneous equation

$$K_\theta f(x) = (1-\theta)K_0 f(x) + \theta K_1 f(x) = C_k \tag{6.7}$$

on S_k for all k. Here $K_0 f(x) = \int_{S_k} \frac{f(y)}{y-x}dy$ on S_k for all k. If K_θ is injective, so is Ξ_θ^n by dividing the function f on S_k by $\sigma_k(x)S_k(x) = d\mu_k^n/dx$. Recall that Tricomi airfol equation shows that K_0 is invertible, see Lemma 4.18, and we have just seen that K_1 is injective. To see that K_θ is still injective for $\theta \in [0,1]$ we notice that we can invert K_1 to deduce that we seek for an Hölder function f $(= K_1 g)$ and a piecewise constant function C so that

$$f(x) = -\frac{1-\theta}{\theta}K_1^{-1}(K_0 f - C)$$

Let us consider this equation for $x \in S_k$ and put $f = K_0^{-1}g$. By the formula for K_1^{-1} and K_0^{-1} we deduce that we seek for constants d, D and a function g so that on S_k :

$$\frac{1}{\sigma_k(x)} PV \int_{S_k} \frac{g(y)+d_k}{x-y} \sigma_k(y) dy = -\frac{1-\theta}{\theta} \frac{1}{\sigma(x)} \sum_\ell PV \int_{S_\ell} \frac{g(y)+D_\ell}{y-x} \sigma(y) dy \,.$$

Here, we used a formula for K_0^{-1} where σ_k was replaced by σ_k^{-1} : this alternative formula is due to Parseval formula [**89**, (2) p.174], see (16) and (18) in [**89**]. Note here that both side vanish at the end points of S_k by the choices of the constants. As a consequence

$$\frac{1}{\sigma_k(x)} \int_{S_k} \frac{g(y)+d_k}{x-y} \sigma_k(y) dy + \frac{1-\theta}{\theta} \frac{1}{\sigma(x)} \sum_\ell \int_{S_\ell} \frac{g(y)+D_\ell}{y-x} \sigma(y) dy$$

is analytic in a neighborhood of S_k.We next integrate over a contour $\mathcal{C}_k$ around S_k to deduce that

$$\begin{aligned}
\int_{S_k} \frac{g(y)+d_k}{x-y} \sigma_k(y) dy &= \frac{1}{2\pi i} \int_{\mathcal{C}_k} \frac{dz}{z-x} \int_{S_k} \frac{g(y)+d_k}{z-y} \sigma_k(y) dy \\
&= -\frac{1-\theta}{\theta} \frac{1}{2\pi i} \int_{\mathcal{C}_k} \frac{dz}{z-x} \frac{\sigma_k(z)}{\sigma(z)} \sum_\ell \int_{S_\ell} \frac{g(y)+D_\ell}{y-z} \sigma(y) dy \\
&= -\frac{1-\theta}{\theta} \int_{S_k} \frac{\sigma_k(y)}{\sigma(y)} \frac{g(y)+D_k}{x-y} \sigma(y) dy \\
&= -\frac{1-\theta}{\theta} \int_{S_k} \frac{g(y)+D_k}{x-y} \sigma_k(y) dy
\end{aligned}$$

where we used that σ_k/σ is analytic in a neighborhood of S_k, as well as the terms coming from the other cuts. Hence we seek for g satisfying

$$\frac{1}{\theta} \int_{S_k} \frac{g(y)+d_k}{x-y} \sigma_k(y) dy = 0$$

for some constant d_k. Tricomi airfol equation shows that this equation has a unique solution which is when $g+d_k$ is a multiple of $1/(\sigma_k)^2$. By our smoothness assumption on g, we deduce that $g+d_k$ must vanish. This implies that $f = K_0^{-1} g$ vanishes by Tricomi. Hence, we conclude that K_θ, and therefore Ξ_θ^n is injective on the space of Hölder continuous functions.

To show that Ξ_θ^n is surjective, it is enough to show that it is surjective when composed with the inverse of the single cut operators $\Xi^n = (\Xi^{n_1}, \ldots, \Xi^{n_p})$, that is that the application given for $x \in S_i = [a_i^n, b_i^n]$ by

$$L_\theta f(x) := n_i f(x) + \theta R f(x), R f(x) = \sum_{j \neq i} n_j \int \frac{(\Xi_j^n)^{-1} f(y)}{x-y} d\mu_j^n(y)$$

is surjective. But R is a kernel operator and in fact it is Hilbert-Schmidt in $L^2(\sigma^{-\epsilon} dx)$ for any $\epsilon > 0$ (here $\sigma(x) = \prod \sqrt{(x-a_i^n)(b_i^n - x)}$). Indeed, on $x \in S_i$, R is a sum of terms of the form

$$\int \frac{(\Xi_j^n)^{-1} f(y)}{x-y} d\mu_j^n(y) = \int \frac{1}{x-y} \frac{1}{S_j(y)} PV \left(\int_{a_j^n}^{b_j^n} \frac{f(t)}{(y-t)} \sigma_j(t) dt \right) d\mu_j^n(y)$$

by Remark 4.19. Even though we have a principal value inside the (smooth) integral we can apply Fubini and notice that

$$\begin{aligned} PV\int \frac{1}{(x-y)(y-t)}\frac{1}{S_j(y)}d\mu_j^n(y) &= \frac{1}{x-t}PV\int\left(\frac{1}{(x-y)}+\frac{1}{(y-t)}\right)d\sigma_j(y) \\ &= 1-\frac{1}{x-t}\sigma_j(x) \end{aligned}$$

where we used that t belongs to S_j but not x to compute the Hilbert transport of σ_j at t and x. Hence, the above term yields

$$\int \frac{(\Xi_j^n)^{-1}f(y)}{x-y}d\mu_j^n(y) = \int_{S_j}\frac{f(t)}{\sigma_j(t)}(1-\frac{1}{x-t}\sigma_j(x))dt, x\in S_i$$

from which it follows that R is a Hilbert-Schmidt operator in $L^2(\sigma^{-\epsilon}dx)$. Hence, R is a compact operator in $L^2(\sigma^{-\epsilon}dx)$. But L_θ is injective in this space. Indeed, for $f\in L^2(\sigma^{-\epsilon}dx)$, $L_\theta f=0$ implies that $f=\theta n_i^{-1}Rf$ is analytic. Writing back $h=(\Xi^n)^{-1}f$, we deduce that $\Xi_\theta^n h=0$ with h Hölder, hence h must vanish by the previous consideration. Hence L_θ is injective. Therefore, by the Fredholm alternative, L_θ is surjective. Hence L_θ is a bijection on $L^2(\sigma^{-\epsilon}dx)$. But note that the above identity shows that R maps $L^2(\sigma^{-\epsilon})$ onto analytic functions, therefore K^{-1} maps Hölder functions with exponent α onto Hölder functions with exponent α. We thus conclude that $\Xi_\theta^n = L_\theta\circ\Xi^n$ is invertible onto the space of Hölder functions. We also see that the inverse has the announced property since for $x\in [a_j^n,b_j^n]$

$$(\Xi_\theta^n)^{-1}g(x)=\Xi^{-1}[g-h],\quad h(x)=\theta\sum_{j\neq i}n_j\int\frac{(\Xi_\theta^n)^{-1}g(y)}{x-y}d\mu_j^n(y)]$$

where h is C^∞. The announced bound follows readily from the bound on one cut as on L^k we have

$$(\Xi_\theta^n)^{-1}f(x)=(\Xi_0^n)^{-1}(f-\theta\sum_{\ell\neq k}\int\frac{(\Xi^{n_\theta})^{-1}f}{x-y}d\mu_\ell^n(y))$$

is such that

$$\|(\Xi_\theta^n)^{-1}f\|_{C^s}\le c_s\|f-\theta\sum_{\ell\neq k}\int\frac{(\Xi^{n_\theta})^{-1}f}{x-y}d\mu_\ell^n(y))\|_{C^{s+2}}\le \tilde{c}_s(\|f\|_{C^{s+2}}+\|(\Xi_\theta^n)^{-1}f\|_\infty)\,.$$

$\diamond$

As Ξ_1^n is invertible with bounded inverse we can apply exactly the same strategy as in the one cut case to prove the central limit theorem :

THEOREM 6.6. *Assume V is analytic and the previous hypotheses hold true. Then there exists $\epsilon>0$ so that for $\max|n_i-\mu([a_i,b_i])|\le\epsilon$, for any f C^k with $k\ge 11$, the random variable $M_N(f):=\sum_{i=1}^N f(\lambda_i)-N\mu^n(f)$ converges in law under $P_{N,n}^{\beta,V}$ towards a Gaussian variable with mean $m_V^n(f)$ and covariance $C_V^n(f,f)$, which are defined as in Theorem 4.27 but with μ^n instead of μ and Ξ^n instead of Ξ.*

We can also obtain the expansion for the partition function

THEOREM 6.7. *Assume V is analytic and the previous hypotheses hold true. Then there exists $\epsilon > 0$ so that for $\max|\hat{n}_i - \mu([a_i, b_i])| \le \epsilon$, $\hat{n}_i = N_i/N$, we have*

$$\ln\left(\frac{N!}{(\hat{n}_1 N)!\cdots(\hat{n}_K N)!} Z^{N,\hat{n}}_{\beta,V}\right) = C^0_\beta N \ln N + C^1_\beta \ln(N)$$

$$+N^2 F^{\hat{n}}_0(V) + N F^{\hat{n}}_1(V) + F^{\hat{n}}_2(V) + o(1) \tag{6.8}$$

with $C^0_\beta = \frac{\beta}{2}$, $C^1_\beta = -(K-1)/2 + \frac{3+\beta/2+2/\beta}{12}$ and for $n_i > 0$, $\sum n_i = 1$,

$$\begin{aligned} F^n_0(V) &= -\mathcal{E}(\mu^n_V) \\ F^n_1(V) &= \left(\frac{\beta}{2}-1\right)\int \ln\left(\frac{d\mu^n_V}{dx}\right) d\mu^n_V - \frac{\beta}{2} n_i \ln n_i + f_1 \end{aligned}$$

where f_1 depends only on the boundary points of the support. $F^n_2(V)$ is a continuous function of n. Above the error term is uniform on n in a neighborhood of n^*.

PROOF. The proof is again by interpolation. We first remove the interaction between cuts by introducing for $\theta \in [0,1]$

$$\mathrm{d}P^{\beta,\theta,V}_{N,\hat{n}}(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z^{\beta,\theta,V}_{N,\hat{n}}} \prod_{h\neq h'} e^{N^2 \frac{\beta}{2}\theta \int \ln|x-y| d(\hat{\mu}^N_h - \mu^{\hat{n}}_h)(x) d(\hat{\mu}^N_{h'} - \mu^{\hat{n}}_{h'})(y)} \prod dP^{\beta, V^{\hat{n}}_{eff}}_{N,\hat{n}_h}$$

where $P^{\beta,V^{\hat{n}}_{eff}}_{N,\hat{n}_h}$ is the β ensemble on S_h with potential given by the effective potential. We still have a similar large deviation principle for the $\hat{\mu}^N_h$ under $P^{\beta,\theta,V}_{N,\hat{n}}$ and the minimizer of the rate function is always $\mu^{\hat{n}}_h$. Hence we are always in a off critical situation. Moreover, we can write the Dyson-Schwinger equations for this model : it is easy to see that the master operator is Ξ^n_θ of Lemma 6.5 which we have proved to be invertible. Therefore, we deduce that the covariance and the mean of linear statistics are in a small neighborhood of $C^{\theta,\hat{n}}_V$ and $m^{\theta,\hat{n}}_V$. It is not hard to see that this convergence is uniform in θ.

Hence, we can proceed and compute

$$\ln \frac{Z^{\beta,1,V}_{N,\hat{n}}}{Z^{\beta,0,V}_{N,\hat{n}}} = N^2 \int_0^1 P^{\beta,\theta,V}_{N,\hat{n}}\left(\sum_{h<h'} \beta \int \ln|x-y| d(\hat{\mu}^N_h - \mu^{\hat{n}}_h)(x) d(\hat{\mu}^N_{h'} - \mu^{\hat{n}}_{h'})(y)\right) d\theta$$

Indeed, using the Fourier transform of the logarithm we have

$$\begin{aligned} &N^2 P^{\beta,\theta,V}_{N,\hat{n}}\left((\int \ln|x-y| d(\hat{\mu}^N_h - \mu^{\hat{n}}_h)(x) d(\hat{\mu}^N_{h'} - \mu^{\hat{n}}_{h'})(y)\right) \\ &\quad = \int \frac{1}{t} P^{\beta,\theta,V}_{N,\hat{n}}\left((N\int e^{itx} d(\hat{\mu}^N_h - \mu^{\hat{n}}_h)(x) (N \int e^{-ity} d(\hat{\mu}^N_{h'} - \mu^{\hat{n}}_{h'})(y))\right) dt \end{aligned}$$

where the above RHS is close to

$$[C^{\theta,\hat{n}}_V(e^{it\cdot}, e^{-it\cdot}) + |m^{\theta,\hat{n}}_V(e^{it\cdot})|^2]$$

Hence, decoupling the cuts in this way only provides a term of order one in the partition function. It is not hard to see that it will be a continuous function of the filling fraction (as the inverse of $\Xi^{\hat{n}}_\theta$ is uniformly continuous in n). Finally we can use the expansion of the one cut case of Theorem 4.28 to expand $Z^{\beta,0,V}_{N,\hat{n}}$ to conclude. ◇

6.2. Central limit theorem for the full model

To tackle the model with random filling fraction, we need to estimate the ratio of the partition functions according to (6.1). Recall that $n_i^* = \mu([a_i, b_i])$. We can now extend the definition of the partition function to non-rational values of the filling fractions by using Theorem 6.7. Then we have

THEOREM 6.8. *Under the previous hypotheses, for* $\max |n_i - n_i^*| \le \epsilon$*, there exists a positive definite form* Q *and a vector* v *such that*

$$\begin{aligned} D(n) &:= \frac{(Nn_1^*)! \cdots (Nn_K^*)!}{(Nn_1! \cdots (Nn_K)!} \frac{Z_{\beta,V}^{N,n}}{Z_{\beta,V}^{N,n^*}} \\ &= \exp\{-\frac{1}{2} Q(N(n-n^*))(1+O(\epsilon)) + \langle N(n-n^*), v\rangle + o(1)\}\,, \end{aligned}$$

where $Z_{\beta,V}^{N,n^*}/(Nn_1^*)! \cdots (Nn_K^*)!$ *is defined thanks to the expansion of* (6.8) *whenever* n^*N *takes non-integer values (note here the right hand side makes sense for any filling fraction* n*).* $O(\epsilon)$ *is bounded by* $C\epsilon$ *uniformly in* N*. We have* $Q = -D^2 F_0^n(V)|_{n=n^*}$ *and* $v_i = \partial_{n_i} F_1^n(V)|_{n=n^*}$*. As a consequence, since the probability that the filling fractions* $\hat{n}$ *are equal to* n *is proportional to* $D(n)$*, we deduce that the distribution of* $N(\hat{n} - n^*) - Q^{-1}v$ *is equivalent to a centered discrete Gaussian variable with values in* $-Nn^* - Q^{-1}v + \mathbb{Z}$ *and covariance* Q^{-1}*.*

Note here that Nn^* is not integer in general so that $N(\hat{n} - n^*) - Q^{-1}v$ does not live in a fixed space : this is why the distribution of $N(\hat{n}-n^*) - Q^{-1}v$ does not converge in general. As a corollary of the previous theorem, we immediatly have that

COROLLARY 6.9. Let f be C^{11}. Then

$$\begin{aligned} &\mathbb{E}_{P_{\beta,V}^N}[e^{\sum f(\lambda_i) - N\mu(f)}] = \exp\{\frac{1}{2} C_V^{n^*}(f,f) + m_V^{n^*}(f)\} \\ &\times \sum_n \frac{\exp\{-\frac{1}{2}Q(N(n-n^*)) + \langle N(n-n^*), v + \partial_n \mu^n|_{n=n^*}(f)\rangle\}}{\sum_n \exp\{-\frac{1}{2}Q(N(n-n^*)) + \langle N(n-n^*), v\rangle\}}(1+o(1)) \end{aligned}$$

We notice that we have a usual central limit theorem as soon as $\partial_n \mu^n|_{n=n^*}(f)$ vanishes (in which case the second term vanishes), but otherwise the discrete Gaussian variations of the filling fractions enter into the game. This term comes from the difference $N\mu(f) - N\mu^n(f)$.

As is easy to see, the last thing we need to show to prove these results is that

LEMMA 6.10. *Assume* V *analytic, off-critical. Then*

- $n \to \mu^n(f)$ *is* C^1 *and* $C_V^n(f,f), m_V^n(f)$ *are continuous in* n*,*
- $n \to F_i^n(V)$ *is* C^{2-i} *in a neighborhood of* n^**,*
- $Q = -D^2 F_0^n(V)|_{n=n^*}$ *is definite positive.*

Let us remark that this indeed implies Corollary 6.9 and Theorem 6.8 since by Theorem 6.7 we have for $|n_i - n_i^*| \le \epsilon$

$$
\begin{aligned}
\ln \frac{(Nn_1^*! \cdots (Nn_K^*)!}{(Nn_1)! \cdots (Nn_K)!} \frac{Z^{N,n}_{\beta,V}}{Z^{N,n^*}_{\beta,V}} &= N^2\{F_0^n(V) - F_0^{n^*}(V)\} + N(F_1^n(V) - F_1^{n^*}(V)) \\
&\quad + (F_2^n(V) - F_2^{n^*}(V)) + o(1) \\
&= -\frac{1}{2} Q(N(n-n^*), N(n-n^*))(1+O(\epsilon)) \\
&\quad + \partial_n F_1^n(V)|_{n=n^*}.(N(n-n^*))
\end{aligned}
$$

where we noticed that $\partial_n F_0^n(V)$ vanishes at n^* since n^* minimizes F_0^n and Q is definite positive. Hence we obtain the announced estimate on the partition function. About Corollary 6.9 we have by (6.1) and by conditionning on filling fractions

$$
\begin{aligned}
\mathbb{E}_{P^N_{\beta,V}}[e^{\sum f(\lambda_i) - N\mu_V(f)}] &= \mathbb{E}\left[e^{N(\mu^{\hat n}(f) - \mu^{n^*}(f))} \mathbb{E}_{P^{N,\hat n}_{\beta,V}}[e^{\sum f(\lambda_i) - N\mu^{\hat n}(f)}]\right] \\
&\simeq \mathbb{E}\left[e^{N\langle \hat n - n^*, \partial_n \mu^n(f)|_{n=n^*}\rangle} \mathbb{E}_{P^{N,\hat n}_{\beta,V}}[e^{\sum f(\lambda_i) - N\mu^{\hat n}(f)}]\right](1+o(1))
\end{aligned}
$$

So we only need to prove Lemma 6.10.

Proof. *$n \to \mu^n$ is twice continuously differentiable.* We have already seen in the proof of Lemma 6.4 that $n \to \mu^n$ is Lipschitz for the distance D for n in a neighborhood of n^*. This implies that $\nu_\epsilon = \epsilon^{-1}(\mu^{n+\epsilon\kappa} - \mu^n)$ is tight (for the distance D and hence the weak topology). Let us consider a limit point ν and its Stieltjes transform $G_\nu(z) = \int (z-x)^{-1} d\nu(x)$. Along this subsequence, the proof of Lemma 6.4 also shows that $\epsilon^{-1}(a_i^{n+\epsilon\kappa} - a_i^n)$ has a limit (and similarly for b^n, as well as H_i^n). Hence, we see that ν is absolutely continuous with respect to Lebesgue measure, with density blowing up at most like a square root at the boundary. By (6.3) in Theorem 6.3 we deduce that

$$G_\nu(E + i0) + G_\nu(E - i0) = 0$$

for all E inside the support of μ^n. This implies that $\sqrt{\prod (z-a_i^n)(b_i^n - z)}G_\nu(z)$ has no discontinuities in the cut, hence is analytic. Finally, G_ν goes to zero at infinity like $1/z^2$ so that $\sqrt{\prod (z-a_i^n)(b_i^n - z)}G_\nu(z)$ is a polynomial of degree at most $p-2$. Its coefficients are uniquely determined by the $p-1$ equations fixing the filling fractions since for a contour C_i^n around $[a_i^n, b_i^n]$

$$\int_{C_i^n} G_{\mu^n}(z)dz = n_i \Rightarrow \int_{C_i^n} G_\nu(z)dz = \kappa_i\,.$$

There is a unique solution to such equations. As it is linear in κ, it is given by

$$G_\nu(z) = \sum \kappa_i \omega_i^n(z) \tag{6.9}$$

where $\omega_i^n(z) = P_i^n(z)/\sqrt{\prod (z-a_i^n)(b_i^n - z)}$ satisfy

$$\int_{C_j^n} \omega_i^n(z)dz = \delta_{i,j} \tag{6.10}$$

and P_i^n are polynomials of degree smaller or equal than $p-2$. Hence G_ν is uniquely determined as well as ν, we conclude that $n \to \mu^n$ is differentiable, as well as a_i^n, b_i^n. The latter implies that $n \to \omega_i^n$ is as well differentiable and hence $n \to G_{\mu^n}$ is twice continuously differentiable. In turn, we conclude that a_i^n, b_i^n, H_i^n are twice continuously differentiable with respect to n, and therefore so is the density of μ^n.

$C_V^n(f,f), m_V^n(f)$ *are continuous in* n. From the continuity of $d\mu^n/dx$ we deduce that Ξ^n is continuous, and since Ξ^n has uniformly bounded inverse (provided we take sufficiently smooth functions), we deduce that $(\Xi^n)^{-1}$ is continuous in n, from which the continuity of $C_V^n(f,f), m_V^n(f)$ follows for smooth enough f.

$n \to F_i^n(V)$ *is* C^{2-i}, $i = 0,1,2$. For $i = 1$, by the formulas of Theorem 6.6, it is a straightforward consequence of the fact that $\mathrm{d}\mu^n/dx$ is continuously differentiable and its differential is integrable. It amounts to show that the inverse of the operators Ξ_θ^n are continuous in n, but again this is due to the continuity of the endpoints and the explicit formulas we have.

$D^2 F_0^n(V)$ *is well defined and definite negative at* $n = n^*$. Set

$$\nu_\eta = \lim_{t\to 0} \frac{\mu^{n+t\eta} - \mu^n}{t} \tag{6.11}$$

By the formula for J in terms of the effective potential

$$\begin{aligned} F_0^{n^*+t\eta}(V) - F_0^{n^*}(V) &= \left(J[\mu^{n^*+t\eta}] - J[\mu^{n^*}]\right) \\ &= -\frac{\beta}{2}\Big(D^2[\mu^{n^*+t\eta}, \mu^{n^*}] - \iint V_{\mathrm{e}ff}^{n^*}(x) d(\mu^{n^*+t\kappa} - \mu^{n^*})(x)\Big) \end{aligned}$$

where we used that at $n = n^*$ the constants in the effective potential are all equal and that $\sum \eta_i = 0$. Since $V_{\mathrm{e}ff}^{n^*}$ vanishes on $\cup[a_i, b_i]$ as well as its derivative and the derivatives of $\epsilon \to \mu_{n+\epsilon\eta}$ are smooth and supported in $\cup[a_i, b_i]$, we deduce that $F^0_{*n+t\eta}$ is a $\mathcal{C}^2$ function of t and its Hessian is :

$$\partial_t^2 F_0^{n^*+t\eta}|_{t=0} = -\frac{\beta}{2} D^2[\nu^*, \nu^*] \tag{6.12}$$

where $\nu^* = \partial_t \mu^{n^*+t\eta}|_{t=0}$. $D^2 F_0$ vanishes only when ν^* vanishes, which implies $\eta = 0$ by (6.9), since no non trivial combination of the ω_i^n can vanish uniformly by (6.10). Therefore, the Hessian is definite negative. $\diamond$

CHAPTER 7

Several matrix-ensembles

Topological expansions have been used a lot in physics to relate enumeration problems with random matrices. Considering several matrix models allows to deal with much more complicated combinatorial questions, that is colored maps. In this section we show how the previous arguments based on Dyson-Schwinger equations allow to study these models in perturbative situations. In fact, large deviations questions are still open in the several matrices case and convergence of the trace of several matrices has only been proved in general perturbative situations [**54**] or for very specific models such as the Ising model corresponding to a simple AB interaction [**45, 74, 72, 58, 59**].

7.0.0.1. *Non-commutative laws.* We let $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ denote the set of polynomials in m-non commutative indeterminates with complex coefficients. We equip it with the involution $*$ so that for any $i_1, \ldots, i_k \in \{1, \ldots, m\}$, for any complex number z, we have

$$(zX_{i_1} \cdots X_{i_k})^* = \bar{z} X_{i_k} \cdots X_{i_1}\,.$$

For $N \times N$ Hermitian matrices $(A_1, \cdots, A_m)$, let us define the linear form $\hat{\mu}_{A_1,\cdots,A_m}$ from $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ into $\mathbb{C}$ by

$$\hat{\mu}_{A_1,\cdots,A_m}(P) = \frac{1}{N} \mathrm{Tr}\left(P(A_1, \cdots, A_m)\right)$$

where Tr is the standard trace $\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii}$. If $A_1, \ldots, A_m$ are random, we denote by

$$\bar{\mu}_{A_1,\cdots,A_m}(P) := \mathbb{E}[\hat{\mu}_{A_1,\cdots,A_m}(P)].$$

$\hat{\mu}_{A_1,\cdots,A_m}, \bar{\mu}_{A_1,\cdots,A_m}$ will be seen as elements of the algebraic dual $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$. $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ is equipped with its weak topology.

Definition 7.1. A sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ converges weakly towards $\mu \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ iff for any $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\lim_{n\to\infty} \mu_n(P) = \mu(P).$$

Lemma 7.2. *Let C be a finite constant and n be an integer number. Set*

$$K_n(C) = \{\mu \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*; |\mu(X_{\ell_1} \cdots X_{\ell_r})| \le C^r \quad \forall \ell_i \in \{1, \cdots, m\}, r \in \mathbb{N}, r \le n\}.$$

Then, any sequence $(\mu_n)_{n\in\mathbb{N}}$ so that $\mu_n \in K_{m_n}(C)$ is sequentially compact if m_n goes to infinity with n, i.e. has a subsequence $(\mu_{\phi(n)})_{n\in\mathbb{N}}$ which converges weakly. We denote in short $K(C)$ or $K_\infty(C)$ the set of such sequances.

Proof. Since $\mu_n(X_{\ell_1} \cdots X_{\ell_r}) \in \mathbb{C}$ is uniformly bounded, it has converging subsequences. By a diagonalisation procedure, since the set of monomials is countable, we can ensure that for a subsequence $(\phi(n), n \in \mathbb{N})$, the terms

$\mu_{\phi(n)}(X_{\ell_1}\cdots X_{\ell_r}), \ell_i \in \{1,\cdots,m\}, r\in\mathbb{N}$ converge simultaneously. The limit defines an element of $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ by linearity. $\diamond$

The following is a triviality, that we however recall since we will use it several times.

Corollary 7.3. Let C be a finite non negative constant and m_n a sequence going to infinity at infinity. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence such that $\mu_n \in K_{m_n}(C)$ which has a unique limit point. Then $(\mu_n)_{n\in\mathbb{N}}$ converges towards this limit point.

Proof. Otherwise we could choose a subsequence which stays at positive distance of this limit point, but extracting again a converging subsequence gives a contradiction. Note as well that any limit point will belong automatically to $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$. $\diamond$

We shall call in these notes non-commutative laws elements of $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ which satisfy

$$\mu(PP^*)\ge 0,\quad \mu(PQ)=\mu(QP), \mu(1)=1$$

for all polynomial functions P,Q. This is a very weak point of view which however is sufficient for our purpose. The name 'law' at least is justified when $m=1$, in which case $\hat\mu^N$ is the empirical measure of the eigenvalues of the matrix A, and hence a probability measure on $\mathbb{R}$, whereas the non-commutativity is clear when $m\ge 2$. There are much deeper reasons for this name when considering C^*-algebras and positivity, and we refer the reader to [**93**] or [**3**].

The laws $\hat\mu_{A_1,\cdots,A_m}, \bar\mu_{A_1,\cdots,A_m}$ are obviously non-commutative laws. Since these conditions are closed for the weak topology, we see that any limit point of $\hat\mu^N, \overline{\mu}^N$ will as well satisfy these properties. A linear functional on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ which satisfies such conditions is called a tracial state. This leads to the notion of C^*-algebras and representations of the laws as moments of non-commutative operators on C^*-algebras. We however do not want to detail this point in these notes.

7.0.1. Non-commutative derivatives. First, for $1\le i\le m$, let us define the non-commutative derivatives ∂_i with respect to the variable X_i. They are linear maps from $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ to $\mathbb{C}\langle X_1,\cdots,X_m\rangle^{\otimes 2}$ given by the Leibniz rule

$$\partial_i PQ = \partial_i P\times(1\otimes Q)+(P\otimes 1)\times\partial_i Q$$

and $\partial_i X_j = \mathbf{1}_{i=j}1\otimes 1$. Here, $\times$ is the multiplication on $\mathbb{C}\langle X_1,\cdots,X_m\rangle^{\otimes 2}$; $P\otimes Q\times R\otimes S = PR\otimes QS$. So, for a monomial P, the following holds

$$\partial_i P = \sum_{P=RX_iS} R\otimes S$$

where the sum runs over all possible monomials R,S so that P decomposes into RX_iS. We can iterate the non-commutative derivatives; for instance $\partial_i^2 : \mathbb{C}\langle X_1,\cdots,X_m\rangle\to\mathbb{C}\langle X_1,\cdots,X_m\rangle\otimes\mathbb{C}\langle X_1,\cdots,X_m\rangle\otimes\mathbb{C}\langle X_1,\cdots,X_m\rangle$ is given for a monomial function P by

$$\partial_i^2 P = 2\sum_{P=RX_iSX_iQ} R\otimes S\otimes Q.$$

We denote by $\sharp : \mathbb{C}\langle X_1,\cdots,X_m\rangle^{\otimes 2}\times\mathbb{C}\langle X_1,\cdots,X_m\rangle\to\mathbb{C}\langle X_1,\cdots,X_m\rangle$ the map $P\otimes Q\sharp R = PRQ$ and generalize this notation to $P\otimes Q\otimes R\sharp(S,V) = PSQVR$. So $\partial_i P\sharp R$ corresponds to the derivative of P with respect to X_i in the direction R, and similarly $2^{-1}[\partial_i^2 P\sharp(R,S)+\partial_i^2 P\sharp(S,R)]$ the second derivative of P with respect to X_i in the directions R,S.

We also define the so-called cyclic derivative D_i. If m is the map $m(A \otimes B) = BA$, we define $D_i = m \circ \partial_i$. For a monomial P, $D_i P$ can be expressed as

$$D_i P = \sum_{P = R X_i S} S R.$$

7.0.2. Non-commutative Dyson -Schwinger equations. Let $X_1^N, \ldots, X_m^N$ be m independent GUE matrices and set $\hat\mu^N = \hat\mu_{X_1^N, \ldots, X_1^N}$ to be their non-commutative law. Let $P_0, \cdots, P_r$ be r polynomials in k non-commutative variables. Then for all $i \in \{1, \ldots, m\}$

$$\begin{aligned} \mathbb{E}[\hat\mu^N(X_i P_0) \prod_{j=1}^r \hat\mu^N(P_j)] &= \mathbb{E}[\hat\mu^N \otimes \hat\mu^N(\partial_i P_0) \prod_{j=1}^r \hat\mu^N(P_j)] \\ &+ \frac{1}{N^2} \sum_{j'=1}^r \mathbb{E}[\hat\mu^N(P_0 D_i P_{j'}) \prod_{j \neq j'} \hat\mu^N(P_j)] \end{aligned} \tag{7.1}$$

The proof is a direct application of integration by parts and is left to the reader. The main point is that our definitions yield

$$\partial_{X_{ij}^k} \operatorname{Tr}(P(X)) = (D_k P)_{ji}, \quad \partial_{X_{ij}^k} (P(X))_{i'j'} = (\partial_k P)_{i'i, jj'} .$$

7.0.3. Independent GUE matrices.

7.0.3.1. *Voiculescu's theorem.* The aim of this section is to prove that if $X^{N,\ell}, 1 \leq \ell \leq k$ are independent GUE matrices

THEOREM 7.4. *[Voiculescu* **[92]***] For any monomial q in the unknowns $X_1, \ldots, X_m$,*

$$\lim_{N \to \infty} \mathbb{E}[\frac{1}{N} \operatorname{Tr}\left(q(X_1^N, X_2^N, \ldots, X_m^N)\right)] = \sigma^m(q)$$

where $\sigma^m(q)$ is the number $\mathcal{M}_0(q)$ of planar maps build on a star of type q.

REMARK 7.5. σ^m, once extended by linearity to all polynomials, is called the law of m free semi-circular variables because it is the unique non-commutative law so that the moments of a single variable are given by the Catalan numbers satisfying

$$\sigma^m\left((X_{\ell_1}^{m_1} - \sigma(x^{\ell_1})) \cdots (X_{\ell_p}^{m_p} - \sigma(x^{\ell_p}))\right) = 0 ,$$

for any choice of $\ell_j, 1 \leq j \leq p$, such that $\ell_p \neq \ell_{p+1}$.

PROOF. By the non-commutative Dyson-Schwinger equation with $P_j = 1$ for $j \geq 1$, we have for all i

$$\mathbb{E}[\hat\mu^N(X_i X_{\ell_1} \cdots X_{\ell_k})] = \sum_{j : \ell_j = i} \mathbb{E}[\hat\mu^N(X_{\ell_1} \cdots X_{\ell_{j-1}}) \hat\mu^N(X_{\ell_{j+1}} \cdots X_{\ell_k})]$$

Let us assume that for all $k \leq K$ there exists C_K finite such that for any $\ell_1, \ldots, \ell_k \in \{1, \ldots, m\}$ so that $\sum \ell_i \leq K$

$$|\mathbb{E}[\hat\mu^N(X_{\ell_1} \cdots X_{\ell_k})]| \leq C_k \tag{7.2}$$

$$\mathbb{E}[\left(\hat\mu^N\left(X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k}\right) - \mathbb{E}[\hat\mu^N\left(X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k}\right)]\right)^2] \leq C_k / N^2 \tag{7.3}$$

Then we deduce that the family $\mathbb{E}[\hat\mu^N\left(X_{\ell_1} X_{\ell_2} \cdots X_{\ell_k}\right)]$ is tight and its limit points $\tau(X_{\ell_1} \cdots X_{\ell_k})$ satisfy

$$\tau(X_{\ell_1} \cdots X_{\ell_k}) = \sum_{j : \ell_j = \ell_1} \tau(X_{\ell_2} \cdots X_{\ell_{j-1}}) \tau(X_{\ell_{j+1}} \cdots X_{\ell_k})$$

and $\tau(1) = 1$, $\tau(X_\ell) = 0$. There is a unique solution to this equation. It is given by $\{\mathcal{M}_0(X_{\ell_1} \cdots X_{\ell_k}, 1), \ell_i \in \{1, \ldots, m\}\}$ since the later satisfies the same equation. Indeed, it is easily seen that the number of planar maps on a trivial star 1 can be taken to be equal to one, and there is none with a star with only one half-edge. Moreover, the number $\mathcal{M}_0(X_{\ell_1} \cdots X_{\ell_k}, 1)$ of planar maps build on $X_{\ell_1} \cdots X_{\ell_k}$ can be decomposed according to the matching of the half-edge of the root. Because the maps are planar, such a matching cut the planar map in to independent planar maps. Hence

$$\mathcal{M}_0(X_{\ell_1} \cdots X_{\ell_k}, 1) = \sum_{j:\ell_j = \ell_1} \mathcal{M}_0(X_{\ell_2} \cdots X_{\ell_{j-1}}, 1)\mathcal{M}_0(X_{\ell_{j+1}} \cdots X_{\ell_k}, 1)\,.$$

The proof of (7.2) is a direct consequence of non-commutative Hölder inequality and the bound obtained in the first chapter for one matrix. We leave (7.3) to the reader : it can be proved by induction over K using the Dyson-Schwinger equation exactly as in the one matrix case, see Lemma 2.3.

$\diamond$

7.0.3.2. *Central limit theorem.*

THEOREM 7.6. *Let $P_1, \ldots, P_r$ be polynomial in $X_1, \ldots, X_m$ and set $Y(P) = N(\hat{\mu}^N(P) - \sigma^m(P))$. Then $(Y(P_1), \ldots, Y(P_k))$ converges towards a centered Gaussian vector with covariance*

$$C(P_1, P_2) = \sum_{i=1}^m \sigma^m(D_i \Xi^{-1} P_1 D_i P_2)\,,$$

with $\Xi P = \sum_i [\partial_i P \# X_i - (\sigma^m \otimes I + I \otimes \sigma^m)(\partial_i D_i P)]$.

Notice above that Ξ is invertible on the space of polynomials with null constant term. Indeed, for any monomial q, the first part of Ξ is the degree operator

$$\sum_i \partial_i q \# X_i = \deg(q) q$$

whereas the second part reduces the degree, so that the sum is invertible.

PROOF. The proof is the same as for one matrix and proceeds by induction based on (7.1). We first observe that $m_N(P) = \mathbb{E}[\hat{\mu}^N(P)] - \sigma(P)$ is of order $1/N^2$ by induction over the degree of P thanks to (7.2) and (7.3). We then show the convergence of the covariance thanks to the Dyson-Schwinger equation (7.1) with $r = 1$ and $P_1 = P - \mathbb{E}[P]$, and $P_0 = D_i P$, which yields after summation over i :

$$N^2 \mathbb{E}[\hat{\mu}^N(\Xi P_0)(\hat{\mu}^N(P) - \mathbb{E}[\hat{\mu}^N(P)])] = \sum_{j=1}^m \mathbb{E}[\hat{\mu}^N(D_i P_0 D_i P)] + N^2 R_N(P)$$

where

$$R_N(P) = \sum_i \mathbb{E}[(\hat{\mu}^N - \sigma_m)^{\otimes 2}(\partial_i \circ D_i P_0)\hat{\mu}^N(P)]$$

Since P is centered, this is of order at most $1/N^3$ by (7.2) and (7.3). Hence, letting N going to infinity and inverting Ξ shows the convergence of the covariance towards C.

Finally, to prove the central limit theorem we deduce from (7.1) that, if $Y(P) = N(\hat{\mu}^N(P) - \sigma^m(P))$, we have

$$\begin{aligned} G_N(P, P_1, \ldots, P_r) &= \mathbb{E}[N(\hat{\mu}^N - \sigma^m)(P) \prod_{j=1}^r Y(P_j)] \\ &= N\mathbb{E}[(\hat{\mu}^N - \sigma^m) \otimes (\hat{\mu}^N - \sigma^m)(\sum_i \partial_i D_i \Xi^{-1} P) \prod_{i=1}^r Y(P_i)] \\ &\quad + \sum_{j=1}^r \mathbb{E}[\sum_{i=1}^m \hat{\mu}^N(D_i \Xi^{-1} P D_i P_j) \prod_{\ell \neq j} Y(P_\ell)] \,. \end{aligned}$$

By induction over the total degree of the P_i's, and using the previous estimate, we can show that the first term goes to zero. Hence, we deduce by induction that $G_N(P, P_1, \ldots, P_r)$ converges towards $G(P, P_1, \ldots, P_r)$ solution of

$$G(P, P_1, \ldots, P_r) = \sum_{j=1}^r \sigma^m(D_i \Xi^{-1} P D_i P_j) G(P, P_1, \ldots, P_{j-1}, P_{j+1}, \ldots P_r) \,,$$

which is Wick formula for Gaussian moments. $\diamond$

7.0.4. Several interacting matrices models. In this section, we shall be interested in laws of interacting matrices of the form

$$d\mu_V^N(X_1, \cdots, X_m) := \frac{1}{Z_V^N} e^{-N\mathrm{Tr}(V(X_1, \cdots, X_m))} d\mu^N(X_1) \cdots d\mu^N(X_m)$$

where Z_V^N is the normalizing constant

$$Z_V^N = \int e^{-N\mathrm{Tr}(V(X_1, \cdots, X_m))} d\mu^N(X_1) \cdots d\mu^N(X_m)$$

and V is a polynomial in m non-commutative unknowns. In the sequel, we fix n monomials q_i non-commutative monomials;

$$q_i(X_1, \cdots, X_m) = X_{j_1^i} \cdots X_{j_{r_i}^i}$$

for some $j_l^k \in \{1, \cdots, m\}$, $r_i \geq 1$, and consider the potential given by

$$V_{\mathbf{t}}(X_1, \cdots, X_m) = \sum_{i=1}^n t_i q_i(X_1, \cdots, X_m)$$

where $\mathbf{t} = (t_1, \ldots, t_n)$ are n complex numbers such that $V_{\mathbf{t}}$ is self-adjoint. Moreover, $d\mu^N(X)$ denotes the standard law of the **GUE**, that is

$$d\mu^N(X) = Z_N^{-1} 1_{X \in \mathcal{H}_N^{(2)}} e^{-\frac{N}{2}\mathrm{Tr}(X^2)} \prod_{1 \leq i \leq j \leq N} d\Re(X_{ij}) \prod_{1 \leq i < j \leq N} d\Im(X_{ij}).$$

This part is motivated by a work of 't Hooft [**86**] and large developments which occurred thereafter in theoretical physics. 't Hooft in fact noticed that if $V = V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with fixed monomials q_i of m non-commutative unknowns and if we see $Z_V^N = Z_{\mathbf{t}}^N$ as a function of $\mathbf{t} = (t_1, \cdots, t_n)$

$$\ln Z_{\mathbf{t}}^N := \sum_{g \geq 0} N^{2-2g} F_g(\mathbf{t}) \tag{7.4}$$

where

$$F_g(\mathbf{t}) := \sum_{k_1,\cdots,k_n \in \mathbb{N}^k} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g((q_i,k_i)_{1\le i\le n})$$

is a generating function of the number $\mathcal{M}_g((q_i,k_i)_{1\le i\le k})$ of maps with genus g build over k_i stars of type q_i, $1 \le i \le n$. A map is a connected oriented graph which is embedded into a surface. Its genus g is by definition the smallest genus of a surface in which it can be embedded in such a way that edges do not cross and the faces of the graph (which are defined by following the boundary of the graph) are homeomorphic to a disc. Intuitively, the genus of a surface is the maximum number of simple closed curves that can be drawn on it without disconnecting it. The genus of a map is related with the number of vertices, edges and faces of the map. The faces of the map are the pieces of the surface in which it is embedded which are enclosed by the edges of the graph. Then, the Euler characteristic $2-2g$ is given by the number of faces plus the number of vertices minus the number of edges.

The vertices of the maps we shall consider have the structure of a star; a star of type q, for some monomial $q = X_{\ell_1}\cdots X_{\ell_k}$, is a vertex with valence $\deg(q)$ and oriented colored half-edges with one marked half edge of color ℓ_1, the second of color ℓ_2 etc until the last one of color ℓ_k. $\mathcal{M}_g((q_i,k_i)_{1\le i\le n})$ is then the number of maps with k_i stars of type q_i, $1 \le i \le n$. The equality (7.4) obtained by 't Hooft **[86]** was only formal, i.e means that all the derivatives on both sides of the equality coincide at $\mathbf{t} = 0$. This result can then be deduced from Wick formula which gives the expression of arbitrary moments of Gaussian variables.

Adding to V a term $t\,q$ for some monomial q and identifying the first order derivative with respect to t at $t = 0$ we derive from (7.4)

$$\int \hat{\mu}^N(q)d\mu^N_{V_{\mathbf{t}}} = \sum_{g\ge 0} N^{-2g} \sum_{k_1,\cdots,k_n\in\mathbb{N}^k} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_g((q_i,k_i)_{1\le i\le n},(q,1)). \tag{7.5}$$

Even though the expansions (7.4) and (7.5) were first introduced by 't Hooft to compute the matrix integrals, the natural reverse question of computing the numbers $\mathcal{M}_g((q_i,k_i)_{1\le i\le n})$ by studying the associated integrals over matrices encountered a large success in theoretical physics. In the course of doing so, one would like for instance to compute the limit $\lim_{N\to\infty} N^{-2}\ln Z^N_{\mathbf{t}}$ and claim that the limit has to be equal to $F_0(\mathbf{t})$. There is here the claim that one can interchange derivatives and limit, a claim that we shall study in this chapter.

We shall indeed prove that the formal limit can be strenghten into a large N expansion. This requires that integrals are finite which could fail to happen for instance with a potential such as $V(X) = X^3$. We could include such potential to the cost of adding a cutoff $1_{\|X_i\|\le M}$ for some sufficiently large (but fixed) M. This introduces however boundary terms that we prefer to avoid hereafter. Instead, we shall assume that

$$(X_k(ij))_{i,j,k} \to \operatorname{Tr}\left(V_{\mathbf{t}}(X_1,\ldots,X_m) + \frac{1}{4}\sum X_k^2\right) \tag{7.6}$$

is convex for all N. We denote by U the set of parameters $\mathbf{t} = (t_1,\ldots,t_n)$ so that (7.6) holds. Note that this is true when $\mathbf{t} = 0$. This implies that the Hessian of $-\ln \frac{d\mu^N_{V_{\mathbf{t}}}}{dx}$ is uniformly bounded below by $-N/4I$, that is is uniformly log-concave.

This property will provide useful a priori bounds. We also denote B_ϵ the set of parameters $\mathbf{t} = (t_1, \ldots, t_n)$ so that $\|\mathbf{t}\|_\infty = \max |t_i|$ is bounded above by ϵ. In the sequel, we denote by $\|t\|_1 = \sum |t_i|$.

Then, we shall prove that for $\mathbf{t} \in U \cap B_\epsilon$, ϵ small enough,

$$\bar{\mu}^N_{V_{\mathbf{t}}}[P] = \mu^N_{V_{\mathbf{t}}}[\hat{\mu}^N(P)] = \sigma^0_{V_{\mathbf{t}}}(P) + \frac{1}{N^2}\sigma^1_{V_{\mathbf{t}}}(P) + o(N^{-2})$$

where $\sigma^g_{V_{\mathbf{t}}}(q) = \sum_{k_1,\cdots,k_n \in \mathbb{N}^k} \prod_{i=1}^k \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g((q_i, k_i)_{1\le i\le k}, (q,1))$ for monomial functions q for $g = 0$ or 1.

This part summarizes results from **[54]** and **[55]**. The full expansion (i.e higher order corrections) was obtained by E. Maurel Segala (see **[73]**).

7.0.4.1. *First order expansion for the free energy.* We prove here (see Theorem 7.16) that

$$\lim_{N\to\infty} \frac{1}{N^2} \ln \int e^{\sum_{i=1}^n t_i N \mathrm{Tr}(q_i(X_1,\cdots,X_m))} d\mu^N(X_1)\cdots d\mu^N(X_m)$$
$$= \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_{i=1}^n \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_0((q_i,k_i), 1 \le i \le n)$$

provided $V_{\mathbf{t}}$ satisfies (7.6) and the parameters t_i's are sufficiently small. To prove this result we first show that, under the same assumptions, $\bar{\mu}^N_t(q) = \mu^N_{\sum t_i q_i}(N^{-1}\mathrm{Tr}(q))$ converges as N goes to infinity towards a limit which is as well related with map enumeration (see Theorem 7.12).

The central tool in our asymptotic analysis will be again the Dyson-Schwinger's equations. They are simple emanation of the integration by parts formula (or, somewhat equivalently, of the symmetry of the Laplacian in $L^2(dx)$). These equations will be shown to pass to the large N limit and be then given as some asymptotic differential equation for the limit points of $\bar{\mu}^N_t = \mu^N_{V_{\mathbf{t}}}[\hat{\mu}^N]$. These equations will in turn uniquely determine these limit points in some small range of the parameters. We will then show that the limit points have to be given as some generating function of maps.

7.0.4.2. *Finite dimensionnal Dyson-Schwinger's equations.* We can generalize the Dyson -Schwinger equations that we proved in Section 7.0.2 for independent GUE matrices to the interacting case as follows.

Property 7.7. For all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, all $i \in \{1, \cdots, m\}$,

$$\mu^N_{V_{\mathbf{t}}}\left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)\right) = \mu^N_{V_{\mathbf{t}}}\left(\hat{\mu}^N((X_i + D_i V_{\mathbf{t}})P)\right)$$

Proof. Using repeatidely Stein's lemma which says that for any differentiable function f on $\mathbb{R}$,

$$\int f(x) x e^{-\frac{x^2}{2}} dx = \int f'(x) e^{-\frac{x^2}{2}} dx,$$

we find, since

$$\begin{aligned} A_l(rs) e^{-\frac{|\Re(A_l(rs))|^2}{2} - \frac{|\Im(A_l(rs))|^2}{2}} &= -(\partial_{\Re(A_l(rs))} + i\partial_{\Im(A_l(rs))}) e^{-\frac{|\Re(A_l(rs))|^2}{2} - \frac{|\Im(A_l(rs))|^2}{2}} \\ &= -\partial_{\bar{A}_l(rs)} e^{-\frac{|\Re(A_l(rs))|^2}{2} - \frac{|\Im(A_l(rs))|^2}{2}} \end{aligned}$$

with $\partial_{\bar{A}_l(rs)} A_k(ij) = \partial_{A_l(sr)} A_k(ij) = 1_{k=l} 1_{rs=ji}$, that

$$\begin{aligned}
\int \frac{1}{N}\mathrm{Tr}(A_k P) d\mu^N_{V_{\mathbf{t}}}(\mathbb{A}) &= \frac{1}{N^2}\sum_{i,j=1}^N \int \partial_{A_k(ji)}(Pe^{-N\mathrm{Tr}(V_{\mathbf{t}})})_{ji} \prod d\mu^N(A_i) \\
&= \frac{1}{N^2}\sum_{i,j=1}^N \int \Bigg(\sum_{P=QX_kR} Q_{jj}R_{ii} \\
&\quad - N\sum_{l=1}^n \sum_{q_l=QX_kR} t_l \sum_{h=1}^N P_{ji}Q_{hj}R_{ih} \Bigg) d\mu^N_{V_{\mathbf{t}}}(\mathbb{A}) \\
&= \int \left(\frac{1}{N^2}(\mathrm{Tr}\otimes\mathrm{Tr})(\partial_k P) - \frac{1}{N}\mathrm{Tr}(D_k V_{\mathbf{t}} P) \right) d\mu^N_{V_{\mathbf{t}}}(\mathbb{A})
\end{aligned}$$

where $\mathbb{A} = (A_1, \dots, A_m)$. This yields

$$\int \left(\hat{\mu}^N((X_k + D_k V_{\mathbf{t}})P) - \hat{\mu}^N \otimes \hat{\mu}^N(\partial_k P) \right) d\mu^N_{V_{\mathbf{t}}}(\mathbb{A}) = 0. \tag{7.7}$$

$\diamond$

7.0.4.3. *A priori estimates.* $\mu^N_{V_{\mathbf{t}}}$ is a probability measure with uniformly log-concave density. This provides very useful a priori inequalities such as concentration inequalities and Brascamp-Lieb inequalities. We recall below the main consequences we shall use and refer to [**3**] and my course in Saint Flour for details.

We assume $V_{\mathbf{t}} = \sum t_i q_i$ satisfies (7.6), that is $\mathbf{t} = (t_1, \dots, t_n) \in U$. Brascamp-Lieb inequalities allow to compare expectation of convex functions with those under the Gaussian law, for which we have a priori bounds on the norm of matrices. From this we deduce, see [**55**] for details, that

LEMMA 7.8. *For ϵ small enough, there exists M_0 finite so that for all $\mathbf{t} \in U \cap B_\epsilon$, $V_{\mathbf{t}} = \sum t_i q_i$ there exists a positive constant c such that for all i and $s \geq 0$*

$$\mu^N_{V_{\mathbf{t}}}\left(\|X_i\| \geq s + M_0 - 1\right) \leq e^{-cNs}.$$

As a consequence, for $\delta > 0$, for all all $r \leq N/2$ and all $\ell_i, i \leq r$

$$\mathbb{E}[|\hat{\mu}^N(X_{\ell_1} \cdots X_{\ell_r})|] \quad \leq \quad (M_0 + \delta)^r. \tag{7.8}$$

Concentration inequalities are deduced from log-Sobolev and Herbst's argument [**55**, section 2.3] :

LEMMA 7.9. *There exists $\epsilon > 0$ and $c > 0$ so that for $\mathbf{t} \in U \cap B_\epsilon$ for any polynomial P*

$$\mu^N_{V_{\mathbf{t}}}\left(\{|\hat{\mu}^N(P) - \mathbb{E}[\hat{\mu}^N(P)]| \geq \|P\|^L_{M_0}\delta\} \cap \{\|X_i\| \leq M_0 + 1\} \right) \leq e^{-cN^2\delta^2}$$

where $\|P\|^L_A = \sup_{\|X_i\| \leq A} (\sum_{k=1}^m \|D_k P(X) D_k P^*(X)\|_\infty)^{1/2}$ if the supremum is taken over m-tuples of $N \times N$ self-adjoint matrices $X = (X_1, \dots, X_m)$ and all N.

Note that if $P = \sum \alpha_q q$, $\|P\|^L_A \leq \left(\sum |\alpha_q|^2 \deg q^2 A^{2\deg(q)}\right)^{1/2}$.

7.0.4.4. *Tightness and limiting Dyson-Schwinger's equations.* We say that $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies the Dyson-Schwinger equation with potential V, denoted in short **SD[V]**, if and only if for all $i \in \{1, \cdots, m\}$ and $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\tau(I) = 1, \quad \tau \otimes \tau(\partial_i P) = \tau((D_i V + X_i)P) \qquad \mathbf{SD[V]}.$$

We shall now prove that

PROPERTY 7.10. There exists $\epsilon > 0$ so that for all $\mathbf{t} \in U \cap B_\epsilon$ $(\bar{\mu}_t^N, N \in \mathbb{N})$ is tight. Any limit point τ satisfies $\mathbf{SD[}V_\mathbf{t}\mathbf{]}$ and belongs to $K(M_0)$, with M_0 as in Lemma 7.8 and $K(M)$ defined in Lemma 7.2.

PROOF. By Lemma 7.8 we know that $\bar{\mu}_t^N = \mu_{V_\mathbf{t}}^N[\hat{\mu}^N]$ belongs to the compact set $K(M_0)$ (the restriction on moments with degree going to infinity with N being irrelevant) hence this sequence is tight. Any limit point τ belongs as well to $K(C_0)$. Moreover, the DS equation (7.7), together with the concentration property of Lemma 7.9, implies that

$$\tau((X_k + D_k V)P) = \tau \otimes \tau(\partial_k P). \tag{7.9}$$

$\diamond$

7.0.4.5. *Uniqueness of the solutions to Dyson-Schwinger's equations for small parameters.* The main result of this paragraph is

THEOREM 7.11. *For all $R \geq 1$, there exists $\epsilon > 0$ so that for $\|\mathbf{t}\|_\infty = \max_{1 \leq i \leq n} |t_i| < \epsilon$, there exists at most one solution $\tau_\mathbf{t} \in K(R)$ to $\mathbf{SD[}V_\mathbf{t}\mathbf{]}$.*

Remark : Note that if $V = 0$, our equation becomes

$$\tau(X_i P) = \tau \otimes \tau(\partial_i P).$$

Because if P is a monomial, $\tau \otimes \tau(\partial_i P) = \sum_{P = P_1 X_i P_2} \tau(P_1)\tau(P_2)$ with P_1 and P_2 with degree smaller than P, we see that the equation $\mathbf{SD[0]}$ allows to define uniquely $\tau(P)$ for all P by induction. The solution can be seen to be exactly $\tau(P) = \sigma^m(P)$, σ^m the law of m free semi-circular found in Theorem 7.4. When $V_\mathbf{t}$ is not zero, such an argument does not hold a priori since the right hand side will also depend on $\tau(D_i q_j P)$, with $D_i q_j P$ of degree strictly larger than $X_i P$. However, our compactness assumption $K(R)$ gives uniqueness because it forces the solution to be in a small neighborhood of the law $\tau_0 = \sigma^m$ of m free semi-circular variables, so that perturbation analysis applies. We shall see in Theorem 7.13 that this solution is actually the one which is related with the enumeration of maps.

PROOF. Let us assume we have two solutions τ and τ' in $K(R)$. Then, by the equation $\mathbf{SD[V_t]}$, for any monomial function P of degree $l-1$, for $i \in \{1, \cdots, m\}$,

$$(\tau - \tau')(X_i P) = ((\tau - \tau') \otimes \tau)(\partial_i P) + (\tau' \otimes (\tau - \tau'))(\partial_i P) - (\tau - \tau')(D_i V_\mathbf{t} P)$$

We define for $l \in \mathbb{N}$

$$\Delta_l(\tau, \tau') = \sup_{\text{monomial } P \text{ of degree } \leq l} |\tau(P) - \tau'(P)|\,.$$

Using $\mathbf{SD[V_t]}$ and noticing that if P is of degree $l-1$,

$$\partial_i P = \sum_{k=0}^{l-2} p_k^1 \otimes p_{l-2-k}^2$$

where p_k^i, $i = 1, 2$ are monomial of degree k or the null monomial, and $D_i V_\mathbf{t}$ is a finite sum of monomials of degree smaller than $D-1$, we deduce

$$\Delta_l(\tau, \tau') = \max_{P \text{ monomial of degree } \leq l-1} \max_{1 \leq i \leq m} \{|\tau(X_i P) - \tau'(X_i P)|\}$$

$$\leq 2\sum_{k=0}^{l-2}\Delta_k(\tau,\tau')R^{l-2-k} + C\|\mathbf{t}\|_\infty \sum_{p=0}^{D-1}\Delta_{l+p-1}(\tau,\tau')$$

with a finite constant C (which depends on n only). For $\gamma > 0$, we set

$$d_\gamma(\tau,\tau') = \sum_{l\geq 0}\gamma^l \Delta_l(\tau,\tau').$$

Note that in $(\mathbf{K(R)})$, this sum is finite for $\gamma < (R)^{-1}$. Summing the two sides of the above inequality times γ^l we arrive at

$$d_\gamma(\tau,\tau') \leq 2\gamma^2(1-\gamma R)^{-1}d_\gamma(\tau,\tau') + C\|\mathbf{t}\|_\infty \sum_{p=0}^{D-1}\gamma^{-p+1}d_\gamma(\tau,\tau').$$

We finally conclude that if $(R, \|\mathbf{t}\|_\infty)$ are small enough so that we can choose $\gamma \in (0, R^{-1})$ so that

$$2\gamma^2(1-\gamma R)^{-1} + C\|\mathbf{t}\|_\infty \sum_{p=0}^{D-1}\gamma^{-p+1} < 1$$

then $d_\gamma(\tau,\tau') = 0$ and so $\tau = \tau'$ and we have at most one solution. Taking $\gamma = (2R)^{-1}$ shows that this is possible provided

$$\frac{1}{4R^2} + C\|\mathbf{t}\|_\infty \sum_{p=0}^{D-1}(2R)^{p-1} < 1$$

so that when $\|\mathbf{t}\|_\infty$ goes to zero, we see that we need R to be at most of order $\|\mathbf{t}\|_\infty^{-\frac{1}{D-2}}$.

$\diamond$

7.0.4.6. *Convergence of the empirical distribution.* We can finally state the main result of this section.

THEOREM 7.12. *There exists $\epsilon > 0$ and $M_0 \in \mathbb{R}^+$ (given in Lemma 7.8) so that for all $\mathbf{t} \in U \cap B_\epsilon$, $\hat{\mu}^N$ (resp. $\bar{\mu}_t^N$) converges almost surely (resp. everywhere) towards the unique solution of $\mathbf{SD[}V_\mathbf{t}\mathbf{]}$ such that*

$$|\tau(X_{\ell_1}\cdots X_{\ell_r})| \leq M_0^r$$

for all choices of $\ell_1, \cdots, \ell_r$.

PROOF. By Property 7.10, the limit points of $\bar{\mu}_t^N$ belong to $\mathbf{K(}M_0\mathbf{)}$ and satisfies $\mathbf{SD[}V_\mathbf{t}\mathbf{]}$. Since M_0 does not depend on $\mathbf{t}$, we can apply Theorem 7.11 to see that if $\mathbf{t}$ is small enough, there is only one such limit point. Thus, by Corollary 7.3 we can conclude that $(\bar{\mu}_t^N, N \in \mathbb{N})$ converges towards this limit point. From concentration inequalities we have that

$$\mu_V^N(|(\hat{\mu}^N - \bar{\mu}_t^N)(P)|^2) \leq BC(P,M)N^{-2} + C^{2d}N^2e^{-\alpha MN/2}$$

insuring by Borel-Cantelli's lemma that

$$\lim_{N\to\infty}(\hat{\mu}^N - \bar{\mu}_t^N)(P) = 0 \quad a.s$$

resulting with the almost sure convergence of $\hat{\mu}^N$. $\diamond$

7.0.4.7. *Combinatorial interpretation of the limit.* In this part, we are going to identify the unique solution $\tau_{\mathbf{t}}$ of Theorem 7.11 as a generating function for planar maps. Namely, we let for $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$ and P a monomial in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\mathcal{M}_{\mathbf{k}}(P) = \text{card}\{ \text{ planar maps with } k_i \text{ labelled stars of type } q_i \text{ for } 1 \leq i \leq n$$
$$\text{and one of type } P\} = \mathcal{M}_0((P,1),(q_i,k_i)_{1\leq i\leq n}).$$

This definition extends to $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ by linearity. Then, we shall prove that

THEOREM 7.13. (1) *The family* $\{\mathcal{M}_{\mathbf{k}}(P), \mathbf{k} \in \mathbb{N}, P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle\}$ *satisfies the induction relation : for all* $i \in \{1, \cdots, m\}$*, all* $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$*, all* $\mathbf{k} \in \mathbb{N}^n$*,*

$$\mathcal{M}_{\mathbf{k}}(X_iP) = \sum_{\substack{0\leq p_j\leq k_j\\ 1\leq j\leq n}} \prod_{j=1}^n C_{k_j}^{p_j} \mathcal{M}_{\mathbf{p}} \otimes \mathcal{M}_{\mathbf{k}-\mathbf{p}}(\partial_i P) + \sum_{1\leq j\leq n} k_j \mathcal{M}_{\mathbf{k}-1_j}([D_i q_j]P) \tag{7.10}$$

where $1_j(i) = 1_{i=j}$ *and* $\mathcal{M}_{\mathbf{k}}(1) = 1_{\mathbf{k}=0}$*. (7.10) defines uniquely the family* $\{\mathcal{M}_{\mathbf{k}}(P), \mathbf{k} \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle\}$*.*

(2) *There exists* A, B *finite constants so that for all* $\mathbf{k} \in \mathbb{N}^n$*, all monomial* $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$*,*

$$|\mathcal{M}_{\mathbf{k}}(P)| \leq \mathbf{k}! A^{\sum_{i=1}^n k_i} B^{deg(P)} \prod_{i=1}^n C_{k_i} C_{deg(P)} \tag{7.11}$$

with $\mathbf{k}! := \prod_{i=1}^n k_i!$ *and* C_p *the Catalan numbers.*

(3) *For* $\mathbf{t}$ *in* $B_{(4A)^{-1}}$*,*

$$\mathcal{M}_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(P)$$

is absolutely convergent. For $\mathbf{t}$ *small enough,* $\mathcal{M}_{\mathbf{t}}$ *is the unique solution of* $\mathbf{SD}[V_{\mathbf{t}}]$ *which belongs to* $\mathbf{K(4B)}$*.*

By Theorem 7.11 and Theorem 7.12, we therefore readily obtain that

COROLLARY 7.14. For all $c > 0$, there exists $\eta > 0$ so that for $\mathbf{t} \in U_c \cap B_\eta$, $\hat{\mu}^N$ converges almost surely and in expectation towards

$$\tau_{\mathbf{t}}(P) = \mathcal{M}_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(P)$$

Let us remark that by definition of $\hat{\mu}^N$, for all P, Q in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\hat{\mu}^N(PP^*) \geq 0 \quad \hat{\mu}^N(PQ) = \hat{\mu}^N(QP).$$

These conditions are closed for the weak topology and hence we find that

COROLLARY 7.15. There exists $\eta > 0$ ($\eta \geq (4A)^{-1}$) so that for $\mathbf{t} \in B_\eta$, $\mathcal{M}_{\mathbf{t}}$ is a linear form on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ such that for all P, Q

$$\mathcal{M}_{\mathbf{t}}(PP^*) \geq 0 \quad \mathcal{M}_{\mathbf{t}}(PQ) = \mathcal{M}_{\mathbf{t}}(QP) \quad \mathcal{M}_{\mathbf{t}}(1) = 1.$$

Remark. This means that $\mathcal{M}_{\mathbf{t}}$ is a tracial state. The traciality property can easily be derived by symmetry properties of the maps. However, the positivity property $\mathcal{M}_{\mathbf{t}}(PP^*) \geq 0$ is far from obvious but an easy consequence of the matrix models approximation. This property will be seen to be useful to actually solve the combinatorial problem (i.e. find an explicit formula for $\mathcal{M}_{\mathbf{t}}$).

Proof of Theorem 7.13.

(1) *Proof of the induction relation* (7.10).

- We first check them for $\mathbf{k} = \mathbf{0} = (0, \cdots, 0)$. By convention, there is one planar map with a single vertex, so $\mathcal{M}_{\mathbf{0}}(1) = 1$. We now check that

$$\mathcal{M}_{\mathbf{0}}(X_i P) = \mathcal{M}_{\mathbf{0}} \otimes \mathcal{M}_{\mathbf{0}}(\partial_i P) = \sum_{P = p_1 X_i p_2} \mathcal{M}_{\mathbf{0}}(p_1)\mathcal{M}_{\mathbf{0}}(p_2)$$

  But this is clear since for any planar map with only one star of type $X_i P$, the half-edge corresponding to X_i has to be glued with another half-edge of P, hence if X_i is glued with the half-edge X_i coming from the decomposition $P = p_1 X_i p_2$, the map is split into two (independent) planar maps with stars p_1 and p_2 respectively (note here that p_1 and p_2 inherites the structure of stars since they inherite the orientation from P as well as a marked half-edge corresponding to the first neighbour of the glued X_i.)
- We now proceed by induction over the $\mathbf{k}$ and the degree of P; we assume that (7.10) is true for $\sum k_i \leq M$ and all monomials, and for $\sum k_i = M + 1$ when $\deg(P) \leq L$. Note that $\mathcal{M}_{\mathbf{k}}(1) = 0$ for $|\mathbf{k}| \geq 1$ since we can not glue a vertex with no half-edges with any star. Hence, this induction can be started with $L = 0$. Now, consider $R = X_i P$ with P of degree less than L and the set of planar maps with a star of type $X_i Q$ and k_j stars of type q_j, $1 \leq j \leq n$, with $|\mathbf{k}| = \sum k_i = M + 1$. Then,

  $\diamond$ either the half-edge corresponding to X_i is glued with an half-edge of P, say to the half-edge corresponding to the decomposition $P = p_1 X_i p_2$; we see that this cuts the map M into two disjoint planar maps M_1 (containing the star p_1) and M_2 (resp. p_2), the stars of type q_i being distributed either in one or the other of these two planar maps; there will be $r_i \leq k_i$ stars of type q_i in M_1, the rest in M_2. Since all stars all labelled, there will be $\prod C_{k_i}^{r_i}$ ways to assign these stars in M_1 and M_2.

  Hence, the total number of planar maps with a star of type $X_i P$ and k_i stars of type q_i, such that the marked half-edge of $X_i P$ is glued with an half-edge of P is

$$\sum_{P = p_1 X_i p_2} \sum_{\substack{0 \leq r_i \leq k_i \\ 1 \leq i \leq n}} \prod_{i=1}^{n} C_{k_i}^{r_i} \mathcal{M}_{\mathbf{r}}(p_1)\mathcal{M}_{\mathbf{k}-\mathbf{r}}(p_2) \tag{7.12}$$

  $\diamond$ Or the half-edge corresponding to X_i is glued with an half-edge of another star, say q_j; let's say with the edge coming from the decomposition of q_j into $q_j = q_1 X_i q_2$. Then, we can see that once we are

giving this gluing of the two edges, we can replace X_iP and q_j by q_2q_1P.
We have k_j ways to choose the star of type q_j and the total number of such maps is

$$\sum_{q_j=q_1X_iq_2} k_j\mathcal{M}_{\mathbf{k}-1_j}(q_2q_1P)$$

Note here that $\mathcal{M}_{\mathbf{k}}$ is tracial. Summing over j, we obtain by linearity of $\mathcal{M}_{\mathbf{k}}$

$$\sum_{j=1}^{n} k_j\mathcal{M}_{\mathbf{k}-1_j}([D_iq_j]P) \tag{7.13}$$

(7.12) and (7.13) give (7.10). Moreover, it is clear that (7.10) defines uniquely $\mathcal{M}_{\mathbf{k}}(P)$ by induction.

(2) *Proof of* (7.11). To prove the second point, we proceed also by induction over $\mathbf{k}$ and the degree of P. First, for $\mathbf{k}=\mathbf{0}$, $\mathcal{M}_{\mathbf{0}}(P)$ is the number of colored maps with one star of type P which is smaller than the number of planar maps with one star of type $x^{\text{deg P}}$ since colors only add constraints. Hence, we have, with C_k the Catalan numbers,

$$\mathcal{M}_{\mathbf{k}}(P) \le C_{[\frac{\deg(P)}{2}]} \le C_{\deg(P)}$$

showing that the induction relation is fine with $A=B=1$ at this step. Hence, let us assume that (7.11) is true for $\sum k_i \le M$ and all polynomials, and $\sum k_i = M+1$ for polynomials of degree less than L. Since $\mathcal{M}_{\mathbf{k}}(1)=0$ for $\sum k_i \ge 1$ we can start this induction. Moreover, using (7.10), we get that, if we denote $\mathbf{k}! = \prod_{i=1}^n k_i!$,

$$\begin{aligned}\frac{\mathcal{M}_{\mathbf{k}}(X_iP)}{\mathbf{k}!} &= \sum_{\substack{0\le p_i\le k_i\\ 1\le j\le n}}\sum_{P=P_1X_iP_2}\frac{\mathcal{M}_{\mathbf{p}}(P_1)}{\mathbf{p}!}\frac{\mathcal{M}_{\mathbf{k}-\mathbf{p}}(P_2)}{(\mathbf{k}-\mathbf{p})!}\\ &\quad+\sum_{\substack{1\le j\le n\\ k_j\neq 0}}\frac{\mathcal{M}_{\mathbf{k}-\mathbf{1_j}}((D_iq_jP)}{(\mathbf{k}-1_j)!}\end{aligned}$$

Hence, taking P of degree less or equal to L and using our induction hypothesis, we find that with D the maximum of the degrees of q_j

$$\begin{aligned}\left|\frac{\mathcal{M}_{\mathbf{k}}(X_iP)}{\mathbf{k}!}\right| &\le \sum_{\substack{0\le p_j\le k_j\\ 1\le j\le n}}\sum_{P=P_1X_iP_2}A^{\sum k_i}B^{degP-1}\prod_{i=1}^{n}C_{p_i}C_{k_i-p_i}C_{degP_1}C_{degP_2}\\ &\quad+D\sum_{1\le l\le n}A^{\sum k_j-1}\prod_j C_{k_j}B^{degP+degq_l-1}C_{degP+degq_l-1}\\ &\le A^{\sum k_i}B^{degP+1}\prod_i C_{k_i}C_{degP+1}\left(\frac{4^n}{B^2}+D\frac{\sum_{1\le j\le n}B^{degq_j-2}4^{degq_j-2}}{A}\right)\end{aligned}$$

It is now sufficient to choose A and B such that

$$\frac{4^n}{B^2}+D\frac{\sum_{1\le j\le n}B^{degq_j-2}4^{degq_j-2}}{A}\le 1$$

(for instance $B = 2^{n+1}$ and $A = 4nDB^{D-2}4^{D-2}$ if D is the maximal degree of the q_j) to verify the induction hypothesis works for polynomials of all degrees (all L's).

(3) *Properties of $\mathcal{M}_{\mathbf{t}}$.* From the previous considerations, we can of course define $\mathcal{M}_{\mathbf{t}}$ and the serie is absolutely convergent for $|\mathbf{t}| \le (4A)^{-1}$ since $C_k \le 4^k$. Hence $\mathcal{M}_{\mathbf{t}}(P)$ depends analytically on $\mathbf{t} \in B_{(4A)^{-1}}$. Moreover, for all monomial P,

$$|\mathcal{M}_{\mathbf{t}}(P)| \le \sum_{k\in\mathbb{N}^n} \prod_{i=1}^n (4t_i A)^{k_i}(4B)^{degP} \le \prod_{i=1}^n (1-4At_i)^{-1}(4B)^{degP}.$$

so that for small t, $\mathcal{M}_{\mathbf{t}}$ belongs to $\mathbf{K(4B)}$.

(4) $\mathcal{M}_{\mathbf{t}}$ *satisfies* $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$. This is derived by summing (7.10) written for all $\mathbf{k}$ and multiplied by the factor $\prod(t_i)^{k_i}/k_i!$. From this point and the previous one (note that B is independent from $\mathbf{t}$), we deduce from Theorem 7.11 that for sufficiently small $\mathbf{t}$, $\mathcal{M}_{\mathbf{t}}$ is the unique solution of $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ which belongs to $\mathbf{K(4B)}$.

$\diamond$

7.0.4.8. *Convergence of the free energy.*

THEOREM 7.16. *There exists $\epsilon > 0$ so that for $t \in U \cap B_\epsilon$*

$$\lim_{N\to\infty} \frac{1}{N^2} \ln \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = \sum_{\mathbf{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\le i\le n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}.$$

Moreover, the right hand side is absolutely converging. Above $\mathcal{M}_{\mathbf{k}}$ denotes the number of planar maps build over k_i stars of type q_i, $1 \le i \le n$.

Proof. Note that if V satisfies (7.6), then for any $\alpha \in [0,1]$, αV also satisfies (7.6). Set

$$F_N(\alpha) = \frac{1}{N^2} \ln Z_N^{V_{\alpha\mathbf{t}}}.$$

Then,

$$\frac{1}{N^2} \ln \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = F_N(1) - F_N(0).$$

Moreover

$$\partial_\alpha F^N(\alpha) = -\mu_{V_{\alpha\mathbf{t}}}^N\left(\hat\mu^N(V_{\mathbf{t}})\right). \tag{7.14}$$

By Theorem 7.12, we know that for all $\alpha \in [0,1]$, we have

$$\lim_{N\to\infty} \mu_{V_{\alpha\mathbf{t}}}^N\left(\hat\mu^N(V_{\mathbf{t}})\right) = \tau_{\alpha\mathbf{t}}(V_{\mathbf{t}})$$

whereas by (7.8), we know that $\mu_{V_{\alpha\mathbf{t}}}^N\left(\hat\mu^N(q_i)\right)$ stays uniformly bounded. Therefore, a simple use of dominated convergence theorem shows that

$$\lim_{N\to\infty} \frac{1}{N^2} \ln \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = -\int_0^1 \tau_{\alpha\mathbf{t}}(V_{\mathbf{t}})d\alpha = -\sum_{i=1}^n t_i \int_0^1 \tau_{\alpha\mathbf{t}}(q_i)d\alpha. \tag{7.15}$$

Now, observe that by Corollary 7.14, that with $1_i = (0,\ldots,1,0,\ldots,0)$ with the 1 in i^{th} position,

$$\begin{aligned}\tau_{\mathbf{t}}(q_i) &= \sum_{\mathbf{k}\in\mathbb{N}^n}\prod_{1\le j\le n}\frac{(-t_j)^{k_j}}{k_j!}\mathcal{M}_{\mathbf{k}+1_i}\\ &= -\partial_{t_i}\sum_{\mathbf{k}\in\mathbb{N}^n\backslash\{0,\cdots,0\}}\prod_{1\le j\le n}\frac{(-t_j)^{k_j}}{k_j!}\mathcal{M}_{\mathbf{k}}\end{aligned}$$

so that (7.15) results with

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N^2}\ln\frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} &= -\int_0^1\partial_\alpha[\sum_{\mathbf{k}\in\mathbb{N}^n\backslash\{0,\cdots,0\}}\prod_{1\le j\le n}\frac{(-\alpha t_j)^{k_j}}{k_j!}\mathcal{M}_{\mathbf{k}}]d\alpha\\ &= -\sum_{\mathbf{k}\in\mathbb{N}^n\backslash\{0,\cdots,0\}}\prod_{1\le j\le n}\frac{(-t_j)^{k_j}}{k_j!}\mathcal{M}_{\mathbf{k}}.\end{aligned}$$

$\diamond$

7.0.5. Second order expansion for the free energy. We here prove that

$$\frac{1}{N^2}\ln\left(\int e^{\sum_{i=1}^n t_i N\mathrm{Tr}(q_i(X_1,\cdots,X_m))}d\mu^N(X_1)\cdots d\mu^N(X_m)\right)$$
$$=\sum_{g=0}^{1}\frac{1}{N^{2g-2}}\sum_{k_1,\cdots,k_n\in\mathbb{N}}\prod_{i=1}^n\frac{(t_i)^{k_i}}{k_i!}\mathcal{M}_g((q_i,k_i),1\le i\le n)+o(\frac{1}{N^2})$$

for some parameters t_i small enough and such that $\sum t_i q_i$ satisfies (7.6). As for the first order, we shall prove first a similar result for $\bar\mu_t^N$. We will first refine the arguments of the proof of Theorem 7.11 to estimate $\bar\mu_t^N-\tau_{\mathbf{t}}$. This will already prove that $(\bar\mu_t^N-\tau_{\mathbf{t}})(P)$ is at most of order N^{-2}. To get the limit of $N^2(\bar\mu_t^N-\tau_{\mathbf{t}})(P)$, we will first obtain a central limit theorem for $\hat\mu^N-\tau_{\mathbf{t}}$ which is of independent interest. The key argument in our approach, besides further uses of integration by parts-like arguments, will be the inversion of the master operator. This can not be done in the space of polynomial functions, but in the space of some convergent series. We shall now estimate differences of $\hat\mu^N$ and its limit. So, we set

$$\begin{aligned}\hat\delta_{\mathbf{t}}^N &= N(\hat\mu^N-\tau_{\mathbf{t}})\\ \bar\delta_{\mathbf{t}}^N &= \int\hat\delta^N d\mu_V^N = N(\bar\mu_{\mathbf{t}}^N-\tau_{\mathbf{t}})\\ \tilde\delta_{\mathbf{t}}^N &= N(\hat\mu^N-\bar\mu_{\mathbf{t}}^N)=\hat\delta_{\mathbf{t}}^N-\bar\delta_{\mathbf{t}}^N.\end{aligned}$$

7.0.5.1. *Rough estimates on the size of the correction $\tilde\delta^N$.* In this section we improve on the perturbation analysis performed in section 7.0.4.5 in order to get the order of

$$\bar\delta_{\mathbf{t}}^N(P)=N(\bar\mu_{\mathbf{t}}^N(P)-\tau_{\mathbf{t}})(P)$$

for all monomial P.

PROPOSITION 7.17. There exists $\epsilon > 0$ so that for $\mathbf{t} \in U \cap B_\epsilon$, for all integer number N, and all monomial functions P of degree less than N,

$$|\bar{\delta}_{\mathbf{t}}^N(P)| \leq \frac{C^{\deg (P)}}{N}.$$

PROOF. The starting point is the finite dimensional Dyson -Schwinger equation of Property 7.7

$$\mu_V^N(\hat{\mu}^N[(X_i + D_i V)P]) = \mu_V^N\left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)\right) \tag{7.16}$$

Therefore, since τ satisfies the Dyson-Schwinger equation **SD[V]**, we get that for all polynomial P,

$$\bar{\delta}_{\mathbf{t}}^N(X_i P) = -\bar{\delta}_{\mathbf{t}}^N(D_i V_{\mathbf{t}} P) + \bar{\delta}_{\mathbf{t}}^N \otimes \bar{\mu}_{\mathbf{t}}^N(\partial_i P) + \tau_{\mathbf{t}} \otimes \bar{\delta}_{\mathbf{t}}^N(\partial_i P) + r(N,P) \tag{7.17}$$

with

$$r(N,P) := N^{-1}\mu_{V_{\mathbf{t}}}^N\left(\tilde{\delta}_{\mathbf{t}}^N \otimes \tilde{\delta}_{\mathbf{t}}^N(\partial_i P)\right).$$

By Lemma 7.9, if P is a monomial of degree d, $r(N,P)$ is at most of order $d^3 M_0^{d-1}/N$. We set

$$\mathbf{D}_d^N = \max_{P \text{ monomial of degree } \leq d} |\bar{\delta}_{\mathbf{t}}^N(P)|.$$

Observe that by (7.8), for $\epsilon > 0$ and any monomial of degree d less than $N/2$,

$$|\bar{\mu}_t^N(P)| \leq (M_0 + \epsilon)^d, \quad |\tau_{\mathbf{t}}(P)| \leq M_0^d.$$

Thus, by (7.17), writing $D_i V = \sum t_j D_i q_j$, we get that for $d < N/2$

$$\mathbf{D}_{d+1}^N \leq \max_{1 \leq i \leq m} \sum_{j=1}^n |t_j| \mathbf{D}_{d+\deg(D_i q_j)}^N + 2\sum_{l=0}^{d-1}(M_0+\epsilon)^{d-l-1}\mathbf{D}_l^N + \frac{1}{N} d^3 M_0^d$$

We next define for $\kappa \leq 1$

$$\mathbf{D}^N(\kappa, \epsilon) = \sum_{k=1}^{N/2} \kappa^k \mathbf{D}_k^N.$$

We obtain, if D is the maximal degree of V,

$$\begin{aligned} \mathbf{D}^N(\kappa) &\leq [n\|\mathbf{t}\|_\infty + 2(1-(M_0+\epsilon)\kappa)^{-1}\kappa^2]\mathbf{D}^N(\kappa) \\ &\quad + n\|\mathbf{t}\|_\infty \sum_{k=N/2+1}^{N/2+D} \kappa^{k-D}\mathbf{D}_k^N + \sum_{k=1}^{N/2} \kappa^k \frac{1}{N} k^3 (M_0+\epsilon)^k \end{aligned} \tag{7.18}$$

where we choose κ small enough so that $\eta = (M_0 + \epsilon)\kappa < 1$. In this case the sum of the last two terms is of order $1/N$. Since D_k^N is bounded by $2N(M_0+\epsilon)^k$, $\sum_{k=N/2+1}^{N/2+D} \kappa^{k-D}\mathbf{D}_k^N$ is of order $N\kappa^{-D}\eta^{N/2}$ is going to zero. Then, for κ small, we deduce

$$\mathbf{D}^N(\kappa) \leq C(\kappa, \epsilon) N^{-1}$$

and so for all monomial P of degree $d \leq N/2$,

$$|\bar{\delta}_{\mathbf{t}}^N(P)| \leq C(\kappa, \epsilon)\kappa^{-d} N^{-1}.$$

$\diamond$

To get the precise evaluation of $N\bar{\delta}_{\mathbf{t}}^N(P)$, and of the full expansion of the free energy, we use loop equations, and therefore introduce the corresponding master operator and show how to invert it.

7.0.5.2. *Higher order loop equations.* To get the central limit theorem we derive the higher order Dyson-Schwinger equations. To this end introduce the Master operator. It is the linear map on polynomials given by

$$\Xi P = \sum_{i=1}^{m} \partial_i P \# X_i + \sum_{i=1}^{d} \partial_i P \# D_i V_t - (1 \otimes \tau_{\mathbf{t}} + \tau_{\mathbf{t}} \otimes 1)\partial_i . D_i P \, .$$

Recall here that if P is a monomial $\sum_{i=1}^m \partial_i P \# X_i = \deg(P) P$. Using the traciality of $\hat{\delta}_t^N$ and again integration by parts we find that

LEMMA 7.18. *For all monomials $p_0, \dots p_k$ we have*

$$\begin{aligned}
&\mu_{V_{\mathbf{t}}}^N \left(\hat{\delta}_{\mathbf{t}}^N [\Xi p_0] \prod_{i=1}^k \hat{\delta}_{\mathbf{t}}^N (p_i) \right) \\
&= \sum_{j=1}^k \sum_{i=1}^m \hat{\mu}_{V_{\mathbf{t}}}^N \left(\hat{\mu}^N (D_i p_0 D_i p_j) \prod_{\ell \neq j} \hat{\delta}_{\mathbf{t}}^N (p_\ell) \right) \\
&+ \frac{1}{N} \sum_{i=1}^m \mu_{V_{\mathbf{t}}}^N \left(\hat{\delta}_{\mathbf{t}}^N \otimes \hat{\delta}_{\mathbf{t}}^N [\partial_i \circ D_i p_0] \prod_{i=1}^k \hat{\delta}_{\mathbf{t}}^N (p_i) \right)
\end{aligned}$$

7.0.5.3. *Inverting the master operator.* Note that when $\mathbf{t} = 0$, Ξ is invertible on the space of self-adjoint polynomials with no constant terms, which we denote $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$. The idea is therefore to invert Ξ for $\mathbf{t}$ small. If P is a polynomial and q a non-constant monomial we will denote $\ell_q(P)$ the coefficient of q in the decomposition of P in monomials. We can then define a norm $\|.\|_A$ on $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$ for $A > 1$ by

$$\|P\|_A = \sum_{\deg q \neq 0} |\ell_q(P)| A^{\deg q}.$$

In the formula above, the sum is taken on all non-constant monomials. We also define the operator norm given, for T from $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$ to $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$, by

$$|||T|||_A = \sup_{\|P\|_A = 1} \|T(P)\|_A.$$

Finally, let $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0_A$ be the completion of $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$ for $\|.\|_A$. We say that T is continuous on $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0_A$ if $|||T|||_A$ is finite. We shall prove that Ξ is continuous on $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0_A$ with continuous inverse when $\mathbf{t}$ is small.

We define a linear map Σ on $\mathbb{C}\langle X_1, \cdots, X_m \rangle$ such that for all monomials q of degree greater or equal to 1

$$\Sigma(q) = \frac{q}{\deg q}.$$

Moreover, $\Sigma(q) = 0$ if $\deg q = 0$. We let Π be the projection from $\mathbb{C}\langle X_1, \cdots, X_m \rangle_{sa}$ onto $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$ (i.e $\Pi(P) = P - P(0, \cdots, 0)$). We now define some operators on $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$ i.e. from $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$ into $\mathbb{C}\langle X_1, \cdots, X_m \rangle^0$, we set

$$\Xi_1 : P \longrightarrow \Pi\left(\sum_{k=1}^{m} \partial_k \Sigma P \sharp D_k V\right)$$

$$\Xi_2 : P \longrightarrow \Pi\left(\sum_{k=1}^{m} (\tau_{\mathbf{t}} \otimes I + I \otimes \tau_{\mathbf{t}})(\partial_k D_k \Sigma P)\right).$$

We denote

$$\Xi_0 = I - \Xi_2 \Rightarrow \Pi \circ \Xi \circ \Sigma = \Xi_0 + \Xi_1 ,$$

where I is the identity on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^0$. Note that the images of the operators Ξ_i's and $\Pi \circ \Xi \circ \Sigma$ are indeed included in $\mathbb{C}\langle X_1, \cdots, X_m\rangle_{sa}$ since V is assumed self-adjoint.

Lemma 7.19. *With the previous notations,*

1. *For $t \in U$, the operator Ξ_0 is invertible on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^0$.*
2. *There exists $A_0 > 0$ such that for all $A > A_0$, the operators Ξ_2, Ξ_0 and Ξ_0^{-1} are continuous on $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$ and their norm $|||.|||_A$ are uniformly bounded for $\mathbf{t}$ in $B_\eta \cap U$.*
3. *For all $\epsilon, A > 0$, there exists $\eta_\epsilon > 0$ such for $\|\mathbf{t}\|_\infty < \eta_\epsilon$, Ξ_1 is continuous on $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$ and $|||\Xi_1|||_A \leq \epsilon$.*
4. *For all $A > A_0$, there exists $\eta > 0$ such that for $\mathbf{t} \in B_\eta \cap U$, $\Pi \circ \Xi \circ \Sigma$ is continuous, invertible with a continuous inverse on $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$. Besides the norms of $\Pi \circ \Xi$ and $(\Pi \circ \Xi)^{-1}$ are uniformly bounded for $\mathbf{t}$ in B_η.*
5. *For any $A > M_0$, there is a finite constant C such that*

$$\|P\|_{M_0}^{L} \leq C\|P\|_A .$$

The norm $\|.\|_{M_0}^{L}$ was defined in Lemma 7.9.

Proof.

1. Recall that $\Xi_0 = I - \Xi_2$, whereas since Ξ_2 reduces the degree of a polynomial by at least 2,

$$P \to \sum_{n \geq 0} (\Xi_2)^n (P)$$

is well defined on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^0$ as the sum is finite for any polynomial P. This clearly gives an inverse for Ξ_0.

2. First remark that a linear operator T has a norm less than C with respect to $\|.\|_A$ if and only if for all non-constant monomial q,

$$\|T(q)\|_A \leq C A^{\deg q}.$$

Recall that $\tau_{\mathbf{t}}$ is uniformly bounded (see Lemma 7.10) and let $C_0 < +\infty$ be such that $|\tau_{\mathbf{t}}(q)| \leq C_0^{\deg q}$ for all monomial q. Take a monomial $q =$

$X_{i_1}\cdots X_{i_p}$, and assume that $A>2C_0$,

$$\begin{aligned}\|\Pi\left(\sum_k (I\otimes\tau_{\mathbf{t}})\partial_k D_k\Sigma q\right)\|_A &\le p^{-1}\sum_{\substack{k,q=q_1X_kq_2,\\ q_2q_1=r_1X_kr_2}}\|r_1\tau_{\mathbf{t}}(r_2)\|_A\\ &\le p^{-1}\sum_{\substack{k,q=q_1X_kq_2,\\ q_2q_1=r_1X_kr_2}} A^{\deg r_1}C_0^{\deg r_2}\le\frac{1}{p}\sum_{n=0}^{p-1}\sum_{l=0}^{p-2}A^lC_0^{p-l-2}\\ &\le A^{p-2}\sum_{l=0}^{p-2}\left(\frac{C_0}{A}\right)^{p-2-l}\le 2A^{-2}\|q\|_A\end{aligned}$$

where in the second line, we observed that once $\deg(q_1)$ is fixed, q_2q_1 is uniquely determined and then r_1,r_2 are uniquely determined by the choice of l the degree of r_1. Thus, the factor $\frac{1}{p}$ is compensated by the number of possible decomposition of q i.e. the choice of the degree of q_1. If $A>2$, $P\to\Pi\left(\sum_k(I\otimes\tau_{\mathbf{t}})\partial_kD_k\Sigma P\right)$ is continuous of norm strictly less than $\frac{1}{2}$. And a similar calculus for $\Pi\left(\sum_k(\tau_{\mathbf{t}}\otimes I)\partial_kD_k\Sigma\right)$ shows that Ξ_2 is continuous of norm strictly less than 1. It follows immediately that Ξ_0 is continuous. Recall now that

$$\Xi_0^{-1}=\sum_{n\ge 0}\Xi_2^n.$$

As Ξ_2 is of norm strictly less than 1, Ξ_0^{-1} is continuous.

(3) Let $q=X_{i_1}\cdots X_{i_p}$ be a monomial and let D be the degree of V

$$\begin{aligned}\|\Xi_1(q)\|_A &\le \frac{1}{p}\sum_{k,q=q_1X_kq_2}\|q_1D_kVq_2\|_A\le\frac{1}{p}\sum_{k,q=q_1X_kq_2}\|\mathbf{t}\|_\infty DnA^{p-1+D-1}\\ &= \|\mathbf{t}\|_\infty DnA^{D-2}\|q\|_A.\end{aligned}$$

It is now sufficient to take $\eta_\epsilon<(nDA^{D-2})^{-1}\epsilon$.

(4) We choose $\eta<(nDA^{D-2})^{-1}|||\Xi_0^{-1}|||_A^{-1}$ so that when $|\mathbf{t}|\le\eta$,

$$|||\Xi_1|||_A|||\Xi_0^{-1}|||_A<1.$$

By continuity, we can extend Ξ_0, Ξ_1, Ξ_2, $\Pi\circ\Xi\circ\Sigma$ and Ξ_0^{-1} on the space $\mathbb{C}\langle X_1,\cdots,X_m\rangle_A^0$. The operator

$$P\to\sum_{n\ge 0}(-\Xi_0^{-1}\Xi_1)^n\Xi_0^{-1}$$

is well defined and continuous. And this is clearly an inverse of

$$\Pi\circ\Xi\circ\Sigma=\Xi_0+\Xi_1=\Xi_0(I+\Xi_0^{-1}\Xi_1).$$

Finally, we notice that Σ^{-1} is bounded from $\mathbb{C}\langle X_1,\cdots,X_m\rangle_A^0$ to $\mathbb{C}\langle X_1,\cdots,X_m\rangle_{A'}^0$ for $A_0<A'<A$, and hence up to take A slightly larger $\Pi\Xi=(\Pi\Xi\Sigma)\circ\Sigma^{-1}$ is continuous on $\mathbb{C}\langle X_1,\cdots,X_m\rangle_A^0$ as well as its inverse.

(5) The last point is trivial.

$\diamond$

7.0.5.4. *Central limit theorem.*

THEOREM 7.20. *Take $\mathbf{t} \in U \cap B_\eta$ for η small enough and $A > M_0 \wedge A_0$. Then For all $P_1, \ldots, P_k$ in $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$, $(\hat{\delta}^N(P_1), \ldots, \hat{\delta}^N(P_k))$ converges in law to a centered Gaussian vector with covariance*

$$\sigma^{(2)}(P,Q) := \sum_{i=1}^m \tau(D_i \Xi^{-1} P D_i Q)\,.$$

Proof. It is enough to prove the result for monomials P_i (which satisfy $P_i(0) = 0$). We know by the previous part that for A large enough there exists $Q_1 \in \mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$ so that $P_1 = \Pi \circ \Xi \circ \Sigma Q_1$. But the space $\mathbb{C}\langle X_1, \cdots, X_m\rangle^0$ is dense in $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$ by construction. Thus, there exists a sequence Q_1^p in $\mathbb{C}\langle X_1, \cdots, X_m\rangle^0$ such that

$$\lim_{p\to\infty} \|Q_1 - Q_1^p\|_A = 0.$$

Let us define $R_p = \Xi \circ \Sigma Q_1 - \Xi \circ \Sigma Q_1^p$ in $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$. By the previous section, it goes to zero for $\|.\|_{A'}$ for $A' \in (A_0, A)$, but also for $\|.\|_{M_0}^L$ for $A > M_0$. But, by Lemma 7.9 and 7.8 we find that since $\hat{\delta}_N^t$ has mass bounded by N, for any polynomial P and $\delta > 0$ and r integer number smaller than $N/2$

$$\begin{aligned}
\mu_{V_{\mathbf{t}}}^N\left(|\hat{\delta}_N^t(R)|^r\right) &\le \mu_{V_{\mathbf{t}}}^N\left(|\hat{\delta}_N^t(R)|^r 1_{\cap_i\{\|X_i\|\le M_0\}}\right) \\
&\quad + \mu_{V_{\mathbf{t}}}^N\left(|\hat{\delta}_N^t(R)|^{2r}\right)^{1/2} \mu_{V_{\mathbf{t}}}^N\left(\cup_i\{\|X_i\| \ge M_0\}\right)^{1/2} \\
&\le (\|R\|_{M_0}^L)^r \int r x^{r-1} e^{-cx^2} dx + (N(2M_0+2))^r e^{-cN}\,.
\end{aligned}$$

We deduce by taking $R = R_p$ that for all $r \in \mathbb{N}$

$$\lim_{p\to\infty} \limsup_{N\to\infty} \mu_{V_{\mathbf{t}}}^N(|\hat{\delta}_N^t(R_p)|^r) = 0\,.$$

Therefore Lemma 7.18 implies that there exists $o(p)$ going to zero when p goes to infinity such that

$$\begin{aligned}
\mu_{V_t}^N\left(\hat{\delta}_{\mathbf{t}}^N[P] \prod_{i=1}^k \hat{\delta}_{\mathbf{t}}^N(q_i)\right) &= \mu_{V_t}^N\left(\hat{\delta}_{\mathbf{t}}^N[\Xi Q_p] \prod_{i=1}^k \hat{\delta}_{\mathbf{t}}^N(q_i)\right) + o(p) \\
&= \sum_{j=1}^k \sum_{i=1}^m \hat{\mu}_{V_t}^N\left(\hat{\mu}^N(D_i Q_p D_i q_j) \prod_{\ell\neq j} \hat{\delta}_{\mathbf{t}}^N(q_\ell)\right) \\
&+ \frac{1}{N} \sum_{i=1}^m \mu_{V_t}^N\left(\hat{\delta}_{\mathbf{t}}^N \otimes \hat{\delta}_{\mathbf{t}}^N[\partial_i \circ D_i Q_p] \prod_{i=1}^k \hat{\delta}_{\mathbf{t}}^N(q_i)\right) + o(p) \\
&\simeq \sum_{j=1}^k \sum_{i=1}^m \tau_{\mathbf{t}}(D_i Q_p D_i q_j) \mu_{V_t}^N\left(\prod_{\ell\neq j} \hat{\delta}_{\mathbf{t}}^N(q_\ell)\right) + o(p) \\
&\simeq \sum_{j=1}^k \sum_{i=1}^m \tau_{\mathbf{t}}(D_i(\Xi^{-1}P) D_i q_j) \mu_{V_t}^N\left(\prod_{\ell\neq j} \hat{\delta}_{\mathbf{t}}^N(q_\ell)\right) + o(p)
\end{aligned}$$

where in the last line we used that $\|D_iQ_n - D_iQ\|_{A_0}$ goes to zero and that $\tau_{\mathbf{t}}$ is continuous for this norm. The result follows then by induction over k since again we recognize Wick formula.

EXERCISE 7.21. Show that for P, Q two monomials,

$$\sigma^{(2)}(P,Q) = \sum \prod \frac{(-t_i)^{\ell_i}}{\ell_i!} M_0(P,Q,(q_i,\ell_i))$$

is the generating function for the enumeration of planar maps with two stars of type P, Q and ℓ_i of type q_i, $1 \le i \le n$.

7.0.5.5. *Second order correction to the free energy.* We now deduce from the Central Limit Theorem the precise asymptotics of $N\bar{\delta}^N(P)$ and then compute the second order correction to the free energy.

Let ϕ_0 and ϕ be the linear forms on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ which are given, if P is a monomial, by

$$\phi(P) = \sum_{i=1}^{m} \sum_{P=P_1X_iP_2X_iP_3} \sigma^{(2)}(P_3P_1, P_2)\,.$$

Note that ϕ vanishes if the degree of P is less than 2.

PROPOSITION 7.22. Take $\mathbf{t} \in U$ small enough. Then, for any polynomial P,

$$\lim_{N\to\infty} N\bar{\delta}^N(P) = \phi(\Xi^{-1}\Pi P).$$

Proof. Again, we base our proof on the finite dimensional Dyson-Schwinger equation (7.16) which, after centering, reads for $i \in \{1, \cdots, m\}$,

(7.19)
$$N^2\mu_V^N\left((\hat{\mu}^N - \tau_{\mathbf{t}})[(X_i + D_iV)P - (I\otimes\tau_{\mathbf{t}} + \tau_{\mathbf{t}}\otimes I)\partial_iP\right) = \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\partial_iP)]\right)$$

Taking $P \to D_i\Pi P$ and summing over $i \in \{1, \cdots, m\}$, we thus have

$$N^2\mu_V^N\left((\hat{\mu}^N - \tau_{\mathbf{t}})(\Xi P)\right) = \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\sum_{i=1}^{m}\partial_i \circ D_iP)\right) \tag{7.20}$$

By Theorem 7.20 we see that

$$\lim_{N\to\infty}\mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\sum_{i=1}^{m}\partial_i \circ D_i\Pi P)\right) = \phi(P)$$

which gives the asymptotics of $N\bar{\delta}^N(\Xi P)$ for all P in the imgae of Ξ.

To generalize the result to arbitrary P, we proceed as in the proof of the full central limit theorem. We take a sequence of polynomials Q_n wich goes to $Q = \Xi^{-1}P$ when n go to ∞ for the norm $\|.\|_A$. We denote $R_n = P - \Xi Q_n = \Xi(Q - Q_n)$. Note that as P and Q_n are polynomials then R_n is also a polynomial.

$$N\bar{\delta}^N(P) = N\bar{\delta}^N(\Xi Q_n) + N\bar{\delta}^N(R_n)$$

According to Proposition 7.17, for any monomial P of degree less than $N^{1-\epsilon}$,

$$|N\bar{\delta}^N(P)| \le C^{\deg(P)}.$$

So if we take the limit in N, for any monomial P,

$$\limsup_N |N\bar{\delta}^N(P)| \le C^{\deg(P)}$$

and if P is a polynomial, Lemma 7.9 yields for $C < A$

$$\limsup_N |N\bar{\delta}^N(P)| \le \|P\|_C^L \le \|P\|_A.$$

We now fix n and take the large N limit,

$$\limsup_N |N\bar{\delta}^N(P - \Xi Q_n)| = \limsup_N |N\bar{\delta}^N(R_n)| \le \|R_n\|_A.$$

If we take the limit in n the right term vanishes and we are left with :

$$\lim_N N\bar{\delta}^N(P) = \lim_n \lim_N N\bar{\delta}^N(Q_n) = \lim_n \phi(Q_n).$$

It is now sufficient to show that ϕ is continuous for the norm $\|.\|_A$. But $P \to \sum_{i=1}^m \partial_i \circ D_i P$ is continuous from $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle_{A-1}^0$ and σ^2 is continuous for $\|.\|_{A-1}$ provided A is large enough. This proves that ϕ is continuous and then can be extended on $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A^0$. Thus

$$\lim_N N\bar{\delta}^N(P) = \lim_n \phi(Q_n) = \phi(Q).$$

$\diamond$

THEOREM 7.23. *Take* $\mathbf{t} \in U$ *small enough. Then*

$$\ln \frac{Z_N^{V_{\mathbf{t}}}}{Z_N^0} = N^2 F_{\mathbf{t}} + F_{\mathbf{t}}^1 + o(1)$$

with

$$F_{\mathbf{t}} = \int_0^1 \tau_{\alpha\mathbf{t}}(V_{\mathbf{t}})d\alpha$$

and

$$F_{\mathbf{t}}^1 = \int_0^1 \phi_{\alpha\mathbf{t}}(\Xi_{\alpha\mathbf{t}}^{-1} V_{\mathbf{t}})ds.$$

Proof. As for the proof of Theorem 7.16, we note that $\alpha V_{\mathbf{t}} = V_{\alpha\mathbf{t}}$ is c-convex for all $\alpha \in [0,1]$ We use (7.14) to see that

$$\partial_\alpha \ln Z_{V_{\alpha\mathbf{t}}}^N = \mu_{\alpha\mathbf{t}}^N(\hat{\mu}^N(V_{\mathbf{t}}))$$

so that we can write

$$\begin{aligned} \ln \frac{Z_{V_{\alpha\mathbf{t}}}^N}{Z_0^N} &= N^2 \int_0^1 \mu_{V_{\alpha\mathbf{t}}}^N(\hat{\mu}^N(V_{\mathbf{t}}))d\alpha \\ &= N^2 F_{\mathbf{t}} + \int_0^1 [N\bar{\delta}_{\alpha\mathbf{t}}^N(V_{\mathbf{t}})]ds \qquad (7.21) \end{aligned}$$

with

$$F_{\mathbf{t}} = \int_0^1 \tau_{\alpha\mathbf{t}}(V_{\mathbf{t}})d\alpha.$$

Proposition 7.22 and (7.21) finish the proof of the theorem since by Proposition 7.17, all the $N\bar{\delta}^N(q_i)$ can be bounded independently of N and $t \in B_\eta \cap U$ so that dominated convergence theorem applies. $\diamond$

EXERCISE 7.24. Show that $F_{\mathbf{t}}^1$ is a generating function for maps of genus one.

7.0.5.6. *More general laws on matrices : the orthogonal and unitary group.* One can also wonder how to generalize the topological expansion to other settings. In [**29, 56**], we considered unitary random matrices following the Haar measure on the unitary or orthogonal group and showed the convergence of the free energy in a perturbative regime. In [**58**] we could prove convergence of the free energy in the special case of the Harich-Chandra-Itzykson-Zuber integral

$$HCIZ(A,B) := \int e^{N\mathrm{Tr}(U^*AUB)}dU.$$

However the free energy is there given by a variational equation and is not related to a topological expansion. Moreover, it relies on the very special structure of this integral and in particular with its relation with the large deviation of the spectral measure of a Hermitian Brownian motion starting at B. We will not describe this result more precisely here but rather detail the results of [**29**] which generalize the strategy of the last chapter to the unitary matrices following the Haar measure.

The result is as follows. We consider matrix integrals given by

$$I_N(V, A_i^N) := \int e^{N\mathrm{Tr}(V(U_i,U_i^*,A_i^N,1\le i\le m))}dU_1\cdots dU_m \tag{7.22}$$

where $(A_i^N, 1 \le i \le m)$ are $N\times N$ deterministic uniformly bounded matrices, dU denotes the Haar measure on the unitary group $\mathcal{U}(N)$ (normalized so that $\int_{\mathcal{U}(N)} dU = 1$) and V is a polynomial function in the non-commutative variables $(U_i, U_i^*, A_i^N, 1 \le i \le m)$. We assume that the joint distribution of the $(A_i^N, 1 \le i \le m)$ converges ; namely for all polynomial function P in m non-commutative unknowns

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}(P(A_i^N, 1\le i\le m)) = \tau(P) \tag{7.23}$$

for some linear functional τ on the set of polynomials. To deal only with probability measures μ_V^N, we assume that the polynomial V is such that $\mathrm{Tr}(V(U_i, U_i^*, A_i^N, 1 \le i \le m))$ is real for all $U_i \in \mathcal{U}(N)$, all Hermitian matrices A_i^N, for all $i \in \{1,\cdots,m\}$ and $N \in \mathbb{N}$.

Under these very general assumptions, the formal convergence of the integrals was already studied by B. Collins [**30**]. The following Theorem is a precise description of our results which gives an asymptotic convergence :

THEOREM 7.25. *Under the above hypotheses and if we further assume that the spectral radius of the matrices $(A_i^N, 1 \le i \le m, N \in \mathbb{N})$ is uniformly bounded (by say M), then there exists $\varepsilon = \varepsilon(M, V) > 0$ so that for $z \in [-\varepsilon, \varepsilon]$, the limit*

$$F_{V,\tau}(z) := \lim_{N\to\infty} \frac{1}{N^2} \ln \int_{\mathcal{U}(N)^m} e^{zN\mathrm{Tr}(V(U_i,U_i^*,A_i^N,1\le i\le m))}dU_1\cdots dU_m$$

exists. Moreover, $F_{V,\tau}(z)$ is an analytic function of $z \in \mathbb{C}\cap B(0,\varepsilon) = \{z \in \mathbb{C} : |z| \le \varepsilon\}$. Furthermore, for all polynomial P there exists a limit

$$\tau_V(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(U_i, U_i^*, A_i^N,)_{1\le i\le m}) \frac{e^{zN\mathrm{Tr}(V(U_i,U_i^*,A_i^N,1\le i\le m))}dU_1\cdots dU_m}{I_N(V, A_i^N)}\,.$$

7.0.5.7. *Idea of the proof.* The strategy is again to find and study the Dyson-Schwinger (or loop) equations under the associated Gibbs measure

$$\mu_V^N(dU_1, \cdots, dU_m) = \frac{1}{Z^N} e^{zN\mathrm{Tr}(V(U_i, U_i^*, A_i^N, 1\le i\le m))} dU_1 \cdots dU_m \,.$$

To describe this equation let us first define derivatives on polynomials in these matrices by the linear form such that

$$\partial_i A_j = 0, \quad \partial_i U_j = 1_{i=j} U_j \otimes 1 \quad \partial_i U_j^* = -1_{i=j} 1 \otimes U_j^*, \quad \forall j,$$

and satisfying the Leibnitz rule, namely, for monomials P, Q,

$$\partial_i(PQ) = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q. \tag{7.24}$$

Here, $\times$ denotes the product $P_1 \otimes Q_1 \times P_2 \otimes Q_2 = P_1 P_2 \otimes Q_1 Q_2$. We also let D_i be the corresponding *cyclic* derivatives such that if $m(A \otimes B) = BA$, then $D_i = m \circ \partial_i$.

If q is a monomial, we more specifically have

$$\partial_i q = \sum_{q=q_1 U_i q_2} q_1 U_i \otimes q_2 - \sum_{q=q_1 U_i^* q_2} q_1 \otimes U_i^* q_2 \tag{7.25}$$

$$D_i q = \sum_{q=q_1 U_i q_2} q_2 q_1 U_i - \sum_{q=q_1 U_i^* q_2} U_i^* q_2 q_1. \tag{7.26}$$

Then, using the invariance by multiplication of the Haar measure one can prove [**29**] the following Dyson -Schwinger equation :

$$\mu_V^N(\frac{1}{N}\mathrm{Tr} \otimes \frac{1}{N}\mathrm{Tr}(\partial_i P)) + \mu_V^N(\frac{z}{N}\mathrm{Tr}(P D_i V)) = 0.$$

This is proved by noticing that if we set $U_p(t) = U_p e^{itB}$ for a Hermitian matrix $B = B^*$, and leave the other U_i's unchanged, then for all k, ℓ

$$\partial_t \int (P(U_i(t), U_i^*(t), A_i^N, 1 \le i \le m))_{k\ell} e^{zN\mathrm{Tr}(V(U_i(t), U_i^*(0), A_i^N, 1\le i\le m))} dU_1 \cdots dU_m = 0$$

This reads

$$\int [(\partial_p P \sharp B)_{k\ell} + z P_{k\ell} \mathrm{Tr}(D_p V B)] d\mu_N^V = 0$$

Taking $B = 1_{k\ell} + 1_{\ell k}$ and $i(1_{k\ell} - 1_{\ell k})$ shows that we can by linearity choose $B = 1_{k\ell}$ even though this is not self-adjoint which yields the result after summation over k and ℓ.

Because $SO(N)$ and $SU(N)$ are compact groups with large Ricci curvature, it can be proved (see [**3**, Theorem 4.4.27] for the case $V = 0$ and [**29**] for the general case) that under μ_V^N, $\frac{1}{N}\mathrm{Tr}(P(U_i, U_i^*, A_i, 1 \le i \le m))$ concentrates. We deduce that for any polynomial P,

$$\lim_{N\to\infty} \left\{ \frac{1}{N}\mathrm{Tr} \otimes \frac{1}{N}\mathrm{Tr}(\partial_i P) + \frac{z}{N}\mathrm{Tr}(D_i V P) \right\} = 0 \quad \mu_V^N \; a.s.$$

In particular, any limit point μ of $\hat{\mu}^N$ under μ_V^N satisfies the Dyson-Schwinger equation

$$\mu \otimes \mu(\partial_i P) + z\mu(D_i V P) = 0 \tag{7.27}$$

for all polynomial P and $\mu|_{(A_i)_{1\le i\le m}} = \tau$. Uniqueness of the solutions to this equation for small $|z|$ can be proved as in section 7.0.4.5, see [**29**]. It implies the almost-sure convergence of $\frac{1}{N}\mathrm{Tr}(P(U_i, U_i^*, A_i, 1 \le i \le m))$ for all polynomials.

7.0.5.8. *discussion.* The study of the fluctuations and large N expansions around this limit were achieved in [**56**] : the strategy is similar to the Gaussian case. The combinatorics of the moments is not as nice as in the Gaussian case, besides some attempts in [**29**]. In the particular case of lattice Gauge theory, similar arguments were used by S. Chatterfjee to show the convergence of the free energy and then identify the combinatorics of Wilson loops [**25**].

CHAPTER 8

Universality for β-models

In this part, we discuss how to prove universality of the local fluctuations for β-models based on approximate transport map ideas developped in [**5, 50**]. We will therefore consider again the β-ensembles $P_N^{\beta,V}$ introduced in Section 4 and will restrict ourselves to the setting of that section, namely the case where the equilibrium measure has a connected support and its density vanishes like a square root at the boundary. Thus, the global fluctuations are known and we now focus on the local fluctuations, such as the fluctuations of the largest eigenvalue or of the spacing between two eigenvalues. Our goal is to show that there is universality in the sense that the local fluctuations are the same than when the potential is quadratic. In fact, this is enough since local fluctuations could be studied in the case $V = x^2$. When $\beta = 1, 2, 4$, this was done by Riemann Hilbert techniques [**87, 88, 75**]. By using tridiagonal representation of the joint law of the eigenvalues of β ensembles with $V(x) = x^2$, derived by Dumitriu and Edelman [**39**], local fluctuations could be studied for general β [**91, 78**]. We are going to see here how to show that the same local fluctuations are true if we take another potential V, provided it is smooth enough, and so that the equilibrium measure has a connected support. Universality in the β-ensembles was first addressed in [**18**] (in the bulk, $\beta > 0$, $V \in C^4$), then in [**19**] (at the edge, $\beta \geq 1$, $V \in C^4$) and [**65**] (at the edge, $\beta > 0$, V convex polynomial) and finally in [**81**] (in the bulk, $\beta > 0$, V analytic, multi-cut case included). The approach we propose here, which was developed in [**5**], is based on the construction of approximate transport maps. More precisely, let us consider $P_N^{\beta,V}$ the previous β-ensemble distribution on $\mathbb{R}^N$. The goal is to construct a map $T^N : \mathbb{R}^N \to \mathbb{R}^N$ such that $T^N = T_0^{\otimes N} + \frac{1}{N} T_1^N$ where T_0 is smooth and increasing as well as T_1^N and so that

$$\lim_{N\to\infty} \|P_N^{\beta,V} - T^N \# P_N^{\beta,x^2}\|_{TV} = 0\,. \tag{8.1}$$

Here, we denote by $T\#\mu$ the push-forward of the measure μ by T given for any test function f by

$$\int f(x) dT\#\mu(x) = \int f(T(y)) d\mu(y)\,.$$

The name of approximate transport map emphasizes that the above equality only holds at the large N limit, and not for all N as a usual transport map would do. Eventhough existence of transport maps is well known for any given N, smoothness and dependency on the dimension of such maps is unknown in general. For this reason, we shall instead construct *explicit* approximate transport maps, for which we can investigate both regularity and dependency on the dimension. The main point is that the existence of approximate transport maps as in (8.1) implies universality. Roughly speaking, if T_1^N is bounded, we see that if λ_i^V are the ordered eigenvalues under $P_N^{\beta,V}$,

$$N^{2/3}(\lambda_N^V - T_0(2)) \simeq_{dist} N^{2/3}(T^N(\lambda_N^{x^2}) - T_0(2)) \simeq T_0'(2)N^{2/3}(\lambda_N^{x^2} - 2) + O(\frac{1}{N^{1/3}})\,.$$

This shows that the largest eigenvalue λ_N^V fluctuates around $T_0(2)$ as in the quadratic case. Moreover, if T_1^N is Hölder $\alpha \in (0,1)$,

$$N(\lambda_{i+1}^V - \lambda_i^V) = NT_0'(\lambda_i^{x^2})(\lambda_{i+1}^{x^2} - \lambda_i^{x^2}) + O(|\lambda_{i+1}^{x^2} - \lambda_i^{x^2}|^\alpha)$$

where the last term is negligible. Similarly, correlation functions can be considered. Hence, we also get universality in the bulk. Let us state the results more precisely. We shall make the following hypotheses, as in Section 4 :

HYPOTHESIS 8.1. (Off criticality and one-cut hypothesis) We assume that μ_{V_0} and μ_{V_1} have a connected support and are off-critical, that is, there exists a constant $\bar{c} > 0$ such that, for $t = 0, 1$,

$$\frac{d\mu_{V_t}}{dx} = S_t(x)\sqrt{(x-a_t)(b_t-x)} \qquad \text{with } S_t \geq \bar{c} \text{ a.e. on } [a_t, b_t].$$

HYPOTHESIS 8.2. (Large deviations control) For $t = 0, 1$, the function

$$x \mapsto U_{V_t}(x) := V_t(x) - \beta \int d\mu_{V_t}(y) \ln|x-y| \tag{8.2}$$

achieves its minimal value on $[a,b]^c$ at its boundary $\{a,b\}$

Then we shall prove

THEOREM 8.3. *Assume that V, W are C^{25} and satisfy Hypotheses 8.1 and 8.2. Then there exists a map $T^N = (T^{N,1}, \ldots, T^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ which satisfies* (8.1) *and has the form*

$$T^{N,i}(\hat{\lambda}) = T_0(\lambda_i) + \frac{1}{N}T_1^{N,i}(\hat{\lambda}) \quad \forall\, i = 1, \ldots, N, \qquad \hat{\lambda} := (\lambda_1, \ldots, \lambda_N),$$

where $T_0 : \mathbb{R} \to \mathbb{R}$ and $T_1^{N,i} : \mathbb{R}^N \to \mathbb{R}$ are smooth and satisfy uniform (in N) regularity estimates. More precisely, T^N is of class C^{23} and we have the decomposition $T_1^{N,i} = X_1^{N,i} + \frac{1}{N}X_2^{N,i}$ where

$$\sup_{1\leq k\leq N} \|X_1^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})} \leq C\ln N, \qquad \sup_{1\leq k\leq N} \|X_2^{N,k}\|_{L^2(\mathbb{P}_N^{\beta,V})} \leq CN^{1/2}(\ln N)^2, \tag{8.3}$$

for some constant $C > 0$ independent of N. In addition, with probability greater than $1 - N^{-N/C}$, we have for all $k, k' \in \{1, \ldots, N\}$

$$|X_1^{N,k}(\hat{\lambda}) - X_1^{N,k'}(\hat{\lambda})| \leq C\,\ln N\sqrt{N}|\lambda_k - \lambda_{k'}|. \tag{8.4}$$

As we shall see in Section 8.0.7, this result implies universality as follows (compare with [**19**, Theorem 2.4]) :

THEOREM 8.4. *Assume V, W are C^{31}, and let T_0 be as in Theorem 8.3 above. Denote $\tilde{P}_V^N$ the distribution of the increasingly ordered eigenvalues λ_i under $P_N^{\beta,V}$. There exists a constant $\hat{C} > 0$, independent of N, such that the following two facts hold true :*

(1) *Let $M \in (0,\infty)$ and $m \in \mathbb{N}$. Then, for any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ supported inside $[-M,M]^m$,*

$$\Big| \int f\big(N(\lambda_{i+1}-\lambda_i),\dots,N(\lambda_{i+m}-\lambda_i)\big)\, dP_N^{\beta,V+W} - \int f\big(T_0'(\lambda_i)N(\lambda_{i+1}-\lambda_i),\dots,T_0'(\lambda_i)N(\lambda_{i+m}-\lambda_i)\big) dP_N^{\beta,V} \Big| \leq \hat{C}\,\frac{(\ln N)^3}{N}\|f\|_\infty + \hat{C}\left(\sqrt{m}\,\frac{(\ln N)^2}{N^{1/2}} + M\,\frac{\ln N}{N^{1/2}} + \frac{M^2}{N}\right)\|\nabla f\|_\infty .$$

(2) *Let a_V (resp. a_{V+W}) denote the smallest point in the support of μ_V (resp. μ_{V+W}), so that* $\mathrm{supp}(\mu_V) \subset [a_V,\infty)$ *(resp.* $\mathrm{supp}(\mu_{V+W}) \subset [a_{V+W},\infty)$*), and let $M \in (0,\infty)$. Then, for any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ supported inside $[-M,M]^m$, we have*

$$\Big| \int f\big(N^{2/3}(\lambda_1 - a_{V+W}),\dots,N^{2/3}(\lambda_m - a_{V+W})\big)\, dP_N^{\beta,V+W} - \int f\Big(N^{2/3}T_0'(a_V)\big(\lambda_1 - a_V\big),\dots,N^{2/3}T_0'(a_V)\big(\lambda_m - a_V\big)\Big)\, dP_N^{\beta,V} \Big| \leq \hat{C}\,\frac{(\ln N)^3}{N}\|f\|_\infty + \hat{C}\left(\sqrt{m}\,\frac{(\ln N)^2}{N^{5/6}} + M\,\frac{\ln N}{N^{5/6}} + \frac{M^2}{N^{4/3}} + \frac{\ln N}{N^{1/3}}\right)\|\nabla f\|_\infty .$$

The same bound holds around the largest point in the support of μ_V.

Universality for correlation functions (which involves taking unbounded test functions) can also be treated provided they are averaged on a small eigenvalue interval, see [**50**].

8.0.1. Approximate Monge-Ampère equation. In this section, we explain a strategy to construct approximate transport maps and how it is related to the previous analysis of Dyson-Schwinger equations. Namely we consider two probability measures on $\mathbb{R}^d$

$$\mu_{V_i}(dx_1,\cdots,dx_d) = \frac{1}{Z_{V_i}}\exp\{-V_i(x)\}dx_1\cdots dx_d,\ i=0,1$$

and we would like to find a map $T_1 : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

$$\|\mu_{V_1} - T_1\#\mu_{V_0}\|_{TV} \leq \varepsilon$$

for $\varepsilon > 0$ small. The idea is to start from the observation that a transport map should satisfy the transport equation :

$$-V_0(x) - \ln Z_{V_0} = -V_1(T_1(x)) + \ln \mathrm{Jac}(T_1(x)) - \ln Z_{V_1}\,. \tag{8.5}$$

Such an equation is difficult to solve, in particular due to the singularity of the logarithm. A way to linearize it is to consider a flow of transport maps T_t such that $T_t\#\mu_{V_0} = \mu_{V_t}$. We take $V_t = tV_1 + (1-t)V_0$. The transport maps T_t should also satisfy the transport equation (8.5). Assuming that they are smooth functions of t, we differentiate this equation in the time variable and compose by T_t^{-1} to deduce that if $Y_t = \partial_t T_t \circ T_t^{-1}$, we have

$$0 = \langle \nabla V_t(x), Y_t(x)\rangle + (V_1 - V_0)(x) + \mathrm{div}(Y_t(x)) - \partial_t \ln Z_{V_t}\,.$$

We denote hereafter

$$D_{V_t} f(x) := -\langle \nabla V_t(x), f(x)\rangle + \operatorname{div}(f(x))\,.$$

The idea of approximate transport map is to allow some flexibility by asking only that

$$R_t := D_{V_t} Y_t + (V_1 - V_0)(x) - \partial_t \ln Z_{V_t} \tag{8.6}$$

is small. We then will see in Property 8.5 that this implies that T_t is almost a transport map. Indeed, we have

PROPERTY 8.5. Let Y be a smooth vector field and let R be given by (8.6). Let T_t be solution of

$$\partial_t T_t(x) = Y_t(T_t(x)), \qquad T_0 = Id$$

Then for any measurable function χ bounded uniformly by one, any $t \in [0,1]$

$$\left|\int \chi(x) d\mu_{V_t}(x) - \int \chi(T_t(x)) d\mu_{V_0}\right| \le \int_0^t \int |R_s(x)| d\mu_{V_s}(x) ds$$

PROOF. Denoting $\rho_t(x) = \frac{d\mu_{V_t}(x)}{dx}$, we have

$$\int \chi(x) d\mu_{V_t}(x) = \int \chi(T_t(x)) \mathrm{Jac}(T_t) \rho_t(T_t(x)) dx$$

we deduce that

$$\begin{aligned} &\left|\int \chi(x) d\mu_{V_t}(x) - \int \chi(T_t(x)) d\mu_{V_0}\right| \\ &\le \|\chi\|_\infty \int |\rho_0(x) - \mathrm{Jac}(T_t)\rho_t(T_t(x))| dx =: A_t \end{aligned}$$

Moreover since the derivative of a norm is bounded above by the norm of the derivative, we find that

$$\begin{aligned} \partial_t A_t &\le \int |\partial_t(\mathrm{Jac}(T_t)\rho_t(T_t(x)))| dx \\ &= \int |R_t(T_t(x))| \mathrm{Jac}(T_t) \rho_t(T_t(x)) dx \\ &= \int |R_t(x)| d\mu_{V_t}(x) \end{aligned}$$

$\diamond$

The previous Lemma can be generalized to include test functions with supremum norm blowing up like ε^{-k} for some finite k. This is for instance the case of test functions defining correlation functions which are of the kind $\sum_{i_1\neq i_2 \cdots \neq i_k} f(x_{i_1}, \ldots, x_{i_k})$ and have uniform norm bounded by N^k (here $\epsilon^{-1} = N$).

LEMMA 8.6. *Let $R_.$ be defined in* (8.6). *Assume that, for any $q < \infty$, there exists a constant C_q such that*

$$\|R_t\|_{L^q(\mu_{V_t})} \le C_q\, \varepsilon (\ln \varepsilon^{-1})^3 \qquad \forall\, t \in [0,1]\,. \tag{8.7}$$

Let $\chi : \mathbb{R}^d \to \mathbb{R}^+$ be a nonnegative measurable function satisfying $\|\chi\|_\infty \le \varepsilon^{-k}$ for some $k \ge 0$. Then, for any $\eta > 0$ there exists a constant $C_{k,\eta}$, independent of χ, such that

$$\left|\ln\biggl(1+\int \chi\, d\mu_{V_1}\biggr) - \ln\biggl(1+\int \chi\circ T\, d\mu_{V_0}\biggr)\right| \le C_{k,\eta}\,\varepsilon^{1-\eta}.$$

PROOF. Let ρ_t denote the density of μ_{V_t} with respect to the Lebesgue measure $\mathcal{L}$. Then, by a direct computation one can check that ρ_t, Y, and $\mathcal{R}_t = R_t(Y)$ are related by the following formula :

$$\partial_t\rho_t + \operatorname{div}(Y_t\rho_t) = \mathcal{R}_t\rho_t. \tag{8.8}$$

Now, given a smooth function $\chi : \mathbb{R}^d \to \mathbb{R}^+$ satisfying $\|\chi\|_\infty \le \varepsilon^{-k}$ we define

$$\chi_t := \chi\circ T_1\circ (T_t)^{-1} \qquad \forall\, t\in[0,1]. \tag{8.9}$$

Note that with this definition $\chi_1 = \chi$. Also, since $\chi_t\circ T_t$ is constant in time, differentiating with respect to t we deduce that

$$0 = \frac{d}{dt}\bigl(\chi_t\circ T_t\bigr) = \Bigl(\partial_t\chi_t + Y_t\cdot\nabla\chi_t\Bigr)\circ T_t,$$

hence χ_t solves the transport equation

$$\partial_t\chi_t + Y_t\cdot\nabla\chi_t = 0, \qquad \chi_1 = \chi. \tag{8.10}$$

Combining (8.8) and (8.10), we compute

$$\begin{aligned}
\frac{d}{dt}\int \chi_t\,\rho_t\,d\mathcal{L} &= \int \partial_t\chi_t\,\rho_t\,d\mathcal{L} + \int \chi_t\,\partial_t\rho_t\,d\mathcal{L}\\
&= -\int Y_t\cdot\nabla\chi_t\,\rho_t\,d\mathcal{L} - \int \chi_t\operatorname{div}(Y_t\rho_t)\,d\mathcal{L} + \int \chi_t\,R_t\,\rho_t\,d\mathcal{L}\\
&= \int \chi_t\,R_t\,\rho_t\,d\mathcal{L}.
\end{aligned}$$

We want to control the last term. To this aim we notice that, since $\|\chi\|_\infty \le \varepsilon^{-k}$, it follows immediately from (8.9) that $\|\chi_t\|_\infty \le \varepsilon^{-k}$ for any $t\in[0,1]$. Hence, using Hölder inequality and (8.7), for any $p>1$ we can bound

$$\begin{aligned}
\left|\int \chi_t\,R_t\,\rho_t\,d\mathcal{L}\right| &\le \|\chi_t\|_{L^p(\mu_{V_t})}\|R_t\|_{L^q(\mu_{V_t})} \le \|\chi_t\|_\infty^{\frac{p-1}{p}}\|\chi_t\|_{L^1(\mu_{V_t})}^{1/p}\|R_t\|_{L^q(\mu_{V_t})}\\
&\le \varepsilon^{-\frac{k(p-1)}{p}}\|\chi_t\|_{L^1(\mu_{V_t})}^{1/p}\|R_t\|_{L^q(\mu_{V_t})}\\
&\le C_q\varepsilon^{-\frac{k(p-1)}{p}}\varepsilon(\ln\varepsilon^{-1})^3\|\chi_t\|_{L^1(\mu_{V_t})}^{1/p},
\end{aligned}$$

where $q := \frac{p}{p-1}$. Hence, given $\eta > 0$, we can choose $p := 1 + \frac{\eta}{2k}$ to obtain

$$\left|\int \chi_t\,R_t\,\rho_t\,d\mathcal{L}\right| \le C_q\,\varepsilon^{1-\eta}\|\chi_t\|_{L^1(\mu_{V_t})}^{1/p} \le C\,\varepsilon^{1-\eta}\Bigl(1+\|\chi_t\|_{L^1(\mu_{V_t})}\Bigr), \tag{8.11}$$

where C depends only on C_q, k, and η. Therefore, setting

$$Z(t) := \int \chi_t\,\rho_t\,d\mathcal{L} = \|\chi_t\|_{L^1(\mu_{V_t})}$$

(recall that $\chi_t \ge 0$), we proved by (8.11) that

$$|\dot Z(t)| \le C\,\varepsilon^{1-\eta}\bigl(1+Z(t)\bigr),$$

which implies that

$$\left|\ln\bigl(1+Z(1)\bigr)-\ln\bigl(1+Z(0)\bigr)\right| \le C\,\varepsilon^{1-\eta}.$$

This proves the desired result when χ is smooth. By approximation the result extends to all measurable functions $\chi:\mathbb{R}\to\mathbb{R}^+$ satisfying $\|\chi\|_\infty \le \varepsilon^{-k}$, concluding the proof. $\diamond$

Later on, ε will be the inverse of the dimension and so we will try to find a function Y which asymptotically satisfies the transport equation with small reminder (8.6) satisfying a bound as (8.7). This program is quite similar to what we have been doing when solving asymptotically the Dyson-Schwinger equation. Indeed, in the case of Dyson-Schwinger equations we tried to solve asymptotically

$$\mathbb{E}_{\mu_V}[D_V Y]=0$$

for well chosen functions Y (namely functions of the empirical measure) whereas here we try to find Y asymptotically solving

$$R_t := D_{V_t}Y_t + (V_1-V_0)(x) - \partial_t \ln Z_{V_t} \ll 1.$$

It will turn out that we will seek for Y given as a function of the empirical measure. It is therefore no surprise that in both cases the Master operator will appear in the first large N limit of the equations we are trying to solve, and that its inversion will be the key to our analysis.

8.0.2. Propagating the hypotheses. The central idea of is to build transport maps as flows, and in fact to build transport maps between $P_N^{\beta,V}$ and P_N^{β,V_t} where $t\mapsto V_t$ is a smooth function so that $V_0=V$ and $V_1=V+W$. In order to have a good interpolation between V and $V+W$, it will be convenient to assume that the supports of the two equilibrium measures μ_V and μ_{V+W} are the same. This can always be done up to an affine transformation. Indeed, if $L:\mathbb{R}\to\mathbb{R}$ is the affine transformation which maps $[a_1,b_1]$ (the support of μ_{V_1}) onto $[a_0,b_0]$ (the support of μ_{V_0}), we first construct a transport map from $\mathbb{P}_N^{\beta,V}$ to $L_\#^{\otimes N}\mathbb{P}_N^{\beta,V+W}=\mathbb{P}_N^{\beta,V+\tilde W}$ where

$$\tilde W = V\circ L^{-1} + W\circ L^{-1} - V, \tag{8.12}$$

and then we simply compose our transport map with $(L^{-1})^{\otimes N}$ to get the desired map from $P_N^{\beta,V}$ to $P_N^{\beta,V+W}$. Hence, without loss of generality we will hereafter assume that μ_V and μ_{V+W} have the same support. We then consider the interpolation μ_{V_t} with $V_t=V+tW$, $t\in[0,1]$. We have :

LEMMA 8.7. *If Hypotheses 8.1 and 8.2 are fulfilled for $t=0,1$, then Hypothesis 8.1 and 8.2 are fulfilled for all $t\in[0,1]$. Moreover, we may assume without loss of generality that V goes to infinity as fast as we want up to modify $P_N^{\beta,V}$ and $P_N^{\beta,V+W}$ by a negligible error (in total variation).*

The first point is a direct consequence of the fact that if μ_{V_0} and μ_{V_1} have the same support, then $t\mu_{V_1}+(1-t)\mu_{V_0}$ is the equilibrium measure for V_t according to the characterization of the equilibrium measure. The second point is that our hypothesis 8.2 insures that the eigenvalues will stay in a neighborhood of the support so that we can always modify the potential at a positive distance of this support.

Thanks to the above lemma and the discussion immediately before it, we can assume that μ_V and μ_{V+W} have the same support, that W is bounded, and that V goes to infinity faster than x^p for some $p > 0$ large enough.

8.0.3. Monge-Ampère equation. Given the two probability densities P_N^{β,V_t} to P_N^{β,V_s} with $0 \le t \le s \le 1$, by optimal transport theory it is well-known that there exists a (convex) function $\phi_{t,s}^N$ such that $\nabla\phi_{t,s}^N$ pushes forward P_N^{β,V_t} onto P_N^{β,V_s} and which satisfies the Monge-Ampère equation

$$\det(D^2\phi_{t,s}^N) = \frac{\rho_t}{\rho_s(\nabla\phi_{t,s}^N)}, \qquad \rho_\tau := \frac{dP_N^{\beta,V_\tau}}{d\lambda_1 \ldots d\lambda_N}\,.$$

Because $\phi_{t,t}(x) = |x|^2/2$ (since $\nabla\phi_{t,t}$ is the identity map), we differentiate the above equation with respect to s and set $s = t$ to get

$$\frac{1}{\beta}\Delta\psi_t^N = c_t^N - \sum_{i<j}\frac{\partial_i\psi_t^N - \partial_j\psi_t^N}{\lambda_i - \lambda_j} + N\sum_i W(\lambda_i) + N\sum_i V_t'(\lambda_i)\partial_i\psi_t^N, \tag{8.13}$$

where $\psi_t^N := \partial_s\phi_{t,s}^N|_{s=t}$ and

$$c_t^N := -N\int\sum_i W(\lambda_i)\, dP_N^{\beta,V_t} = \partial_t \ln Z_{V_t}^N\,.$$

Although this is a formal argument, it suggests to us a way to construct maps $T_{0,t}^N : \mathbb{R}^N \to \mathbb{R}^N$ sending $P_N^{\beta,V}$ onto P_N^{β,V_t} : indeed, if $T_{0,t}^N$ sends $P_N^{\beta,V}$ onto P_N^{β,V_t} then $\nabla\phi_{t,s}^N \circ T_{0,t}^N$ sends $P_N^{\beta,V}$ onto P_N^{β,V_s}. Hence, we may try to find $T_{0,s}^N$ of the form $T_{0,s}^N = \nabla\phi_{t,s}^N \circ T_{0,t}^N + o(s-t)$. By differentiating this relation with respect to s and setting $s = t$ we obtain $\partial_t T_{0,t}^N = \nabla\psi_t^N(T_{0,t}^N)$.

Thus, to construct a transport map T^N from $P_N^{\beta,V}$ onto $P_N^{\beta,V+W}$ we could first try to find ψ_t^N by solving (8.13), then solve the ODE $\dot X_t^N = \nabla\psi_t^N(X_t^N)$, and finally set $T^N := X_1^N$. We notice that, in general, T^N is not an optimal transport map for the quadratic cost.

Unfortunately, finding an exact solution of (8.13) enjoying "nice" regularity estimates that are uniform in N seems extremely difficult. So, instead, we make an ansatz on the structure of ψ_t^N (see (8.16) below) : the idea is that at first order eigenvalues do not interact, then at order $1/N$ eigenvalues interact at most by pairs, and so on. As we shall see, in order to construct a function which enjoys nice regularity estimates and satisfies (8.13) up to a error that goes to zero as $N \to \infty$, it will be enough to stop the expansion at $1/N$. Actually, while the argument before provides us the right intuition, we notice that there is no need to assume that the vector field generating the flow X_t^N is a gradient, so we will consider general vector fields $\mathbf{Y}_t^N = (\mathbf{Y}_{1,t}^N, \ldots, \mathbf{Y}_{N,t}^N) : \mathbb{R}^N \to \mathbb{R}^N$ that approximately solve

$$\frac{1}{\beta}\mathrm{div}\mathbf{Y}_t^N = c_t^N - \sum_{i<j}\frac{\mathbf{Y}_{i,t}^N - \mathbf{Y}_{j,t}^N}{\lambda_i - \lambda_j} + N\sum_i W(\lambda_i) + N\sum_i V_t'(\lambda_i)\mathbf{Y}_{i,t}^N, \tag{8.14}$$

8.0.4. Constructing an approximate solution to (8.14). Fix $t \in [0,1]$ and define the random measures

$$\hat\mu^N := \frac{1}{N}\sum_i \delta_{\lambda_i} \quad \text{and} \quad M_N := \sum_i \delta_{\lambda_i} - N\mu_{V_t}. \tag{8.15}$$

As we explained in the previous section, a natural ansatz to find an approximate solution of (8.13) given by

$$\begin{aligned}\psi_t^N(\lambda_1,\dots,\lambda_N) \quad := \quad & \int \Big[\psi_{0,t}(x)+\frac{1}{N}\psi_{1,t}(x)\Big]\,dM_N(x) \\ & +\frac{1}{2N}\iint \psi_{2,t}(x,y)\,dM_N(x)\,dM_N(y),\end{aligned} \tag{8.16}$$

for some functions $\psi_{0,t},\psi_{1,t}:\mathbb{R}\to\mathbb{R}$ and $\psi_{2,t}:\mathbb{R}^2\to\mathbb{R}$, where (without loss of generality) $\psi_{2,t}(x,y)=\psi_{2,t}(y,x)$.

Since we do not want to use gradient of functions but general vector fields (as this gives us more flexibility), in order to find an ansatz for an approximate solution of (8.14) we compute first the gradient of ψ :

$$\partial_i\psi_t^N=\psi'_{0,t}(\lambda_i)+\frac{1}{N}\psi'_{1,t}(\lambda_i)+\frac{1}{N}\xi_{1,t}^N(\lambda_i,M_N),$$

with $\xi_{1,t}^N(x,M_N):=\int\partial_1\psi_{2,t}(x,y)\,dM_N(y)$. This suggests us the following ansatz for the components of $\mathbf{Y}_t^N$:

$$\mathbf{Y}_{i,t}^N(\lambda_1,\dots,\lambda_N):=\mathbf{y}_{0,t}(\lambda_i)+\frac{1}{N}\mathbf{y}_{1,t}(\lambda_i)+\frac{1}{N}\boldsymbol{\xi}_t(\lambda_i,M_N), \tag{8.17}$$

where $\boldsymbol{\xi}_t(x,M_N):=\int\mathbf{z}_t(x,y)\,dM_N(y)$, and the functions $\mathbf{y}_{0,t},\mathbf{y}_{1,t}:\mathbb{R}\to\mathbb{R}$, $\mathbf{z}_t:\mathbb{R}^2\to\mathbb{R}$ need to be chosen.

Here and in the following, given a function of two variables ψ, we write $\psi\in C^{s,v}$ to denote that it is s times continuously differentiable with respect to the first variable and v times with respect to the second.

The aim of this section is to prove the following result :

Proposition 8.8. Assume $V,W\in C^r$ with $r\geq 31$. Then, there exist $\mathbf{y}_{0,t}\in C^{r-3}$, $\mathbf{y}_{1,t}\in C^{r-9}$, and $\mathbf{z}_t\in C^{s,v}$ for $s+v\leq r-6$, such that

$$\mathcal{R}_t^N:=\left(c_t^N-\sum_{i<j}\frac{\mathbf{Y}_{i,t}^N-\mathbf{Y}_{j,t}^N}{\lambda_i-\lambda_j}+N\sum_i W(\lambda_i)+N\sum_i V_t'(\lambda_i)\mathbf{Y}_{i,t}^N\right)-\frac{1}{\beta}\mathrm{div}\mathbf{Y}_t^N$$

satisfies for all $p\geq 1$

$$\|\mathcal{R}_t^N\|_{L^p(P_N^{\beta,V_t})}\leq C_p\,\frac{(\ln N)^3}{N}$$

for some positive constant C independent of $t\in[0,1]$.

The proof of this proposition is pretty involved, and will take the rest of the section.

8.0.4.1. *Finding an equation for* $\mathbf{y}_{0,t},\mathbf{y}_{1,t},\mathbf{z}_t$. Using (8.17) we compute

$$\mathrm{div}\mathbf{Y}_t^N=N\int\mathbf{y}'_{0,t}(x)\,d\hat\mu^N(x)+\int\mathbf{y}'_{1,t}(x)\,d\hat\mu^N(x)+\int\partial_1\boldsymbol{\xi}_t(x,M_N)\,d\hat\mu^N(x)+\boldsymbol{\eta}(\hat\mu^N),$$

where, given a measure ν, we set

$$\boldsymbol{\eta}(\nu)\quad:=\quad\int\partial_2\mathbf{z}_t(y,y)\,d\nu(y).$$

Therefore, recalling that $M_N = N(\hat{\mu}^N - \mu_{V_t})$, we get that

$$\begin{aligned}
\mathcal{R}_t^N &= -\frac{N^2}{2}\iint \frac{\mathbf{y}_{0,t}(x)-\mathbf{y}_{0,t}(y)}{x-y}\,d\hat{\mu}^N(x)\,d\hat{\mu}^N(y)\\
&+N^2\int (V_t'(x)\,\mathbf{y}_{0,t}(x)+W(x))\,d\hat{\mu}^N(x)\\
&-\frac{N}{2}\iint \frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y}\,d\hat{\mu}^N(x)\,d\hat{\mu}^N(y)+N\int V_t'(x)\,\mathbf{y}_{1,t}(x)\,d\hat{\mu}^N(x)\\
&-\frac{N}{2}\iint \frac{\boldsymbol{\xi}_t(x,M_N)-\boldsymbol{\xi}_t(y,M_N)}{x-y}\,d\hat{\mu}^N(x)\,d\hat{\mu}^N(y)+N\int V_t'(x)\,\boldsymbol{\xi}_t(x,M_N)\,d\hat{\mu}^N(x)\\
&-\frac{1}{N}\boldsymbol{\eta}(M_N)-N\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\int \mathbf{y}_{0,t}'(x)\,d\hat{\mu}^N(x)-\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\int \mathbf{y}_{1,t}'(x)\,d\hat{\mu}^N(x)\\
&-\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\int \partial_1\boldsymbol{\xi}_t(x,M_N)\,d\hat{\mu}^N(x)-\boldsymbol{\eta}(\mu_{V_t})+\tilde{c}_t^N,
\end{aligned}$$

where $\tilde{c}_t^N$ is a constant and we use the convention that, when we integrate a function of the form $\frac{f(x)-f(y)}{x-y}$ with respect to $\hat{\mu}^N\otimes\hat{\mu}^N$, the diagonal terms give $f'(x)$.

We now observe that $\hat{\mu}^N$ converges towards μ_{V_t} as $N\to\infty$ minimizing the corresponding large deviation rate function I_{V_t} (see Theorem 4.4). Hence, considering $\mu_\varepsilon := (x+\varepsilon f)_{\#}\mu_{V_t}$ and writing that $I_{V_t}(\mu_\varepsilon)\ge I_{V_t}(\mu_{V_t})$, by taking the derivative with respect to ε at $\varepsilon=0$ we get

$$\int V_t'(x)f(x)\,d\mu_{V_t}(x)=\frac{1}{2}\iint \frac{f(x)-f(y)}{x-y}\,d\mu_{V_t}(x)\,d\mu_{V_t}(y) \tag{8.18}$$

for all smooth bounded functions $f:\mathbb{R}\to\mathbb{R}$.

This fact allows us to recenter $\hat{\mu}^N$ by μ_{V_t} in the formula above : more precisely, if we set

$$\Xi_t f(x) := -\int \frac{f(x)-f(y)}{x-y}\,d\mu_{V_t}(y)+V_t'(x)f(x), \tag{8.19}$$

then

$$\begin{aligned}
N^2\int V_t'(x)\,f(x)\,d\hat{\mu}^N(x)&-\frac{N^2}{2}\iint \frac{f(x)-f(y)}{x-y}\,d\hat{\mu}^N(x)\,d\hat{\mu}^N(y)\\
&=N\int \Xi_t f(x)\,dM_N(x)-\frac{1}{2}\iint \frac{f(x)-f(y)}{x-y}\,dM_N(x)\,dM_N(y)
\end{aligned}$$

Applying this identity to $f=\mathbf{y}_{0,t},\mathbf{y}_{1,t},\boldsymbol{\xi}_t(\cdot,M_N)$ and recalling the definition of $\boldsymbol{\xi}_t(\cdot,M_N)$ (see (8.17)), we find

$$\begin{aligned}
\mathcal{R}_t^N \;=\; & N\int [\Xi_t\mathbf{y}_{0,t}+W](x)\,dM_N(x)\\
&+\int\left(\Xi_t\mathbf{y}_{1,t}(x)+\Big(\frac{1}{2}-\frac{1}{\beta}\Big)\Big[\mathbf{y}_{0,t}'(x)+\int \partial_1\mathbf{z}_t(z,x)d\mu_{V_t}(z)\Big]\right)dM_N(x)\\
&+\iint dM_N(x)\,dM_N(y)\left(\Xi_t\mathbf{z}_t(\cdot,y)[x]-\frac{1}{2}\frac{\mathbf{y}_{0,t}(x)-\mathbf{y}_{0,t}(y)}{x-y}\right)+C_t^N+E_N,
\end{aligned}$$

where

$$\Xi_t\mathbf{z}_t(\cdot,y)[x]=-\int \frac{\mathbf{z}_t(x,y)-\mathbf{z}_t(\tilde{x},y)}{x-\tilde{x}}\,d\mu_{V_t}(\tilde{x})+V_t'(x)\,\mathbf{z}_t(x,y),$$

C_t^N is a deterministic term, and E_N is a remainder that we will prove to be negligible :

$$\begin{aligned}
E_N \quad := \quad & -\frac{1}{N}\int \partial_2 \mathbf{z}_t(x,x)\, dM_N(x) - \frac{1}{N}\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\int \mathbf{y}'_{1,t}(x)\, dM_N(x) \\
& -\frac{1}{N}\Big(\frac{1}{\beta}-\frac{1}{2}\Big)\iint \partial_1 \mathbf{z}_t(x,y)\, dM_N(x)\, dM_N(y) \\
& -\frac{1}{2N}\iint \frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y}\, dM_N(x)\, dM_N(y) \\
& -\frac{1}{2N}\iiint \frac{\mathbf{z}_t(x,y)-\mathbf{z}_t(\tilde{x},y)}{x-\tilde{x}}\, dM_N(x)\, dM_N(y)\, dM_N(\tilde{x})\,. \qquad (8.20)
\end{aligned}$$

Hence, for $\mathcal{R}_t^N$ to be small we want to impose

$$\begin{aligned}
\Xi_t \mathbf{y}_{0,t} \quad &= \quad -W + c, \\
\Xi_t \mathbf{z}_t(\cdot,y)[x] \quad &= \quad -\frac{\beta}{2}\frac{\mathbf{y}_{0,t}(x)-\mathbf{y}_{0,t}(y)}{x-y} + c(y), \qquad (8.21)\\
\Xi_t \mathbf{y}_{1,t} \quad &= \quad -\Big(\frac{\beta}{2}-1\Big)\Big[\mathbf{y}'_{0,t} + \int \partial_1 \mathbf{z}_t(z,\cdot)\, d\mu_{V_t}(z)\Big] + c',
\end{aligned}$$

where c, c' are some constant to be fixed later, and $c(y)$ does not depend on x.

8.0.4.2. *Inverting the operator* Ξ. As a consequence of Lemma 4.18 we find the following result (recall that $\psi \in C^{s,v}$ means that ψ is s times continuously differentiable with respect to the first variable and v times with respect to the second).

LEMMA 8.9. *Let $r \geq 9$. If $W, V \in C^r$, we can choose $\mathbf{y}_{0,t}$ of class C^{r-3}, $\mathbf{z}_t \in C^{s,v}$ for $s+v \leq r-6$, and $\mathbf{y}_{1,t} \in C^{r-9}$. Moreover, these functions (and their derivatives) go to zero at infinity like $1/V'$ (and its corresponding derivatives).*

PROOF. Note that V_t is of class C^r as both V, W are. By Lemma 4.18 we have $\mathbf{y}_{0,t} = \Xi_t^{-1} W \in C^{r-3}$. For $\mathbf{z}_t$, we can rewrite

$$\begin{aligned}
\Xi_t \mathbf{z}_t(\cdot,y)[x] \quad &= \quad -\frac{1}{2}\int_0^1 \mathbf{y}'_{0,t}(\alpha x + (1-\alpha)y)\, d\alpha + c(y) \\
&= \quad -\frac{1}{2}\int_0^1 [\mathbf{y}'_{0,t}(\alpha x + (1-\alpha)y) + c_\alpha(y)]\, d\alpha
\end{aligned}$$

where we choose $c_\alpha(y)$ to be the unique constant provided by Lemma 4.18 which ensures that $\Xi_t^{-1}[\mathbf{y}'_{0,t}(\alpha x + (1-\alpha)y) + c_\alpha(y)]$ is smooth. This gives that $c(y) = \int_0^1 c_\alpha(y)\, d\alpha$. Since Ξ_t^{-1} is a linear integral operator, we have

$$\mathbf{z}_t(x,y) = -\frac{1}{2}\int_0^1 \Xi_t^{-1}[\mathbf{y}'_{0,t}(\alpha\,\cdot + (1-\alpha)y)](x)\, d\alpha\,.$$

As the variable y is only a translation, it is not difficult to check that $\mathbf{z}_t \in C^{s,v}$ for any $s+v \leq r-6$. It follows that

$$-\Big(\frac{1}{2}-\frac{1}{\beta}\Big)\Big[\mathbf{y}'_{0,t} + \int \partial_1 \mathbf{z}_t(z,\cdot)\, d\mu_{V_t}(z)\Big] + c'$$

is of class C^{r-7} and therefore by Lemma 4.18 we can choose $\mathbf{y}_{1,t} \in C^{r-9}$, as desired.

The decay at infinity is finally again a consequence of Lemma 4.18. $\diamond$

8.0.4.3. *Getting rid of the random error term E_N.* We show that the $L^1_{P_N^{\beta,V_t}}$-norm of the error term E_N defined in (8.20) goes to zero. In fact, this is a direct consequence of the central limit theorem of Section 4, se Lemma 4.23, as well as the concentration bounds we obtained there :

COROLLARY 8.10. Assume that $V, W \in C^r$ with $r \geq 4$. Then for all $p \geq 1$ there exists a finite constant C_p such that, for all $\lambda \in \mathbb{R}$ and all $t \in [0,1]$,

$$\int |M_N(e^{i\lambda \cdot})|^{2p} dP_N^{\beta,V_t} \quad \leq \quad C[\ln N(1+|\lambda|^4)]^{2p} \,, \tag{8.22}$$

We can now estimate E_N.

The linear term can be handled in the same way as we shall do now for the quadratic and cubic terms (which are actually more delicate), so we just focus on them.

We have two quadratic terms in M_N which sum up into

$$E_N^1 = -\frac{1}{N}\Big(\frac{1}{\beta} - \frac{1}{2}\Big) \iint \partial_1 \mathbf{z}_t(x,y)\, dM_N(x)\, dM_N(y)$$

$$-\frac{1}{2N} \iint \frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x-y}\, dM_N(x)\, dM_N(y).$$

Writing

$$\frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x-y} = \int_0^1 \mathbf{y}'_{1,t}(\alpha x + (1-\alpha)y)\, d\alpha = \int_0^1 \Big(\int \widehat{\mathbf{y}'_{1,t}}(\xi) e^{i(\alpha x + (1-\alpha)y)\xi} d\xi \Big)\, d\alpha$$

we see that

$$\iint \frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x-y}\, dM_N(x)\, dM_N(y) = \int d\xi\, \widehat{\mathbf{y}'_{1,t}}(\xi) \int_0^1 d\alpha\, M_N(e^{i\alpha\xi \cdot}) M_N(e^{i(1-\alpha)\xi \cdot}),$$

so using (8.22) we get

$$\int |E_N^1|\, dP_N^{\beta,V_t} \leq C\, \frac{(\ln N)^2}{N} \Big(\int d\xi\, |\hat{\mathbf{y}}_{1,t}|(\xi)\, |\xi|\, \big(1+|\xi|^4\big)^2$$

$$+ \iint d\xi\, d\zeta\, |\hat{\mathbf{z}}_t|(\xi,\zeta)\, |\xi|\, \big(1+|\xi|^4\big)\big(1+|\zeta|^4\big) \Big).$$

It is easy to see that the right hand side is finite if $\mathbf{y}_{1,t}$ and $\mathbf{z}_t$ are smooth enough (recall that these functions and their derivatives decay fast at infinity). More precisely, to ensure that

$$|\hat{\mathbf{y}}_{1,t}|(\xi)\, |\xi|\, \big(1+|\xi|^4\big)^2 \leq \frac{C}{1+|\xi|^2} \in L^1(\mathbb{R})$$

and

$$|\hat{\mathbf{z}}_t|(\xi,\zeta)\, |\xi|\, \big(1+|\xi|^4\big)\big(1+|\zeta|^4\big) \leq \frac{C}{1+|\xi|^3+|\zeta|^3} \in L^1(\mathbb{R}^2),$$

we need $\mathbf{y}_{1,t} \in C^{11}$ and $\mathbf{z}_t \in C^{8,7}$, so (recalling Lemma 8.9) $V, W \in C^{25}$ is enough to guarantee that the right hand side is finite.

Using (8.22), and Hölder inequality, we can similarly bound the expectation of the cubic term

$$\begin{aligned} E_N^2 &= \frac{1}{2N}\iiint \frac{\mathbf{z}_t(x,y)-\mathbf{z}_t(\tilde{x},y)}{x-\tilde{x}}\,dM_N(x)\,dM_N(y)\,dM_N(\tilde{x}) \\ &= i\frac{1}{2N}\iint d\xi\,d\zeta\,\widehat{\partial_1\mathbf{z}_t}(\xi,\zeta)\int_0^1 d\alpha\,M_N(e^{i\alpha\xi\cdot})M_N(e^{i(1-\alpha)\xi\cdot})M_N(e^{i\zeta\cdot}) \end{aligned}$$

to get

$$\int |E_N^2|\,dP_N^{\beta,V_t} \le C\,\frac{(\ln N)^3}{N}\iint d\xi\,d\zeta\,|\hat{\mathbf{z}}_t(\xi,\zeta)|\,|\xi|\left(1+|\xi|^4\right)^2\left(1+|\zeta|^4\right).$$

Again the right hand side is finite if $\mathbf{z}_t \in C^{12,7}$, which is ensured by Lemma 8.9 if V, W are of class C^{31}.

8.0.5. Control on the deterministic term C_t^N. By what we proved above we have

$$\int |\mathcal{R}_t^N - C_t^N|\,dP_N^{\beta,V_t} \le C\,\frac{(\ln N)^3}{N},$$

thus, in particular,

$$\left|C_t^N - \mathbb{E}[\mathcal{R}_t^N]\right| \le C\,\frac{(\ln N)^3}{N}.$$

Notice now that, by construction,

$$\mathcal{R}_t^N = -\mathcal{L}\mathbf{Y}_t^N + N\sum_i W(\lambda_i) + c_t^N$$

with $c_t^N = -\mathbb{E}[N\sum_i W(\lambda_i)]$ and

$$\mathcal{L}\mathbf{Y} := \mathrm{div}\mathbf{Y} + \beta\sum_{i<j}\frac{\mathbf{Y}^i - \mathbf{Y}^j}{\lambda_i - \lambda_j} - N\sum_i V'(\lambda_i)\mathbf{Y}^i.$$

Also, an integration by parts shows that, under $P_N^{\beta,V}$, $\mathbb{E}[\mathcal{L}\mathbf{Y}] = 0$ for any vector field $\mathbf{Y}$. This implies that $\mathbb{E}[\mathcal{R}_t^N] = 0$, therefore $|C_t^N| \le C\,\frac{(\ln N)^3}{N}$.

This concludes the proof of Proposition 8.8.

8.0.6. Reconstructing the transport map via the flow. We finally need to study the properties of the flow generated by the vector field $\mathbf{Y}_t^N$ defined in (8.17). As we shall see, we will need to assume that $W, V \in C^r$ with $r \ge 16$.

We consider the flow of $\mathbf{Y}_t^N$ given by

$$X_t^N : \mathbb{R}^N \to \mathbb{R}^N, \qquad \dot{X}_t^N = \mathbf{Y}_t^N(X_t^N).$$

Recalling the form of $\mathbf{Y}_t^N$ (see (8.17)) it is natural to expect that we can give an expansion for X_t^N. More precisely, let us define the flow of $\mathbf{y}_{0,t}$,

$$X_{0,t} : \mathbb{R} \to \mathbb{R}, \qquad \dot{X}_{0,t} = \mathbf{y}_{0,t}(X_{0,t}), \quad X_{0,0}(\lambda) = \lambda\,. \tag{8.23}$$

Observe that we expect $X_{0,1}$ to be a transport map of μ_{V_0} onto μ_{V_t}. Let $X_{1,t}^N = (X_{1,t}^{N,1}, \dots, X_{1,t}^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ be the solution of the linear ODE

$$\begin{aligned} \dot{X}_{1,t}^{N,k}(\lambda_1,\dots,\lambda_N) &= \mathbf{y}_{0,t}'(X_{0,t}(\lambda_k)) \cdot X_{1,t}^{N,k}(\lambda_1,\dots,\lambda_N) + \mathbf{y}_{1,t}(X_{0,t}(\lambda_k)) \\ &\quad + \int \mathbf{z}_t(X_{0,t}(\lambda_k), y)\, dM_N^{X_{0,t}}(y) \\ &\quad + \frac{1}{N}\sum_{j=1}^N \partial_2 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big) \cdot X_{1,t}^{N,j}(\lambda_1,\dots,\lambda_N) \end{aligned} \tag{8.24}$$

with the initial condition $X_{1,t}^N = 0$, where $M_N^{X_{0,t}}$ is defined as

$$\int f(y)\, dM_N^{X_{0,t}}(y) := \sum_{i=1}^N \Big[f(X_{0,t}(\lambda_i)) - \int f\, d\mu_{V_t} \Big] \qquad \forall\, f \in C_c(\mathbb{R}).$$

If we set

$$X_{0,t}^N(\lambda_1,\dots,\lambda_N) := \big(X_{0,t}(\lambda_1),\dots,X_{0,t}(\lambda_N)\big),$$

then the following result holds.

Lemma 8.11. *Assume that $W, V \in C^r$ with $r \geq 16$. Then the flow $X_t^N = (X_t^{N,1}, \dots, X_t^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ is of class C^{r-9} and the following properties hold : Let $X_{0,t}$ and $X_{1,t}^N$ be as in (8.23) and (8.24) above, and define $X_{2,t}^N : \mathbb{R}^N \to \mathbb{R}^N$ via the identity*

$$X_t^N = X_{0,t}^N + \frac{1}{N} X_{1,t}^N + \frac{1}{N^2} X_{2,t}^N.$$

Then

$$\sup_{1\le k\le N} \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})} \le C \ln N, \qquad \|X_{2,t}^N\|_{L^2(\mathbb{P}_N^{\beta,V})} \le C N^{1/2} (\ln N)^2, \tag{8.25}$$

where

$$\|X_{i,t}^N\|_{L^2(\mathbb{P}_N^{\beta,V})} = \left(\int |X_{i,t}^N|^2 d\mathbb{P}_N^{\beta,V}\right)^{1/2}, \qquad |X_{i,t}^N| := \sqrt{\sum_{j=1,\dots,N} |X_{i,t}^{N,j}|^2}, \quad i = 0,1,2.$$

In addition, there exists a constant $C > 0$ such that, with probability greater than $1 - N^{-N/C}$,

$$\max_{1\le k,k'\le N} |X_{1,t}^{N,k}(\lambda_1,\dots,\lambda_N) - X_{1,t}^{N,k'}(\lambda_1,\dots,\lambda_N)| \le C \ln N \sqrt{N} |\lambda_k - \lambda_{k'}|. \tag{8.26}$$

Proof. Since $\mathbf{Y}_t^N \in C^{r-9}$ (see Lemma 8.9) it follows by Cauchy-Lipschitz theory that X_t^N is of class C^{r-9}.

Using the notation $\hat\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{R}^N$ and

$$X_t^{N,k,\sigma}(\hat\lambda) := X_{0,t}(\lambda_k) + \sigma \frac{X_{1,t}^{N,k}}{N}(\hat\lambda) + \sigma \frac{X_{2,t}^{N,k}}{N^2}(\hat\lambda) = (1-\sigma) X_{0,t}(\lambda_k) + \sigma X_t^{N,k}(\hat\lambda)$$

and defining the measure $M_N^{X_t^{N,s}}$ so that forall $f \in C_c(\mathbb{R})$,

$$\int f(y)\, dM_N^{X_t^{N,s}}(y) = \sum_{i=1}^N \Big[f\big((1-s)X_{0,t}(\lambda_i) + s X_t^{N,i}(\hat\lambda)\big) - \int f\, d\mu_{V_t} \Big]. \tag{8.27}$$

By a Taylor expansion we get an ODE for $X_{2,t}^N$:

$$
\begin{aligned}
\dot{X}_{2,t}^{N,k}(\hat{\lambda}) \quad = \quad & \int_0^1 \mathbf{y}'_{0,t}\Big(X_t^{N,k,s}(\hat{\lambda})\Big)\, ds \cdot X_{2,t}^{N,k}(\hat{\lambda}) \\
& +N \int_0^1 \Big[\mathbf{y}'_{0,t}\Big(X_t^{N,k,s}(\hat{\lambda})\Big) - \mathbf{y}'_{0,t}\Big(X_{0,t}(\lambda_k)\Big)\Big]\, ds \cdot X_{1,t}^{N,k}(\hat{\lambda}) \\
& + \int_0^1 \mathbf{y}'_{1,t}\Big(X_t^{N,k,s}(\hat{\lambda})\Big)\, ds \cdot \Big(X_{1,t}^{N,k}(\hat{\lambda}) + \frac{X_{2,t}^{N,k}(\hat{\lambda})}{N}\Big) \\
& + \int_0^1 \Big[\int \partial_1 \mathbf{z}_t\Big(X_t^{N,k,s}(\hat{\lambda}), y\Big)\, dM_N^{X_t^{N,s}}(y) \\
& \qquad - \int \partial_1 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\, dM_N^{X_{0,t}}(y)\Big]\, ds \cdot \Big(X_{1,t}^{N,k}(\hat{\lambda}) + \frac{X_{2,t}^{N,k}(\hat{\lambda})}{N}\Big) \\
& + \int \partial_1 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\, dM_N^{X_{0,t}}(y) \cdot \Big(X_{1,t}^{N,k}(\hat{\lambda}) + \frac{X_{2,t}^{N,k}(\hat{\lambda})}{N}\Big) \\
& + \sum_{j=1}^N \int_0^1 \Big[\partial_2 \mathbf{z}_t\Big(X_t^{N,k,s}(\hat{\lambda}), X_t^{N,j,s}(\hat{\lambda})\Big) - \partial_2 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big)\Big]\, ds \cdot X_{1,t}^{N,j}(\hat{\lambda}) \\
& + \sum_{j=1}^N \int_0^1 \Big[\partial_2 \mathbf{z}_t\Big(X_t^{N,k,s}(\hat{\lambda}), X_t^{N,j,s}(\hat{\lambda})\Big)\Big]\, ds \cdot \frac{X_{2,t}^{N,j}(\hat{\lambda})}{N},
\end{aligned}
\tag{8.28}
$$

with the initial condition $X_{2,0}^{N,k} = 0$. Using that

$$\|\mathbf{y}_{0,t}\|_{C^{r-3}(\mathbb{R})} \le C$$

(see Lemma 8.9) we obtain

$$\|X_{0,t}\|_{C^{r-2}(\mathbb{R})} \le C. \tag{8.29}$$

We now start to control $X_{1,t}^N$. First, simply by using that M_N has mass bounded by $2N$ we obtain the rough bound $|X_{1,t}^{N,k}| \le C\,N$. Inserting this bound into (8.28) one easily obtain the bound $|X_{2,t}^{N,k}| \le C\,N^2$. We now prove finer estimates.

Since $X_{0,t}$ and $x \mapsto \mathbf{z}_t(y,x)$ are Lipschitz (uniformly in y), it follows by Lemma 4.14 that there exists a finite constant C such that, with probability greater than $1 - N^{-N/C}$,

$$\Big\|\int \mathbf{z}_t(\cdot,\lambda)\, dM_N^{X_{0,t}}(\lambda)\Big\|_\infty \le C\, \ln N\sqrt{N}. \tag{8.30}$$

Hence it follows easily from (8.24) by applying twice Gronwall Lemma that

$$\max_k \|X_{1,t}^{N,k}\|_\infty \le C \ln N\sqrt{N} \tag{8.31}$$

outside a set of probability bounded by $N^{-N/C}$.

In order to control $X_{2,t}^N$ we first estimate $X_{1,t}^N$ in $L^4(\mathbb{P}_N^{\beta,V})$: using (8.24) again, we get

$$\begin{aligned}(8.32)\quad &\frac{d}{dt}\Big(\max_k \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})}\Big)\\ &\le C\bigg(\max_k \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})} + 1 + \bigg\|\int \mathbf{z}_t(X_{0,t}(\lambda_k),y)\,dM_N^{X_{0,t}}(y)\bigg\|_{L^4(\mathbb{P}_N^{\beta,V})}\bigg).\end{aligned}$$

To bound $X_{1,t}^N$ in $L^4(\mathbb{P}_N^{\beta,V})$ and then to be able to estimate $X_{2,t}^N$ in $L^2(\mathbb{P}_N^{\beta,V})$, we will use the following estimates :

Lemma 8.12. *For any $k=1,\dots,N$,*

$$\bigg\|\int \mathbf{z}_t(X_{0,t}(\lambda_k),y)\,dM_N^{X_{0,t}}(y)\bigg\|_{L^4(\mathbb{P}_N^{\beta,V})} \le C\ln N, \tag{8.33}$$

$$\bigg\|\int \partial_1\mathbf{z}_t\Big(X_{0,t}(\lambda_k),y\Big)\,dM_N^{X_{0,t}}(y)\bigg\|_{L^4(\mathbb{P}_N^{\beta,V})} \le C\ln N. \tag{8.34}$$

Proof. We write the Fourier decomposition of $\eta_t(x,y) := \mathbf{z}_t(X_{0,t}(x),X_{0,t}(y))$ to get

$$\int \eta_t(x,y)\,dM_N(y) = \int \hat{\eta}_t(x,\xi)\int e^{i\xi y}\,dM_N(y)\,d\xi\,.$$

Since $\mathbf{z}_t \in C^{u,v}$ for $u+v\le r-6$ and $X_{0,t}\in C^{r-2}$ (see (8.29)), we deduce that

$$|\hat{\eta}_t(x,\xi)| \le \frac{C}{1+|\xi|^{r-6}},$$

so that we get

$$\begin{aligned}\bigg\|\sup_x\bigg|\int \eta_t(x,y)\,dM_N(y)\bigg|\bigg\|_{L^4(\mathbb{P}_N^{\beta,V})} &\le \int \Big\|\hat{\eta}_t(\cdot,\xi)\Big\|_\infty \bigg\|\int e^{i\xi y}dM_N(y)\bigg\|_{L^4(\mathbb{P}_N^{\beta,V})} d\xi\\ &\le C\ln N\int \Big\|\hat{\eta}_t(\cdot,\xi)\Big\|_\infty \left(1+|\xi|^7\right)\,d\xi\\ &\le C\ln N,\end{aligned}$$

provided $r>14$. The same arguments work for $\partial_1\mathbf{z}_t$ provided $r>15$. Since by assumption $r\ge 16$, this concludes the proof. ◇

Inserting (8.33) into (8.32) we get

$$\|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})} \le C\ln N \qquad \forall\, k=1,\dots,N, \tag{8.35}$$

which proves the first part of (8.25).

We now bound the time derivative of the L^2 norm of $X_{2,t}^N$: using that M_N has mass bounded by $2N$, in (8.28) we can easily estimate

$$\bigg|N\int_0^1 \Big[\mathbf{y}'_{0,t}\Big(X_t^{N,k,s}(\hat{\lambda})\Big) - \mathbf{y}'_{0,t}\Big(X_{0,t}(\lambda_k)\Big)\Big]\,ds\cdot X_{1,t}^{N,k}(\hat{\lambda})\bigg| \le C|X_{1,t}^{N,k}|^2 + \frac{C}{N}|X_{1,t}^{N,k}|\,|X_{2,t}^{N,k}|,$$

$$\begin{aligned}&\int_0^1\bigg|\int \partial_1\mathbf{z}_t\Big(X_t^{N,k,s}(\hat{\lambda}),y\Big)\,dM_N^{X_t^{N,s}}(y) - \int \partial_1\mathbf{z}_t\Big(X_{0,t}(\lambda_k),y\Big)\,dM_N^{X_{0,t}}(y)\bigg|\,ds\\ &\qquad\le C|X_{1,t}^{N,k}| + \frac{C}{N}|X_{2,t}^{N,k}| + \frac{C}{N}\sum_j\bigg(|X_{1,t}^{N,j}| + \frac{1}{N}|X_{2,t}^{N,j}|\bigg),\end{aligned}$$

$$\sum_{j=1}^N \int_0^1 \left| \partial_2 \mathbf{z}_t \Big(X_t^{N,k,s}(\hat{\lambda}), X_t^{N,j,s}(\hat{\lambda}) \Big) - \partial_2 \mathbf{z}_t \Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j) \Big) \right| ds \, |X_{1,t}^{N,j}| \\ \leq \frac{C}{N} \sum_j \Big(|X_{1,t}^{N,j}|^2 + \frac{1}{N} |X_{2,t}^{N,j}| \, |X_{1,t}^{N,j}| \Big),$$

hence

$$\begin{aligned}
\frac{d}{dt} \|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})} &= 2 \int \sum_k X_{2,t}^{N,k} \cdot \dot{X}_{2,t}^{N,k} \, d\mathbb{P}_N^{\beta,V} \\
&\leq C \int \sum_k |X_{2,t}^{N,k}|^2 \, d\mathbb{P}_N^{\beta,V} + C \int \sum_k |X_{1,t}^{N,k}|^2 |X_{2,t}^{N,k}| d\mathbb{P}_N^{\beta,V} \\
&\quad + \frac{C}{N} \int \sum_k |X_{1,t}^{N,k}| |X_{2,t}^{N,k}|^2 d\mathbb{P}_N^{\beta,V} + C \int \sum_k |X_{1,t}^{N,k}| |X_{2,t}^{N,k}| \, d\mathbb{P}_N^{\beta,V} \\
&\quad + \frac{C}{N^2} \int \sum_k |X_{2,t}^{N,k}|^3 \, d\mathbb{P}_N^{\beta,V} + \frac{C}{N} \int \sum_{k,j} |X_{1,t}^{N,j}| \, |X_{1,t}^{N,k}| \, |X_{2,t}^{N,k}| \, d\mathbb{P}_N^{\beta,V} \\
&\quad + \frac{C}{N^3} \int \sum_{k,j} |X_{2,t}^{N,k}|^2 \, |X_{2,t}^{N,j}| \, d\mathbb{P}_N^{\beta,V} \\
&\quad + \sum_k \int X_{2,t}^{N,k} \cdot \int_0^1 \left[\int \partial_1 \mathbf{z}_t \Big(X_{0,t}(\lambda_k), y \Big) \, dM_N^{X_{0,t}}(y) \right] ds \cdot X_{1,t}^{N,k} \, d\mathbb{P}_N^{\beta,V} \\
&\quad + \frac{C}{N} \int \sum_{k,j} |X_{1,t}^{N,j}|^2 \, |X_{2,t}^{N,k}| \, d\mathbb{P}_N^{\beta,V} \\
&\quad + \frac{C}{N^2} \int \sum_{k,j} |X_{2,t}^{N,k}| \, |X_{2,t}^{N,j}| \, |X_{1,t}^{N,j}| \, d\mathbb{P}_N^{\beta,V} \\
&\quad + \frac{C}{N} \int \sum_{k,j} |X_{2,t}^{N,k}| \, |X_{2,t}^{N,j}| \, d\mathbb{P}_N^{\beta,V}.
\end{aligned}$$

Using the trivial bounds $|X_{1,t}^{N,k}| \leq C\,N$ and $|X_{2,t}^{N,k}| \leq C\,N^2$, (8.34), and elementary inequalities such as, for instance,

$$\sum_{k,j} |X_{1,t}^{N,j}| \, |X_{1,t}^{N,k}| \, |X_{2,t}^{N,k}| \leq \sum_{k,j} \Big(|X_{1,t}^{N,j}|^4 + |X_{1,t}^{N,k}|^4 + |X_{2,t}^{N,k}|^2 \Big),$$

we obtain

(8.36)
$$\frac{d}{dt} \|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})} \leq C \Big(\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})} + \int \sum_k |X_{1,t}^{N,k}|^4 \, d\mathbb{P}_N^{\beta,V} \\ + \int \sum_k |X_{1,t}^{N,k}|^2 \, d\mathbb{P}_N^{\beta,V} + \sum_k \ln N \, \|X_{2,t}^{N,k}\|_{L^2(\mathbb{P}_N^{\beta,V})} \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})} \Big).$$

We now observe that, thanks to (8.35), that the last term is bounded by

$$\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})} + (\ln N)^2 \sum_k \|X_{1,t}^{N,k}\|^2_{L^4(\mathbb{P}_N^{\beta,V})} \leq \|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})} + C\,N (\ln N)^4.$$

Hence, using that $\|X_{1,t}^{N,k}\|_{L^2(\mathbb{P}_N^{\beta,V})} \le \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_N^{\beta,V})}$ and (8.35) again, the right hand side of (8.36) can be bounded by $C\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})}+C\,N(\ln N)^4$ and a Gronwall argument gives

$$\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_N^{\beta,V})} \le C\,N(\ln N)^4,$$

thus

$$\|X_{2,t}^N\|_{L^2(\mathbb{P}_N^{\beta,V})} \le C\,N^{1/2}(\ln N)^2,$$

concluding the proof of (8.25).

We now prove (8.26) : using (8.24) we have

$$\begin{aligned}
&|\dot X_{1,t}^{N,k}(\hat\lambda) - \dot X_{1,t}^{N,k'}(\hat\lambda)|\\
\le\;& |\mathbf{y}'_{0,t}(X_{0,t}(\lambda_k)) - \mathbf{y}'_{0,t}(X_{0,t}(\lambda_{k'}))|\,|X_{1,t}^{N,k}(\hat\lambda)|\\
&+|\mathbf{y}'_{0,t}(X_{0,t}(\lambda_{k'}))|\,|X_{1,t}^{N,k}(\hat\lambda) - X_{1,t}^{N,k'}(\hat\lambda)| + |\mathbf{y}_{1,t}(X_{0,t}(\lambda_k)) - \mathbf{y}_{1,t}(X_{0,t}(\lambda_{k'}))|\\
&+\left|\int \Big(\mathbf{z}_t(X_{0,t}(\lambda_k),y) - \mathbf{z}_t(X_{0,t}(\lambda_{k'}),y)\Big)\,dM_N^{X_{0,t}}(y)\right|\\
&+\frac{1}{N}\sum_{j=1}^N\int_0^1\Big|\partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_k),X_{0,t}(\lambda_j)\Big) - \partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_{k'}),X_{0,t}(\lambda_j)\Big)\Big|\,ds\,|X_{1,t}^{N,j}(\hat\lambda)|.
\end{aligned}$$

Using that $|X_{0,t}(\lambda_k) - X_{0,t}(\lambda_{k'})| \le C|\lambda_k - \lambda_{k'}|$, the bound (8.31), the Lipschitz regularity of $\mathbf{y}'_{0,t}$, $\mathbf{y}_{1,t}$, $\mathbf{z}_t$, and $\partial_2\mathbf{z}_t$, and the fact that

$$\left\|\int \partial_1\mathbf{z}_t(\cdot,\lambda)\,dM_N^{X_{0,t}}(\lambda)\right\|_\infty \le C\,\ln N\sqrt{N}$$

with probability greater than $1-N^{-N/C}$ (see Lemma 4.14), we get

$$|\dot X_{1,t}^{N,k}(\hat\lambda) - \dot X_{1,t}^{N,k'}(\hat\lambda)| \le C|X_{1,t}^{N,k}(\hat\lambda) - X_{1,t}^{N,k'}(\hat\lambda)| + C\ln N\sqrt{N}|\lambda_k-\lambda_{k'}|$$

outside a set of probability less than $N^{-N/C}$, so (8.26) follows from Gronwall. $\diamond$

8.0.7. Transport and universality. In this section we prove Theorem 8.4 on universality using the regularity properties of the approximate transport maps obtained in the previous sections.

Proof of Theorem 8.4. Let us first remark that the map T_0 from Theorem 8.3 coincides with $X_{0,1}$, where $X_{0,t}$ is the flow defined in (8.23). Also, notice that $X_1^N:\mathbb{R}^N\to\mathbb{R}^N$ is an approximate transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_N^{\beta,V+W}$ (see Lemma 8.6 and Proposition 8.8). Set $\hat X_1^N := X_{0,1}^N + \frac{1}{N}X_{1,1}^N$, with $X_{0,t}^N$ and $X_{1,t}^N$ as in Lemma 8.11. Since $X_1^N - \hat X_1^N = \frac{1}{N^2}X_{2,1}^N$, recalling (8.25) and using Hölder inequality to control the L^1 norm with the L^2 norm, we see that

$$\begin{aligned}
\left|\int g(\hat X_1^N)\,dP_N^{\beta,V} - \int g(X_1^N)\,dP_N^{\beta,V}\right| &\le \|\nabla g\|_\infty\frac{1}{N^2}\int |X_{2,1}^N|\,dP_N^{\beta,V}\\
&\le \|\nabla g\|_\infty\frac{1}{N^2}\|X_{2,1}^N\|_{L^2(\mathbb{P}_V)}\\
&\le C\|\nabla g\|_\infty\frac{(\ln N)^2}{N^{3/2}}. \qquad (8.37)
\end{aligned}$$

This implies that also $\hat X_1^N:\mathbb{R}^N\to\mathbb{R}^N$ is an approximate transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_N^{\beta,V+W}$. In addition, we see that $\hat X_1^N$ preserves the order of the λ_i with large probability. Indeed, hence differentiating (8.23) we get that the spatial derivative $X'_{0,t}$ of $X_{0,t}$ verifies

$$\partial_t X'_{0,t} = \mathbf{y}'_{0,t}(X_{0,t})X'_{0,t}\big|, \qquad X'_{0,0} = 1,$$

so that

$$X'_{0,t} = \exp\{\int_0^t \mathbf{y}'_{0,s}(X_{0,s})ds\}\,.$$

Hence, the flow of $\mathbf{y}_{0,t}$ which is Lipschitz with some constant L,

$$e^{-Lt} \le \left|X'_{0,t}\right| \le e^{Lt}\,.$$

Since $X'_{0,0} = 1$, it follows by continuity that $X'_{0,t}$ must remain positive for all times and it satisfies

$$e^{-Lt} \le X'_{0,t} \le e^{Lt}, \tag{8.38}$$

from which we deduce that

$$e^{-Lt}\big(\lambda_j - \lambda_i\big) \le X_{0,t}(\lambda_j) - X_{0,t}(\lambda_i) \le e^{Lt}\big(\lambda_j - \lambda_i\big), \qquad \forall\, \lambda_i < \lambda_j.$$

In particular,

$$e^{-L}\big(\lambda_j - \lambda_i\big) \le X_{0,1}(\lambda_j) - X_{0,1}(\lambda_i) \le e^{L}\big(\lambda_j - \lambda_i\big).$$

Hence, using the notation $\hat\lambda = (\lambda_1, \dots, \lambda_N)$, since

$$\left|\frac{1}{N}X^{N,j}_{1,t}(\hat\lambda) - \frac{1}{N}X^{N,i}_{1,t}(\hat\lambda)\right| \le C\,\frac{\ln N}{\sqrt{N}}|\lambda_i - \lambda_j|$$

with probability greater than $1 - N^{-N/C}$ (see (8.26)), we get for some $C' > 0$

$$\frac{1}{C'}\big(\lambda_j - \lambda_i\big) \le \hat X^{N,j}_1(\hat\lambda) - \hat X^{N,i}_1(\hat\lambda) \le C'\big(\lambda_j - \lambda_i\big)$$

with probability greater than $1 - N^{-N/C}$.

We now make the following observation : the ordered measures $\tilde P^N_V$ and $\tilde P^{\beta,V+W}_N$ are obtained as the image of $P^{\beta,V}_N$ and $\mathbb{P}^{\beta,V+W}_N$ via the map $R : \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$[R(x_1, \dots, x_N)]_i := \min_{\sharp J = i} \max_{j \in J} x_j.$$

Notice that this map is 1-Lipschitz for the sup norm.

Hence, if g is a function of m-variables we have $\|\nabla(g \circ R)\|_\infty \le \sqrt{m}\|\nabla g\|_\infty$, so by Lemma 8.6, Proposition 8.8, and (8.37), we get

$$\left|\int g\circ R(\hat X^N_1)\, dP^{\beta,V}_N - \int g\circ R\, dP^{\beta,V+W}_N\right| \le C\frac{(\ln N)^3}{N}\,\|g\|_\infty + C\sqrt{m}\,\frac{(\ln N)^2}{N^{3/2}}\,\|\nabla g\|_\infty.$$

Since $\hat X^N_1$ preserves the order with probability greater than $1 - N^{-N/C}$, we can replace $g \circ R(N\hat X^N_1)$ with $g(N\hat X^N_1 \circ R)$ up to a very small error bounded by

$\|g\|_\infty N^{-N/C}$. Hence, since $R_\# P_N^{\beta,V} = \tilde{P}_V^N$ and $R_\# P_N^{\beta,V+W} = \tilde{P}_N^{\beta,V+W}$, we deduce that, for any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$,

$$\Big| \int f\big(N(\lambda_{i+1} - \lambda_i), \dots, N(\lambda_{i+m} - \lambda_i)\big)\, d\tilde{P}_N^{\beta,V+W} \\ - \int f\Big(N\big(\hat{X}_1^{N,i+1}(\hat\lambda) - \hat{X}_1^{N,i}(\hat\lambda)\big), \dots, N\big(\hat{X}_1^{N,i+m}(\hat\lambda) - \hat{X}_1^{N,i}(\hat\lambda)\big)\Big)\, d\tilde{P}_N^{\beta,V} \Big| \\ \leq C\, \frac{(\ln N)^3}{N}\, \|f\|_\infty + C\,\sqrt{m}\, \frac{(\ln N)^2}{N^{1/2}}\, \|\nabla f\|_\infty.$$

Recalling that

$$\hat{X}_1^{N,j}(\hat\lambda) = X_{0,1}(\lambda_j) + \frac{1}{N} X_{1,1}^{N,j}(\hat\lambda),$$

we observe that, as $X_{0,1}$ is of class C^2,

$$X_{0,1}(\lambda_{i+k}) - X_{0,1}(\lambda_i) = X'_{0,1}(\lambda_i)\,(\lambda_{i+k} - \lambda_i) + O(|\lambda_{i+k} - \lambda_i|^2).$$

Also, by (8.26) we deduce that, out of a set of probability bounded by $N^{-N/C}$,

$$|X_{1,1}^{N,i+k}(\hat\lambda) - X_{1,1}^{N,i}(\hat\lambda)| \leq C\, \ln N \sqrt{N}\, |\lambda_{i+k} - \lambda_i|\,. \tag{8.39}$$

As $X'_{0,1}(\lambda_i) \geq e^{-L}$ (see (8.38)) we conclude that

$$\frac{1}{N}\, |X_{1,1}^{N,i+k}(\hat\lambda) - X_{1,1}^{N,i}(\hat\lambda)| \leq C\, |X'_{0,1}(\lambda_i)\,(\lambda_{i+k} - \lambda_i)|\, \frac{\ln N}{N^{1/2}}$$

and

$$O(|\lambda_{i+k} - \lambda_i|^2) = O\big(\big|X'_{0,1}(\lambda_i)\,(\lambda_{i+k} - \lambda_i)\big|^2\big)$$

hence with probability greater than $1 - N^{-N/C}$ it holds

$$\begin{aligned} \hat{X}_1^{N,i+k}(\hat\lambda) \quad &- \quad \hat{X}_1^{N,i}(\hat\lambda) \\ &= \quad X'_{0,1}(\lambda_i)\,(\lambda_{i+k} - \lambda_i)\left[1 + O\left(\frac{\ln N}{N^{1/2}}\right) + O\big(\big|X'_{0,1}(\lambda_i)\,(\lambda_{i+k} - \lambda_i)\big|\big)\right]. \end{aligned}$$

Since we assume f supported in $[-M, M]^m$, the domain of integration is restricted to $\hat\lambda$ such that $\{N X'_{0,t}(\lambda_i)\,(\lambda_i - \lambda_{i+k})\}_{1 \leq k \leq m}$ is bounded by $2M$ for N large enough, therefore

$$\hat{X}_1^{N,i+k}(\hat\lambda) - \hat{X}_1^{N,i}(\hat\lambda) = X'_{0,1}(\lambda_i)\,(\lambda_{i+k} - \lambda_i) + O\left(2M \frac{\ln N}{N^{3/2}}\right) + O\left(\frac{4M^2}{N^2}\right),$$

from which the first bound follows easily.

For the second point we observe that $a_{V+W} = X_{0,1}(a_V)$ and, arguing as before,

$$\Big| \int f\big(N^{2/3}(\lambda_1 - a_{V+W}), \dots, N^{2/3}(\lambda_m - a_{V+W})\big)\, d\tilde{P}_N^{\beta,V+W} \\ - \int f\Big(N^{2/3}\big(\hat{X}_1^{N,1}(\hat\lambda) - X_{0,1}(a_V)\big), \dots, N^{2/3}\big(\hat{X}_1^{N,m}(\hat\lambda) - X_{0,1}(a_V)\big)\Big)\, dP_N^{\beta,V} \Big| \\ \leq C\, \frac{(\ln N)^3}{N}\, \|f\|_\infty + C\,\sqrt{m} \frac{(\ln N)^2}{N^{5/6}}\, \|\nabla f\|_\infty.$$

Since, by (8.25),

$$\begin{aligned}\hat{X}_1^{N,i}(\lambda) &= X_{0,1}(\lambda_i) + O_{L^4(\mathbb{P}_N^{\beta,V})}\left(\frac{\ln N}{N}\right)\\ &= X_{0,1}(a_V) + X'_{0,1}(a_V)\,(\lambda_i - a_V) + O\big(|\lambda_i - a_V|^2\big) + O_{L^4(P_N^{\beta,V})}\left(\frac{\ln N}{N}\right),\end{aligned}$$

we conclude as we did for the first point. $\diamond$

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