Introduction to stochastic analysis

A. Guionnet[1]

[2] Department of Mathematics, MIT,
77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA.

Abstract

These lectures notes are notes in progress designed for course 18176 which gives an introduction to stochastic analysis. They are designed to reflect exactly the content of the course, rather than propose original material. I am very much indebted to Nadine Guillotin who lend me her latex files of her own (french) lectures notes which I have used thoroughly. I also used Nathanael Berestick i lectures notes, as well as books by Daniel revuz and Marc Yor (Continuous martingales and Brownian motion), Jean-Francois Le Gall (Mouvement Brownien, martingales et calcul stochastique), by Ioannis Karatzas and Steven E. Shreve (Brownian motion and Stochastic calculus), by Bernt Oksendal (Stochastic Differential equations).

Contents

[1]guionnet@math.mit.edu

notations, classical (admitted) notions

- A measurable space $(\Omega, \mathcal{G})$ is given by
  -the sample space Ω, an arbitrary non-empty set $\emptyset$,
  -the σ-algebra $\mathcal{G}$(also called σ-field) a set of elements of Ω such that: $\mathcal{G}$ contains the empty set $\emptyset$, $\mathcal{G}$ is closed under complements (if $A \in Fa, \Omega \backslash A \in \mathcal{G}$), $\mathcal{G}$ is closed under countable unions (if $A_i \in \mathcal{G}$, $\cup_i A_i \in \mathcal{G}$)
- A function $f : X \rightarrow Y$ between two measurable spaces $(X, \mathcal{G})$ and $(Y, \mathcal{G})$ is measurable iff for all $B \in \mathcal{G}$, $f^{-1}(B) \in \mathcal{G}$. We will use that pointwise limits of measurable functions are measurable (exercise).
- Convergence in law : A sequence $\mu_n, n \geq 0$ of probability measures on a measurable space $(\Omega, \mathcal{G})$ converges in law towards a probability measure μ iff for any bounded continuous function F on $(\Omega, \mathcal{G})$
$$\lim_{n\to\infty} \int F d\mu_n = \int F d\mu$$
- The monotone convergence theorem asserts that if $f_n \geq 0$, $f_n \leq f_{n+1}$ grows $\mathbb{P}$-as to f then
$$\lim_{n\to\infty} \int f_n d\mathbb{P} = \int f d\mathbb{P}$$
- The bounded convergence theorem asserts that if f_n is a sequence of uniformly bounded functions converging $\mathbb{P}$ as to f then
$$\lim_{n\to\infty} \int f_n d\mathbb{P} = \int f d\mathbb{P}\,.$$
- Borel-Cantelli lemma states that if A_n is a sequence of measurable sets of a measurable space $(\Omega, \mathcal{G})$ equipped with a probability measure $\mathbb{P}$ such that $\sum \mathbb{P}(A_n^c) < \infty$, then
$$\mathbb{P}(\limsup A_n) = 1 \qquad \limsup A_n = \cup_{n\geq 0} \cap_{p\geq n} A_p\,.$$
- $\simeq$ denotes asymptotic equality (in general, $A_n \simeq B_n$ iff $A_n - B_n$ goes to zero, but it can also mean that A_n/B_n goes to one)

1. Brownian motion and stochastic processes

Stochastic processes theory is the study of random phenomena depending on a time variable. Maybe the most famous is the Brownian motion first described by R. Brown, who observed around 1827 that tiny particles of pollen in water have an extremely erratic motion. It was observed by Physicists that this was due to an important number of random shocks undertaken by the particles from the (much smaller) water molecules in motion in the liquid. A. Einstein established in 1905 the first mathematical basis for Brownian motion, by showing that it must be an isotropic Gaussian process. The first rigorous mathematical construction of Brownian motion is due to N. Wiener in 1923, after the work of L. Bachelier in 1900 who is considered to be the first to introduce this notion.

1.1. Microscopic approach. In order to motivate the introduction of this object, we first begin by a microscopical depiction of Brownian motion. Suppose $(X_n, n \geq 0)$ is a sequence of $\mathbb{R}^d$ valued random variables with mean 0 and covariance matrix $\sigma^2 I$, which is the identity matrix in d dimensions, for some $\sigma^2 > 0$. Namely, if $X_1 = (X_1^1, \ldots, X_d^1)$, we have

$$E[X_i^1] = 0, E[X_i^1 X_j^1] = \sigma^2 \delta_{ij}, 1 \leq i, j \leq d$$

We interpret X_n as the spatial displacement resulting from the shocks due to water molecules during the n-th time interval, and the fact that the covariance matrix is scalar stands for an isotropy assumption (no direction of space is privileged). From this, we let $S_n = X_1^1 + \cdots + X_n^1$ and we embed this discrete-time process into continuous time by letting

$$B_t^{(n)} := (\frac{1}{\sqrt{n}} S_{[nt]}, t \geq 0)$$

Let $\|.\|_2$ be the Euclidean norm on $\mathbb{R}^d$ and for $t > 0$ and $x \in \mathbb{R}^d$, define

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp(-\frac{\|x\|_2^2}{2t})$$

which is the density of the Gaussian distribution $N(0, t\mathrm{Id})$ with mean 0 and covariance matrix $t\mathrm{Id}$. By convention, the Gaussian law $N(m, 0)$ is the Dirac mass at m.

PROPOSITION 1.1. *Let $0 < t_1 \leq t_2 < \cdots < t_k$. Then the finite marginal distributions of $B^{(n)}$ with respect to times $t_1, \ldots, t_k$ converge weakly as n goes to infinity. More precisely, if F is a bounded continuous function, and letting $x_0 = 0, t_0 = 0$,*

$$\lim_{n\to\infty} \mathbb{E}[F(B_{t_1}^{(n)}, \ldots, B_{t_k}^{(n)})] = \int F(x_1, \ldots, x_k) \prod_{1\leq i\leq k} p_{\sigma^2(t_i - t_{i-1})(x_i - x_{i-1})} dx_i$$

PROOF. The proof boils done to the central limit theorem as

$$B_{t_i}^{(n)} - B_{t_{i-1}}^{(n)} = \frac{1}{\sqrt{n}} \sum_{j=[nt_i]+1}^{[nt_{i+1}]} X_j$$

are independent and converges in law towards centered Gaussian vectors with covariance $\sigma^2(t_{i+1} - t_i)$ by the central limit theorem. The latter can be checked by computing the Fourier transform given for any real parameters $\xi_j \in \mathbb{R}^d$ by

$$\mathbb{E}[e^{i\sum_{j=1}^k \xi_i.(B^{(n)}_{t_j}-B^{(n)}_{t_{j-1}})}] = \prod_{j=1}^k \mathbb{E}[e^{i\xi_j(B^{(n)}_{t_j}-B^{(n)}_{t_{j-1}})}]$$

while

$$\lim_{n\to\infty} \mathbb{E}[e^{i\xi_j.(B^{(n)}_{t_j}-B^{(n)}_{t_{j-1}})}] = e^{-\frac{\sigma^2}{2}(t_{j+1}-t_j)}$$

as can easily be checked (at least if the X_i's have a moment of order $2+\epsilon$ for some $\epsilon > 0$) as

$$\begin{aligned} \mathbb{E}[e^{i\xi_i.(B^{(n)}_{t_i}-B^{(n)}_{t_{i-1}})}] &= \prod_{j=[nt_i]+1}^{[nt_{i+1}]} \mathbb{E}[e^{\frac{1}{\sqrt{n}}\xi_i.X_j}] \\ &\simeq \prod_{j=[nt_i]+1}^{[nt_{i+1}]} (1-n^{-1}\sigma^2\|\xi_i\|_2^2/2) \simeq e^{-\frac{\sigma^2}{2}(t_{i+1}-t_i)} \end{aligned}$$

□

This suggests that $B^{(n)}$ should converge to a process B whose increments are independent and Gaussian with covariances dictated by the above formula. The precise sense of this convergence as well as the state space in which the limit should live is the object of the next subsections. The limit of $B^{(n)}$ should be described as follows:

Definition 1.2. *An $\mathbb{R}^d$-valued stochastic process $(B_t, t \geq 0)$ is called a standard Brownian motion if it is a continuous process, that satisfies the following conditions:*

1. $B_0 = 0$ *a.s.,*
2. *for every $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_k$, the increments $(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}})$ are independent,*
3. *for every $t, s \geq 0$, the law of $B_{t+s} - B_t$ is Gaussian with mean 0 and covariance sId.*

The properties (1), (2), (3) exactly amount to say that the finite-dimensional marginals of a Brownian motion are given by the formula of Proposition 1.1. Therefore the law of the Brownian motion is uniquely determined.

1.2. Equivalent processes, indistinguishable processes. The previous section yields several remarks; how can we construct a random continuous process with given marginals ? how does it compares to other constructions ? How can we speak about the law of the Brownian motion ? etc etc In this section we make all these definitions more precise. We will denote throughout $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space. $\mathbb{T}$ will be the space time, often $\mathbb{T} = \mathbb{R}^+$. $(E, \mathcal{E})$ is the measurable space of state.

Definition 1.3. A stochastic process with values in $(E, \mathcal{E})$ based on $(\Omega, \mathcal{G}, \mathbb{P})$ is a family $(X_t)_{t\in\mathbb{T}}$ of random variables from $(\Omega, \mathcal{G}, \mathbb{P})$ into $(E, \mathcal{E})$.

To any $\omega \in \Omega$, we associate the map

$$\begin{array}{ccc} \mathbb{T} & \to & E \\ t & \to & X_t(\omega) \end{array}$$

called the trajectory of $(X_t)_{t\in\mathbb{T}}$ associated with ω.

To simplify, we will hereafter restrict ourselves to the case $\mathbb{T} = \mathbb{R}^+$, $E = \mathbb{R}^d$ and $\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$.

We say that $(X_t)_{t\in\mathbb{T}}$ is $\mathbb{P}$-a.s right (resp. $\mathbb{P}$-a.s left, resp. $\mathbb{P}$-a.s) continuous if for almost all $\omega \in \Omega$, the trajectory of $(X_t)_{t\in\mathbb{T}}$ associated with ω is right (resp. left, resp.) continuous.

We will say that two stochastic processes describe the same random phenomenon if they are equivalent in the following sense

Definition 1.4. *Let $(X_t)_{t\in\mathbb{T}}$ and $(X'_t)_{t\in\mathbb{T}}$ be two processes with values in the same state space $(E, \mathcal{E})$ with $(X_t)_{t\in\mathbb{T}}$ (resp. $(X'_t)_{t\in\mathbb{T}}$) based on $(\Omega, \mathcal{G}, \mathbb{P})$(resp. $(\Omega', \mathcal{G}', \mathbb{P}')$). We say that $(X_t)_{t\in\mathbb{T}}$ and $(X'_t)_{t\in\mathbb{T}}$ are equivalent if for all $n \geq 1$, for all $t_1, \ldots, t_n \in \mathbb{T}$, all $B_1, \cdots, B_n \in \mathcal{E}$,*

$$\mathbb{P}\left(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\right) = \mathbb{P}'\left(X'_{t_1} \in B_1, \ldots, X'_{t_n} \in B_n\right) .$$

We also say that these processes are " a version of each other" or a version of the same process.

Note that this defines an equivalence relation.

The family of the random variables $(X_{t_1}, \ldots, X_{t_n})$ for $t_i \in \mathbb{T}$ is called the family of the finite dimensional marginals of $(X_t)_{t\in\mathbb{T}}$. Two processes are equivalent if they have same finite marginal distributions. Note however that this does not imply in general that $X_t = X'_t$ almost surely for every t as the set of parameters $\mathbb{T}$ is not countable, unless the processes under study possess some regularity. The latter property refers to indistinguishable processes

Definition 1.5. *Two processes $(X_t)_{t\in\mathbb{T}}$ and $(X'_t)_{t\in\mathbb{T}}$ defined on the same probability space $(\Omega, \mathcal{G}, \mathbb{P})$ are indisguishable if*

$$\mathbb{P}(X_t(\omega) = X'_t(\omega) \qquad \forall t \in \mathbb{T}) = 1$$

Note that, up to indistinguishability there exists at most one continuous modification of a given process $(X_t, t \geq 0)$. We will say that a process X is a modification of another process X' if they are indisguishable.

1.3. Kolmogorov's criterion. Kolmogorov's criterion is a fundamental result which guarantees the existence of a continuous version (but not necessarily indistinguishable version) based solely on an L^p control of the two-dimensional distributions. We will apply it to Brownian motion below, but it is useful in many other contexts.

THEOREM 1.6. *(Kolmogorov's continuity criterion) Let $X_t, t \in \mathbb{R}^+$ be a stochastic process with values in $\mathbb{R}^d$. Suppose there exist $\alpha > 0, \beta > 0, C > 0$ so that*

$$\mathbb{E}[\|X_t - X_s\|^\alpha] \leq C|t-s|^{1+\beta}$$

for some norm $\|.\|$ on $\mathbb{R}^d$. Then, there exists a modification $(X_t')_{t\in\mathbb{R}^+}$ of $(X_t)_{t\in\mathbb{R}^+}$ which is almost surely continuous, and even ε Hölder for $\varepsilon < \beta/\alpha$.

As a direct application we deduce that

COROLLARY 1.7. *If $(X_t)_{t\in\mathbb{R}^+}$ is a d-dimensional Brownian motion defined by Definition 1.2, there exists a a modification $(X_t')_{t\in\mathbb{R}^+}$ of $(X_t)_{t\in\mathbb{R}^+}$ with continuous (and even ε-Hölder with $\varepsilon < 1/2$) trajectories.*

Indeed it follows from the fact that for all integer number n,

$$\mathbb{E}[\|X_t - X_s\|_2^{2n}] = C_n(t-s)^n$$

with C_n the $2n$th moment of a centered d-dimensionnal Gaussian variable with variance one, so that Kolmogorov Theorem holds with $\varepsilon < (n-1)/2n$.

PROOF. It is enough to restrict ourselves to $\mathbb{T} = [0,1]$ up to put

$$B_t = \sum_{i=1}^{[t]} B_1^i + B_{t-[t]}^{[t]}$$

with B^i independent copies of the Brownian motion on $[0,1]$. Let $D_n = \{k2^{-n}, 0 \leq k \leq 2n\}$ denote the dyadic numbers of $[0,1]$ with level n, so D_n increases as n increases. Then letting $\varepsilon < \beta/\alpha$, Tchebychev's inequality gives for $0 \leq k \leq 2^n$

$$\mathbb{P}\left(\|X_{k2^{-n}} - X_{(k+1)2^{-n}}\| \geq 2^{-n\varepsilon}\right) \leq 2^{n\varepsilon\alpha}\mathbb{E}[\|X_{k2^{-n}} - X_{(k+1)2^{-n}}\|^\alpha] \leq C2^{n\varepsilon\alpha - n(1+\beta)}$$

Summing over k we deduce that

$$\mathbb{P}\left(\max_{0\leq k\leq 2^n} \|X_{k2^{-n}} - X_{(k+1)2^{-n}}\| \geq 2^{-n\varepsilon}\right) \leq C2^n 2^{n\varepsilon\alpha - n(1+\beta)} \leq C2^{n\varepsilon\alpha - n\beta}$$

which is summable. Therefore, Borel Cantelli's lemma implies that there exists $N(\omega)$ almost surely finite so that for $n \geq N(\omega)$,

$$\max_{0\leq k\leq 2^n} \|X_{k2^{-n}} - X_{(k+1)2^{-n}}\| \leq 2^{-n\varepsilon}.$$

We claim that this implies that for every $s,t \in D = \cup D_n$,

$$\|X_s - X_t\| \leq M(\omega)|s-t|^\varepsilon$$

for some almost surely finite constant $M(\omega)$. Indeed take $s, t \in D$ so that $2^{-r-1} \leq |s-t| \leq 2^{-r}$ for some $r \geq N(\omega)$. We can always write the diadyc decomposition of t and s

$$t = \eta_0 2^{-r} + \sum_{i=1}^{m} 2^{-r-i}\eta_i \qquad s = \eta_0 2^{-r} - \sum_{i=1}^{p} 2^{-r-i}\eta_i'$$

for some $\eta_i, \eta_i' \in \{0,1\}$ and set

$$t_j = k2^{-r} + \sum_{i=1}^{j} 2^{-r-i}\eta_i \qquad s_j = k2^{-r} - \sum_{i=1}^{j} 2^{-r-i}\eta_i'$$

to deduce from the triangle inequality that, as $X_t = X_{t_0} + \sum_{i=1}^{m}(X_{t_i} - X_{t_{i-1}})$, with $X_{t_0} = X_{s_0}$,

$$\begin{aligned} \|X_t - X_s\| &\leq \|X_{t_0} - X_{s_0}\| + \sum_{i=1}^{m} \|X_{t_i} - X_{t_{i-1}}\| + \sum_{i=1}^{l} \|X_{s_i} - X_{s_{i-1}}\| \\ &\leq \sum_{i=1}^{m} 2^{-(r+i)\varepsilon} + \sum_{i=1}^{l} 2^{-(r+i)\varepsilon} \\ &\leq C(\varepsilon)2^{-r\varepsilon} \leq C(\varepsilon)|t-s|^{\varepsilon} \end{aligned}$$

as $|t-s| \geq 2^{-r-1}$. Therefore the process $(X_t(\omega), t \in D)$ is uniformly continuous, and even ε-Hölder, for all ω such that $N(\omega) < \infty$ Since D is an everywhere dense set in [0,1], this process admits a unique continuous extension $\tilde{X}(\omega)$ on $[0,1]$, which is also ε-Hölder. It is defined by $\tilde{X}_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega)$, where $(t_n, n \geq 0)$ is any D- valued sequence converging to t. On the exceptional set where $(X_d, d \in D)$ is not uniformly continuous (that is $N(\omega) = +\infty$), we let $\tilde{X}_t(\omega) = 0$ so $\tilde{X}(\omega)$ is continuous. It remains to show that $\tilde{X}$ is a version of X. But by Fatou's lemma, if t_n is a sequence of diadyc numbers converging to t, we have

$$\mathbb{E}[\|X_t - \tilde{X}_t\|^p] \leq \liminf \mathbb{E}[\|X_{t_n} - X_t\|^p] = 0$$

So that indeed the finite marginals of X coincide with those of $\tilde{X}$. □

From now on we will consider exclusively a continuous modification of Brownian motion, which is unique up to indistinguishability. Hence, we have constructed a Brownian motion B which can be seen as an application from a probability space $(\Omega, \mathbb{P})$ into the space $C(\mathbb{R}^+, \mathbb{R})$ of continuous function from $\mathbb{R}^+$ into $\mathbb{R}$. The Wiener measure, or law of the Brownian motion, is by definition the image of $\mathbb{P}$ by this application; it is therefore a probability measure on $C(\mathbb{R}^+, \mathbb{R})$. In the next part, we study this measure, as a warming up to what we will soon develop for more general processes.

1.4. Behaviour of Brownian motion trajectories. In this paragraph we are given a Brownian motion $(B_t)_{t\geq 0} : \Omega \to C(\mathbb{R}^+, \mathbb{R})$ and study the properties of its trajectories.

1.4.1. *Generic properties.* Here we derive some information on the shape of the trajectories. A very useful result is the so-called 0-1 Blumenthal law which states as follows.

LEMMA 1.8. *For all $t \geq 0$ let $\mathcal{F}_t$ be the sigma algebra generated by $\{B_s, s \leq t\}$, that is the smallest σ-algebra on Ω that contains all pre-images of measurable subsets of Ω for times $s \leq t$. Let $\mathcal{F}_0^+ = \cap_{s>0}\mathcal{F}_s$. Then any $A \in \mathcal{F}_0^+$ is such that $P(A) = 0$ or* 1.

PROOF. Take $A \in \mathcal{F}_0^+$ and $0 < t_1 < \cdots < t_n$, and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded continuous function. Then, by continuity as B_ϵ goes to zero with ϵ

$$\begin{aligned}
\mathbb{E}[1_A f(B_{t_1}, \ldots, B_{t_n})] &= \lim_{\epsilon\to 0} E[1_A f(B_{t_1} - B_\epsilon, \ldots, B_{t_n} - B_\epsilon)] \\
&= \lim_{\epsilon\to 0} P(A)\mathbb{E}[f(B_{t_1} - B_\epsilon, \ldots, B_{t_n} - B_\epsilon)] \\
&= P(A)\mathbb{E}[1_A f(B_{t_1}, \ldots, B_{t_n})]
\end{aligned}$$

where we used the Markov property (see the second point in Definition 1.2). Hence, $\mathcal{F}_0^+$ is independent of $\sigma(B_{t_1}, \ldots, B_{t_n})$, and thus of $\sigma(B_s, s > 0)$. Finally, $\sigma(B_s, s > 0) = \sigma(B_s, s \geq 0)$ as B_0 is the limit of B_t as t goes to zero. On the other hand $\mathcal{F}_0^+ \subset \sigma(B_s, s > 0)$, and therefore we have proved that $\mathcal{F}_0^+$ is independent of itself, which yields the result. □

As a corollary, we derive the following property

PROPOSITION 1.9. *We almost surely have for all $\varepsilon > 0$*

$$\sup_{0\leq s\leq \varepsilon} B_s > 0, \qquad \inf_{0\leq s\leq\varepsilon} B_s < 0\,.$$

For all $a \in \mathbb{R}$, let $T_a = \inf\{t \geq 0 : B_t = a\}$ (with the convention that this is infinite if $\{B_t = a\} =$). Then almost surely T_a is finite for all $a \in \mathbb{R}$. As a consequence

$$\liminf_{s\to\infty} B_s = -\infty, \qquad \limsup_{s\to\infty} B_s = +\infty$$

PROOF. Note that $\sup_{0<s\leq\varepsilon} B_s$ is measurable as B_s is continuous so that $\sup_{s\in[0,\varepsilon]} B_s = \sup_{s\in[0,\varepsilon]\cap\mathbb{Q}} B_s$. This type of argument will be repeated in many places hereafter. We put for some sequence ε_p going to zero with p

$$A = \cap\{\sup_{0\leq s\leq\varepsilon_p} B_s > 0\}$$

A belongs to $\mathcal{F}_0^+$ as a decreasing intersection of events in $\mathcal{F}_{\varepsilon_p}$ and

$$P(A) = \lim_{p\to\infty} \mathbb{P}(\sup_{0\leq s\leq\varepsilon_p} B_s > 0) \geq \lim_{p\to\infty} \mathbb{P}(B_{\varepsilon_p} > 0) \geq 1/2$$

implying with Blumenthal law that $P(A) = 1$. By changing B into $-B$ we obtain the statement for the inf. To prove the last result observe that we have proved

$$1 = \mathbb{P}(\sup_{0\le s\le 1} B_s > 0) = \lim_{\delta\downarrow 0}\mathbb{P}(\sup_{0\le s\le 1} B_s > \delta) = \lim_{\delta\downarrow 0}\mathbb{P}(\sup_{0\le s\le 1} B_{s\delta^{-4}} > \delta^{-1})$$

where we used that $(cB_{t/\sqrt{c}}, t \ge 0)$ has the law of the Brownian motion for any $c > 0$ (see Exercise 2.19). But

$$\mathbb{P}(\sup_{0\le s\le 1} B_{s\delta^{-4}} > \delta^{-1}) = \mathbb{P}(\sup_{0\le s\le \delta^{-4}} B_s > \delta^{-1}) \le \mathbb{P}(\sup_{0\le s\le \infty} B_s > \delta^{-1})$$

and hence we conclude that

$$\mathbb{P}(\sup_{0\le s\le \infty} B_s > \delta^{-1}) = 1$$

for all $\delta > 0$. The same result with the infimum is derived by replacing B by $-B$. The fact that T_a is almost surely finite follows from the continuity of the trajectories, which takes all values in $(-\infty, +\infty)$. □

1.4.2. *Regularity.* Note that in fact Corollary 1.7 is optimal in the sense that

THEOREM 1.10. *Let B be a continuous modification of the Brownian motion. Let $\gamma > 1/2$. Then*

$$\mathbb{P}\left(\forall t \ge 0 : \limsup_{h\to 0^+} \frac{|B_{t+h} - B_t|}{h^\gamma} = +\infty\right) = 1\,.$$

PROOF. We first observe that

$$\left\{\exists t \ge 0 : \limsup_{h\to 0^+} \frac{|B_{t+h} - B_t|}{h^\gamma} < +\infty\right\} \subset \cup_{p,k,m=1}^{\infty}\left\{\exists t \in [0,m] : |B_{t+h} - B_t| \le ph^\gamma, \forall h \in (0, 1/k)\right\}$$

so that it is enough to show that for any $\delta > 0$

$$\mathbb{P}\left(\exists t \in [0,m] : |B_{t+h} - B_t| \le ph^\gamma, \forall h \in (0,\delta)\right) = 0$$

and in turn if $A_{i,n} = \{\exists t \in [i/n, (i+1)/n] : |B_{t+h} - B_t| \le ph^\gamma, \forall h \in (0,\delta)\}$,

$$\lim_{n\to\infty} \sum_{i=0}^{mn-1} \mathbb{P}\left(A_{i,n}\right) = 0\,.$$

Fix a large constant $K > 0$ to be chosen suitably later. We wish to exploit the fact that on the event $A_{i,n}$ many increments must be small. The trick is to be able to fix in advance the times at which these increments will be too small. More precisely, on $A_{i,n}$, as long as $n \ge (K+1)/\delta$, for all $1 \le j \le K$ so that $t - (i+j)/n \le K/n \le \delta$

$$|B_t - B_{\frac{i+j}{n}}| \le p(\frac{K+1}{n})^\gamma$$

and therefore by the triangular inequality

$$|B_{\frac{i+j-1}{n}} - B_{\frac{i+j}{n}}| \leq 2p(\frac{K+1}{n})^\gamma$$

Hence

$$\begin{aligned} \mathbb{P}(A_{i,n}) &\leq \mathbb{P}\left(\cap_{j=2}^K \{ |B_{\frac{i+j-1}{n}} - B_{\frac{i+j}{n}}| \leq 2p(\frac{K+1}{n})^\gamma \} \right) \\ &= \mathbb{P}\left(|B_{\frac{i+j-1}{n}} - B_{\frac{i+j}{n}}| \leq 2p(\frac{K+1}{n})^\gamma \right)^{K-1} \end{aligned}$$

with $B_{\frac{i+j-1}{n}} - B_{\frac{i+j}{n}}$ with law $N/\sqrt{n}$ for a standard Gaussian variable N so that

$$\mathbb{P}\left(\{ |B_{\frac{i+j-1}{n}} - B_{\frac{i+j}{n}}| \leq 2p(\frac{K+1}{n})^\gamma \} \right) = \mathbb{P}(|N| \leq \sqrt{n}2p(\frac{K+1}{n})^\gamma) \simeq C\sqrt{n}2p(\frac{K+1}{n})^\gamma$$

for some finite constant C as long as $\sqrt{n}2p(\frac{K+1}{n})^\gamma$ is small, that is $\gamma > 1/2$. Hence, keeping K fixed we find a finite constant C so that

$$\mathbb{P}(A_{i,n}) \leq C^K n^{(\frac{1}{2}-\gamma)(K-1)}$$

and therefore

$$\sum_{i=1}^{mn-1} \mathbb{P}(A_{i,n}) \leq C^K n^{(\frac{1}{2}-\gamma)(K-1)} mn$$

which goes to zero when n goes to infinity as soon as K is chosen big enough. □

We will later spend a lot of time to give a precise and rigorous construction of the stochastic integral, for as large a class of processes as possible, subject to continuity. This level of generality has a price, which is that the construction can appear quite technical without shedding any light on the sort of processes we are talking about. The real difficulty in the construction of the integral is in how to make sense of an integral against the Brownian motion, denoted

$$\int_0^t H_s dB_s$$

as $B_.$ is at best Hölder 1/2. To do that we will need to use randomness and martingale theory. We will enlarge our scope and consider more general processes than the Brownian motion soon. Before doing so we introduce (and hopefully motivate) some notions that we will discuss later in a wider scope, namely strong Markov property and stopping times.

1.4.3. *Strong Markov property.* We have already seen that the Wiener law satisfies the Markov property:

" For all $s \geq 0$, the process $B_{t+s} - B_s, t \geq 0$ is a Brownian motion independent of $\sigma(B_r, r \leq s)$."

The goal of this paragraph would be to extend this result to the case where s is itself a random variable. To do so, we need to resterict ourselves to the so-called stopping times; a random variable T with values in $[0, \infty]$ is a stopping time if, for all $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t = \sigma(B_s, s \leq t)$. We define

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : \forall t \geq 0, A \cap \{T \leq t\} \in \mathcal{F}_t\}$$

We set

$$1_{T<\infty} B_T(\omega) = \begin{cases} B_{T(\omega)}(\omega) & \text{if } T(\omega) < \infty \\ 0 & \text{otherwise.} \end{cases}$$

$1_{T<\infty} B_T$ is $\mathcal{F}_T$ measurable. Indeed by continuity of the trajectories

$$1_{T<\infty} B_T = \lim_{n\to\infty} \sum_{k=1}^{\infty} 1_{k2^{-n} \leq T < (k+1)2^{-n}} B_{k2^{-n}} = \lim_{n\to\infty} \sum_{k=1}^{\infty} 1_{T<(k+1)2^{-n}} (1_{k2^{-n} \leq T} B_{k2^{-n}}) \,,$$

where $1_{k2^{-n} \leq T} B_{k2^{-n}}$ is $\mathcal{F}_T$ measurable.

Theorem 1.11. *(Strong Markov Property) Let T be a stopping time such that $\mathbb{P}(T < \infty) > 0$. Then, the process*

$$B_t^{(T)} = 1_{T<\infty}(B_{T+t} - B_T), t \geq 0$$

is a Brownian motion independent of $\mathcal{F}_T$ under $\mathbb{P}(.|T < \infty)$.

Proof. We first assume $T < \infty$ a.s. and show that if $A \in \mathcal{F}_T$, for all bounded continuous function f

$$\mathbb{E}[1_A f(B_{t_1}^{(T)}, \ldots, B_{t_k}^{(T)})] = \mathbb{P}(A)\mathbb{E}[1_A f(B_{t_1}, \ldots, B_{t_k})]$$

which is enough to prove the statement. We denote $[T]_n = ([2^n T] + 1)2^{-n}$ with $[a]$ the integer part of a real a. Observe that by continuity of the trajectories,

$$f(B_{t_1}^{(T)}, \ldots, B_{t_k}^{(T)}) = \lim_{n\to\infty} f(B_{t_1}^{([T]_n)}, \ldots, B_{t_k}^{([T]_n)})$$

so that by dominated convergence theorem, for all bounded continuous function f

$$\begin{aligned} & \mathbb{E}[1_A f(B_{t_1}^{(T)}, \ldots, B_{t_k}^{(T)})] \\ = & \lim_{n\to\infty} \mathbb{E}[1_A f(B_{t_1}^{([T]_n)}, \ldots, B_{t_k}^{([T]_n)})] \\ = & \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{E}[1_A 1_{(k-1)2^{-n} < T \leq k2^{-n}} f(B_{t_1+k2^{-n}} - B_{k2^{-n}}, \ldots, B_{t_k+k2^{-n}} - B_{k2^{-n}})] \end{aligned}$$

For $A \in \mathcal{F}_T$, $A \cap \{(k-1)2^{-n} < T \le k2^{-n}\} = (A \cap \{T \le k2^{-n}\}) \cap \{T \le (k-1)2^{-n}\}^c$ is $\mathcal{F}_{k2^{-n}}$ measurable. Hence, the usual Markov property implies that

$$\mathbb{E}[1_A 1_{(k-1)2^{-n} < T \le k2^{-n}} f(B_{t_1+k2^{-n}} - B_{k2^{-n}}, \dots, B_{t_k+k2^{-n}} - B_{k2^{-n}})]$$
$$= \mathbb{E}[1_A 1_{(k-1)2^{-n} < T \le k2^{-n}}]\mathbb{E}[f(B_{t_1+k2^{-n}} - B_{k2^{-n}}, \dots, B_{t_k+k2^{-n}} - B_{k2^{-n}})]$$

from which the result follows. The same arguments can be followed when $T < \infty$ with positive probability. □

A nice application of the strong Markov property is the reflexion principle :

Theorem 1.12. *For all $t > 0$, denote by $S_t = \sup_{s \le t} B_s$. Then, if $a \ge 0$ and $b \le a$ we have*

$$\mathbb{P}(\max_{s \le t} B_s \ge a, B_t \le b) = \mathbb{P}(B_t \ge 2a - b)$$

In particular $\max_{s \le t} B_s$ *has the same law as* $|B_t|$.

Proof. We apply the strong Markov property with the stopping time

$$T_a = \inf\{t \ge 0 : B_t = a\}$$

We have already seen that T_a is finite almost surely. Moreover

$$\mathbb{P}(\max_{s \le t} B_s \ge a, B_t \le b) = \mathbb{P}(T_a \le t, B^{(T_a)}_{t-T_a} \le b - a)$$

By the strong Markov property, $B^{(T_a)}_{t-T_a}$ is independent of T_a and also has the same law as $-B^{(T_a)}_{t-T_a}$ where B is a Brownian motion. Hence we get

$$\begin{aligned} \mathbb{P}(T_a \le t, B^{(T_a)}_{t-T_a} \le b - a) &= \mathbb{P}(T_a \le t, -B^{(T_a)}_{t-T_a} \le b - a) \\ &= \mathbb{P}(T_a \le t, -B_t + B_{T_a} \le b - a) \\ &= \mathbb{P}(T_a \le t, B_t \ge 2a - b) = \mathbb{P}(B_t \ge 2a - b) \end{aligned}$$

where we used that $2a - b \ge a$ in the last line. For the last point we notice that for $a \ge 0$,

$$\mathbb{P}(S_t \ge a) = \mathbb{P}(S_t \ge a, B_t \ge a) + \mathbb{P}(S_t \ge a, B_t \le a) = 2\mathbb{P}(B_t \ge a) = \mathbb{P}(|B_t| \ge a)$$

□

2. Processes with independent increments

We will often consider stochastic processes with independent increments

Definition 2.1. *A stochastic process $(X_t)_{t \in \mathbb{T}}$ based on $(\Omega, \mathcal{G}, \mathbb{P})$ with values in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a process with independent increments (abbreviated I.I.P) iff*

(1) $X_0 = 0 \quad$ *a.s.,*

(2) *For all $n \geq 2$, for all $t_1, \dots, t_n \in \mathbb{R}^+$ so that $t_1 < t_2 < \cdots < t_n$, the random variables*

$$X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$$

are independent.

A stochastic process is a stationary process with independent increments (abbreviated S.I.I.P) if it is a I.I.P so that for all $s, t \in \mathbb{R}^+$, $0 \leq s < t$, $X_t - X_s$ has the same law than X_{t-s}.

When $\mathbb{T} = \mathbb{N}$, stationarity is described by the fact that there exists a sequence $(Z_i)_{i \in \mathbb{N}}$ of i.i.d. variables so that

$$S_n = Z_1 + \cdots + Z_n$$

In this case S_n is also called a random walk.

A family $(\mu_t)_{t \in \mathbb{R}^{+*}}$ of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called a convolution semi-group if for all $s, t \in [0, +\infty)$,

$$\mu_{s+t} = \mu_s * \mu_t \,.$$

PROPOSITION 2.2. *If $(X_t)_{t \in \mathbb{R}^+}$ is a S.I.I.P and μ_t is the law of X_t, $(\mu_t)_{t \in \mathbb{R}^+}$ is a convolution semi-group. It is called the convolution semi-group of $(X_t)_{t \in \mathbb{R}^+}$.*

More generally if $(X_t)_{t \in \mathbb{R}^+}$ is a I.I.P so that for all $s < t$ $X_t - X_s$ has law $\mu_{s,t}$ then for any $s < t < u$, we have

$$\mu_{s,u} = \mu_{s,t} * \mu_{t,u} \,.$$

PROOF. We write

$$X_u - X_s = (X_u - X_t) + (X_t - X_s)$$

and use the independence of $X_u - X_t$ and $X_t - X_s$ to conclude.

□

For SIIP we have an easier way to characterize the equivalence relation defined in 1.4

PROPOSITION 2.3. *a) If X and X' are two SIIP with the same convolution semi-group, they are equivalent.*

b) More generally, if X and X' are two IIP so that for all $s < t$, $X_t - X_s$ and $X'_t - X'_s$ have the same distribution, then they are equivalent.

Let us give some examples

1) $\mu = \delta_{at}$

2)μ_t is the Poisson law $P_{\lambda t}$ with parameter λt for all $t > 0$ ($P_x(k) = e^{-x} x^k / k!$). We will call *Poisson process with parameter* $\lambda > 0$ the SIIP with such convolution semi-group.

3)μ_t is the centered Gaussian law with covariance t. Check that the SIIP with such convolution semi-group is the Brownian motion.

More generally we will say that a stochastic process is a Gaussian real process iff

- It takes its values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,
- For all $n \geq 1$, all $t_1, \dots, t_n \in \mathbb{R}^+$, the random variable $(X_{t_1}, \dots, X_{t_n})$ is Gaussian.

Note that in this case the semi-group is determined by the mean

$$m(t) = \mathbb{E}[X_t]$$

and the covariance

$$C(t,s) = \mathbb{E}[X_t X_s] - m(t)m(s)\,,$$

as so is any Gaussian law.

Note that any covariance $C(t,s)$ is positive semi-definite, namely $C(s,t) = C(t,s)$ $\forall s,t \in \mathbb{T}$ and $\forall\ n \geq 1, \forall\ t_1, \dots, t_n \in \mathbb{T}, \forall\ \lambda_1, \dots, \lambda_n \in \mathbb{R}$,

$$\sum_{i,j=1}^{n} c(t_i, t_j)\lambda_i \lambda_j \geq 0.$$

Proposition 2.4. *A real stochastic process is a real Brownian motion iff it is a centered Gaussian real process with covariance*

$$\mathbb{E}[X_t X_s] = t \wedge s$$

Proof. $\Rightarrow$ If X is a real Brownian motion,

-$X_0 = 0$,

-For all $t > 0$, X_t follows $N(0,t)$

-For all $n \geq 2$ and all $t_1, \dots, t_n$ so that $t_1 < t_2 < \cdots < t_n$, $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ are independent Gaussian.

Hence X is a SIIP. Finally, for $s \leq t$

$$\mathbb{E}[X_s X_t] = \mathbb{E}[X_s^2] + \mathbb{E}[(X_t - X_s)X_s] = \mathbb{E}[X_s^2] = s = s \wedge t$$

by independence and centering.

$\Leftarrow$ Let X be a centered Gaussian process with covariance $s \wedge t$. The first thing we need to check is that the increments are independent ; but this follows from the vanishing of the covariance

$$\mathbb{E}[(X_{t_2} - X_{t_1})(X_{t_4} - X_{t_3})] = 0 \quad \text{if } t_1 < t_2 \leq t_3 < t_4$$

Hence X is a IIP. To check stationnarity it is enough to check that the covariances are stationary. But

$$\mathbb{E}[(X_t - X_s)^2] = t + s - 2s \wedge t = t - s$$

which completes the argument. □

2.1. Law of a stochastic process, canonical process. Let $(E, \mathcal{E})$ be a measurable space. Let $\mathbb{T}$ be a non empty set. We denote

$$E^{\mathbb{T}} = \{x = (x_t)_{t\in\mathbb{T}} : x_t \in E, \forall t \in \mathbb{T}\}.$$

We will call product σ-algebra on $E^{\mathbb{T}}$ (associated to the σ-algebra $\mathcal{E}$ and $\mathbb{T}$) the smallest σ-algebra on $E^{\mathbb{T}}$ so that the coordinate mappings :

$$\gamma_t : x = (x_s)_{s\in\mathbb{T}} \mapsto x_t$$

are measurable as $t \in \mathbb{T}$. It is denoted by $\mathcal{E}^{\otimes\mathbb{T}}$. We call measurable product space associated with $(E, \mathcal{E})$ and $\mathbb{T}$ the space $(E^{\mathbb{T}}, \mathcal{E}^{\otimes\mathbb{T}}) = (E, \mathcal{E})^{\mathbb{T}}$.

Proposition 2.5. *Let $(\Omega, \mathcal{F})$ be a measurable space. Let U be a map from Ω into $E^{\mathbb{T}}$.Then, U is measurable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})^{\mathbb{T}}$ iff $\forall\ t \in \mathbb{T}$, $\gamma_t \circ U$ is measurable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})$.*

Proof. $\Rightarrow$: If U is measurable, then $\gamma_t \circ U$ is measurable as the composition of measurable maps.
$\Leftarrow$: Reciprocally, if $\forall\ t \in \mathbb{T}$, $\gamma_t \circ U$ is measurable, then $(\gamma_t \circ U)^{-1}(A) \in \mathcal{F}, \forall\ A \in \mathcal{E}$. But, $(\gamma_t \circ U)^{-1}(A) = U^{-1}((\gamma_t)^{-1}(A))$. Thus, $U^{-1}(B) \in \mathcal{F}$ for all

$$B \in \mathcal{D} := \{(\gamma_t)^{-1}(A) : t \in \mathbb{T}, A \in \mathcal{E}\}.$$

Since $\mathcal{E}^{\otimes\mathbb{T}}$ is the sigma-algebra generated by $\mathcal{D}$, this implies that

$$U^{-1}(B) \in \mathcal{F}$$

for all $B \in \mathcal{E}^{\otimes\mathbb{T}}$. □

Let $(X_t)_{t\in\mathbb{T}}$ be a family of maps from Ω into E.
We denote by X the map :

$$\begin{array}{ccc} \Omega & \longrightarrow & E^{\mathbb{T}} \\ \omega & \longmapsto & (X_t(\omega))_{t\in\mathbb{T}}. \end{array}$$

We then have, recall the definition 1.3 of stochastic processes,

Corollary 2.6. *$(X_t)_{t\in\mathbb{T}}$ is a stochastic process with values in $(E, \mathcal{E})$ iff the map X is measurable from $(\Omega, \mathcal{F})$ into $(E, \mathcal{E})^{\mathbb{T}}$.*

Proof. Follows from the previous Proposition. □

According to the last corollary, we can identify the stochastic process $(X_t)_{t\in\mathbb{T}}$ and the measurable map X. In the following we will set $X = (X_t)_{t\in\mathbb{T}}$.

We denote *law, on $(E, \mathcal{E})^{\mathbb{T}}$, of X* the push-forward of the probability measure $\mathbb{P}$ by the measurable map X. We denote it $\mathbb{P}_X$.

Proposition 2.7. *Two stochastic processes $(X_t)_{t\in\mathbb{T}}$ and $(X'_t)_{t\in\mathbb{T}}$ (based respectively on $(\Omega,\mathcal{F},\mathbb{P})$ and $(\Omega',\mathcal{F}',\mathbb{P}')$) with values in $(E,\mathcal{E})$ are equivalent iff they have the same law on $(E,\mathcal{E})^{\mathbb{T}}$.*

Proof. $\Leftarrow$: If $\mathbb{P}_X = \mathbb{P}'_{X'}$, then the processes are equivalent as : for any $t_1,\dots,t_n \in \mathbb{T}$, if $A = \prod_{t\in\mathbb{T}} A_t$ with

$$A_t = \begin{cases} B_i \in \mathcal{E} \text{ if } t = t_i, 1 \le i \le n \\ E \text{ if } t \notin \{t_1,\dots,t_n\}, \end{cases}$$

we have :

$$X^{-1}(A) = \{X_{t_1} \in B_1,\dots,X_{t_n} \in B_n\}.$$
$$X'^{-1}(A) = \{X'_{t_1} \in B_1,\dots,X'_{t_n} \in B_n\}.$$

Therefore,

$$\mathbb{P}'(X'_{t_1} \in B_1,\dots,X'_{t_n} \in B_n) = \mathbb{P}'_{X'}(A) = \mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(X_{t_1} \in B_1,\dots,X_{t_n} \in B_n).$$

$\Rightarrow$: If $(X_t)_{t\in\mathbb{T}}$ and $(X'_t)_{t\in\mathbb{T}}$ are equivalent, we have : $\mathbb{P}_X(A) = \mathbb{P}'_{X'}(A)$ for any cylinder A in $E^{\mathbb{T}}$. But, $\mathcal{E}^{\otimes\mathbb{T}} = \sigma(\mathcal{C})$ with $\mathcal{C}$ the cylinder family (see exercise 2.23). But, $\mathcal{C}$ is stable under finite intersection and therefore following exercise 2.16 (Monotone class Thm), see also Lemma 9.2, we deduce that $\mathbb{P}_X = \mathbb{P}'_{X'}$. □

Let $X = (X_t)_{t\in\mathbb{T}}$ be a stochastic process, based on $(\Omega,\mathcal{F},\mathbb{P})$, with values in $(E,\mathcal{E})$.

The canonical process $(Y_t)_{t\in\mathbb{T}}$, on $(E,\mathcal{E})^{\mathbb{T}}$, associated with $(X_t)_{t\in\mathbb{T}}$ is the stochastic process based on $(E^{\mathbb{T}},\mathcal{E}^{\otimes\mathbb{T}},\mathbb{P}_X)$ defined by :

$$Y_t(x) = \gamma_t(x) = x_t, \forall x = (x_s)_{s\in\mathbb{T}} \in E^{\mathbb{T}}.$$

Proposition 2.8. *$(X_t)_{t\in\mathbb{T}}$ and its canonical process $(Y_t)_{t\in\mathbb{T}}$ are equivalent.*

Proof. Let $t_1,\dots,t_n \in \mathbb{T}$ and $B_1,\dots,B_n \in \mathcal{E}$. If $A = \prod_{t\in\mathbb{T}} A_t$ with

$$A_t = \begin{cases} B_i \text{ if } t = t_i, 1 \le i \le n \\ E \text{ if } t \notin \{t_1,\dots,t_n\}, \end{cases}$$

we have :

$$X^{-1}(A) = \{X_{t_1} \in B_1,\dots,X_{t_n} \in B_n\}.$$

We also have

$$A = \{x; Y_{t_1}(x) \in B_1,\dots,Y_{t_n}(x) \in B_n\}.$$

Therefore,

$$\mathbb{P}(X_{t_1} \in B_1,\dots,X_{t_n} \in B_n) = \mathbb{P}_X(A) = \mathbb{P}_X(Y_{t_1} \in B_1,\dots,Y_{t_n} \in B_n).$$

□

2.2. Canonical process with given finite partitions. Let $\mathbb{T}$ be an infinite set. We denote $\mathcal{I}$ the set of finite (non empty) subsets of $\mathbb{T}$. Let $(E, \mathcal{E})$ be a measurable space. Let $(\mathbb{P}_I)_{I \in \mathcal{I}}$ be a family of probability measures indexed by $\mathcal{I}$, so that for all $I \in \mathcal{I}$, $\mathbb{P}_I$ is a probability measure on $(E, \mathcal{E})^{\text{card}(I)} = (E^{\text{card}(I)}, \mathcal{E}^{\text{card}(I)})$.

We say that $(\mathbb{P}_I)_{I \in \mathcal{I}}$ is *a compatible system* (or a projective system) if : for all $I \in \mathcal{I}$, any $J \in \mathcal{I}$ so that $J \subset I$, $\mathbb{P}_J$ is the push-forward of $\mathbb{P}_I$ by the map

$$\Pi_{I,J} : (x_t)_{t \in I} \longrightarrow (x_s)_{s \in J}.$$

Let $X = (X_t)_{t \in \mathbb{T}}$ be a stochastic process, based on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $(E, \mathcal{E})$. If, for $I = \{t_1, \ldots, t_n\} \in \mathcal{I}$, we denote $\mathbb{P}_I$ the law of the random variable $(X_{t_1}, \ldots, X_{t_n})$, then $(\mathbb{P}_I)_{I \in \mathcal{I}}$ is the family of the finite partitions of the stochastic process $X = (X_t)_{t \in \mathbb{T}}$. Moreover, we have :

Proposition 2.9. *The finite partitions of $X = (X_t)_{t \in \mathbb{T}}$ are a compatible system.*

Proof. If $I = \{t_1, \ldots, t_n\} \supset J = \{t_{i_1}, \ldots, t_{i_k}\}$, with $t_i \in \mathbb{T}, \forall i = 1, \ldots, n, n \geq 2$, $1 \leq k < n, 1 \leq i_1 < i_2 < \ldots < i_k \leq n$. We have :

$$\begin{aligned} \mathbb{P}_J(B_1 \times \ldots \times B_k) &= \mathbb{P}(X_{t_{i_1}} \in B_1, \ldots, X_{t_{i_k}} \in B_k) \\ &= \mathbb{P}(X_{t_j} \in E, \forall j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}, X_{t_{i_1}} \in B_1, \ldots, X_{t_{i_k}} \in B_k) \\ &= \mathbb{P}_I(\Pi_{I,J}^{-1}(B_1 \times \ldots \times B_k)) \end{aligned}$$

□

Reciprocally, let us be given a compatible system of probability measures $(\mathbb{P}_I)_{I \in \mathcal{I}}$ (with for all $I \in \mathcal{I}$, $\mathbb{P}_I$ a probability measure $(E, \mathcal{E})^{\text{card}(I)}$).
We set :
$\Omega = E^{\mathbb{T}} = \{\omega = (\omega_t)_{t \in \mathbb{T}} : \omega_t \in E, \forall t \in \mathbb{T}\}$.
$\mathcal{F} = \mathcal{E}^{\otimes \mathbb{T}}$.
$Y_t(\omega) = \omega_t$.
Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$, $(Y_t)_{t \in \mathbb{T}}$ can be seenas a stochastic process based on $(\Omega, \mathcal{F}, \mathbb{P})$. It is called the associated canonical process associated with $\mathbb{P}$ (on $(E^{\mathbb{T}}, \mathcal{E}^{\otimes \mathbb{T}}, \mathbb{P})$).

Theorem 2.10. [Kolmogorov] *(admitted) If E is a Polish space and if $\mathcal{E}$ is the Borel σ-algebra of E, there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}) := (E, \mathcal{E})^{\mathbb{T}}$ so that the canonical process $(Y_t)_{t \in \mathbb{R}^+}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ has $(P_I)_{I \in \mathcal{I}}$ has finite partitions family.*

Applications :

Corollary 2.11. *a). To any convolution semi-group $(\mu_t)_{t \in]0,+\infty[}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ corresponds a SIIP $(Y_t)_{t \in \mathbb{R}^+}$, which is unique up to equivalence, so that for all*

$t\in]0,+\infty[$, μ_t is the law of Y_t.
b). To any family $(\mu_{s,t})_{0\le s<t}$ of probability measures on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ satisfying :

$$\forall s,t,u \text{ so that } 0\le s<t<u,\ \mu_{s,u}=\mu_{s,t}*\mu_{t,u},$$

corresponds an IIP $(Y_t)_{t\in\mathbb{R}^+}$, unique up to equivalence so that $\forall s,t\in\mathbb{R}^+$ so that $0\le s<t$, Y_t-Y_s has law $\mu_{s,t}$.

Remark : a). allows in particular to show existence of the homogeneous Poisson process and of the Brownian motion, and b) that of inhomogeneous Poisson process.

Proof. It is enough to prove b). : Let $I=\{t_1,\dots,t_n\}$ with $0\le t_1<t_2<\ldots<t_n$, $\mathbb{P}_I$ the push-forward of $\mu_{0,t_1}\otimes\mu_{t_1,t_2}\otimes\ldots\otimes\mu_{t_{n-1},t_n}$, by the map ϕ_n : $(x_1,\dots,x_n)\mapsto(x_1,x_1+x_2,\dots,x_1+x_2+\ldots+x_n)$.
Let us show that $(\mathbb{P}_I)_I$ is compatible :
Let $J=\{t_{i_1},\dots,t_{i_k}\}$ with $1\le i_1<i_2<\ldots<i_k\le n$, (and $k<n$). We have :

$$\mathbb{P}_J=\phi_k\cdot(\mu_{0,t_{i_1}}\otimes\mu_{t_{i_1},t_{i_2}}\otimes\ldots\otimes\mu_{t_{i_{k-1}},t_{i_k}}),$$

that is that the push-forward by ϕ_k of the probability measure $\mu_{0,t_{i_1}}\otimes\mu_{t_{i_1},t_{i_2}}\otimes\ldots\otimes\mu_{t_{i_{k-1}},t_{i_k}}$.
But,

$$\mu_{0,t_{i_1}}\otimes\mu_{t_{i_1},t_{i_2}}\otimes\ldots\otimes\mu_{t_{i_{k-1}},t_{i_k}}=\gamma\cdot(\mu_{0,t_1}\otimes\mu_{t_1,t_2}\otimes\ldots\otimes\mu_{t_{n-1},t_n}),$$

with

$$\gamma(x_1,\dots,x_n)=(\sum_{0<i\le i_1}x_i,\sum_{i_1<i\le i_2}x_i,\dots,\sum_{i_{k-1}<i\le i_k}x_i).$$

Therefore

$$\mathbb{P}_J=(\phi_k\circ\gamma)\cdot(\mu_{0,t_1}\otimes\mu_{t_1,t_2}\otimes\ldots\otimes\mu_{t_{n-1},t_n}).$$

It is easy to see that $\phi_k\circ\gamma=\Pi\circ\phi_n$ where $\Pi:=\Pi_{I,J}$. We hence deduce that

$$\mathbb{P}_J=\Pi\cdot(\phi_n\cdot(\mu_{0,t_1}\otimes\mu_{t_1,t_2}\otimes\ldots\otimes\mu_{t_{n-1},t_n}))=\Pi_{I,J}\cdot\mathbb{P}_I.$$

□

Corollary 2.12. *Let m be a map from $\mathbb{T}$ into $\mathbb{R}$ and let c be a semi-definite positive function from $\mathbb{T}^2$ into $\mathbb{R}$. There exists a real Gaussian process, unique up to equivalence, with mean m and covariance c.*

Proof. Let $I=\{t_1,\dots,t_n\}$ with $0\le t_1<t_2<\ldots<t_n$, $\mathbb{P}_I$ the Gaussian law on $(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n))$ with mean $(m(t_1),\dots,m(t_n))$ and covariance matrix $(c(t_i,t_j))_{i,j=1,\dots n}$. Let $J\subset I$, with $\Pi_{I,J}\cdot\mathbb{P}_I$ and $\mathbb{P}_J$ be two Gaussian probability measures with same mean and covariance. Hence, $(\mathbb{P}_I)_{I\in\mathcal{I}}$ is compatible and we can conclude by Kolmogorov theorem. □

Example : Take $\mathbb{T} = \mathbb{R}^+$. The function c is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ by $c(s,t) = s \wedge t$ is a covariance function as $\forall s_1, \dots, s_n$ so that $0 \leq s_1 < s_2 < \dots < s_n$, the matrix $(s_i \wedge s_j)_{i,j=1,\dots,n}$ is positive semi-definite as the covariance matrix of $(U_1, U_1 + U_2, \dots, U_1 + \dots + U_n)$ where the variables $(U_1, \dots, U_n)$ are independent and with centered Gaussian law $\mathcal{N}(0, s_1)$ for U_1 and $\mathcal{N}(0, s_k - s_{k-1})$ for U_k.

2.3. Point processes, Poisson processes. In this last paragraph, we consider random partitions of points in $]0, +\infty[$.

A point process on $]0, +\infty[$ is a sequence $(S_k)_{k\geq 1}$ of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that we have : $0 < S_1(\omega) < S_2(\omega) < \dots < S_k(\omega) < \dots$ and $\lim_{k\to+\infty} S_k(\omega) = +\infty$ for all $\omega \in \Omega$. The S_k represent the arrival time of a random phenomenon(cf the arrival times of clients etc) ...
We set : $Z_1 = S_1$ and for all $k \geq 2$, $Z_k = S_k - S_{k-1}$ (delay between two successive arrivals). So for all $n \geq 1$, $S_n = \sum_{k=1}^{n} Z_k$.
To any point process $(S_k)_{k\geq 1}$ on $]0, +\infty[$, we associate a stochastic process called *random counting function* $(N_t)_{t\in\mathbb{R}^+}$ given by: $N_0(\omega) = 0$ for all $\omega \in \Omega$ and

$$N_t(\omega) = \sum_{n=1}^{+\infty} \mathbf{1}_{\{S_n(\omega)\leq t\}}(\omega)$$

the number of arrivals during the time interval $]0, t]$.
As $\lim_{k\to+\infty} S_k(\omega) = +\infty$, we have $N_t(\omega) < +\infty$ for all $t > 0$ and all $\omega \in \Omega$. Moreover, $(N_t)_{t\in\mathbb{R}^+}$ takes values in $\mathbb{N}$ and has non decreasing, right continuous, trajectoires, following a stair shape with jumps no larger than one unit. The data of the point process $(S_k)_{k\geq 1}$ is equivalent to that of $(N_t)_{t\in\mathbb{R}^+}$ since $(S_0 \equiv 0)$

$$\{N_t = n\} = \{S_n \leq t < S_{n+1}\}$$

We also have for all $n \in \mathbb{N}^*$, $\{N_t < n\} = \{S_n > t\}$.

THEOREM 2.13. *If the random variables Z_k are independent, exponentially distributed with parameter λ, then we have :*
a) For all $t > 0$, the random variable N_t has Poisson distribution with parameter λt.
b) $(N_t)_{t\in\mathbb{R}^+}$ is a SIIP.

PROOF. a). The random variable S_n is the sum of n independent random exponential variables with parameter λ, and therefore follows a Gamma distribution with density $g_n(x) = \lambda^n \frac{x^{n-1}}{(n-1)!} e^{-\lambda x} \mathbf{1}_{\mathbb{R}^+}(x)$. Therefore,

$$\begin{aligned}
\mathbb{P}(N_t < n) &= \frac{\lambda^n}{(n-1)!} \int_t^{+\infty} x^{n-1} e^{-\lambda x}\, dx \\
&= e^{-\lambda t}\Big(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \dots + \frac{(\lambda t)^{n-1}}{(n-1)!}\Big)
\end{aligned}$$

Hence, $\mathbb{P}(N_t = 0) = \mathbb{P}(N_t < 1) = e^{-\lambda t}$ and so for all $n \geq 1$,

$$\mathbb{P}(N_t = n) = \mathbb{P}(N_t < n+1) - \mathbb{P}(N_t < n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Moreover, N is a SIIP as we can write

$$N_t - N_s = \sum_{n=N_s+1}^{\infty} 1_{S_n - S_{N_s} \leq t-s}$$

where $S_{N_s} = s$ and conditionally to N_s, $S_n - S_{N_s}$ has the same law than S_{n-N_s} and is independent from N_s. Hence, the law of $N_t - N_s$ conditionally to N_s is the same as the law of N_{t-s}, or in other words N is a SIIP. □

We call *standard Poisson process* any real stochastic process $(X_t)_{t\in\mathbb{R}^+}$ so that $(X_t)_{t\in\mathbb{R}^+}$ is a Poisson process so that $X_0 \equiv 0$ and all trajectories are non-decreasing, right-continuous and with jump bounded by one. As a reciprocal to the previous theorem we have

THEOREM 2.14. *Assume that the counting function $(N_t)_{t\in\mathbb{R}^+}$ of the point process $(S_k)_{k\geq 1}$ is a SIIP.*
a) There exists $\lambda > 0$ so that the random variable Z_1 is exponential with parameter λ.
b) For any $t > 0$, the random variable N_t is exponential with parameter λt.
c) The sequence $(Z_k)_{k\geq 1}$ is i.i.d with exponential distribution with parameter λ.

PROOF. a). Noticing that $\{Z_1 > t\} = \{N_t = 0\}$, we deduce

$$\begin{aligned}
\mathbb{P}(Z_1 > t+s) &= \mathbb{P}(N_{t+s} = 0) \\
&= \mathbb{P}(N_{t+s} - N_s = 0, N_s = 0) \\
&= \mathbb{P}(N_{t+s} - N_s = 0)\mathbb{P}(N_s = 0) \\
&= \mathbb{P}(N_t = 0)\mathbb{P}(N_s = 0) \\
&= \mathbb{P}(Z_1 > t)\mathbb{P}(Z_1 > s)
\end{aligned}$$

Hence the fonction $t \to \mathbb{P}(Z_1 > t)$ taking values in $[0,1]$, non increasing and so that $\mathbb{P}(Z_1 > 0) = 1$, there exists $\lambda > 0$ so that for all $t \in \mathbb{R}^+$,

$$\mathbb{P}(Z_1 > t) = e^{-\lambda t} = \int_t^{+\infty} \lambda e^{-\lambda x} dx.$$

Moreover, let $\varepsilon \in (0,t)$ so that, as N is a SIIP,
(2.1)

$$\mathbb{P}(N_t = n) - \mathbb{P}(N_{t-\varepsilon} = n) = (\mathbb{P}(N_\varepsilon = 0) - 1)\mathbb{P}(N_{t-\varepsilon} = n) + \sum_{y=1}^{n} \mathbb{P}(N_\varepsilon = y)\mathbb{P}(N_{t-\varepsilon} = n-y)$$

We next claim that again by the SIIP property, $\mathbb{P}(N_\varepsilon \geq 2) \leq C\varepsilon^2$. Indeed, we can write

$$p_\varepsilon = \mathbb{P}(N_\varepsilon \geq 2) \leq \mathbb{P}(N_{\varepsilon/2} \geq 2) + \mathbb{P}(N_{\varepsilon/2} \geq 1)^2 \leq e^{-\lambda\varepsilon/2} p_{\varepsilon/2} + \frac{\lambda^2\varepsilon^2}{4}$$

where we used $\mathbb{P}(N_{\varepsilon/2} \geq 1) = 1 - e^{-\varepsilon\lambda/2} \leq \varepsilon\lambda/2$. The result follows by iteration. We therefore deduce from (**??**) that

$$\left| \varepsilon^{-1}(\mathbb{P}(N_t = n) - \mathbb{P}(N_{t-\varepsilon} = n)) - \varepsilon^{-1}(\mathbb{P}(N_\varepsilon = 0) - 1)\mathbb{P}(N_{t-\varepsilon} = n) \right.$$
$$\left. + \varepsilon^{-1}\mathbb{P}(N_\varepsilon \geq 1)\mathbb{P}(N_{t-\varepsilon} = n-1) \right| \leq C\varepsilon \tag{2.2}$$

We thus see that $t \to \mathbb{P}(N_t = n)$ is continuous and even differentiable with, by letting ε going to zero,

$$\partial_t \mathbb{P}(N_t = n) = -\lambda\mathbb{P}(N_t = n) + \lambda\mathbb{P}(N_t = n-1)\,.$$

It follows by induction over n that $\mathbb{P}(N_t = n) = e^{-\lambda t}(\lambda t)^n/n!$ for all integer number n, which proves the second point. Hence, the law of $(N_t)_{t\geq 0}$ is uniquely determined and so is the law of $(S_n)_{n\geq 0}$. As it corresponds to the sum of i.i.d exponential variables, we are done.

□

EXERCISES

Exercise 2.15. Prove the following monotone class theorem :

Let $\mathcal{C}$ be a family of subsets of Ω, non empty and stable under finite intersection. Then the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$ coincides with the smallest family $\mathcal{D}$ of subsets of Ω containing $\mathcal{C}$, with $\Omega \in \mathcal{D}$, which is stable under difference and increasing limit.

Exercise 2.16. Prove the following corollary to the previous monotone class theorem:

Let $(\Omega, \mathcal{F})$ be a mesurable space and let $\mathcal{C}$ be a family of subsets of Ω contained in $\mathcal{F}$, stable under finite intersection and such that $\sigma(\mathcal{C}) = \mathcal{F}$.
Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$ which coincide on $\mathcal{C}$ (i.e. such that $\mathbb{P}(A) = \mathbb{Q}(A)$, $\forall A \in \mathcal{C}$). Then we have $\mathbb{P} = \mathbb{Q}$.

EXERCISE 2.17. Show that the d-dimensionnal stochastic process $(X_t)_{t\in\mathbb{R}^+}$ iff if we set $\mathcal{F}_t^o = \sigma(X_s; s \le t)$, we have : $X_0 = 0$ $\mathbb{P}$-a.s. and if $\forall s,t \in \mathbb{R}^+$ so that $s < t$, the random variable $X_t - X_s$ is independent from the σ-algebra $\mathcal{F}_s^o$.
(Indication : Use the monotone class theorem to show that if $(X_t)_{t\in\mathbb{R}^+}$ is a IIP the random variable $X_t - X_s$ is independent from $\mathcal{F}_s^o$).

EXERCISE 2.18. A stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is *self-similar* (of order 1) if, for all $\lambda > 0$, the stochastic processes $(X_{\lambda t})_{t\in\mathbb{R}^+}$ and $(\lambda\, X_t)_{t\in\mathbb{R}^+}$ are equivalent.
Show that if $(B_t)_{t\in\mathbb{R}^+}$ is a real Brownian motion, the process $(B_{t^2})_{t\in\mathbb{R}^+}$ is self-similar of order one.

EXERCISE 2.19. Show that if $(B_t)_{t\in\mathbb{R}^+}$ is a real Brownian motion, the following stochastic processes are real Brownian motions :
a)$(-B_t)_{t\in\mathbb{R}^+}$.
b) $(c\, B_{t/c^2})_{t\in\mathbb{R}^+}$, $\forall c > 0$.
c) $(X_t)_{t\in\mathbb{R}^+}$ defined by : $X_0 = 0$ and by $X_t = t\, B_{1/t}$, $\forall t > 0$.

EXERCISE 2.20. Let $(B_t)_{t\in\mathbb{R}^+}$ a real Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. We set for $t \in [0,1]$, $Y_t = B_t - t\, B_1$ et $Z_t = Y_{1-t}$.
a) Show that $(Y_t)_{t\in[0,1]}$ et $(Z_t)_{t\in[0,1]}$ are centered Gaussian process and compare their finite dimensional laws.
b) We set for $t \in \mathbb{R}^+$, $W_t = (t+1)Y_{t/(1+t)}$. Show that $(W_t)_{t\in\mathbb{R}^+}$ is a real Brownian motion.

EXERCISE 2.21. Let $(B_t)_{t\in\mathbb{R}^+}$ be a real Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\lambda > 0$. We set for $t \in \mathbb{R}^+$, $U_t = e^{-\lambda t} B_{e^{2\lambda t}}$.
a) Show that $(U_t)_{t\in\mathbb{R}^+}$ is a centered Gaussian process and determine its covariance c.
b) Deduce from the form c that $(U_t)_{t\in\mathbb{R}^+}$ is *stationnary*, i.e. that :
$\forall n \ge 1$, $\forall t_1, \ldots, t_n \in \mathbb{R}^+$, $\forall s > 0$, with $0 \le t_1 < t_2 < \ldots < t_n$, the random variables $(X_{t_1}, \ldots, X_{t_n})$ and $(X_{t_1+s}, \ldots, X_{t_n+s})$ have the same law.

EXERCISE 2.22. Let $d \ge 2$ and denote by $< .,. >$ the scalar product on $\mathbb{R}^d$ and $||.||$ the Euclidean norm. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider d real independent Brownian motions $(B_t^1)_{t\in\mathbb{R}^+}, (B_t^2)_{t\in\mathbb{R}^+}, \ldots, (B_t^d)_{t\in\mathbb{R}^+}$ and we set for

$t \in \mathbb{R}^+$, $B_t = (B_t^1, \dots, B_t^d)$. $(B_t)_{t\in\mathbb{R}^+}$ is a d-dimensionnal Brownian .
a) Show that $\forall x = (x_1, \dots, x_d) \in \mathbb{R}^d$ such that $||x|_2^2 = \sum x_i^2| = 1$, the real stochastic process $(< B_t, x >)_{t\in\mathbb{R}^+}$ is a real Brownian motion.
b) Take $d = 2$ and set $X_t = (X_t^1, X_t^2)$ with

$$X_t^1 = B^1_{\frac{2t}{3}} - B^2_{\frac{t}{3}} \quad \text{and } X_t^2 = B^2_{\frac{2t}{3}} + B^1_{\frac{t}{3}}$$

If $x = (x_1, x_2) \in \mathbb{R}^2$ has norm one, what can we say about the process $(< X_t, x >)_{t\in\mathbb{R}^+}$? Are the stochastic processes $(X_t^1)_{t\in\mathbb{R}^+}$ and $(X_t^2)_{t\in\mathbb{R}^+}$ independent ? Are they real Brownian motions ?
c) If $(X_t)_{t\in\mathbb{R}^+} = (X_t^1, \dots, X_t^d)_{t\in\mathbb{R}^+}$ is a d dimensionnal stochastic process such that $x \in \mathbb{R}^d$ has norm one, $(< X_t, x >)_{t\in\mathbb{R}^+}$ is a real Brownian motion, are $(X_t^1)_{t\in\mathbb{R}^+}, \dots, (X_t^d)_{t\in\mathbb{R}^+}$ independent Brownian motions ?

Exercise 2.23. Show that the product σ-algebra $\mathcal{E}^{\otimes\mathbb{T}}$ on $E^{\mathbb{T}}$ coincides with the σ-algebra generated by cylinders of $E^{\mathbb{T}}$, that is by the sets $B = \prod_{t\in\mathbb{T}} A_t$ where $A_t \in \mathcal{E}, \forall\, t \in \mathbb{T}$ and $A_t = E$ except for a finite number of times t.

3. Martingales

3.1. Filtration. Adapted process. Martingale. *A filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$ (i.e. $s < t$, $\mathcal{F}_s \subset \mathcal{F}_t$). A measurable space $(\Omega, \mathcal{G})$ endowed with a filtration $(\mathcal{G}_t)_{t\geq 0}$ is said to be filtered.

The filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is said to be right-continuous if $\forall t \in \mathbb{R}^+$, we have

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s.$$

To any filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$, we can associate a right-continuous filtration denoted $(\mathcal{F}_{t^+})_{t\in\mathbb{R}^+}$ given by

$$\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s.$$

We say that $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is complete (for $\mathbb{P}$) if $\mathcal{F}_0$ contains all the neglectable ensembles of $\mathcal{G}$(for $\mathbb{P}$). If $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, we associate to it its completed filtration (for $\mathbb{P}$): $(\bar{\mathcal{F}}_t)_{t\in\mathbb{R}^+}$ by adding to each $\mathcal{F}_t$ the neglectable sets of $\mathcal{G}$. We assume in general that $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is complete up to replacing it by its completed filtration.

Let $(X_t)_{t\in\mathbb{R}^+}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $(E, \mathcal{E})$. *The natural filtration* $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$ *associated to* $(X_t)_{t\in\mathbb{R}^+}$ is given by:

$$\mathcal{F}_t^0 = \sigma(X_s : s \leq t), \forall t \in \mathbb{R}^+.$$

Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be a filtration on $(\Omega,\mathcal{F},\mathbb{P})$. A stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is said $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$*-adapted* if for all $t\in\mathbb{R}^+$, the random variable X_t is $\mathcal{F}_t-$measurable. Any stochastic process is clearly adapted to its natural filtration. A stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-adapted if for all $t\in\mathbb{R}^+$, $\mathcal{F}_t^0\subset\mathcal{F}_t$. If $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is complete for $\mathbb{P}$, if $(X_t)_{t\in\mathbb{R}^+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-adapted and $(Y_t)_{t\in\mathbb{R}^+}$ is a modification of $(X_t)_{t\in\mathbb{R}^+}$, then $(Y_t)_{t\in\mathbb{R}^+}$ is also adapted.

A real stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a *supermartingale with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ if :

i) $(X_t)_{t\in\mathbb{R}^+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-adapted.

ii) $\forall t\in\mathbb{R}^+$, the random variable X_t is integrable.

iii) $\forall s\in\mathbb{R}^+$, $\forall t\in\mathbb{R}^+$, so that $s\leq t$, we have :

$$X_s\geq\mathbb{E}(X_t|\mathcal{F}_s),\quad \mathbb{P}-a.s.$$

A real stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a *a submartingale with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ if $(-X_t)_{t\in\mathbb{R}^+}$ is a super-martingale, that is :

i) $(X_t)_{t\in\mathbb{R}^+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-adapted.

ii) $\forall t\in\mathbb{R}^+$, the random variable the random variable X_t is integrable.

iii) $\forall s\in\mathbb{R}^+$, $\forall t\in\mathbb{R}^+$, so that $s\leq t$, we have :

$$X_s\leq\mathbb{E}(X_t|\mathcal{F}_s),\quad \mathbb{P}-a.s.$$

A real stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a *a martingale with respect to* $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ if it is both a supermartingale and a submartingale, that is

i) $(X_t)_{t\in\mathbb{R}^+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-adapted.

ii) $\forall t\in\mathbb{R}^+$, the random variable the random variable X_t is integrable.

iii) $\forall s\in\mathbb{R}^+$, $\forall t\in\mathbb{R}^+$, so that $s\leq t$, we have :

$$X_s=\mathbb{E}(X_t|\mathcal{F}_s),\quad \mathbb{P}-a.s.$$

Remark :
a) . If $(X_t)_{t\in\mathbb{R}^+}$ is a sub (resp. super, resp.) the function $t\to\mathbb{E}(X_t)$ is decreasing (resp. increasing, resp constant)

b). If $(X_t)_{t\in\mathbb{R}^+}$ is a sub(resp. super)martingale so that the function $t\to\mathbb{E}(X_t)$ is constant then $(X_t)_{t\in\mathbb{R}^+}$ is a martingale. (Exercise !)

Example :
Let $U\in L^1$ and set $M_t=\mathbb{E}(U|\mathcal{F}_t),\forall t\in\mathbb{R}^+$, then $(M_t)_{t\in\mathbb{R}^+}$ is a martingale.

A d-dimensionnal stochastic process $(X_t)_{t\in\mathbb{R}^+}$ is a $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-*IIP*iff:
i). $X_0 = 0$, $\mathbb{P}$-a.s.
ii). $(X_t)_{t\in\mathbb{R}^+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$- adapted.
iii). $\forall s \in \mathbb{R}^+$, $\forall t \in \mathbb{R}^+$, so that $s \leq t$, the random variable $X_t - X_s$ is independent from $\mathcal{F}_s$.
A d-dimensionnal stochastic process $(X_t)_{t\in\mathbb{R}^+}$is a $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-*SIIP* if it is a $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-IIP so that :
iv). $\forall s \in \mathbb{R}^+$, $\forall t \in \mathbb{R}^+$, so that $s \leq t$, the random variable $X_t - X_s$ has the same law as X_{t-s}.

THEOREM 3.1. *If $(X_t)_{t\in\mathbb{R}^+}$ is a real $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-IIP and if $\forall t \in \mathbb{R}^+$, the random variable X_t is centered and integrable, then $(X_t)_{t\in\mathbb{R}^+}$ is a $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-martingale.*

PROOF. If $s \leq t$ with $s, t \in \mathbb{R}^+$, we have :

$$\mathbb{E}(X_t - X_s|\mathcal{F}_s) = \mathbb{E}(X_t - X_s) = \mathbb{E}(X_t) - \mathbb{E}(X_s) = 0, \mathbb{P} - a.s.$$

But

$$\mathbb{E}(X_t - X_s|\mathcal{F}_s) = \mathbb{E}(X_t|\mathcal{F}_s) - X_s, \mathbb{P} - a.s.$$

Therefore

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s, \mathbb{P} - a.s.$$

□

As a corollary one easily deduces the following

COROLLARY 3.2. *a). If $(B_t)_{t\in\mathbb{R}^+}$ is a real Brownian motion, then $(B_t)_{t\in\mathbb{R}^+}$ is a martingale with respect to its natural filtration $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$ and also for the natural completed filtration $(\bar{\mathcal{F}}_t^0)_{t\in\mathbb{R}^+}$.*
b). If $(N_t)_{t\in\mathbb{R}^+}$ is a Poisson process with parameter $\lambda > 0$ then $(N_t - \lambda t)_{t\in\mathbb{R}^+}$ is a martingale with respect to the natural filtration of $(N_t)_{t\in\mathbb{R}^+}$.

We have also

PROPOSITION 3.3. *a). If $(B_t)_{t\in\mathbb{R}^+}$ is a real Brownian motion then $(B_t^2 - t)_{t\in\mathbb{R}^+}$ is a martingale for the natural completed filtration of $(B_t)_{t\in\mathbb{R}^+}$.*
For all $\alpha \neq 0$, $(\exp(\alpha B_t - \frac{\alpha^2}{2}t))_{t\in\mathbb{R}^+}$ is a martingale for the natural completed filtration $(\bar{\mathcal{F}}_t^0)_{t\in\mathbb{R}^+}$.

b). If $(N_t)_{t\in\mathbb{R}^+}$ is a Poisson process with parameter $\lambda > 0$ then $((N_t - \lambda t)^2 - \lambda t)_{t\in\mathbb{R}^+}$ is a martingale with respect to the natural filtration $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$ of $(N_t)_{t\in\mathbb{R}^+}$.
For all $\alpha \neq 0$, $(\exp(\alpha N_t - (e^\alpha - 1)\lambda t))_{t\in\mathbb{R}^+}$ is as well a martingale for $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$.

Proof. a)Clearly $B_t^2 \in L^1$ and moreover

$$\mathbb{E}[B_t^2 - B_s^2|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2 + 2B_s(B_t - B_s)|\mathcal{F}_s]$$

As $B_t - B_s$ is independent of $\mathcal{F}_s$ and centered we deduce that

$$\mathbb{E}[B_t^2 - B_s^2|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2] = t - s$$

from which the result follows. Similarly for all α $\exp\{\alpha B_t\} \in L^1$ and

$$\mathbb{E}[e^{\alpha B_t}|\mathcal{F}_s] = e^{\alpha B_s}\mathbb{E}[e^{\alpha(B_t - B_s)}] = e^{\alpha B_s} e^{\frac{\alpha^2}{2}(t-s)}$$

The proof of b) is similar.

We next prove an important inequality due to Doob.

THEOREM 3.4. [Doob's inequality]
Let $(X_t)_{t\in\mathbb{R}^+}$ be a right-continuous non-negative submartingale or a right-continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$, a).then for all $t > 0$, and $c > 0$,

$$\mathbb{P}(\sup_{s\in[0,t]} X_s \geq c) \leq \frac{\mathbb{E}(|X_t|)}{c}.$$

b). Assume as well that for all $t \in \mathbb{R}^+$, $X_t \in L^p$, with $p > 1$ given, then for all $t > 0$, all $c > 0$,

$$\mathbb{P}(\sup_{s\in[0,t]} |X_s| \geq c) \leq \frac{\mathbb{E}(|X_t|^p)}{c^p}.$$

c). Under the hypotheses of b). we deduce that $\sup_{s\in[0,t]} |X_s| \in L^p$ and

$$|| \sup_{s\in[0,t]} |X_s| ||_p \leq C||X_t||_p,$$

where $C = p/(p-1)$

PROOF. The proof of a) will be deduced from the following Lemma

LEMMA 3.5. *Let $(Y_k)_{k\in\mathbb{N}}$ be a submartingale with respect to the filtration $(\mathcal{G}_k)_{k\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, then for all $m \geq 1$, all $c > 0$,*

$$\mathbb{P}(\max_{0\leq k\leq m} Y_k \geq c) \leq \frac{\mathbb{E}(|Y_m| 1_{\max_{0\leq k\leq m} Y_k \geq c})}{c} \leq \frac{\mathbb{E}(|Y_m|)}{c}.$$

PROOF. Let for $k \geq 1$,

$$A_k = \{Y_0 < c\} \cap \ldots \cap \{Y_{k-1} < c\} \cap \{Y_k \geq c\}$$

and $A_0 = \{Y_0 \geq c\}$. Let $A = \{\max_{0\leq k\leq m} Y_k \geq c\}$. As A is the disjoint union of the A_k's, we get

$$\begin{aligned} c\,\mathbb{P}(A) &= \sum_{k=0}^{m} c\,\mathbb{P}(A_k) \\ &\leq \sum_{k=0}^{m} \mathbb{E}(Y_k \mathbf{1}_{A_k}) \end{aligned}$$

Fix $k \geq 0$, $A_k \in \mathcal{G}_k$, so

$$\begin{aligned}\mathbb{E}(Y_k \mathbf{1}_{A_k}) &\leq \mathbb{E}(\mathbb{E}(Y_m|\mathcal{G}_k)\mathbf{1}_{A_k}) \\ &\leq \mathbb{E}(\mathbb{E}(Y_m \mathbf{1}_{A_k}|\mathcal{G}_k)) \\ &\leq \mathbb{E}(\mathbb{E}(|Y_m|\mathbf{1}_{A_k}|\mathcal{G}_k)) = \mathbb{E}(|Y_m|\mathbf{1}_{A_k})\end{aligned}$$

Hence

$$\begin{aligned}c\ \mathbb{P}(A) &\leq \sum_{k=0}^{m} \mathbb{E}(|Y_m|\mathbf{1}_{A_k}) \\ &= \mathbb{E}(|Y_m|\mathbf{1}_A) \leq \mathbb{E}(|Y_m|)\end{aligned}$$

□

We apply the Lemma for $n \geq 1$, to

$$Y_k^{(n)} = X_{\frac{k}{2^n}}$$

with $\mathcal{G}_k = \mathcal{F}_{\frac{k}{2^n}}$, $\forall k \in \mathbb{N}$.

- If $t \in D$ the set of the dyadic numbers of $\mathbb{R}^+$, we obtain, letting n going to $+\infty$, that

$$\mathbb{P}(\sup_{s\in[0,t]\cap D} X_s \geq c) \leq \frac{\mathbb{E}(|X_t|)}{c}.$$

But as $(X_t)_{t\in\mathbb{R}^+}$ is right continuous, $\sup_{s\in[0,t]} X_s = \sup_{s\in[0,t]\cap D} X_s$, so that the point a) follows.

- If $t \notin D$, we use a sequence $(t_n)_{n\geq 1}$ in D so that $t_n \downarrow t$,

$$\sup_{s\in[0,t]} X_s = \lim_n \downarrow \sup_{s\in[0,t_n]} X_s.$$

Letting n going to infinity in

$$\mathbb{P}(\sup_{s\in[0,t_n]} X_s \geq c) \leq \frac{\mathbb{E}(|X_{t_n}|)}{c},$$

we obtain

$$\mathbb{P}(\sup_{s\in[0,t]} X_s \geq c) \leq \frac{\liminf_{n\to\infty} \mathbb{E}(|X_{t_n}|)}{c},$$

and in fact $\liminf_{n\to\infty} \mathbb{E}(|X_{t_n}|) = \mathbb{E}[|X_t|]$. Indeed, we may assume without loss of generality that $t_n \leq t+1$. Then,

$$\mathbb{E}(|X_{t_n}|) \leq \mathbb{E}[|X_t|] + \varepsilon + \mathbb{E}[(|X_{t_n}| - |X_t|)1_{|X_{t_n}|\geq|X_t|+\varepsilon})$$

whereas as $X$is a submartingale and we assume $t_n \leq t+1$

$$\mathbb{E}[(|X_{t_n}| - |X_t|)1_{|X_{t_n}|\geq|X_t|+\varepsilon}) \leq \mathbb{E}[(|X_{t+1}| - |X_t|)1_{|X_{t_n}|\geq|X_t|+\varepsilon})\,.$$

Since $|X_{t+1}| - |X_t| \in L^1$ and $\mathbb{P}(|X_{t_n}| \geq |X_t| + \varepsilon)$ goes to zero as n goes to infinity by right continuity, the conclusion follows.

The point b) is a direct application of the point a) as if X is a martingale or a non-negative subartingale, as in both cases $|X_t|^p$ is a non-negative submartingale since by Hölder' inequality, for $s \le t$

$$|X_s|^p \le |\mathbb{E}[X_t|\mathcal{F}_s]|^p \le \mathbb{E}[|X_t|^p|\mathcal{F}_s]$$

so that b) follows by applying a) to $(|X_t|^p)_{t\in\mathbb{R}^+}$.

To deduce c), observe that for any fixed k, with $S = \sup_{s\in[0,t]} |X_s| \wedge k$,

$$\mathbb{E}[S^p] \quad = \quad \mathbb{E}[\int_0^S px^{p-1}dx] = \int_0^k pP(S \ge x)x^{p-1}dx \tag{3.1}$$

so that b) implies that for all $p' < p$, there exists a finite constant c so that

$$\mathbb{E}[S^{p'}] \le c\|X_t\|_p$$

Letting k going to infinity and invoking the monotone convergence theorem yields the estimate with $p' < p$. To derived it as announced we need to show the bound

$$xP(S \ge x) \le \mathbb{E}[|X_t|1_{S\ge x}] \tag{3.2}$$

This inequality was proved in Lemma 3.5 in the discrete case. To show that it extends to the continuous setting we can proceed by discrete approximation exactly as in the previous proof to deduce that if $t \in D$, (3.2) holds, and then for all t by density as if t_n is a sequence of dyadic numbers decreasing to t,

$$\mathbb{E}[|X_{t_n}|1_{\sup_{s\in[0,t_n]}|X_s|\wedge k\ge x}] \le \liminf_{n\to\infty} \mathbb{E}[(|X_t|+\epsilon)1_{\sup_{s\in[0,t_n]}|X_s|\wedge k\ge x}] + \mathbb{E}[(|X_{t+1}|-|X_t|)1_{|X_{t_n}|\ge|X_t|+\epsilon}]$$

As $1_{\sup_{s\in[0,t_n]}|X_s|\wedge k\ge x}$ and $1_{|X_{t_n}|\ge|X_t|+\epsilon}$ go to $1_{\sup_{s\in[0,t]}|X_s|\wedge k\ge x}$ and 0 respectively while $|X_t$ and $|X_{t+1}| - |X_t|$ are in L^1 we conclude that

$$\liminf_{n\to\infty} \mathbb{E}[|X_{t_n}|1_{\sup_{s\in[0,t_n]}|X_s|\wedge k\ge x}] \le \mathbb{E}[|X_t|1_{\sup_{s\in[0,t]}|X_s|\wedge k\ge x}]$$

so that (3.2) extends to all $t \in \mathbb{R}^+$.

From (3.2) once plugged into (4.4) we deduce

$$\mathbb{E}[S^p] \le \int_0^k p\mathbb{E}[|X_t|1_{S\ge x}]x^{p-2}dx = \frac{p}{p-1}\mathbb{E}[|X_t|S^{p-1}] \le \frac{p}{p-1}\mathbb{E}[|X_t|^p]^{1/p}\mathbb{E}[S^p]^{\frac{p-1}{p}}$$

by Hölder inequality. Hence we conclude that

$$\mathbb{E}[S^p] \le (\frac{p}{p-1})^p\mathbb{E}[|X_t|^p]$$

Letting k going to infinity and applying monotone convergence theorem concludes the argument.

□

Application of Theorem 3.4 (a):

PROPOSITION 3.6. *Let $(B_t)_{t\in\mathbb{R}^+}$ be a real Brownian motion and set*

$$S_t = \sup_{s\in[0,t]} B_s.$$

Then for all $a>0$,

$$\mathbb{P}(S_t \geq a\ t) \leq \exp(-\frac{a^2 t}{2}).$$

PROOF. We may assume without loss of generality that the trajectories of B are continuous by Corollary 1.7. Let's use the martingales $(M_t^{(\alpha)})_{t\in\mathbb{R}^+}$ given for $\alpha>0$, by :

$$M_t^{(\alpha)} = \exp(\alpha B_t - \frac{\alpha^2}{2} t).$$

We have :

$$\begin{aligned} \exp(\alpha\ S_t - \frac{\alpha^2}{2}t) &= \exp(\alpha(\sup_{s\in[0,t]} B_s) - \frac{\alpha^2}{2}t) \\ &\leq \sup_{s\in[0,t]} M_s^{(\alpha)}. \end{aligned}$$

As for $\alpha>0$, $x\to\exp(\alpha x)$ is increasing, we have

$$\begin{aligned} \mathbb{P}(S_t \geq a\ t) &= \mathbb{P}(\exp(\alpha\ S_t - \frac{\alpha^2}{2}t) \geq \exp(\alpha a t - \frac{\alpha^2}{2}t)) \\ &\leq \mathbb{P}(\sup_{s\in[0,t]} M_s^{(\alpha)} \geq \exp(\alpha a t - \frac{\alpha^2}{2}t)) \\ &\leq \exp(-\alpha a t + \frac{\alpha^2}{2}t)\ \mathbb{E}(|M_t^{(\alpha)}|) \text{ by the first Doob inequality} \\ &= \exp(-\alpha a t + \frac{\alpha^2}{2}t)\ \mathbb{E}(M_0^{(\alpha)}) = \exp(-\alpha a t + \frac{\alpha^2}{2}t) \end{aligned}$$

But $\inf_{\alpha>0}(-\alpha a t + \frac{\alpha^2}{2}t) = -\frac{a^2 t}{2}$, so that the result follows □

4. Stopping time.

Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be a filtration on $(\Omega,\mathcal{F},\mathbb{P})$. *A stopping time with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is a map $T:\Omega\to[0,+\infty]$ so that for all $t\in\mathbb{R}^+$, $\{T\leq t\}\in\mathcal{F}_t$.*

We denote $\mathcal{T}$ the family of stopping times. We set :

$$\mathcal{F}_\infty = \sigma(\bigcup_{t\in\mathbb{R}^+} \mathcal{F}_t).$$

Let T be a stopping time for $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$. We call σ-algebra of the events anterior to T and denote it $\mathcal{F}_T$, the family of elements A in $\mathcal{F}_\infty$ so that :

$$\forall t\in\mathbb{R}^+, A\cap\{T\leq t\}\in\mathcal{F}_t.$$

We verify that $\mathcal{F}_T$ is indeed a σ-algebra.

Properties:
a). If $T \equiv t$, $T$is a stopping time.
b). If $T \in \mathcal{T}$ and if $S = T + t$ with $t \in \mathbb{R}^+$, then $S \in \mathcal{T}$.
c). If $T \in \mathcal{T}$, then T is $\mathcal{F}_T$-mesurable.
d). If $S, T \in \mathcal{T}$, and if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.
e). If $S, T \in \mathcal{T}$, then $S \wedge T \in \mathcal{T}$ and $S \vee T \in \mathcal{T}$.

Remark :
We have the following result :
$T : \Omega \to [0, +\infty]$ is a $(\mathcal{F}_{t^+})_{t\in\mathbb{R}^+}$-stopping time iff

$$\forall t \in \,]0, +\infty[, \{T < t\} \in \mathcal{F}_t.$$

Examples of stopping time:
Let $A \in \mathcal{B}(\mathbb{R}^d)$ and $(X_t)_{t\in\mathbb{R}^+}$ be a stochastic d-dimensionnal process. We set

$$T_A(\omega) = \inf\{t > 0 : X_t(\omega) \in A\}$$

($+\infty$ if $\{-\} = \emptyset$).
T_A is called the *hitting time of A*.

Proposition 4.1. *Let A be open. If $(X_t)_{t\in\mathbb{R}^+}$ is right continuous then T_A is a stopping time for the natural filtration $(\mathcal{F}^0_{t^+})_{t\in\mathbb{R}^+}$.*

Proposition 4.2. *Let A be closed. If $(X_t)_{t\in\mathbb{R}^+}$ is continuous then the random variable D_A defined by*

$$D_A(\omega) = \inf\{t \geq 0 : X_t(\omega) \in A\}$$

is a stopping time for $(\mathcal{F}^0_t)_{t\in\mathbb{R}^+}$. D_A is called entry time in A .

The proof of these propositions follows from writing by right continuity

$$\{T_A < t\} = \cup_{s\in[0,t[\cap\mathbb{Q}}\{X_s \in A\} \in \mathcal{F}^0_t$$

whereas by continuity

$$\{D_A \leq t\} = \cup_{s\in[0,t]\cap\mathbb{Q}}\{X_s \in F\} \in \mathcal{F}^0_t\,.$$

Let $(X_t)_{t\in\mathbb{R}^+}$ be a d-dimensionnal stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $(X_t)_{t\in\mathbb{R}^+}$ is *strongly adapted* if for all $T \in \mathcal{T}$, the map $\omega \to X_{T(\omega)}(\omega)\mathbf{1}_{\{T(\omega)<\infty\}}$ is $\mathcal{F}_T$-measurable. If $(X_t)_{t\in\mathbb{R}^+}$ is strongly adapted , then $(X_t)_{t\in\mathbb{R}^+}$ is adapted. We next give conditions so that $(X_t)_{t\in\mathbb{R}^+}$ is strongly adapted. We say that $(X_t)_{t\in\mathbb{R}^+}$ is *progressively measurable* if for all $t > 0$, the map $(s, \omega) \to X_s(\omega)$ is measurable on $([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_t)$ in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.If $(X_t)_{t\in\mathbb{R}^+}$ is progressively measurable, then $(X_t)_{t\in\mathbb{R}^+}$ is adapted.

Theorem 4.3. *If $(X_t)_{t\in\mathbb{R}^+}$ is progressively measurable, then it is strongly adapted.*

PROPOSITION 4.4. *If $(X_t)_{t\in\mathbb{R}^+}$ is adapted, right-continuous, then $(X_t)_{t\in\mathbb{R}^+}$ is progressively measurable.*

PROOF. Let $t>0$ and define

$$Y^{(n)}(s,\omega)=\sum_{k=1}^{n}X_{\frac{kt}{n}}(\omega)\mathbf{1}_{[\frac{(k-1)t}{n},\frac{kt}{n}[}(s)+X_t(\omega)\mathbf{1}_{\{t\}}(s).$$

Then

$$(s,\omega)\to Y^{(n)}(s,\omega)$$

is measurable (as sum of measurable maps as $(X_t)_{t\in\mathbb{R}^+}$ is adapted).
But, as $(X_t)_{t\in\mathbb{R}^+}$ is right continuous, if n goes to $+\infty$, $Y^{(n)}(s,\omega)$ goes to $X_s(\omega)$ so $(s,\omega)\to X_s(\omega)$ is measurable. □

PROOF. (of the Theorem) We use the following result: A map $U:\Omega\to\mathbb{R}^d$ which vanishes on the event $\{T=+\infty\}$ is $\mathcal{F}_T$-measurable iff for all $t\in\mathbb{R}^+$, $U.\mathbf{1}_{\{T\le t\}}$ is $\mathcal{F}_t-$measurable.

As a consequence, it is enough to show that for all $t>0$, $X_T\mathbf{1}_{\{T\le t\}}$ is $\mathcal{F}_t$-measurable. To see that, write

$$X_T\mathbf{1}_{\{T\le t\}}=X_{T\wedge t}\mathbf{1}_{\{T\le t\}}$$

The random variable $\mathbf{1}_{\{T\le t\}}$ is $\mathcal{F}_t$-measurable and $\omega\to X_{T(\omega)\wedge t}(\omega)$ can be decomposed into $\psi\circ\phi(\omega)$ where $\phi:\omega\to(T(\omega)\wedge t,\omega)$ is measurable from $\mathcal{F}_t$ into $\mathcal{B}([0,t])\otimes\mathcal{F}_t$ and $\psi:(s,\omega)\to X_s(\omega)$ is measurable, by hypothesis, from $\mathcal{B}([0,t])\otimes\mathcal{F}_t$ into $\mathcal{B}(\mathbb{R}^d)$.

4.1. Stopping time theorem for bounded stopping time.

THEOREM 4.5. [(Optional) Stopping theorem] *Let $(X_t)_{t\in\mathbb{R}^+}$ be a right continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$.*
a). For all bounded stopping time S, the random variable X_S is integrable and $\mathcal{F}_S$-measurable.
b). If S and T are two bounded stopping time and if $S\le T$, then

$$X_S=\mathbb{E}(X_T|\mathcal{F}_S).$$

COROLLARY 4.6. *Let $(X_t)_{t\in\mathbb{R}^+}$ be a real right continuous adapted stochastic process. Then , $(X_t)_{t\in\mathbb{R}^+}$ is a martingale iff for all $T\in\mathcal{T}$ bounded , X_T is integrable and $\mathbb{E}(X_T)=\mathbb{E}(X_0)$.*

PROOF. $\Rightarrow$: If $(X_t)_{t\in\mathbb{R}^+}$ is a martingale, according to the stopping theorem, X_T is integrable and $X_0=\mathbb{E}(X_T|\mathcal{F}_0)$ $(S\equiv 0\le T)$, so $\mathbb{E}(X_T)=\mathbb{E}(X_0)$.
$\Leftarrow$: i) Take $T=t$ to check that X_t is integrable for all $t\in\mathbb{R}^+$. We want to show that $\mathbb{P}$-a.s., for all $t\ge s$,

$$X_s=\mathbb{E}(X_t|\mathcal{F}_s)$$

i.e. $\mathbb{E}(X_s\mathbf{1}_A) = \mathbb{E}(X_t\mathbf{1}_A), \forall A \in \mathcal{F}_s$.
Let $A \in \mathcal{F}_s$, and set

$$T = s\ \mathbf{1}_A + t\ \mathbf{1}_{A^c}.$$

T is a bounded stopping time. So, $X_T \in L^1$ and

$$\mathbb{E}(X_0) = \mathbb{E}(X_T) = \mathbb{E}(X_s\mathbf{1}_A) + \mathbb{E}(X_t\mathbf{1}_{A^c}).$$

and so $\mathbb{E}(X_0) = \mathbb{E}(X_t)$ gives $\mathbb{E}(X_s\mathbf{1}_A) = \mathbb{E}(X_t\mathbf{1}_A)$. □

Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. If $(X_t)_{t\in\mathbb{R}^+}$ is a real adapted right continuous stochastic process and if T is a stopping time we denote $(X_t^T)_{t\in\mathbb{R}^+}$ the stochastic process given by

$$X_t^T = X_{t\wedge T}.$$

The process $(X_t^T)_{t\in\mathbb{R}^+}$ is called *the process stopped at time T.*

Corollary 4.7. *Let $(X_t)_{t\in\mathbb{R}^+}$ be a right continuous martingale and T be a stopping time, then $(X_t^T)_{t\in\mathbb{R}^+}$ is a right continuous martingale.*

Proof. $(X_t^T)_{t\in\mathbb{R}^+}$ is clearly right continuous
. To show that it is a martingale, it is enough to show that for any bounded stopping time S, we have : $X_S^T \in L^1$ and $\mathbb{E}(X_S^T) = \mathbb{E}(X_0^T)$, according to the previous corollary. But, $X_S^T = X_{S\wedge T}$ and $S \wedge T$ is a bounded stopping time, so, by the stopping theorem we deduce that $X_{S\wedge T} \in L^1$ and $\mathbb{E}(X_{S\wedge T}) = \mathbb{E}(X_0)$. □

Proof. (Proof of the stopping theorem) : It is enough to show that for any stopping time S bounded by c (i.e. $S(\omega) \le c, \forall\omega \in \Omega$), we have : $X_S \in L^1$ and

$$X_S = \mathbb{E}(X_c|\mathcal{F}_S), \mathbb{P} - a.s.$$

In fact, if S and T are two stopping times such that $S \le T$ bounded by c, then

$$X_S = \mathbb{E}(X_c|\mathcal{F}_S), \mathbb{P} - a.s.$$

and

$$X_T = \mathbb{E}(X_c|\mathcal{F}_T), \mathbb{P} - a.s.$$

and therefore, as $\mathcal{F}_S \subset \mathcal{F}_T$,

$$\mathbb{E}(X_T|\mathcal{F}_S) = \mathbb{E}[\mathbb{E}(X_c|\mathcal{F}_T|\mathcal{F}_S] = \mathbb{E}(X_c|\mathcal{F}_S) = X_S.$$

Let S be a bounded stopping time so that for all $\omega \in \Omega$, $S(\omega) \le c$. Set :

$$S_n(\omega) = \begin{cases} \frac{k}{2^n} & \text{if} \quad \frac{k-1}{2^n} \le S(\omega) < \frac{k}{2^n} \text{ for } 1 \le k < c2^n. \\ c & \text{otherwise} \quad (\text{ i.e. if } S(\omega) = c) \end{cases}$$

For all integer number $n \neq 0$, S_n is a stopping time which takes a finite number of values . Moreover for all $\omega \in \Omega, S_n(\omega) \downarrow S(\omega)$.
We shall use the following

LEMMA 4.8. *Let $(Y_k)_{k\in\mathbb{N}}$ be a martingale with respect to the filtration $(\mathcal{G}_k)_{k\in\mathbb{N}}$.*
a). For all bounded stopping time S, $Y_S \in L^1$.
b). For all bounded stopping time S, T so that $S \leq T$,

$$Y_S = \mathbb{E}(Y_T|\mathcal{G}_S) \quad \mathbb{P}-a.s.$$

The proof is straightforward as if $|S| \leq c$, by definition

$$\mathbb{E}[|Y_S|] \leq \sum_{k=1}^{c} \mathbb{E}[|Y_k|1_{S=k}] \leq c \max_{k\leq c} \mathbb{E}[|Y_k|] \,.$$

Whereas if $S \leq T$ are two stopping times bounded by c, it is enough to show the result for $T = c$ as then

$$Y_S = \mathbb{E}[Y_c|\mathcal{G}_S] \qquad Y_T = \mathbb{E}[Y_c|\mathcal{G}_T]$$

from which as $\mathcal{G}_S \subset \mathcal{G}_T$, the result follows. But

$$\mathbb{E}[Y_c 1_A] = \sum_{\ell\leq c} \mathbb{E}[Y_c 1_A 1_{S=\ell}] = \sum_{\ell\leq c} \mathbb{E}[\mathbb{E}[Y_c|\mathcal{G}_\ell] 1_A 1_{S=\ell}] = \sum_{\ell\leq c} \mathbb{E}[Y_\ell 1_A 1_{S=\ell}] = \mathbb{E}[Y_S 1_A] \,.$$

We apply the lemma to $(Y_k^{(n)})_{k\in\mathbb{N}}$ given by

$$Y_k^{(n)} = \begin{cases} X_{\frac{k}{2^n}} & \text{if } 1 \leq k < c2^n. \\ X_c & \text{if } k \geq c2^n, \end{cases}$$

with

$$\mathcal{G}_k^{(n)} = \begin{cases} \mathcal{F}_{\frac{k}{2^n}} & \text{if } 1 \leq k < c2^n. \\ \mathcal{F}_c & \text{if } k \geq c2^n. \end{cases}$$

For all $n \geq 1$, $(Y_k^{(n)})_{k\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{G}_k^{(n)})_{k\in\mathbb{N}}$. We therefore have

$$X_{S_n} = \mathbb{E}(X_c|\mathcal{F}_{S_n}), \ \mathbb{P}-a.s.$$

by applying the lemma to $(Y_k^{(n)})_{k\in\mathbb{N}}$ and the stopping times :

$$T_n = 2^n S_n, T' = c2^n.$$

For all $\omega \in \Omega$, $X_{S_n}(\omega)$ goes to $X_S(\omega)$ when n goes to infinity (as $S_n \downarrow S$ and by right continuity of the trajectories). If we show that $X_{S_n} \overset{L^1}{\to} X_S$, we will have $\forall A \in \mathcal{F}_S$,

$$\int_\Omega X_S \mathbf{1}_A \, d\mathbb{P} = \lim_n \int_\Omega X_{S_n} \mathbf{1}_A \, d\mathbb{P} = \int_\Omega X_c \mathbf{1}_A \, d\mathbb{P}.$$

The fact that $X_S \in L^1$ and the convergence in L^1 of X_{S_n} to X_S as n goes to infinity is due to the uniform integrability of the sequence $(X_{S_n})_n$ in the sense that

DEFINITION 4.9. A family $(U_i)_{i\in I}$ of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is uniformly integrable if

$$\sup_{i\in I} \int_{\{|U_i|\geq\lambda\}} |U_i| \, d\mathbb{P} \to 0$$

when $\lambda \to +\infty$.

The following are left as exercises.

EXERCISE 4.10. (1) $(U_i)_{i\in I}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ is uniformly integrable iff
i) $\sup_{i\in I}\mathbb{E}(|U_i|)<+\infty$.
ii) *Equicontinuity property: :*
$\forall\epsilon>0,\exists\eta>0$ so that if $A\in\mathcal{F}$ and if $\mathbb{P}(A)\leq\eta$, then

$$\sup_{i\in I}\Big(\int_A |U_i|\,d\mathbb{P}\Big)\leq\epsilon.$$

(2) Any family U_i so that there exists U in L^1 so that for all $i\in I$,

$$|U_i|\leq U$$

(3) Let $(X_n)_{n\geq1}$ be a family of integrable random variables and let X be a random variable. The following conditions are equivalent :
i) X is integrable and when $n\to+\infty$, $X_n\overset{L^1}{\to}X$.
ii) When $n\to+\infty$, $X_n\overset{\mathbb{P}}{\to}X$ and $(X_n)_{n\geq1}$ is uniformly integrable.

To complete the proof of the stopping time theorem, we therefore need to show that $(X_{S_n})_{n\geq1}$ is uniformly integrable. We have :

$$X_{S_n}=\mathbb{E}(X_c|\mathcal{F}_{S_n}),\ \mathbb{P}-a.s.$$

so

$$|X_{S_n}|\leq\mathbb{E}(|X_c|\ |\mathcal{F}_{S_n}),\ \mathbb{P}-a.s.$$

and therefore for all $a>0$,

$$\int_{\{|X_{S_n}|\geq a\}}|X_{S_n}|\,d\mathbb{P}\leq\int_{\{|X_{S_n}|\geq a\}}|X_c|\,d\mathbb{P}.$$

Note that the family $\{|X_c|\}(X_c\in L^1)$ is uniformly integrable and therefore equicontinuous. In fact we can write for all $M\geq0$

$$\int_{\{|X_{S_n}|\geq a\}}|X_c|\,d\mathbb{P}\leq M\mathbb{P}(|X_{S_n}|\geq a)+\int_{\{|X_c|\geq M\}}|X_c|\,d\mathbb{P}$$

where if we set $A_n=\{|X_{S_n}|\geq a\}$, so that Tchebychev (or Markov) inequality yields

$$\mathbb{P}(A_n)\leq\frac{1}{a}\mathbb{E}(|X_{S_n}|)\leq\frac{1}{a}\mathbb{E}(|X_c|).$$

Finally, we conclude

$$\sup_{n\geq1}\Big(\int_{\{|X_{S_n}|\geq a\}}|X_{S_n}|\,d\mathbb{P}\Big)\leq M\frac{1}{a}\mathbb{E}(|X_c|)+\int_{\{|X_c|\geq M\}}|X_c|\,d\mathbb{P}\leq\epsilon$$

if we choose M and then a large enough. □

4.2. Convergence theorem. Stopping theorem for a uniformly integrable martingale. In this paragraph, we study diverse forms of convergence for martingales and give a stopping theorem for possibly unbounded stopping time. Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ a filtration of $(\Omega, \mathcal{F}, \mathbb{P})$.

THEOREM 4.11. [Almost sure convergence theorem]
a) Let $(X_t)_{t\in\mathbb{R}^+}$ be a right continuous submartingale bounded in L^1 (i.e. $\sup_{t\in\mathbb{R}^+}\mathbb{E}(|X_t|) < +\infty$), then $(X_t)_{t\in\mathbb{R}^+}$ converges $\mathbb{P}-a.s.$ when $t\to+\infty$ towards a limit in L^1.
b) Let $p > 1$. If $(X_t)_{t\in\mathbb{R}^+}$ is a right continuous martingale bounded in L^p (i.e. $\sup_{t\in\mathbb{R}^+}\mathbb{E}(|X_t|^p) < +\infty$), then $(X_t)_{t\in\mathbb{R}^+}$ converges $\mathbb{P}-a.s.$ as $t\to+\infty$ towards a limit in L^p.

Remark :
When $(X_t)_{t\in\mathbb{R}^+}$ is a right continuous submartingale, $(X_t)_{t\in\mathbb{R}^+}$ is bounded in L^1 as soon as

$$\sup_{t\in\mathbb{R}^+}\mathbb{E}(X_t^+) < +\infty.$$

(here $x^+ = x\vee 0$). Indeed, we have

$$\mathbb{E}[|X_t|] = \mathbb{E}[X_t^+] + \mathbb{E}[X_t^-] = 2\mathbb{E}[X_t^+] - \mathbb{E}[X_t] \leq 2\mathbb{E}[X_t^+] - \mathbb{E}[X_0]$$

COROLLARY 4.12. *If $(Y_t)_{t\in\mathbb{R}^+}$ is a right continuous positive supermartingale, then Y_t converges $\mathbb{P}$-a.s. as $t\to+\infty$ and the limit is integrable.*

PROOF. We set $X_t = -Y_t$. $(X_t)_{t\in\mathbb{R}^+}$ is a right continuous submartingale and $X_t^+ = (-Y_t)^+ = 0, \forall t\in\mathbb{R}^+$. The proof is then a direct consequence of the theorem and the previous remark. □

PROOF. a). can be shown by using the discrete version of Theorem 9.3 with $\mathbb{T} = \mathbb{N}$ instead of $\mathbb{R}^+$ and setting $Y_k^{(n)} = X_{\frac{k}{2^n}}, \forall n\geq 1, \forall k\geq 0$.
$\forall n\geq 1$, when $k\to+\infty$,

$$Y_k^{(n)} \to Y_\infty^{(n)},\ \mathbb{P}-a.s.$$

with $Y_\infty^{(n)}$ integrable.
Denote D_n the dyadic numbers of order n. As $D_{n+1}\subset D_n$, we see that the $Y_\infty^{(n)}$ are equal $\mathbb{P}-$a.s.. We set : $Y_\infty = Y_\infty^{(n_0)}$. We conclude by using right continuity as $t\to+\infty$,

$$X_t \to Y_\infty,\ \mathbb{P}-a.s..$$

b) Let $p > 1$. By a), when $t\to+\infty$,

$$X_t \to X_\infty \in L^1,\ \mathbb{P}-a.s..$$

Let us show that $X_\infty\in L^p$. Thanks to Fatou's Lemma, we have

$$\mathbb{E}(|X_\infty|^p) \leq \liminf_{t\to+\infty}\mathbb{E}(|X_t|^p) \leq \sup_{t\in\mathbb{R}^+}\mathbb{E}(|X_t|^p) < +\infty$$

(as $|X_t|^p \to |X_\infty|^p$ $\mathbb{P}-a.s.$ when $t\to+\infty$). □

THEOREM 4.13. [Mean convergence theorem of order 1 or p.]
a). Let $(X_t)_{t\in\mathbb{R}^+}$ be a right continuous martingale.
The three following conditions are equivalent :
i). $(X_t)_{t\in\mathbb{R}^+}$ converges in L^1 as $t \to +\infty$.
ii). There exists a random variable $X_\infty \in L^1$ so that :

$$X_t = \mathbb{E}(X_\infty|\mathcal{F}_t), \forall t \in \mathbb{R}^+.$$

iii) $(X_t)_{t\in\mathbb{R}^+}$ is uniformly integrable.
b). Moreover, if $p > 1$ and if $(X_t)_{t\in\mathbb{R}^+}$ is bounded in L^p (i.e. $\sup_{t\in\mathbb{R}^+} \mathbb{E}(|X_t|^p) < +\infty$) then the convergence holds as well in L^p with $X_\infty \in L^p$.

PROOF. a). i) $\Rightarrow$ ii) : Denote X_∞ an integrable random variable so that X_t converges towards X_∞ in L^1 as $t \to +\infty$. Let $0 \le s < t$. We have $\mathbb{P}-$a.s.,

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s.$$

Letting t going to infinity we get

$$\mathbb{E}(X_\infty|\mathcal{F}_s) = X_s \;\; \mathbb{P} - a.s..$$

(We used that the map $U \to \mathbb{E}(U|\mathcal{G})$, $\mathcal{G}$ sub σ algebra of $\mathcal{F}$ is a continuous linear operator from L^1 into L^1 so that $\mathbb{E}(|\mathbb{E}(U|\mathcal{G}) - \mathbb{E}(V|\mathcal{G})|) \le \mathbb{E}(|U - V|))$.
ii) $\Rightarrow$ iii) : Set for $a > 0$ and $t \in \mathbb{R}^+$,

$$A_t(a) = \int_{\{|X_t|\ge a\}} |X_t| \, d\mathbb{P}.$$

We have

$$A_t(a) \le \int_{\{|X_t|\ge a\}} \mathbb{E}(|X_\infty| \; |\mathcal{F}_t) \, d\mathbb{P} = \int_{\{|X_t|\ge a\}} |X_\infty| \, d\mathbb{P}.$$

$|X_\infty|$ being uniformly integrable, it is also equicontinuous that is for all $\epsilon > 0$, exists $\eta > 0$ so that if $A \in \mathcal{F}$ with $\mathbb{P}(A) \le \eta$ then

$$\int_A |X_\infty| \, d\mathbb{P} \le \epsilon.$$

But, by Markov's inequality,

$$\mathbb{P}(|X_t| \ge a) \le \frac{1}{a}\mathbb{E}(|X_t|) \le \frac{1}{a}\mathbb{E}(|X_\infty|) \to 0, a \to +\infty.$$

We deduce that

$$\sup_{t\in\mathbb{R}^+} A_t(a) \to 0, a \to +\infty.$$

iii) $\Rightarrow$ i) : $(X_t)_{t\in\mathbb{R}^+}$ is uniformly integrable, and therefore bounded in L^1. Hence, by the almost sure convergence theorem $X_t \to X_\infty$ $\mathbb{P}$-a.s. when t goes to $+\infty$. Using uniform integrability, we get that X_t converges in L^1, towards X_∞ when t goes to $+\infty$.

b) If $p > 1$ and if

$$\sup_{t\in\mathbb{R}^+} \mathbb{E}(|X_t|^p) < +\infty,$$

then

$$\sup_{t\in\mathbb{R}^+} |X_t| \in L^p.$$

(See Theorem 3.4 c) which extends to the case where $[0,t]$ is replaced by $\mathbb{R}^+$).
$(|X_t|^p)_{t\in\mathbb{R}^+}$ is uniformly integrable as it is bounded by $(\sup_{t\in\mathbb{R}^+}|X_t|)^p \in L^1$. We deduce that $X_\infty \in L^p$ as $|X_\infty|^p \le (\sup_{t\in\mathbb{R}^+}|X_t|)^p \in L^1$ and that X_t converges towards X_∞ in L^p as $t \to +\infty$ by the following result :
Let $(X_n)_{n\ge 1}$ be a sequence of random variables in L^p, the two following statements are equivalent :
i) X_n converges in probability towards X as n goes to infinity and $(|X_n|^p)_n$ is uniformly integrable.
ii) $X \in L^p$ and X_n converges in L^p towards X as n tends to infinity. □

Notation : Let $(X_t)_{t\in\mathbb{R}^+}$ be a right continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ uniformly integrable and let $X_\infty \in L^1$ be the limit (in L^1) of $(X_t)_{t\in\mathbb{R}^+}$. Let S be a stopping time. Define :

$$X_S(\omega) = \begin{cases} X_{S(\omega)}(\omega) & \text{if } S(\omega) < +\infty. \\ X_\infty(\omega) & \text{if } S(\omega) = +\infty. \end{cases}$$

We state (without proof) a more general version of the previous theorem:

Theorem 4.14. [Modified stopping theorem] *Let $(X_t)_{t\in\mathbb{R}^+}$ be a uniformly integrable right continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$.*
a). For any stopping time S, the random variable X_S is integrable (and $\mathcal{F}_S$-measurable).
b). If S and T are two stopping times so that $S \le T$, then

$$X_S = \mathbb{E}(X_T|\mathcal{F}_S),\ \mathbb{P}-a.s.$$

Remark :
The uniform integrability condition is fundamental: the continuous martingale $(M_t^{(\alpha)})_{t\in\mathbb{R}^+}$ defined for $\alpha > 0$ by

$$M_t^{(\alpha)} = \exp(\alpha B_t - \frac{\alpha^2 t}{2})$$

does not verify the conclusion of the stopping theorem. Indeed, as $M_t^{(\alpha)} \to 0, \mathbb{P}-a.s.$ as t goes to $+\infty$, the stopping time

$$T = \inf\{t \ge 0; M_t^{(\alpha)} \le 1/2\}$$

is finite $\mathbb{P}$-a.s. and $\mathbb{E}(M_T^{(\alpha)}) = \frac{1}{2}$. But

$$\mathbb{E}(M_0^{(\alpha)}) = 1.$$

Indeed, $(M_t^{(\alpha)})_{t\in\mathbb{R}^+}$ is bounded in L^1 but not equicontinuous.

5. Finite variation process and Stieltjes integral

5.1. Finite variation functions and Stieltjes integral.

Finite variation functions. Let A be a function defined on $\mathbb{R}^+$ with values in $\mathbb{R}$, right continuous and left limited (cad lag). $A_t \equiv A(t)$. Let $t > 0$. A partition Δ_t of the interval $[0,t]$ is a sequence of points $(t_i)_{i=0,1,\ldots,n}$ so that $t_0 = 0 < t_1 < t_2 < \ldots < t_n = t$. For all $t > 0$, we define

$$V(A)_t = \sup_{\Delta_t} \sum_{t_i \in \Delta_t} |A_{t_{i+1}} - A_{t_i}|$$

Definition 5.1. The function A has finite variation if for all $t \in \mathbb{R}^+$, $V(A)_t$ is finite. The function $t \to V(A)_t$ is called *the (total) variation of A.*

Property 5.2. *a)- The function $t \to V(A)_t$ is increasing.*
b)- The function $t \to V(A)_t$ is right continuous and has left limits. Moreover,

$$V(A)_t = V(A)_{t-} + |\Delta A_t|$$

where $\Delta A_t = A_t - A_{t-}$ is the jump of A at t.
c)- The function A can be rewritten as

$$A_t = A_t^c + \sum_{s \le t} \Delta A_s$$

where A_t^c is the continuous part of A_t. Then,

$$V(A)_t = V(A^c)_t + \sum_{s \le t} |\Delta A_s|.$$

d)- .$(\sum_{s\le t} \Delta A_s)_{t\ge 0}$ is a finite variation function and

$$V(\sum_{s \le t} \Delta A_s) = \sum_{s \le t} |\Delta A_s|.$$

. If A and B are two finite variation functions, then $A + B$ has finite variation and

$$V(A+B) \le V(A) + V(B).$$

Examples : If A is $\mathcal{C}^1$ or monotone, then A has finite variation.

Remark :
A function A has finite variation iff it has locally bounded variations (A function $f = (f(t))_{t\ge 0}$ is locally bounded if for all $t < \infty$, there exists K_t so that $|f(s)| \le K_t$, for all $s \le t$.)

Property 5.3. *Any finite variation function is the difference of two increasing non-negative functions.*

Proof. Let $A^+ = \frac{1}{2}(V(A) + A - A_0)$ and $A^- = \frac{1}{2}(V(A) - A + A_0)$. Clearly, $A^+ + A^- = V(A)$ and $A^+ - A^- = A - A_0$. Let us show that A^+ is increasing : Let $t' > t$,

$$V(A)_{t'} = V(A)_t + V(A - A_t)_{[t,t']}.$$

As $V(A - A_t)_{[t,t']} \geq |A_{t'} - A_t|$,

$$V(A)_{t'} - V(A)_t + A_{t'} - A_t = V(A - A_t)_{[t,t']} + (A_{t'} - A_t) \geq 0.$$

So, A^+ is an increasing function, vanishing in 0 (Idem for A^-). □

Finite variation functions and measures on $\mathbb{R}^{+,*}$. We can associate to any right continuous non decreasing functions A^+ and A^- a positive $\sigma-$finite measure on $(\mathbb{R}^{+,*}, \mathcal{B}(\mathbb{R}^{+,*}))$ denoted μ_{A^+} and μ_{A^-} such that for all $t > 0$,

$$\mu_{A^+}(]0, t]) = A_t^+$$

and

$$\mu_{A^-}(]0, t]) = A_t^-.$$

Hence, we can associate to any finite variation function A a σ-finite measure on $(\mathbb{R}^{+,*}, \mathcal{B}(\mathbb{R}^{+,*}))$ by setting

$$\mu_A = \mu_{A^+} - \mu_{A^-}.$$

In particular for all $t > 0$,

$$\mu_A(]0, t]) = A_t - A_0.$$

For any borelian locally bounded function f from $\mathbb{R}^+$ into $\mathbb{R}$ localement bornee, we can define the Stieljes integral of f with respect to A by :

$$\int_0^t f(s)\, dA_s \equiv \int_0^t f(s)\, \mu_A(ds).$$

Notice that

$$\left|\int_0^t f(s)\, \mu_A(ds)\right| = \left|\int_0^t f(s)\, \mu_{A^+}(ds) - \int_0^t f(s)\, \mu_{A^-}(ds)\right|$$

$$\leq \int_0^t |f(s)|\, (\mu_{A^+} + \mu_{A^-})(ds) \leq \sup_{[0,t]} |f(s)|\; V(A)_t < +\infty.$$

Moreover

$$\mu_A(]0, t[) = A_{t-} - A_0$$

and

$$\mu_A(\{t\}) = \Delta A_t.$$

If f is a bounded function on $[0, t]$, then $V(\int_{]0,s]} f(u)\, dA_u)_{s \leq t}$ is finite and

$$V(\int_{]0,t]} f(u)\, dA_u) = \int_{]0,t]} |f(u)|\, dV(A_u).$$

5.2. Integration by parts.

THEOREM 5.4. *Let A and B be two finite variation functions. Then, for all $t > 0$,*

$$A_t B_t = A_0 B_0 + \int_0^t A_{s-}\, dB_s + \int_0^t B_s\, dA_s \tag{5.1}$$

and

$$A_t B_t = A_0 B_0 + \int_0^t A_{s-}\, dB_s + \int_0^t B_{s-}\, dA_s + \sum_{s\le t} \Delta A_s \Delta B_s. \tag{5.2}$$

PROOF. The second equality comes from the first when noticingg that $B_s = B_{s-} + \Delta B_s$ and $\mu_A(\{s\}) = \Delta A_s$.
Let μ_A and μ_B be the measures associated with A and B.
a) By definition of the product measure,

$$\begin{aligned}\mu_A \otimes \mu_B(]0,t]\times]0,t]) &= \mu_A(]0,t])\mu_B(]0,t]) \\ &= (A_t - A_0)(B_t - B_0)\end{aligned}$$

b) Moreover

$$\begin{aligned}\mu_A \otimes \mu_B(]0,t]\times]0,t]) &= \int_{]0,t]\times]0,t]} \mu_A(dx)\, \mu_B(dy) \\ &= \int_{0<x<y,0<y\le t} \mu_A(dx)\, \mu_B(dy) + \int_{0<y\le x,0<x\le t} \mu_A(dx)\, \mu_B(dy) \\ &= \int_{]0,t]} \mu_A(]0,y[)\, \mu_B(dy) + \int_{]0,t]} \mu_B(]0,x])\, \mu_A(dx) \\ &= \int_{]0,t]} (A_{s-} - A_0)\, dB_s + \int_{]0,t]} (B_s - B_0)\, dA_s \\ &= \int_{]0,t]} A_{s-}\, dB_s + \int_{]0,t]} B_s\, dA_s - A_0(B_t - B_0) - B_0(A_t - A_0) \\ &= A_t B_t - A_0 B_t - B_0 A_t + A_0 B_0, \text{ according to a)},\end{aligned}$$

from which the first result follows. The second follows. □

We easily deduce from the theorem the following corollary :

COROLLARY 5.5. *Let A be a finite variation function such that $A_0 = 0$. Then,*

$$A_t^2 = 2\int_0^t A_{s-}\, dA_s + \sum_{s\le t} (\Delta A_s)^2.$$

5.3. Formule de changement de variable.

THEOREM 5.6. *Let A une finite variation function and $F:\mathbb{R}\to\mathbb{R}$ a $\mathcal{C}^1$ function. Then, $F(A)$ is a finite variation function and, for all $t>0$,*

$$F(A_t)=F(A_0)+\int_0^t F'(A_{s-})\,dA_s+\sum_{s\le t}\Big(F(A_s)-F(A_{s-})-\Delta A_s F'(A_{s-})\Big).$$

Remark :
If A is continuous then

$$F(A_t)=F(A_0)+\int_0^t F'(A_s)\,dA_s.$$

For instance, if $F(x)=e^x$,

$$\exp(A_t)=\exp(A_0)+\int_0^t \exp(A_s)\,dA_s.$$

PROOF. The proof of the theorem goes in two steps:we first show the formula is true for $F(x)=x$ (which is clear) then extend it to polynomial functions by applying integration by parts formula. We then entend the result to C^1 functions by approximating F and F' uniformly by a sequence of polynomial functions. □

PROPERTY 5.7. *Let A be a finite variation function and let Y be defined by*

$$Y_t=Y_0\prod_{s\le t}(1+\Delta A_s)\exp(A_t^c-A_0^c).$$

Then, Y is the unique solution with finite variation of the differential equation

$$Y_t=Y_0+\int_0^t Y_{s-}\,dA_s.$$

PROOF. a)- Notice that if X is a finite variation function, then $t\to\exp(X_t)$ has finite variations :
Let $0=t_0<t_1<\dots t_n=t$,

$$\sum_{i=0}^{n-1}|\exp(X_{t_{i+1}})-\exp(X_{t_i})|\le\sup_{s\le t}(\exp(X_s))\sum_{i=0}^{n-1}|X_{t_{i+1}}-X_{t_i}|$$

$$\Rightarrow\quad V(\exp(X))_t\le\exp(V(X)_t)\ V(X)_t.$$

b)- Set

$$U_t=Y_0\prod_{s\le t}(1+\Delta A_s)$$

and

$$V_t=\exp(A_t^c-A_0^c).$$

U and V are finite variation functions as their logarithm is. The integration by part formula gives

$$Y_t = U_t V_t = Y_0 + \int_0^t U_{s-}\, dV_s + \int_0^t V_s\, dU_s.$$

But, $dV_s = V_s dA_s^c$ and $dU_t = Y_0 \prod_{s<t}(1+\Delta A_s)\Delta A_t = U_{t-} d(\sum_{u\leq t}\Delta A_u)$, so, as $V_s = V_{s_-}$,

$$\begin{aligned} Y_t &= Y_0 + \int_0^t U_{s-} V_s\, dA_s^c + \int_0^t V_{s-} U_{s-}\, d(\sum_{u\leq s} \Delta A_u) \\ &= Y_0 + \int_0^t Y_{s-}\, dA_s \end{aligned}$$

by noticing that $A_s = A_s^c + \sum_{u\leq s}\Delta A_u$.
c)- We finally prove uniqueness. Assume that we have two solutions with finite variations of the differential equation. Denote Z the difference of these two solutions. Then Z is also solution of this differential equation and

$$|Z_t| \leq M_t\ V(A)_t$$

where $M_t = \sup_{u\leq t}|Z_u|$. Iterating the procedure by using the new bounds for Z yields

$$|Z_t| \leq \int_0^t M_s\ V(A)_s\, dV(A)_s \leq M_t\ \frac{V(A)_t^2}{2} \leq \ldots \leq M_t\ \frac{V(A)_t^n}{n!} \to 0$$

as $n \to +\infty$. □

Time change. Let A be an increasing function, therefore with finite variations, so that $A_0 = 0$. We define for each $t \in \mathbb{R}^+$,

$$\tau_t = \begin{cases} \inf\{s : A_s > t\} \text{ if } \{.\ \} \neq \emptyset \\ +\infty \quad \text{if } \{.\ \} = \emptyset. \end{cases}$$

The function $t \to \tau_t$ is called "pseudo-inverse" of the function A.

Property 5.8. *Let f be a borelien positive function borelienne on $[0, +\infty[$ and bounded on $[0, t]$. Then,*

$$\int_{]0,t]} f(s)\, dA_s = \int_0^{A_t} f(\tau_s)\, ds.$$

Proof. We prove the proposition for $f(s) = \mathbf{1}_{]0,u]}(s)$ with $u \leq t$. The generalization to all borelian function is classical.
. $f \circ \tau_s = \mathbf{1}_{]0,u]}(\tau_s) = \mathbf{1}_{\{\tau_s \leq u\}} = \mathbf{1}_{\{A_u \geq s\}}$ Lebesgue a.s..
Using that A is increasing, we find that

$$\int_0^{A_t} \mathbf{1}_{]0,u]}(\tau_s)\, ds = \int_0^{A_t} \mathbf{1}_{\{A_u \geq s\}}\, ds = \int_0^{A_u} ds = A_u.$$

. $\int_{]0,t]} f(s)\, dA_s = \int_{]0,t]} \mathbf{1}_{]0,u]}(s)\, dA_s = \int_{]0,u]} dA_s = A_u - A_0 = A_u.$ □

5.4. Finite variation process. Let $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that we assume complete for $\mathbb{P}$ and $(A_t)_{t\in\mathbb{R}^+}$ an $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-adapted process.

The process A has *finite variations* if $\mathbb{P}$-almost all trajectories $t \to A_t(\omega)$ have finite variations.

Remark :
Note that $V(A)$, A^+ and A^- are adapted processes for $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$.

Let A be a finite variation process. For $t \geq 0$ and $\mathbb{P}-$ almost all $\omega \in \Omega$, we can define (if exists) the stochastic integral $\left(\int_0^t X_s\, dA_s\right)(\omega)$ as the Stieltjes integral

$$\int_0^t X_s(\omega)\, dA_s(\omega).$$

The next proposition gives sufficient conditions for this integral to be well defined.

Property 5.9. *(admitted) Let X be a progressively mesurable process, bounded on any interval $[0,t]$ and let A be a finite variation process. Then $\left(\int_0^t X_s\, dA_s\right)_{t\in\mathbb{R}^+}$ is a adapted finite variation process.*

(On the neglectable set where A is not finite variation, the integral is set to zero).

Example : Let $N = (N_t)_{t\in\mathbb{R}^+}$ be the Poisson process with intensity λ. We have already seen that $(N_t - \lambda t)_{t\in\mathbb{R}^+}$ is a martingale with respect tp the natural filtration $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$ of $(N_t)_{t\in\mathbb{R}^+}$. Let $\alpha > -1$. Let us consider the martingale with respect to the natural filtration $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$

$$L_t^\alpha = \exp[\ln(1+\alpha)N_t - \lambda\alpha t].$$

Observe that the Poisson process can be written

$$N_t = \sum_{s\leq t} \Delta N_s$$

with $\Delta N_s = \mathbf{1}_{\{\Delta N_s \neq 0\}}$. We can write $(L_t^\alpha)_{t\in\mathbb{R}^+}$ as

$$\begin{aligned} L_t^\alpha &= \exp(-\lambda\alpha t)\prod_{s\leq t}(1+\alpha)^{\Delta N_s} \\ &= \exp(-\lambda\alpha t)\prod_{s\leq t}(1+\Delta(\alpha N_s - \alpha\lambda s)) \end{aligned}$$

According to Property 5.7, L_t^α is the unique finite variation solution of the stochastic equation

$$Y_t = Y_0 + \int_0^t Y_{s-} d(\alpha N_s - \alpha\lambda s).$$

In other words

$$\int_0^t L_{s-}^\alpha d(\alpha N_s - \alpha\lambda s) = L_t^\alpha - 1.$$

As $(L_t^\alpha)_{t\in\mathbb{R}^+}$ is a martingale with respect to the natural filtration $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$, we deduce that $\left(\int_0^t L_{s-}^\alpha d(\alpha N_s - \alpha\lambda s)\right)_{t\in\mathbb{R}^+}$ is a martingale with respect to $(\mathcal{F}_t^0)_{t\in\mathbb{R}^+}$. A natural question is the following: being given a martingale M with finite variations, when we can say that $\left(\int_0^t X_s\, dM_s\right)_{t\in\mathbb{R}^+}$ is a martingale? It is not always true. Let us take the standard Poisson process with parameter $\lambda = 1$. The process

$$\Big(\int_0^t (N_s - s)\, d(N_s - s)\Big)_{t\in\mathbb{R}^+}$$

is not a martingale
1)- Indeed by integration by parts formula

$$(N_t - t)^2 = \int_0^t (N_s - s)_-\, d(N_s - s) + \int_0^t (N_s - s)\, d(N_s - s)$$

So,

$$\int_0^t (N_s - s)\, d(N_s - s) = (N_t - t)^2 - \int_0^t (N_{s-} - s)\, d(N_s - s)$$

2)- As $N_s = N_{s-} + \Delta N_s$, we have

$$\int_0^t (N_s - s)\, d(N_s - s) = \int_0^t (N_{s-} - s)\, d(N_s - s) + \int_0^t \Delta(N_s - s)\, d(N_s - s)$$

But $\Delta(N_s - s) = \Delta N_s$, therefore

$$\int_0^t \Delta(N_s - s)\, d(N_s - s) = \sum_{s\le t} (\Delta N_s)^2 = \sum_{s\le t} (\Delta N_s) = N_t.$$

We then deduce the relation

$$\int_0^t (N_s - s)\, d(N_s - s) = \int_0^t (N_{s-} - s)\, d(N_s - s) + N_t.$$

Adding the above formula we get

$$2\int_0^t (N_s - s)\, d(N_s - s) = (N_t - t)^2 - t + N_t - t + 2t.$$

But we already saw that the processes $((N_t - t)^2 - t)_{t\in\mathbb{R}^+}$ and $(N_t - t)_{t\in\mathbb{R}^+}$ are martingales, hence $\left(\int_0^t (N_s - s)\, d(N_s - s)\right)_{t\in\mathbb{R}^+}$ is not a martingale. However

$$2\int_0^t (N_s - s)_-\, d(N_s - s) = (N_t - t)^2 - t - (N_t - t).$$

is a martingale as difference of two martingales. This shows that the right notion is to require the process which is integrated to be adapted. This is the content of the next theorem.

THEOREM 5.10. *Let H be an adapted process, left continuous, bounded on all interval $[0,t]$. Let M be a martingale with integrable variations on all $[0,t]$ (i.e. $\mathbb{E}(V(M)_t) < +\infty$). Then, $\left(\int_0^t H_s\, dM_s\right)_{t\in\mathbb{R}^+}$ is a martingale.*

PROOF. Let $H_s = \alpha_u \mathbf{1}_{]u,v]}(s)$ with $u \le v$, α_u bounded and $\mathcal{F}_u$-measurable. Then,

$$\int_0^t H_s\, dM_s = \alpha_u(M_{v\wedge t} - M_{u\wedge t}) = \alpha_u(M_t^v - M_t^u).$$

(with the notation for stopped martingales).
Let $s \le t$.
a)- Case $u \le s$.

$$\begin{aligned}\mathbb{E}\Big(\int_0^t H_{t'}\, dM_{t'}\Big|\mathcal{F}_s\Big) &= \mathbb{E}(\alpha_u(M_t^v - M_t^u)|\mathcal{F}_s)\\ &= \alpha_u(\mathbb{E}(M_t^v|\mathcal{F}_s) - \mathbb{E}(M_t^u|\mathcal{F}_s))\\ &= \alpha_u(M_s^v - M_s^u)\\ &= \int_0^s H_{t'}\, dM_{t'}.\end{aligned}$$

b)- Case $s < u$.

$$\begin{aligned}\mathbb{E}\Big(\int_0^t H_{t'}\, dM_{t'}\Big|\mathcal{F}_s\Big) &= \mathbb{E}(\alpha_u(M_t^v - M_t^u)|\mathcal{F}_s)\\ &= \mathbb{E}(\mathbb{E}(\alpha_u(M_t^v - M_t^u)|\mathcal{F}_u)|\mathcal{F}_s)\\ &= \mathbb{E}(\alpha_u(M_{v\wedge u} - M_{u\wedge u})|\mathcal{F}_s) = 0\\ &= \int_0^s \alpha_u \mathbf{1}_{]u,v]}(t')\, dM_{t'}.\end{aligned}$$

The result extends by linearity to $H_s = \sum_i \alpha_{u_i} \mathbf{1}_{]u_i,u_{i+1}]}(s)$ with bounded $\mathcal{F}_{u_i}$-measurable α_{u_i}. For $s \le t$, we shall approximate the left continuous process H by H^n of the form :

$$H_s^n = \sum_{i=1}^{p(n)} \alpha^n_{u_i^n} \mathbf{1}_{]u_i^n, u_{i+1}^n]}(s).$$

It is enough to show the following three points to prove the theorem:
i) For all $A \in \mathcal{F}_s$, when $n \to +\infty$,

$$\mathbb{E}(\mathbf{1}_A \int_0^t H_u^n \, dM_u) \longrightarrow \mathbb{E}(\mathbf{1}_A \int_0^t H_u \, dM_u).$$

Note that $dM_u = d\mu_M^+ - d\mu_M^-$, so that it is enough to consider

$$\mathbb{E}(\mathbf{1}_A \int_0^t H_u^n \, d\mu_M^+(u)) = \int_{\Omega \times]0,t]} \mathbf{1}_A H_u^n(\omega) \mu_M^+(\omega, du) \mathbb{P}(d\omega).$$

The convergence follows from bounded convergence theorem.
ii) $(\int_0^t H_s \, dM_s)_{t \in \mathbb{R}^+}$ is adapted with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$. This is clear from property 5.9 as H is progressively measurable (H being adapted and left continuous).
iii) H being bounded by say K_t on all interval $[0, t]$,

$$\mathbb{E}(|\int_0^t H_s \, dM_s|) \leq \mathbb{E}(\int_0^t |H_s| \, dV(M)_s) \leq K_t \; \mathbb{E}(V(M)_t) < +\infty.$$

□

THEOREM 5.11. *A martingale M is continuous and with finite variation iff it is constant.*

PROOF. We can assume that $M_0 = 0$. Fix $t \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Let

$$S_n = \begin{cases} \inf\{s : V(M)_s \geq n\} \text{ if } \{ \, . \, \} \neq \emptyset \text{ for } s \leq t \\ t \text{ if } \{ \, . \, \} = \emptyset. \end{cases}$$

S_n is a stopping time and the martingale M^{S_n} has bounded variations. It is therefore enough to show the theorem for a continuous martingale M such that M and its variation are bounded by a constant K. Let $\Delta = \{t_0 = 0, t_1, \dots, t_p = t\}, t_i < t_{i+1}$ be a partition of $[0, t]$, with $|\tau| = \sup_i |t_{i+1} - t_i|$. Let us compute

$$\mathbb{E}(M_t^2) = \sum_{i=0}^{p-1} \mathbb{E}(M_{t_{i+1}}^2 - M_{t_i}^2).$$

But for all $u \leq v$,

$$\mathbb{E}(M_v^2 - M_u^2) = \mathbb{E}((M_v - M_u)^2).$$

Hence

$$
\begin{aligned}
\mathbb{E}(M_t^2) &= \sum_{i=0}^{p-1} \mathbb{E}((M_{t_{i+1}} - M_{t_i})^2) \\
&\leq \mathbb{E}\Big[(\sup_i |M_{t_{i+1}} - M_{t_i}|) \times \sum_{i=0}^{p-1} |M_{t_{i+1}} - M_{t_i}|\Big] \\
&\leq \mathbb{E}\Big[V(M)_t \sup_i |M_{t_{i+1}} - M_{t_i}|\Big] \\
&\leq K\mathbb{E}\Big[\sup_i |M_{t_{i+1}} - M_{t_i}|\Big] \to 0
\end{aligned}
$$

when $|\tau|$ goes to 0 (by bounded convergence theorem as M is bounded continuous). Hence, $M \equiv 0$.

□

We have seen that Stieltjes integral allows to integrate a process with respect to another one provided it has finite variation. This integral holds trajectory by trajectory. The previous theorem shows that this is not possible if we want to integrate with respect to a continuous martingale, for instance with respect to a real Brownian motion.

6. Continuous local martingales

6.1. Quadratic variation of a bounded continuous martingale. The Brownian motion having infinite variations, cf Theorem 1.10, we can not define its Stieltjes integral. We shall define another type of integral, in the L^2 sense. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ be a probability space and put $X = (X_t)_{t\in\mathbb{R}^+}$ to be a real valued process. Let $\Delta = \{t_0 = 0 < t_1 < \ldots\}$ be a subdivision of $\mathbb{R}^+$ with finitely many points in each $[0, t]$. We define for all $t > 0$, the random variable

$$
T_t^\Delta(X) = \sum_{i=0}^{k-1} (X_{t_{i+1}} - X_{t_i})^2 + (X_t - X_{t_k})^2
$$

where k is such that $t_k \leq t < t_{k+1}$.

Definition 6.1. X has a *finite quadratic variation* if there exists a process $\langle X, X\rangle$ such that for all $t > 0$,

$$
T_t^\Delta(X) \xrightarrow{\mathbb{P}} \langle X, X\rangle_t
$$

when Δ goes to 0.

Theorem 6.2. *A bounded continuous martingale M has a finite quadratic variation $\langle M, M\rangle$. Moreover, $\langle M, M\rangle$ is the unique continuous increasing process, vanishing at the origin and such that $M^2 - \langle M, M\rangle$ is a martingale.*

PROOF. a) - *Uniqueness :* Direct consequence of Theorem 5.11 as if A and B are two such processes $A - B$ is a martingale with finite variation which vanishes at the origin.
b) - *Existence :*
. Observe that if $t_i < s < t_{i+1}$,

$$\mathbb{E}((M_{t_{i+1}} - M_{t_i})^2|\mathcal{F}_s) = \mathbb{E}((M_{t_{i+1}} - M_s)^2|\mathcal{F}_s) + (M_s - M_{t_i})^2.$$

Hence, we see that

$$\mathbb{E}(T_t^\Delta(M) - T_s^\Delta(M)|\mathcal{F}_s) = \mathbb{E}((M_t - M_s)^2|\mathcal{F}_s) = \mathbb{E}(M_t^2 - M_s^2|\mathcal{F}_s). \tag{6.1}$$

We deduce that $(M_t^2 - T_t^\Delta(M))_t$ is a continuous martingale.

. Set $a = t_n > 0$. We shall see that if (Δ_n) is a sequence of subdivisions of $[0, a]$ such that $|\Delta_n|$ goes to 0, then $(T_a^{\Delta_n})_n$ converges in L^2. Let Δ and Δ' be two subdivisions of $[0, a]$, and denote $\Delta\Delta' = \Delta \cup \Delta'$. According to (6.1), the process $X = T^\Delta(M) - T^{\Delta'}(M)$ is a martingale and since by definition of $T_t^{\Delta\Delta'}(X)$, since $(X_t^2 - T_t^{\Delta\Delta'}(X))_t$ is a martingale,

$$\mathbb{E}(X_a^2) = \mathbb{E}\Big((T_a^\Delta(M) - T_a^{\Delta'}(M))^2\Big) = \mathbb{E}(T_a^{\Delta\Delta'}(X)).$$

As $(x + y)^2 \leq 2(x^2 + y^2)$ and

$$(X_{t_{i+1}} - X_{t_i})^2 = \Big[(T_{t_{i+1}}^\Delta(M) - T_{t_i}^\Delta(M)) - (T_{t_{i+1}}^{\Delta'}(M) - T_{t_i}^{\Delta'}(M))\Big]^2,$$

we have

$$T_a^{\Delta\Delta'}(X) \leq 2\{T_a^{\Delta\Delta'}(T^\Delta(M)) + T_a^{\Delta\Delta'}(T^{\Delta'}(M))\}.$$

Hence it suffices to show that

$$\mathbb{E}(T_a^{\Delta\Delta'}(T^\Delta(M)))$$

goes to zero as $|\Delta| + |\Delta'|$ goes to zero : Let $s_k \in \Delta\Delta'$ and $t_l \in \Delta$ so that $t_l \leq s_k < s_{k+1}$. We have

$$\begin{aligned} T_{s_{k+1}}^\Delta(M) - T_{s_k}^\Delta(M) &= (M_{s_{k+1}} - M_{t_l})^2 - (M_{s_k} - M_{t_l})^2 \\ &= (M_{s_{k+1}} - M_{s_k})(M_{s_{k+1}} + M_{s_k} - 2M_{t_l}) \end{aligned}$$

and consequently

$$T_a^{\Delta\Delta'}(T^\Delta(M)) \leq \left(\sup_k |M_{s_{k+1}} + M_{s_k} - 2M_{t_l}|^2\right) T_a^{\Delta\Delta'}(M).$$

By Cauchy-Schwarz inequality, we deduce

$$\mathbb{E}\left(T_a^{\Delta\Delta'}(T^\Delta(M))\right) \leq \mathbb{E}\left(\sup_k |M_{s_{k+1}} + M_{s_k} - 2M_{t_l}|^4\right)^{1/2} \mathbb{E}(T_a^{\Delta\Delta'}(M)^2)^{1/2}.$$

When $|\Delta| + |\Delta'|$ goes to zero, the first term goes to zero by continuity of M. The second is bounded by a constant independent of the choice of Δ and Δ' : indeed, compute

$$
\begin{aligned}
(T_a^\Delta(M))^2 &= \left(\sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^2\right)^2 \\
&= 2\sum_{k=1}^n (T_a^\Delta(M) - T_{t_k}^\Delta(M))(T_{t_k}^\Delta(M) - T_{t_{k-1}}^\Delta(M)) + \sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^4.
\end{aligned}
$$

Thanks to (6.1), we have

$$\mathbb{E}(T_a^\Delta(M) - T_{t_k}^\Delta(M)|\mathcal{F}_{t_k}) = \mathbb{E}((M_a - M_{t_k})^2|\mathcal{F}_{t_k})$$

and therefore

$$
\begin{aligned}
\mathbb{E}(T_a^\Delta(M))^2) &= 2\sum_{k=1}^n \mathbb{E}[(M_a - M_{t_k})^2(T_{t_k}^\Delta(M) - T_{t_{k-1}}^\Delta(M))] + \sum_{k=1}^n \mathbb{E}[(M_{t_k} - M_{t_{k-1}})^4] \\
&\leq \mathbb{E}\left[\left(2\sup_k |M_a - M_{t_k}|^2 + \sup_k |M_{t_k} - M_{t_{k-1}}|^2\right) T_a^\Delta(M)\right]
\end{aligned}
$$

Let us assume that M is bounded by a constant C. Then by (6.1) we see that

$$\mathbb{E}(T_a^\Delta(M))) \leq 4C^2$$

so that

$$\mathbb{E}(T_a^\Delta(M))^2) \leq 12C^2\mathbb{E}(T_a^\Delta(M))) \leq 48C^4.$$

Hence we have shown that the sequence $(T_a^{\Delta_n}(M))_n$ has a limit in L^2, and therefore in probability, that we denote $\langle M, M\rangle_a$, for any sequence (Δ_n) of subdivisions of $[0, a]$ so that $|\Delta_n|$ goes to 0.

. In this last part, we show that the limiting process $(\langle M, M\rangle_t)_t$ has all the announced properties.
- Let $(\Delta_n)_n$ be defined as above. Doob's inequality for the martingale $(T_t^{\Delta_n}(M) - T_t^{\Delta_m}(M))_t$ can be written

$$\mathbb{E}\left[\sup_{t\leq a} |T_t^{\Delta_n}(M) - T_t^{\Delta_m}(M)|^2\right] \leq 4\ \mathbb{E}[(T_a^{\Delta_n}(M) - T_a^{\Delta_m}(M))^2].$$

Hence there exists a subsequence (Δ_{n_k}) such that $T_t^{\Delta_{n_k}}(M)$ converges almost surely uniformly on $[0, a]$ towards the limit $\langle M, M\rangle_t$ which is therefore almost surely continuous.
- The sequence$(\Delta_n)_n$ can be chosen so that Δ_{n+1} is a refinement of the subdivision Δ_n and so that $\bigcup_n \Delta_n$ is dense $[0, a]$. For all $(s, t) \in \bigcup_n \Delta_n$ such that $s < t$, there exists an integer number n_0 so that both s and t belong to all subdivisions Δ_n with $n \geq n_0$. But $T_s^{\Delta_n} \leq T_t^{\Delta_n}$ so that

$$\langle M, M\rangle_s \leq \ \langle M, M\rangle_t$$

and therefore $\langle M, M\rangle$ is increasing on $\bigcup_n \Delta_n$ and therefore on $[0,a]$ as $\langle M, M\rangle$ is continuous.
- Last point : we show that $M^2 - \langle M, M\rangle$ is a martingale by going to the limit $\Delta \to 0$ in (6.1). □

PROPERTY 6.3. *Let M be a bounded continuous martingale, T be a stopping time. We denote by M^T the martingale stopped in T. Then,*

$$\langle M^T, M^T\rangle = \langle M, M\rangle^T.$$

PROOF. . $\langle M^T, M^T\rangle$ is the unique increasing continuous process, which vanishes at the origin and so that

$$(M^2)^T - \langle M^T, M^T\rangle$$

is a martingale.
. $(M^2 - \langle M, M\rangle)^T = (M^2)^T - \langle M, M\rangle^T$ is a martingale and $\langle M, M\rangle^T$ is a continuous increasing process, vanishing at the origin and hence $\langle M^T, M^T\rangle = \langle M, M\rangle^T$. □

6.2. Continuous local martingales.

DEFINITION 6.4. Let M be a continuous real valued process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$. We say that $M$is a *local continuous martingale* if
i) M_0 is integrable.
ii) There exists a sequence $(T_n)_n$ of stopping times, $T_n \uparrow +\infty$ a.s. and M^{T_n} is a uniformly integrable martingale.

Remarks :

(1) Condition ii) can be replaced by :
ii') There exists a stopping time $(T_n)_n$ such that $T_n \uparrow +\infty$ a.s. and such that M^{T_n} is a martingale. Indeed it is enough to stop the martingale at n so that it becomes uniformly integrable.
(2) A martingale is a local martingale (take $T_n = n$).
(3) A stopped local martingale is a local martingale : Take S a stopping time, M a local martingale.
There exists a sequence of stopping times (T_n) such that $T_n \uparrow +\infty$ a.s. and M^{T_n} is a uniformly integrable martingale. $(M^S)^{T_n} = M^{T_n\wedge S}$ is also a uniformly integrable martingale.
(4) The sum of two local martingales is a local martingale.

Exercises :
Show that :

(1) A non negative local martingale is a supermartingale (Use Fatou's lemma).
(2) A local bounded martingale is a martingale.

(3) Let M be a continuous local martingale and let $T_n = \inf\{t : |M_t| \geq n\}$ then M^{T_n} is a bounded martingale and $T_n \uparrow +\infty$ a.s.
(4) There exists local martingales which are not martingales (cf exercise 7.7).

Basic example : The real Brownian motion (B_t) is a martingale, hence a local martingale, which is not uniformly integrable as

$$\sup_t \mathbb{E}(|B_t|) = +\infty.$$

However, there exists a sequence $(T_n)_n$ of stopping times such that $T_n \uparrow +\infty$ a.s. so that B^{T_n} is a uniformly integrable martingale.

THEOREM 6.5. *Let M be a continuous local martingale. There exists a unique process $\langle M, M\rangle$ non decreasing, continuous, vanishing at the origin and so that $M^2 - \langle M, M\rangle$ is a local martingale.*
Moreover,

$$\sup_{s\leq t} |T_s^{\Delta_n}(M) - \langle M, M\rangle_s|$$

converges in probability towards zero as $n \to +\infty$.

PROOF. . Let $(T_n)_n$ be a sequence of stopping times so that $T_n \uparrow +\infty$ a.s. and such that M^{T_n} is bounded. According to Theorem 6.2, for all n, there exists a process $A^n = \langle M^{T_n}, M^{T_n}\rangle$ so that $(M^2)^{T_n} - A^n$ is a uniformly integrable martingale.

$$\left((M^2)^{T_{n+1}} - A^{n+1}\right)^{T_n} = (M^2)^{T_n} - (A^{n+1})^{T_n}$$

so that, by uniqueness,

$$(A^{n+1})^{T_n} = A^n.$$

We can construct a process $\langle M, M\rangle$ non decreasing, continuous, vanishing at the origin so that $\langle M, M\rangle^{T_n} = A^n$ for all integer n. By this construction, $M^2 - \langle M, M\rangle$ is a local martingale since $(M^2 - \langle M, M\rangle)^{T_n} = (M^2)^{T_n} - A^n$ is a uniformly integrable martingale
. Let t be given. Let $\epsilon > 0$ and S be a stopping time so that M^S is bounded, $\mathbb{P}(S \leq t) \leq \delta$. On $[0, S]$, $T^\Delta(M)$ and $\langle M, M\rangle$ coincide with $T^\Delta(M^S)$ and $\langle M^S, M^S\rangle$, so that

$$\mathbb{P}(\sup_{s\leq t} |T_s^\Delta(M) - \langle M, M\rangle_s| > \epsilon) \leq \delta + \mathbb{P}(\sup_{s\leq t} |T_s^\Delta(M^S) - \langle M^S, M^S\rangle_s| > \epsilon)$$

and the last term vanishes when Δ goes to zero. □

COROLLARY 6.6. *Let M, N be two continuous local martingales. There exists a unique continuous process $\langle M, N\rangle$, with finite variation, vanishing at the origin*

so that $MN - \langle M, N\rangle$ is a continuous local martingale.
Moreover, for all t and any sequence of subdivisions $(\Delta_n)_n$ of $[0,t]$ so that $|\Delta_n|\to 0$,

$$\sup_{s\le t} |\widehat{T}_s^{\Delta_n}(M,N) - \langle M,N\rangle_s|$$

converges in probability towards 0 as $n\to+\infty$, where

$$\widehat{T}_s^{\Delta_n}(M,N) = \sum_{t_i\in\Delta_n} (M^s_{t_{i+1}} - M^s_{t_i})(N^s_{t_{i+1}} - N^s_{t_i}).$$

Proof. . We have

$$\widehat{T}_t^{\Delta_n}(M,N) = \frac{1}{4}\Big(T_t^{\Delta_n}(M+N) - T_t^{\Delta_n}(M-N)\Big).$$

We set :

$$\langle M,N\rangle = \frac{1}{4}\Big(\langle M+N, M+N\rangle - \langle M-N, M-N\rangle\Big).$$

But

$$MN = \frac{1}{4}\Big((M+N)^2 - (M-N)^2\Big)$$

so that $MN - \langle M,N\rangle$ is a local continuous martingale according to Theorem 6.5.
. Let A and A' be two continuous processes with finite variation so that $MN - A$ and $MN - A'$ are local martingales. Then, $A - A'$ is a continuous local martingale with finite variations. There exists $T_n \uparrow +\infty$ such that $(A-A')^{T_n}$ is a uniformly integrable martingale with finite variations, hence for all n

$$(A-A')^{T_n} = 0$$

and thus going to the limit, $A \equiv A'$. □

Exercise :
Let M, N be two local martingales locales and T be a stopping time. Then,

$$\langle M^T, N^T\rangle = \langle M, N^T\rangle = \langle M^T, N\rangle = \langle M,N\rangle^T.$$

(Hint : Show that $M(N-N^T)$ is a local martingale).
Remarks :

(1) The map $(M,N)\to\langle M,N\rangle$ is bilinear, symmetric and non negative. It is also non-degenerate in the sense that

$$\langle M,M\rangle = 0 \Leftrightarrow M = M_0 \ \ a.s.$$

(2) Let M and N be two continuous local martingales.
MN is a local martingale $\Leftrightarrow \langle M,N\langle = 0$.

Property 6.7. *[Kunita-Watanabe inequality] Let M, N be two continuous local martingales and H and K be two measurable processes. Then*

$$\int_0^\infty |H_s||K_s||d\langle M,N\rangle_s| \le \left(\int_0^\infty H_s^2\ d\langle M,M\rangle_s\right)^{1/2}\left(\int_0^\infty K_s^2\ d\langle N,N\rangle_s\right)^{1/2}.$$

Proof. Using Cauchy-Schwarz inequality as well as the approximations of $\langle M, M\rangle$ and $\langle M, N\rangle$ obtained in Theorem 6.5, as well as corollary 6.6, we deduce

$$|\langle M, N\rangle_s^t| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}$$

where $\langle M, N\rangle_s^t = \langle M, N\rangle_t - \langle M, N\rangle_s$.
Let $s = t_0 < t_1 < \dots t_p = t$,

$$\begin{aligned}
\sum_{i=1}^p |\langle M, N\rangle_{t_{i-1}}^{t_i}| &\leq \sum_{i=1}^p \sqrt{\langle M, M\rangle_{t_{i-1}}^{t_i}}\sqrt{\langle N, N\rangle_{t_{i-1}}^{t_i}} \\
&\leq \left(\sum_{i=1}^p \langle M, M\rangle_{t_{i-1}}^{t_i}\right)^{1/2} \left(\sum_{i=1}^p \langle N, N\rangle_{t_{i-1}}^{t_i}\right)^{1/2} \\
&= \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}
\end{aligned}$$

so that, since $\int_s^t |d\langle M, N\rangle_u| = \sup \sum_{i=1}^p |\langle M, N\rangle_{t_i}^{t_{i+1}}|$, we deduce that

$$\int_s^t |d\langle M, N\rangle_u| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}$$

We then generalize this inequality to all bounded borelian sets A of $\mathbb{R}^+$ by the monotone convergence theorem

$$\int_A |d\langle M, N\rangle_u| \leq \sqrt{\int_A d\langle M, M\rangle_u}\sqrt{\int_A d\langle N, N\rangle_u}.$$

Let $h = \sum_i \lambda_i \mathbf{1}_{A_i}$ and $k = \sum_i \mu_i \mathbf{1}_{A_i}$ be two nonnegative stepwise constant functions. Then,

$$\begin{aligned}
\int h_s k_s |d\langle M, N\rangle_u| &= \sum_i \lambda_i \mu_i \int_{A_i} |d\langle M, N\rangle_s| \\
&\leq \left(\sum \lambda_i^2 \int_{A_i} d\langle M, M\rangle_s\right)^{1/2} \left(\sum \mu_i^2 \int_{A_i} d\langle N, N\rangle_s\right)^{1/2} \\
&= \left(\int h_s^2 d\langle M, M\rangle_s\right)^{1/2} \left(\int k_s^2 d\langle N, N\rangle_s\right)^{1/2}
\end{aligned}$$

We conclude by using the fact that any non negative measurable function is the increasing limit of stepwise constant functions and the monotone convergence theorem. □

Property 6.8. *A continuous local martingale M is a martingale bounded in L^2 iff*

- $M_0 \in L^2$.
- $\langle M, M\rangle_\infty$ *is* $\mathbb{P}$*-integrable.*

PROOF. . $\Rightarrow$ - M is bounded in L^2 so that $M_0 \in L^2$
-There exists a sequence of stopping times $T_n \uparrow +\infty$ such that for all n, M^{T_n} is a bounded martingale. We have

$$\mathbb{E}(M^2_{T_n \wedge t}) - \mathbb{E}(\langle M, M\rangle_{T_n\wedge t}) = \mathbb{E}(M_0^2). \tag{6.2}$$

As M is bounded in L^2, we can pass to the limit and get that

$$\mathbb{E}(M^2_\infty) - \mathbb{E}(\langle M, M\rangle_\infty) = \mathbb{E}(M_0^2).$$

Hence, $\langle M, M\rangle_\infty$ is integrable.
. $\Leftarrow$ - According to (6.2),

$$\mathbb{E}(M^2_{T_n \wedge t}) \leq \mathbb{E}(\langle M, M\rangle_\infty) + \mathbb{E}(M_0^2) = K < \infty. \tag{6.3}$$

According to Fatou's lemma,

$$\mathbb{E}(M_t^2) \leq \liminf_n \mathbb{E}(M^2_{T_n\wedge t}) \leq K.$$

Hence, M is bounded in L^2.
- Moreover $(M_t^{T_n})_t$ is a martingale, so that

$$\mathbb{E}(M_{T_n\wedge t}|\mathcal{F}_s) = M_{T_n\wedge s} \quad p.s.$$

and we obtain by passing to the limit that

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad p.s..$$

As a consequence, M is a martingale. □

Remark : It is easy to see that $(M^2 - \langle M, M\rangle)_t$ is a uniformly integrable martingale since

$$\sup_t |M_t^2 - \langle M, M\rangle_t| \leq (M^\star_\infty)^2 + \langle M, M\rangle_\infty \in L^1$$

with $M^\star_\infty = \sup_{t\in\mathbb{R}^+} |M_t|$.

COROLLARY 6.9. *A local continuous martingale converges a.s when $t\to\infty$ on the set $\{\langle M, M\rangle_\infty < +\infty\}$.*

The proof is left as an exercise (hint: use the stopping time $T_n = \inf\{t; \langle M, M\rangle_t \geq n\}$ and consider $(M_t^{T_n})_t$).

6.3. Continuous semi-martingales. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ a filtered probability space and let $X = (X_t)_{t\in\mathbb{R}^+}$ be a real valued process.
X is *a continuous semi-martingale* if X can be written

$$X = M + A$$

where M is a continuous local martingale and A is a continuous finite variation process, vanishing at the origin. Such a decomposition is unique. Indeed if we have two such decompositions

$$X = M + A \text{ and } X = M' + A',$$

then $M - M' = A' - A$ is a continuous local martingale with finite variations and therefore is a constant.

PROPERTY 6.10. *A continuous semi-martingale X with decomposition $X = M + A$ has a quadratic variation denoted $\langle X, X\rangle$ and*

$$\langle X, X\rangle = \langle M, M\rangle.$$

PROOF. For all $t > 0$,

$$\begin{aligned} T_t^{\Delta}(X) &= \sum_{i=0}^{k-1}(X_{t_{i+1}} - X_{t_i})^2 + (X_t - X_{t_k})^2 \\ &= \sum_{i=0}^{k-1}(M_{t_{i+1}} - M_{t_i})^2 + (M_t - M_{t_k})^2 + \sum_{i=0}^{k-1}(A_{t_{i+1}} - A_{t_i})^2 + (A_t - A_{t_k})^2 \\ &+ 2\sum_{i=0}^{k-1}(M_{t_{i+1}} - M_{t_i})(A_{t_{i+1}} - A_{t_i}) + 2(M_t - M_{t_k})(A_t - A_{t_k}) \end{aligned}$$

The first term converges in probability towards $\langle M, M\rangle$ as Δ goes to 0. It remains to check that the other terms go to zero as $|\Delta|$ goes to 0.
By continuity of A, we obtain

$$\left|\sum_{i=0}^{k-1}(A_{t_{i+1}} - A_{t_i})^2 + (A_t - A_{t_k})^2\right| \le \sup_i |A_{t_{i+1}} - A_{t_i}|\ V(A)_t \to 0,$$

as $|\Delta| \to 0$.
Similarly, by continuity of M,

$$\left|\sum_{i=0}^{k-1}(M_{t_{i+1}} - M_{t_i})(A_{t_{i+1}} - A_{t_i}) + (M_t - M_{t_k})(A_t - A_{t_k})\right| \le \sup_i |M_{t_{i+1}} - M_{t_i}|\, V(A)_t \to 0,$$

as $|\Delta| \to 0$. □

If $X = M + A$ and $Y = N + B$ are two continuous semi-martingales, we define *the bracket of X and Y* by

$$\langle X, Y\rangle = \langle M, N\rangle = \frac{1}{4}[\langle X + Y, X + Y\rangle - \langle X - Y, X - Y\rangle].$$

Clearly, $\langle X, Y\rangle_t$ is the limit in probability of

$$\sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

f

7. Stochastic Integral

7.1. Stochastic integral with respect to a martingale bounded in L^2. We denote H^2 the space of continuous martingales bounded in L^2 such that $M_0 = 0$. According to Proposition 6.8., if $M \in H^2$, then $\mathbb{E}(\langle M, M\rangle_\infty) < +\infty$. According to Kunita-Watanabe inequality, if $M, N \in H^2$, then $\mathbb{E}(|\langle M, N\rangle_\infty|) < +\infty$. We can therefore define on H^2 a scalar product by

$$\langle M, N\rangle_{H^2} = \mathbb{E}(\langle M, N\rangle_\infty).$$

The norm associated to this scalar product is given by

$$||M||_{H^2} = \mathbb{E}(\langle M, M\rangle_\infty)^{1/2}.$$

PROPERTY 7.1. *The space H^2 is a Hilbert space for the norm*

$$||M||_{H^2} = \mathbb{E}(\langle M, M\rangle_\infty)^{1/2}.$$

PROOF. Let us first show that H^2 is complete for the norm $||.||_{H^2}$. Let $(M^n)_n$ be a Cauchy sequence in H^2 for this norm. According to Proposition 6.8., we have

$$\lim_{m,n\to+\infty} \mathbb{E}[(M^n_\infty - M^m_\infty)^2] = \lim_{m,n\to+\infty} \mathbb{E}[\langle M^n - M^m, M^n - M^m\rangle_\infty] = 0.$$

Doob's inequality gives therefore that

$$\lim_{m,n\to+\infty} \mathbb{E}[\sup_{t\geq 0}(M^n_t - M^m_t)^2] = 0.$$

We can therefore construct a subsequence n_k of integer numbers such that

$$\mathbb{E}\left[\sum_{k=1}^{\infty} \sup_t |M^{n_k}_t - M^{n_{k+1}}_t|\right] \leq \sum_{k=1}^{\infty} \mathbb{E}\left[\sup_t |M^{n_k}_t - M^{n_{k+1}}_t|^2\right]^{1/2} < \infty$$

We deduce that almost surely

$$\sum_{k=1}^{\infty} \sup_t |M^{n_k}_t - M^{n_{k+1}}_t| < \infty$$

and hence the sequence $(M^{n_k}_t)_{t\geq 0}$ converges uniformly on $\mathbb{R}^+$ towards a limit M. We easily see that the limit M is a continuous martingale (Note that $(M^{n_k}_t)_{t\geq 0}$ converges also in L^2 towards M as a Cauchy sequence in L^2). As the variables M^n_t are uniformly bounded in L^2, the martingale M is also bounded in L^2 and hence $M \in H^2$. Finally

$$\lim_{k\to+\infty} \mathbb{E}[\langle M^{n_k} - M, M^{n_k} - M\rangle_\infty] = \lim_{k\to+\infty} \mathbb{E}[(M^{n_k}_\infty - M_\infty)^2] = 0$$

which shows that the subsequence M^{n_k} converges a.s. Hence also (M^n) converges towards M in H_2. □

If $M \in H^2$, we call $\mathcal{L}^2(M)$ the space of progressively measurable processes K such that

$$||K||_M^2 = \mathbb{E}\left[\int_0^\infty K_s^2 \ d\langle M, M\rangle_s\right] < +\infty.$$

For all $A \in \mathcal{P} = \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_\infty$, we set

$$\mathbb{P}_M(A) = \mathbb{E}\left[\int_0^\infty \mathbf{1}_A(s,\omega) \ d\langle M, M\rangle_s(\omega)\right] < +\infty.$$

$\mathbb{P}_M$ is a bounded measure on $\mathcal{P}$ and $\mathcal{L}^2(M)$ is the space of progressively measurable functions with integrable square with respect to $\mathbb{P}_M$. We will denote $L^2(M)$ the space of the equivalence classes of elements of $\mathcal{L}^2(M)$. This is an Hilbert space for the norm $||.||_M$ (The associated scalar product is denoted $(.,.)_M$).
Note that $L^2(M)$ contains the continuous bounded adapted processes.
We denote $\mathcal{E}$ the family of the elementary processes defined by

$$H_s(\omega) = \sum_{i=0}^{p-1} H^{(i)}(\omega)\mathbf{1}_{]t_i,t_{i+1}]}(s)$$

where the $H^{(i)}$ are $\mathcal{F}_{t_i}$-measurables and bounded.

Property 7.2. *For all $M \in H^2$, $\mathcal{E}$ is dense in $L^2(M)$.*

Proof. It is enough to show that if $K \in L^2(M)$ is such that $(K, N)_M = 0$ for all $N \in \mathcal{E}$, then $K = 0$. Let $0 \leq s < t$, and F be a $\mathcal{F}_s$-measurable bounded variable. If $(H, K)_M = 0$ for all $H = F\mathbf{1}_{]s,t]} \in \mathcal{E}$, then

$$\mathbb{E}\left[F\int_s^t K_u d\langle M, M\rangle_u\right] = 0.$$

Set for all $t \geq 0$,

$$X_t = \int_0^t K_u d\rangle M, M\rangle_u.$$

By using Cauchy-Schwarz inequality and the fact that $M \in H^2$ and $K \in L^2(M)$, this integral is a.s. converging and in L^1. Hence, for all F $\mathcal{F}_s$-measurable variable, $\mathbb{E}(F(X_t - X_s)) = 0$. As $X_0 = 0$ and X has finite variation, $X = 0$ according to Theorem 5.11. Hence, almost surely, $K_u = 0$ $d\langle M, M\rangle_u$−a.s., Hence $K = 0$ in $L^2(M)$. □

We start by defining $\int_0^t H_s \, dM_s$ for $H \in \mathcal{E}$ an elementary process defined by

$$H_s(\omega) = \sum_{i=0}^{p-1} H^{(i)}(\omega)\mathbf{1}_{]t_i,t_{i+1}]}(s)$$

where the $H^{(i)}$ are $\mathcal{F}_{t_i}$-measurables and bounded.
We set :

$$(H.M)_t = \int_0^t H_s\, dM_s = \sum_{i=0}^{p-1} H^{(i)}(\omega)(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}).$$

For all $H \in \mathcal{E}$, $(\int_0^t H_s\, dM_s)_{t\geq 0}$ belongs to H^2 if $M \in H^2$.

THEOREM 7.3. *i). The map $H \to H.M$ can be extended into an isometry from $L^2(M)$ into H^2.*
ii). Moreover, $H.M$ is characterized by

$$\langle H.M, N\rangle = H.\langle M, N\rangle, \forall N \in H^2.$$

PROOF. i) If $H \in \mathcal{E}$, then $H.M$ is the sum of the martingales $M_t^{(i)} = H^{(i)}(M_{t\wedge t_{i+1}} - M_{t\wedge t_i})$ such that $\rangle M^{(i)}, M^{(j)}\rangle_t = 0$ if $i \neq j$ and

$$\langle M^{(i)}, M^{(i)}\rangle_t = (H^{(i)})^2(\langle M, M\rangle_{t\wedge t_{i+1}} - \langle M, M\rangle_{t\wedge t_i}).$$

Consequently,

$$\langle H.M, H.M\rangle_t = \sum_{i=0}^{p-1}(H^{(i)})^2(\langle M, M\rangle_{t\wedge t_{i+1}} - \langle M, M\rangle_{t\wedge t_i})$$

and hence

$$\begin{aligned} ||H.M||_{H^2}^2 &= \mathbb{E}\left[\sum_{i=0}^{p-1}(H^{(i)})^2(\langle M, M\rangle_{t_{i+1}} - \langle M, M\rangle_{t_i})\right] \\ &= \mathbb{E}\left[\int_0^\infty H_s^2 d\langle M, M\rangle_s\right] \\ &= ||H||_M^2. \end{aligned}$$

The map $H \to H.M$ is therefore an isometry from $\mathcal{E}$ into H^2. According to proposition 7.2, $\mathcal{E}$ is dense in $L^2(M)$ and H^2 is an Hilbert space, we can entend uniquely a map as an isometry from $L^2(M)$ into H^2.

ii) If $H \in \mathcal{E}$,

$$\langle H.M, N\rangle = \sum_{i=0}^{p-1}\langle M^{(i)}, N\rangle$$

and

$$\langle M^{(i)}, N\rangle_t = H^{(i)}(\langle M, N\rangle_{t\wedge t_{i+1}} - \langle M, N\rangle_{t\wedge t_i}).$$

We therefore deduce that

$$\langle H.M, N\rangle_t = \sum_{i=0}^{p-1} H^{(i)}(\langle M, N\rangle_{t\wedge t_{i+1}} - \langle M, N\rangle_{t\wedge t_i}) = \int_0^t H_s d\rangle M, N\rangle_s.$$

Hence, ii) is proved if $H \in \mathcal{E}$. According to Kunita-Watanabe inequality 6.7, for all $N \in H^2$, the map $X \to \langle X, N\rangle_\infty$ is continuous from H^2 into L^1. If $H^n \in \mathcal{E}$ and $H^n \to H$ are in $L^2(M)$, we have

$$\langle H.M, N\rangle_\infty = \lim_{n\to+\infty}\langle H^n.M, N\rangle_\infty = \lim_{n\to+\infty}(H^n.\langle M, N\rangle)_\infty = (H.\langle M, N\rangle)_\infty$$

where convergence holds in L^1. The last equality comes from Kunita-Watanabe inequality:

$$\mathbb{E}\left[\left|\int_0^\infty (H_s^n - H_s)d\langle M, N\rangle_s\right|\right] \le \mathbb{E}[\langle N, N\rangle_\infty]^{1/2}||H^n - H||_M.$$

Taking N^t instead of N in the equality $\langle H.M, N\rangle_\infty = (H.\langle M, N\rangle)_\infty$, we deduce ii). $H.M$ is characterized by ii), in the sense that if X is another martingale in H^2, we have for all $N \in H^2$,

$$\langle H.M - X, N\rangle = 0$$

and hence, taking $N = H.M - X$, we deduce $X = H.M$. □

Property 7.4. *If $K \in L^2(M)$ and $H \in L^2(K.M)$, then $HK \in L^2(M)$ and*

$$(HK).M = H.(K.M).$$

Proof. First,

$$\langle K.M, K.M\rangle = K^2.\langle M, M\rangle$$

Hence $HK \in L^2(M)$. Moreover, according to Theorem 7.3, for all $N \in H^2$,

$$\begin{aligned}\langle (HK).M, N\rangle &= (HK).\langle M, N\rangle\\ &= H.(K.\langle M, N\rangle)\\ &= H.\langle K.M, N\rangle\\ &= \langle H.(K.M), N\rangle\end{aligned}$$

□

Property 7.5. *If T is a stopping time and if $M \in H^2$,*

$$K.M^T = K\mathbf{1}_{[0,T]}.M = (K.M)^T.$$

Proof. We have $M^T = \mathbf{1}_{[0,T]}.M$ since for all $N \in H^2$,

$$\langle M^T, N\rangle = \langle M, N\rangle^T = \mathbf{1}_{[0,T]}.\langle M, N\rangle = \langle \mathbf{1}_{[0,T]}.M, N\rangle.$$

By the previous proposition,

$$K.M^T = K.(\mathbf{1}_{[0,T]}.M) = K\mathbf{1}_{[0,T]}.M$$

and

$$(K.M)^T = \mathbf{1}_{[0,T]}.(K.M) = \mathbf{1}_{[0,T]}K.M.$$

□

The martingale $H.M$ is called *stochastic integral of H with respect to M* and is denoted

$$\int_0^{\cdot} H_s \, dM_s.$$

7.2. Stochastic integral with respect to a local martingale. Let B be a Brownian motion with values in $\mathbb{R}$. We know that $B \notin H^2$ but B stopped at t belongs to H^2. Hence, we can define $\int_0^t H_s \, dB_s$ for all $t \in \mathbb{R}^+$ in H satisfying

$$\mathbb{E}\left[\int_0^t H_s^2 \, ds\right] < \infty.$$

If M is a continuous local martingale, we denote by $L^2_{loc}(M)$ the set of progressively measurable processes H such that

$$\mathbb{E}\left[\int_0^{T_n} H_s^2 \, d\langle M, M\rangle_s\right] < \infty$$

where T_n is a sequence of stopping time $\uparrow +\infty$ a.s..

Property 7.6. *For all $H \in L^2_{loc}(M)$, there exists a unique continuous local martingale vanishing at the origin, denoted by $H.M$, such that for all continuous local martingale N,*

$$\langle H.M, N\rangle = H.\langle M, N\rangle.$$

Proof. We can construct a sequence of stopping times $(T_n)_n \uparrow +\infty$ such that $M^{T_n} \in H^2$ and $H^{T_n} \in L^2(M^{T_n})$. Hence, for all n, we can define the stopped martingale,

$$X^{(n)} = H^{T_n}.M^{T_n} \in H^2.$$

If we stop $X^{(n+1)}$ at T_n, we obtain

$$\begin{aligned} (X^{(n+1)})^{T_n} &= (H^{T_{n+1}}.M^{T_{n+1}})^{T_n} \\ &= H^{T_{n+1}}\mathbf{1}_{[0,T_n]}.M^{T_{n+1}} \\ &= H\mathbf{1}_{[0,T_n]}.M^{T_n} \end{aligned}$$

We can therefore define $H.M$ by putting

$$(H.M)_t = X_t^{(n)} \text{ on } [0, T_n].$$

$(H.M)_t$ is a local continuous martingale as $(H.M)^{T_n} = X^{(n)} \in H^2$.
We clearly have

$$\langle H.M, N\rangle = H.\langle M, N\rangle$$

since

$$\langle H.M, N\rangle^{T_n} = (H.\langle M, N\rangle)^{T_n}$$

where $(T_n)_n \uparrow +\infty$. □

$H.M$, the stochastic integral of H with respect to M, is denoted by

$$\int_0^{\cdot} H_s \, dM_s.$$

A progressively measurable process H is said to be locally bounded *if there exists a sequence of stopping times $(T_n)_n \uparrow +\infty$ and finite constants C_n such that*

$$|H^{T_n}| \leq C_n.$$

An adapted and continuous process H is progressively measurable : we can choose the stopping times

$$T_n = \inf\{t; |H_t| \geq n\}.$$

The interest of this definition lies in the fact that if H is progressively measurable and locally bounded, then for all continuous local martingales M we have $H \in L^2_{loc}(M)$.

7.3. stochastic integral with respect to a continuous semi-martingale. Let $X = M + A$ be a continuous semi-martingale , H be a locally bounded process, then the *stochastic integral of H with respect to X* is the continuous semi-martingale :

$$H.X = H.M + H.A$$

where $H.M$ is the integral of H with respect to the continuous local martingale M and $H.A$ the integral of H with respect to A in the sense of Stieltjes integral.
The semi-martingale $H.X$ is denoted

$$\int_0^{\cdot} H_s \, dX_s.$$

Stochastic integral properties:

- The map $(H, X) \to H.X$ is bilinear.
- If H and K are locally bounded then $H.(K.X) = (HK).X$.
- For all stopping times T, $(H.X)^T = H\mathbf{1}_{[0,T]}.X = H.X^T$.
- If X is a local martingale (resp. a finite variation process), then $H.X$ is a local martingale (resp. a finite variation process).
- If $H \in \mathcal{E}$ can be written $H_s(\omega) = \sum_{i=0}^{p-1} H^{(i)}(\omega)\mathbf{1}_{]t_i,t_{i+1}]}$ with $H^{(i)}$ $\mathcal{F}_{t_i}$-measurable and bounded, then

$$(H.X)_t = \sum_{i=0}^{p-1} H^{(i)}(X_{t\wedge t_{i+1}} - X_{t\wedge t_i}).$$

We next show a dominated convergence theorem for stochastic integrals

Theorem 7.7. *Let X be a continuous semi-martingale . If $(H^n)_n$ is a sequence of continuous locally bounded processes converging pointwise to 0, and if there exists a locally bounded process H such that $|H^n| \leq H$ for all n, then $H^n.X$ converges towards 0 in probability, uniformly over all compact intervals.*

Proof. We restrict ourselves to the case where X is a continuous local martingale. Let T_n be the sequence of stopping time localizing X, then $(H^n)^{T_m}$ converges towards 0 In $L^2(X^{T_m})$ and according to Theorem 7.3., $(H^n.X)^{T_m}$ converges to 0 in H^2. We conclude as in the proof of Theorem 6.5. □

The next result will be crucial to establish Itô's formula.

Property 7.8. *If H is a left continuous adapted process and if $(\Delta_n)_n$ is a sequence of subdivisions of $[0,t]$ such that $|\Delta_n| \to 0$, then*

$$\int_0^t H_s \ dX_s = \mathbb{P} - \lim_{n\to+\infty} \sum_{t_i^n \ \in\Delta_n} H_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}).$$

Proof. If H is bounded, the right hand side are the stochastic integrals of the elementary processes $\sum_i H^{(i)}\mathbf{1}_{]t_i,t_{i+1}]}$ which converge point wise towards $||H||_\infty$; hence this proposition is a direct consequence of the previous theorem. The general case is obtained by localization. □

7.4. Integration by parts formula. Let X, Y be two continuous semi-martingales.

Property 7.9.

$$i).\quad X_tY_t = X_0Y_0 + \int_0^t X_s \ dY_s + \int_0^t Y_s \ dX_s + \langle X, Y\rangle_t.$$

$$ii).\quad X_t^2 = X_0^2 + 2\int_0^t X_s \ dX_s + < X, X >_t .$$

Example : $X_t = B_t$ the standard Brownian motion, $< B, B >_t = t$, $B_0 = 0$. We then have :

$$B_t^2 - t = 2\int_0^t B_s \ dB_s.$$

In particular $(\int_0^t B_s \ dB_s)_t$ is a martingale.

PROOF. i). If ii) is proved, by applying it to $X+Y$ and $X-Y$, we get :

$$\begin{aligned} X_tY_t &= \frac{1}{4}[(X_t+Y_t)^2-(X_t-Y_t)^2] \\ &= \frac{1}{4}[(X_0+Y_0)^2-(X_0-Y_0)^2] \\ &+ \frac{1}{4}[2\int_0^t (X_s+Y_s)\ d(X_s+Y_s) - 2\int_0^t (X_s-Y_s)\ d(X_s-Y_s)] \\ &+ \frac{1}{4}[< X+Y, X+Y >_t - < X-Y, X-Y >_t] \\ &= X_0Y_0 + \int_0^t X_s\ dY_s + \int_0^t Y_s\ dX_s + \langle X, Y\rangle_t \end{aligned}$$

as by definition

$$\langle X, Y\rangle_t = \frac{1}{4}[\langle X+Y, X+Y\rangle_t - \langle X-Y, X-Y\rangle_t]$$

ii) Let us prove ii). Let $(\Delta_n)_n$ be a sequence of subdivisions of $[0,t]$ such that $|\Delta_n| \to 0$.
Then,

$$\begin{aligned} X_t^2 - X_0^2 &= \sum_i (X_{t_{i+1}^n}^2 - X_{t_i^n}^2) \\ &= 2\sum_i X_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}) + \sum_i (X_{t_{i+1}^n} - X_{t_i^n})^2 \end{aligned}$$

Hence, as $|\Delta_n| \to 0$, $X_t^2 - X_0^2$ converges in probability towards

$$2\int_0^t X_s\ dX_s + \langle X, X\rangle_t.$$

□

7.5. Change of variables formula. Let d be an integer, $X = (X^1, \ldots, X^d)$ a $\mathbb{R}^d$-valued process is a semi-martingale if each of its marginal $X^i, i = 1, \ldots, d$, is a real semi-martingale.

THEOREM 7.10. [Itô's formula] *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function, X a continuous semi-martingale with values in $\mathbb{R}^d$. Then, $F(X)$ is a semi-martingale and*

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X)_s\ dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X)_s\ d\langle X^i, X^j\rangle_s.$$

Example : Let $X_t = (B_t^1, B_t^2)$ where $(B_t^1)_t$, $(B_t^2)_t$ are two independent real Brownian motions. Note that $< B^1, B^2 >= 0$, by independence (so that $(B^1B^2)_t$ is a martingale.)

Proof. We restrict ourselves to the case $d = 1$.
a) Let $G(x)$ be a monomial such that the formula is satisfied for G (For instance, $G(x) = x^2$). Let $F(x) = xG(x)$. By the integration by parts formula, we have

$$F(X_t) = X_t G(X_t) = X_0 G(X_0) + \int_0^t X_s \ dG(X_s) + \int_0^t G(X_s) \ dX_s + \langle X, G(X)\rangle_t.$$

But

$$G(X_t) = G(X_0) + \int_0^t G'(X_s) \ dX_s + \frac{1}{2}\int_0^t G''(X_s) \ d\langle X, X\rangle_s.$$

so that we deduce

$$\int_0^t X_s \ dG(X_s) = \int_0^t X_s G'(X_s) \ dX_s + \frac{1}{2}\int_0^t X_s G''(X_s) \ d\langle X, X\rangle_s$$

as well as

$$\begin{aligned} \langle X, G(X)\rangle_t &= \langle X, \int_0^{\cdot} G'(X_s) \ dX_s\rangle_t + \frac{1}{2}\langle X, \int_0^{\cdot} G''(X_s) \ d\langle X, X\rangle_s\rangle_t \\ &= \int_0^t G'(X_s) \ d\langle X, X\rangle_s \end{aligned}$$

(as $(\int_0^{\cdot} G''(X_s) \ d < X, X >_t)_t$ has finite variation).
Hence,

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t (G(X_s) + X_s G'(X_s)) \ dX_s + \frac{1}{2}\int_0^t (X_s G''(X_s) + 2G'(X_s)) \ d\langle X, X\rangle_s \\ &= F(X_0) + \int_0^t F'(X_s) \ dX_s + \frac{1}{2}\int_0^t F''(X_s) \ d\langle X, X\rangle_s \end{aligned}$$

b) The result trivially extends to polynomials.
c) General case : Let F be a C^2 function. Let $n \in \mathbb{N}$ and $T_n = \inf\{t; |X_t| \geq n\}$. For all n, there exists a sequence of polynomiale $(F^{n,m})_{m\in\mathbb{N}}$ converging uniformly to F on $[-n, n]$ as m goes to infinity. Moreover, $(F^{n,m'})_{m\in\mathbb{N}}$ and $(F^{n,m''})_{m\in\mathbb{N}}$ converge uniformly towards F' and F'' respectivly on $[-n, n]$. For all $t > 0$, on the set $\{T_n \geq t\}$, we therefore have the following convergence as $m \to +\infty$,

$$F^{n,m}(X_t) \to F(X_t)$$

$$F^{n,m'}(X_t) \to F'(X_t)$$

$$F^{n,m''}(X_t) \to F''(X_t)$$

$(F^{n,m})$ being a polynomial, the following formula holds

$$F^{n,m}(X_{t\wedge T_n}) = F^{n,m}(X_0) + \int_0^{t\wedge T_n} F^{n,m'}(X_s) \ dX_s + \frac{1}{2}\int_0^{t\wedge T_n} F^{n,m''}(X_s) \ d\langle X, X\rangle_s.$$

On $\{T_n > 0\}$, we let m going to $+\infty$ and apply the dominated convergence theorem.
□

Remarks : 1- One often writes Itö's formula under the differential form :

$$dF(X_t) = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(X_t)\ dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 F}{\partial x_i \partial x_j}(X_t)\ d\langle X^i, X^j\rangle_t.$$

2- If X is a real semi-martingale, then by Itô's formula, if F is a C^2 function, then $F(X_t)$ is a real semi-martingale.Hence, the set of semi-martingales is invariant under composition by C^2 functions (which is not true for local martingales).

7.6. Applications of Itô's formula. Itô's formula has many applications, a few of which we now review.

7.6.1. *Doléans exponential of a continuous local martingale.*

THEOREM 7.11. *Let X be a continuous local martingale, λ a complex number, then the following equation (in Z)*

$$Z_t = 1 + \int_0^t Z_s\ d(\lambda X_s) \tag{7.1}$$

has a unique solution which is the continuous local martingale :

$$Z_t = \exp\left[\lambda(X_t - X_0) - \frac{\lambda^2}{2} < X, X >_t\right].$$

The process $(Z_t)_{t\geq 0}$ is called Doleans exponential of λX *and denoted*

$$Z = \mathcal{E}(\lambda X).$$

PROOF. a) *Existence :* Assume that $X_0 = 0$ and apply Itô to the function $F(x) = e^x$ and $Y_s = \lambda X_s - \frac{\lambda^2}{2}\langle X, X\rangle_s$:

$$F(Y_t) = F(Y_0) + \int_0^t e^{Y_s}\ dY_s + \frac{1}{2}\int_0^t e^{Y_s}\ d\langle Y, Y\rangle_s$$

that is

$$e^{\lambda X_t - \frac{\lambda^2}{2}\langle X,X\rangle_t} = 1 + \int_0^t e^{\lambda X_s - \frac{\lambda^2}{2}\langle X,X\rangle_s}\ d(\lambda X_s)$$

by noticing that

$$\langle Y, Y\rangle_t = \langle \lambda X, \lambda X\rangle_t = \lambda^2 \langle X, X\rangle_t.$$

b) *Uniqueness :* Let Y be another solution of (7.1). We apply Itô's formula to the function $F(u, v) = u/v$ and $u = Y_s$, $v = \mathcal{E}(\lambda X)_s$:

$$\frac{Y_t}{\mathcal{E}(\lambda X)_t} = 1 + \int_0^t \frac{1}{\mathcal{E}(\lambda X)_s}\ dY_s - \int_0^t \frac{Y_s}{(\mathcal{E}(\lambda X)_s)^2}\ d\mathcal{E}(\lambda X)_s$$

$$-\int_0^t \frac{1}{(\mathcal{E}(\lambda X)_s)^2}\ d < Y, \mathcal{E}(\lambda X) >_s + \frac{1}{2}\int_0^t \frac{2Y_s}{(\mathcal{E}(\lambda X)_s)^3}\ d < \mathcal{E}(\lambda X), \mathcal{E}(\lambda X) >_s .$$

. As Y and $\mathcal{E}(\lambda X)$ are solutions of (7.1), we deduce that $d\mathcal{E}(\lambda X) = \mathcal{E}(\lambda X)d(\lambda X)$ and $dY_s = Y_s d(\lambda X_s)$.
. $\langle Y, \mathcal{E}(\lambda X)\rangle_s = \langle \int Y d(\lambda X), \int \mathcal{E}(\lambda X)\ d(\lambda X)\rangle = (Y\mathcal{E}(\lambda X)).\langle \lambda X, \lambda X\rangle$
Hence

$$d\langle Y, \mathcal{E}(\lambda X)\rangle_s = Y\mathcal{E}(\lambda X)d\langle \lambda X, \lambda X\rangle.$$

. $d\langle \mathcal{E}(\lambda X), \mathcal{E}(\lambda X)\rangle_s = \mathcal{E}(\lambda X)_s^2\ d\langle \lambda X, \lambda X\rangle_s$.
We conclude that $Y_t = \mathcal{E}(\lambda X)_t$ for all t. □

7.6.2. *(Lévy) Characterization of the Brownian motion.* Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space.

THEOREM 7.12. *Let M be a continuous real local martingale, vanishing in 0, such that $< M, M >_t = t$ for all t. Then, M is a $(\mathcal{F}_t)_t$-Brownian motion.*

PROOF. Let $u \in \mathbb{R}$, $\mathcal{E}(iuM_t) = \exp(iuM_t + \frac{u^2}{2}t)$ is a local martingale according to the previous theorem and

$$|\mathcal{E}(iuM_t)| = \exp(\frac{u^2}{2}t),$$

Hence $(\mathcal{E}(iuM_s))_{s\in[0,t]}$ is a bounded martingale (cf previous section). For all s, t such that $s \leq t$, we therefore have

$$\mathbb{E}(\exp(iu(M_t - M_s))|\mathcal{F}_s) = \mathbb{E}\left(\frac{\mathcal{E}(iuM_t)}{\mathcal{E}(iuM_s)}\exp(-\frac{u^2}{2}(t-s))|\mathcal{F}_s\right) = \exp(-\frac{u^2}{2}(t-s)).$$

Consequently , $M_t - M_s$ is a standard Gaussian variable $\mathcal{N}(0, t-s)$ and is independent of $\mathcal{F}_s$. □

7.6.3. *Dubins-Schwarz representation theorem.*

THEOREM 7.13. *Let M be a continuous local martingale on a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$ such that $M_0 = 0$ and $\langle M, M\rangle_\infty = \infty a.s.$. Then, there exists a Brownian motion B such that*

$$\forall t \geq 0, M_t = B_{\langle M,M\rangle_t} \qquad a.s.$$

PROOF. For all $r \geq 0$ we denote

$$\tau_r = \inf\{t \geq 0 : \langle M, M\rangle_t \geq r\}$$

τ_r is a stopping time as an entry time for an adapted process, see Proposition 4.2. Moreover by hypothesis τ_r is finite almost surely. It will be useful to take the convention that $\tau_r \equiv 0$ on the neglectable set $N = \{\langle M, M\rangle_\infty < \infty\}$. As the filtration is complete, τ_r is still a stopping time. Moreover $r\rangle\tau_r$ is increasing, continuous (by continuity of $\langle M, M\rangle$, see Theorem **??**) . We set $B_r = M_{\tau_r}$ for all $r \geq 0$. By Proposition (4.4), M is progressively measurable and hence B is adapted with respect to the filtration $\mathcal{G}_r = \mathcal{G}_{\tau_r}$. More precisely, B is a continuous $\mathcal{G}_r$- martingale and $\langle B, B\rangle_r = \langle M, M\rangle_{\tau_r} = r$. hence by Lévy's characterization, B

is a $\mathcal{G}_r$-Brownian motion. To prove that $B_{\langle M,M\rangle_t} = M_t$ we just need to make sure that M is constant whenever $\langle M, M\rangle$ is. More precisely

LEMMA 7.14. *Almost surely, for all $0 \le a < b$,*

$$M_t = M_a, \quad \forall t \in [a,b] \Leftrightarrow \langle M,M\rangle_b = \langle M,M\rangle_a.$$

PROOF. By continuity of the trajectories, it is enough to show that for any given $a < b$,

$$\{M_t = M_a, \quad \forall t \in [a,b]\} := S_1 = S_2 := \{\langle M,M\rangle_b = \langle M,M\rangle_a\} \qquad a.s.$$

It is clear that $S_1 \subset S_2$ by the construction of $\langle M,M\rangle$ given in Theorem 6.5. To show that $S_2 \subset S_1$ observe that the local martingale

$$N_t = M_{t\wedge b} - M_{t\wedge a} = \int_0^t 1_{[a,b]}(s) dM_s$$

is such that $\langle N,N\rangle_t = \langle M,M\rangle_{t\wedge b} - \langle M,M\rangle_{t\wedge a}$. For $\epsilon > 0$ let T_ϵ be the stopping time

$$T_\epsilon = \inf\{t \ge 0 : \langle N,N\rangle_t \ge \epsilon\}$$

N^{T_ϵ} is a bounded martingale in L^2 and $\mathbb{E}[N_t^{T_\epsilon}] = \mathbb{E}[\langle N,N\rangle_t] \le epsilon$. In particular as $S_2 \subset \{T_\epsilon \ge b\}$, we deduce

$$\mathbb{E}[1_{S_2} N_t^2] = \mathbb{E}[1_{S_2} N_{t\wedge T_\epsilon}^2] \le \epsilon\,.$$

We can finally let ϵ going to zero to conclude that N_t vanishes almost surely on S_2 and hence $S_2 \subset S_1$ almost surely. □

□

7.6.4. *Burkholder' s inequality.*

THEOREM 7.15. *Let M be a continuous martingale such that*

$$\mathbb{E}(< M, M >_T^{p/2}) < \infty$$

where $T > 0$ is given, $M_0 = 0$ and $p \ge 2$. Then,
i) There exists a constant C'_p (independent of M) such that

$$\forall t \le T, \mathbb{E}(|M_t|^p) \le C'_p \mathbb{E}(\langle M,M\rangle_t^{p/2}).$$

ii) There exists a constant C_p (independent of M) such that

$$\mathbb{E}(\sup_{t\le T} |M_t|^p) \le C_p \mathbb{E}(\langle M,M\rangle_T^{p/2}).$$

PROOF. We first assume M bounded. We apply Itô's formula with $F(x) = |x|^p$, $p \ge 2$ and $x = M$.

$$|M_t|^p = p\int_0^t |M_s|^{p-1}\text{sign}(M_s)\; dM_s + \frac{p(p-1)}{2}\int_0^t |M_s|^{p-2}\; d\langle M,M\rangle_s$$

Hence,

$$\begin{aligned}\mathbb{E}(|M_t|^p) &= \frac{p(p-1)}{2}\mathbb{E}\left[\int_0^t |M_s|^{p-2}\, d\langle M,M\rangle_s\right] \\ &\leq \frac{p(p-1)}{2}\mathbb{E}\left[\sup_{s\leq t}|M_s|^{p-2}\, \langle M,M\rangle_t\right] \\ &\leq \frac{p(p-1)}{2}\left(\mathbb{E}\left[\sup_{s\leq t}|M_s|^p\right]\right)^{(p-2)/p}\left(\mathbb{E}\left[\langle M,M\rangle_t^{p/2}\right]\right)^{2/p}\end{aligned}$$

by Hölder's inequality.
Hence,

$$\mathbb{E}(|M_t|^p)^p \leq (\frac{p(p-1)}{2})^p\left(\mathbb{E}\left[\sup_{s\leq t}|M_s|^p\right]\right)^{p-2}\left(\mathbb{E}\left[\langle M,M\rangle_t^{p/2}\right]\right)^2$$

According to Doob's inequality,

$$\mathbb{E}\left[\sup_{s\leq t}|M_s|^p\right] \leq q^p\mathbb{E}(|M_t|^p)$$

for all q such that $1/p+1/q=1$.
Hence,

$$\mathbb{E}(|M_t|^p)^p \leq (\frac{p(p-1)}{2}q^{p-2})^p\mathbb{E}(|M_t|^p)^{p-2}\left(\mathbb{E}\left[\langle M,M\rangle_t^{p/2}\right]\right)^2$$

that is

$$\mathbb{E}(|M_t|^p) \leq (\frac{p(p-1)}{2}q^{p-2})^{p/2}\left(\mathbb{E}\left[\langle M,M\rangle_t^{p/2}\right]\right).$$

The generalization to all continuous martingale is obtained by considering the stopping times

$$T_n = \inf\{t; |M_t| \geq n\}.$$

□

Burkholder's inequality can be strengthened into Burkholder-Davis-Gundy inequalities

Theorem 7.16. *Let M be a continuous local martingale. Then for every $p \in \mathbb{R}^+$ there exists two constants c_p and C_p such that*

$$c_p\mathbb{E}[\langle M,M\rangle^{p/2}] \leq \mathbb{E}[(\sup_{t\geq 0}|M_t|)^p] \leq C_p\mathbb{E}[\langle M,M\rangle^{p/2}]$$

We refer the reader to Revuz-Yor, Chapter IV, section 4.

7.7. Girsanov's theorem. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space. The goal of this paragraph is to study the probability measures $\mathbb{Q}$ which are absolutely continuous with respect to $\mathbb{P}$. We start with the Radon-Nikodym theorem (stated without proof)*

THEOREM 7.17. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space. Let $\mathbb{Q}$ be a probability measure which is absolutely continuous with respect to $\mathbb{P}$: $\mathbb{Q} << \mathbb{P}$ (i.e. for all $A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \to \mathbb{Q}(A) = 0$).*
Then, there exists a random variable $\mathcal{F}$-measurable, positive such that for all $A \in \mathcal{F}$,

$$\mathbb{Q}(A) = \int_A Z \; d\mathbb{P}.$$

Then,

$$Z = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$$

is called the Radon-Nikodym density.

We set, for all $t \geq 0$,

$$Z_t = \mathbb{E}(Z|\mathcal{F}_t).$$

Then, $(Z_t)_t$ is a right continuous martingale and is uniformly integrable. Let T be a stopping time with respect to the filtration $(\mathcal{F}_t)_t$. We set $\mathbb{Q}_T = \mathbb{Q}_{|\mathcal{F}_T}$, $\mathbb{P}_T = \mathbb{P}_{|\mathcal{F}_T}$ the measures $\mathbb{Q}$ and $\mathbb{P}$ restricted to the σ-algebra $\mathcal{F}_T$ of events prior to T. Then one easily shows that

$$Z_T = \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}_T}.$$

We leave the proof to the reader.

LEMMA 7.18. *$(Z_t)_t$ is $\mathbb{Q}$-a.s. positive.*

Assume that $t \to Z_t$ is continuous.

LEMMA 7.19. *Let X be continuous, adapted process. If the process XZ is a $\mathbb{P}$-local martingale then X is a $\mathbb{Q}$-local martingale.*

THEOREM 7.20. *[Girsanov] Let Q be a mesure absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_\infty$. We assume that $(Z_t)_t$ is continuous. Then,*
i). each $\mathbb{P}-$semi-martingale is a $\mathbb{Q}-$semi-martingale.
ii). If M is a $\mathbb{P}-$continuous local martingale and if

$$M' = M - \frac{1}{Z}.\langle M, Z\rangle,$$

then M' is well defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{Q})$ and $(M'_t)_t$ is a $\mathbb{Q}-$local martingale. Moreover,

$$\langle M', M'\rangle = \langle M, M\rangle \qquad \mathbb{Q} - a.s.$$

PROOF. Assume that ii) is proved. If X is a $\mathbb{P}-$semi-martingale which we write as $X = M + A$ (under $\mathbb{P}$), then

$$X = M' + \frac{1}{Z}.\langle M, Z\rangle + A$$

is a $\mathbb{Q}-$semi-martingale.
We next show ii). First, M' is well defined (under $\mathbb{Q}$) as $1/Z$ is $\mathbb{Q}$-locally bounded (According to lemma 7.18). Let us show that $M'Z$ is a $\mathbb{P}-$local martingale. By the integration by parts formula,

$$\begin{aligned} M'_t Z_t &= M'_0 Z_0 + \int_0^t M'_s dZ_s + \int_0^t Z_s \, dM'_s + \langle M', Z\rangle_t \\ &= M'_0 Z_0 + \int_0^t M'_s dZ_s + \int_0^t Z_s \, dM_s. \end{aligned}$$

We use the lemma 7.19 to conclude. □

The two following corollaries show some applications of Girsanov's theorem:

COROLLARY 7.21. *Let $(B_t)_{t\in\mathbb{R}^+}$ be an $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$- standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$. For all $t \in \mathbb{R}^+$, we set*

$$\tilde{B}_t = B_t - \frac{1}{Z}.\langle B, Z\rangle_t.$$

Then, $(\tilde{B}_t)_{t\in\mathbb{R}^+}$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{Q})$.

PROOF. Use theorem 7.12. □

COROLLARY 7.22. *Let L be a continuous local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ such that $L_0 = 0$. We assume that*

$$\mathbb{E}\left[\exp(\frac{1}{2}\langle L, L\rangle_\infty)\right] < \infty.$$

Then,
i) $\mathcal{E}(L)$ is a uniformly integrable martingale.
ii) If we define $\tilde{B}_t = B_t - \langle L, B\rangle_t$, then $(\tilde{B}_t)_t$ is a $\mathbb{Q}-$Brownian motion if $(B_t)_t$ is a $\mathbb{P}-$Brownian motion.

PROOF. The proof of i) is based on exercise 7.23 and ii) is straightforward. □

EXERCISES

EXERCISE 7.23. Let Z be a local non negative continuous defined on a filtered space and with initial value 1.
I.
a) Show that Z is a non-negative supermartingale and that $\mathbb{E}(Z_\infty) \leq 1$.
b) Show that Z is a uniformly integrable martingale off $\mathbb{E}(Z_\infty) = 1$.
II.
We consider Z of the form $Z = \mathcal{E}(M)$ where M is a continuous local martingale ($\mathcal{E}(M)$ denotes Doléans exponential of M) and we assume that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M, M\rangle_\infty\right)\right] < \infty.$$

a) Show that $\langle M, M\rangle_\infty$ is integrable and that M converges a.s. as t goes to infinity.
b) Let $0 < \lambda < 1$. Show that for all $t \geq 0$,

$$\mathcal{E}(\lambda M)_t = (\mathcal{E}(M)_t)^\lambda \exp\left[(\lambda(1-\lambda)/2)\langle M, M >_t\right]$$

and that $\mathcal{E}(\lambda\ M)$ is a uniformly integrable martingale.
(Use Hölder inequality with exponent $p = 1/\lambda$).
c) Deduce by letting λ going to one that

$$\mathbb{E}[\mathcal{E}(M)_\infty] \geq 1$$

and that $\mathcal{E}(M)$ is a uniformly integrable martingale.

EXERCISE 7.24. I. Let $M = (M_t)_{t\geq 0}$ be a real continuous martingale defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, $M_0 = 0$. Let $A = (A_t)_{t\geq 0}$ be a continuous increasing adapted process, vanishing at 0. Consider the process $Z = (Z_t)_{t\geq 0}$ given by :

$$Z_t = \int_0^t \frac{dM_s}{1 + A_s}$$

a) SHow that $M_t = \int_0^t (Z_t - Z_s)\ dA_s + Z_t$.
b) Show that for u such that $0 < u < t$, we have :

$$\left|\frac{M_t}{1 + A_t}\right| \leq \frac{|Z_t|}{1 + A_t} + \frac{\int_0^u (Z_t - Z_s)\ dA_s}{1 + A_t} + \frac{A_t - A_u}{1 + A_t} \sup_{u<s\leq t} |Z_t - Z_s|.$$

c) Assume that $(Z_t)_{t\geq 0}$ converges $\mathbb{P}$-a.s. towards a finite limit as t goes to infinity. Deduce that on the set $\{A_\infty = \infty\}$:

$$\frac{M_t}{1 + A_t} \to 0 \quad \mathbb{P} - a.s. \text{ as } t \to \infty.$$

II. Let $L = (L_t)_{t\geq 0}$ be a real local martingale defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, vanishing in 0. Show that $(L_t)_{t\geq 0}$ converges $\mathbb{P}$-a.s. on the set $\{\langle L, L\rangle_\infty < \infty\}$ towards a finite limit as t goes to infinity.
(Hint : one can considere the local martingale $L^2 - \langle L, L\rangle$ and for each n, the time $T_n = \inf\{t; \langle L, L\rangle_t \geq n\}$ if $\{-\} \neq \emptyset$ and $T_n = \infty$ if $\{-\} = \emptyset$, and then study the

behavior of $L^2 - \langle L, L\rangle$ on the sets $A_p = \{\langle L, L >_\infty < p\}$ where $p \in \mathbb{N}$).

Exercise 7.25. Let M be a local continuous martingale vanishing at the origin. We set

$$L_t = \sup_{s\le t} M_s \quad \text{and} \quad U_t = L_t - M_t.$$

Let a be a non-positive real number and b a positive real number.
a) Applying Itô formula to : $(x, y) \to F(x, y) = xy$ where
$x = \exp(aL - (b^2/2)\langle M, M >)$ and $y = \phi(U_t)$ where ϕ is C^2, show that

$$Z = \phi(U)\exp(aL - (b^2/2)\langle M, M >)$$

is a local martingale if for all $t \ge 0$, we have :

$$\int_0^t (a\phi(U_s) + \phi'(U_s))\; dL_s = 0 \quad \text{and} \qquad \int_0^t (b^2\phi(U_s) + \phi''(U_s))\; d\langle M, M\rangle_s = 0$$

Show that the function ϕ given by : $\phi(x) = b\cosh(bx) - a\sinh(bx)$ can be used.
b) Let $\lambda > 0$. We define the stopping times :

$$T = \begin{cases} \inf\{t; U_t > \lambda\} \text{ if } \;\{-\} \neq \emptyset \\ +\infty \;\text{ otherwise} \end{cases}$$

and

$$S = \begin{cases} \inf\{t; M_t < -\lambda\} \text{ if } \;\{-\} \neq \emptyset \\ +\infty \;\text{ otherwise} \end{cases}$$

Show that Z^T and $\mathcal{E}(aM^S)$ are bounded martingales and that

$$\mathbb{E}\left[\exp(-a^2/2\langle M, M\rangle_S)\right] \ge \frac{1}{\exp(-a\lambda)}.$$

c) Assume that $\langle M, M\rangle_\infty = \infty$ $\mathbb{P}$-a.s., deduce from the above that

$$\mathbb{P}(S < \infty) \ge \frac{1}{\exp(-a\lambda)}$$

and therefore $\mathbb{P}(S < \infty) = 1$ and finally $\mathbb{P}(T < \infty) = 1$.
d) Show that

$$\mathbb{E}[\exp(aL_t - (b^2)/2\langle M, M\rangle_t)] = b/(b\cosh(b\lambda) - a\sinh(b\lambda)).$$

e) Deduce that L_T follows an exponential law with parameter $1/\lambda$.

Exercise 7.26. Let X be a continuous process independent from a σ-algebra $\mathcal{F}$ and such that

$$\mathbb{E}(\sup_{t\le K} |X_t|) < \infty.$$

Let T be a random variable, $\mathcal{F}$-measurable and bounded by K.
Show that

$$\mathbb{E}(X_T|\mathcal{F}) = \phi(T)$$

where $\phi(t) = \mathbb{E}[X_t]$.
(Hint : start with the case where T is a finite combination of indicateor functions.)

Exercise 7.27. Let $(B_t)_{t\in\mathbb{R}^+}$ be a standard real Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$, the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ being right continuous. Being given $a > 0$, we define the stopping time T_a by

$$T_a = \begin{cases} \inf\{t;\ B_t \geq a\} \text{ if } \{\ \} \neq \emptyset \\ +\infty \quad \text{otherwise} \end{cases}$$

1) We denote by $Z = \mathcal{E}(\lambda B^{T_a})$ the Doléans exponential of λB^{T_a} where λ is a positive real number.
a)- Show that Z is a bounded martingale. Using $\mathbb{E}(Z_\infty)$ and letting λ going to 0, show that

$$\mathbb{P}[T_a < \infty] = 1.$$

b)- Compute $\mathbb{E}[\exp(-\lambda T_a)]$.

2) Consider the local martingale $M = \mathcal{E}(iuB)$ the Doléans exponential of iuB where u is a real number.
a)- Show that for all $s, t \in\ [0, +\infty[$ such that $s < t$, for all $K > 0$, we have

$$\mathbb{E}[\exp\{iu(B_{(T_a\wedge k)+t} - B_{(T_a\wedge k)+s})\}|\mathcal{F}_{(T_a\wedge k)+s}] = \exp\{-\frac{u^2}{2}(t-s)\}.$$

(Hint : use the stopped local martingale $\mathcal{E}(iuB^{(T_a\wedge k)+t})$).
b)- Deduce by letting K going to infinity, that the process $B(T_a)$ defined by

$$B(T_a)_t = B_{T_a+t} - B_{T_a}$$

is a Brownian motion with respect to the filtration $(\mathcal{G}_t)_t$ where $\mathcal{G}_t = \mathcal{F}_{T_a+t}$. Show that for all $t > 0$, $B(T_a)_t$ is independent of $\mathcal{F}_{T_a}$.

3) Let $b > a$. Consider T_b defined as T_a.
a)- Show that $T_b - T_a$ is a stopping time with respect to the filtration $(\mathcal{G}_t)_t$.
b)-Using the martingale $\mathcal{E}(\lambda B^{T_b})$ show that

$$\mathbb{E}[\exp\{-\frac{\lambda^2}{2}(T_b - T_a)\}|\mathcal{F}_{T_a}] = \exp\{-\lambda(b-a)\}.$$

c)- Deduce that the process $(T_a)_{a\geq 0}$ has independent increments.

The following questions are independent from 3) but exercise 7.26 is useful for question (4).

4) a)- Show that for all borelian bounded f, we have

$$\mathbb{E}[f(B_t)\mathbf{1}_{\{T_a\leq t\}}] = \mathbb{E}[\mathbf{1}_{\{T_a\leq t\}}\psi(t - T_a)]$$

where $\psi(u) = \mathbb{E}[f(B_u + a)]$.
b)- Noticing that $\mathbb{E}[f(B_u + a)] = \mathbb{E}[f(-B_u + a)]$, deduce from a) that

$$\mathbb{E}[f(B_t)\mathbf{1}_{\{T_a \leq t\}}] = \mathbb{E}[f(2a - B_t)\mathbf{1}_{\{T_a \leq t\}}].$$

5) Show that, if we denote $S_t = \sup_{s \leq t} B_s$, and for $a > 0$, we have

$$\mathbb{P}[B_t \leq a; S_t \geq a] = \mathbb{P}[B_t \geq a; S_t \geq a] = \mathbb{P}[B_t \geq a] = \frac{1}{2}\mathbb{P}[S_t \geq a].$$

Deduce that S_t has the same law as $|B_t|$.
6) Show that for $a \geq b$ and $a \geq 0$, we have

$$\mathbb{P}[B_t \leq b; S_t \geq a] = \mathbb{P}[B_t \geq 2a - b; S_t \geq a] = \mathbb{P}[B_t \geq 2a - b].$$

Show that for $a \leq b$ and $a \geq 0$, we have :

$$\mathbb{P}[B_t \leq b; S_t \geq a] = 2\mathbb{P}[B_t \geq a] - \mathbb{P}[B_t \geq b].$$

7) Verify that the law of the cpuple (B_t, S_t) has density :

$$\mathbf{1}_{\{y \geq 0\}}\mathbf{1}_{\{y \geq x\}}\frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(2y - x)^2}{2t}\right\}$$

Exercise 7.28. Let a, b and X be real valued processes defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})$. The processes a and b are progressively measurable and locally bounded; X is continuous and $X_0 = 0$.
For all function f on $\mathbb{R}$ which is C^2, we set:

$$L_s(\omega)f(x) = \frac{1}{2}a_s^2(\omega)f''(x) + b_s(\omega)f'(x)$$

$$\text{and} \qquad M_t(f) = f(X_t) - \int_0^t L_s f(X_s)\, ds.$$

1) Show that the two following facts are equivalent :
(i) For all C^2 function f, $M(f)$ is a local martingale.
(ii) The process M defined by :

$$M_t = X_t - \int_0^t b_s \; ds$$

is a local martingale with quadratic variation :

$$\langle M, M >_t = \int_0^t a_s^2 \; ds.$$

2) For $\lambda \in \mathbb{R}$, we denote $\mathcal{E}_t^\lambda = \exp\{\lambda X_t - \lambda \int_0^t b_s \; ds - \frac{\lambda^2}{2} \int_0^t a_s^2 \; ds\}$. We consider :
(iii) For all $\lambda \in \mathbb{R}$ réel, $\mathcal{E}^\lambda$ is a local martingale.
a)- Show that (ii) implies (iii).
b)- Show that (iii) implies (i) for all functions of the form $f(x) = \exp(\lambda x)$.
(Hint : consider the process V such that $\exp(\lambda X) = \mathcal{E}^\lambda V$).
c)- Deduce by a density argument that (iii) implies (i).

EXERCISE 7.29. Let $B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$ be a 3-dimensionnal Brownian (i.e. for $i = 1, 2, 3$, $(B_t^{(i)})_{t\in\mathbb{R}^+}$ are 3 independent Brownian motions) starting from $x \neq 0$. For $t \geq 0$, we set :

$$R_t = \sqrt{(B_t^{(1)})^2 + (B_t^{(2)})^2 + (B_t^{(3)})^2}.$$

1) Show that $(\frac{1}{R_t})_{t\geq 0}$ is a local martingale. (Hint: use Itô's formula).
2) Compute for $t \geq 0$, $\mathbb{E}(\frac{1}{R_t})$ et $\mathbb{E}(\frac{1}{R_t^2})$.
3) Deduce that $(\frac{1}{R_t})_{t\geq 1}$ is a local martingale which is uniformly integrable, but not a martingale.

8. Stochastic differential equations (SDE)

The goal of this chapter is to provide a mathematical model for a differential equation perturbed by a noise. Let us start with the ordinary differential equation (ODE)

$$\frac{dX_t}{dt} = b(X_t).$$

These equations describe the evolution of a physical system such as the position of a satellite at time t. The equation which describe its evolution can be thought as random because so many parameters are unknowns, or too complicated to be analyzed. We then add a random noise of the form $\sigma(X_t)dB_t$ where B_t is a Brownian motion and $\sigma(.)$ represents the intensite of the noise which depends on the system state at time t. We arrive at the *stochastic differential equation* (abreviation : S.D.E.) of the following form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

The Itô integral introduced in the previous chapters gives a mathematical meaning to this equation under the form

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s. \tag{8.1}$$

Notice that we have already encounter the (linear)stochastic differential equation :

$$X_t = 1 + \int_0^t X_s\ dB_s$$

whose solution is

$$X_t = \exp\Big(B_t - \frac{t}{2}\Big).$$

More generally the equation

$$X_t = x_0 + a\int_0^t X_s\ dB_s + b\int_0^t X_s\ ds, \quad (x_0, a, b \in \mathbb{R})$$

has solution the stochastic process

$$X_t = x_0 \exp\Big(aB_t + (b - \frac{a^2}{2})t\Big).$$

In general the solution of a SDE is not so easily determined. There exists sufficient conditions on b and σ to insure existence and uniqueness of the solutions to the SDE (8.1). We next discuss them, in the more general setting that they may depend on time. We study the SDE :

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s. \tag{8.2}$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ be a filtered probability space and $(B_t)_{t\in\mathbb{R}^+}$ be a Brownian motion. We fix the interval $[0, T]$ and $s \in [0, T]$. We set

$$\mathcal{F}_{s,t} = \sigma(B_u - B_s; s \le u \le t).$$

Then , $(X_t^x)_{t\in[s,T]}$ is solution of the S.D.E. :

$$X_t^x = x + \int_s^t b(r, X_r^x)\ dr + \int_s^t \sigma(r, X_r^x) dB_r \tag{8.3}$$

if X_t^x is $\mathcal{F}_{s,t}$-measurable for all $t \in [s, T]$ and satisfies (8.3).
Hypothesis :
(H_1) *: Lipschitz condition :*
There exists $L > 0$ such that

$$|b(t, y) - b(t, x)| \le L|x - y|$$
$$|\sigma(t, y) - \sigma(t, x)| \le L|x - y|$$

for all $t \in [0, T]$.
(H_2) *:* The functions $t \to b(t, x)$ and $t \to \sigma(t, x)$ are continuous for all $x \in \mathbb{R}$.
We deduce that there exists $A > 0$ such that for all $x \in \mathbb{R}$ and $t \in [0, T]$,

$$|b(t, x)| + |\sigma(t, x)| \le A(1 + |x|).$$

THEOREM 8.1. *Under the hypotheses (H_1) and (H_2), the stochastic differential equation (8.2) has a unique solution for any initial condition $x \in L^p$, for any $p \ge 2$.*

PROOF. *i) Existence of a solution to*(8.2) *:*
We use the successive approximation method :

$$X_t^{(0)} = x, X_t^{(1)} = x + \int_s^t b(r, x)\ dr + \int_s^t \sigma(r, x) dB_r,$$
$$\dots$$
$$X_t^{(n)} = x + \int_s^t b(r, X_r^{(n-1)})\ dr + \int_s^t \sigma(r, X_r^{(n-1)}) dB_r.$$

We thus have

$$X_t^{(n+1)} - X_t^{(n)} = \int_s^t \Big[b(r, X_r^{(n)}) - b(r, X_r^{(n-1)})\Big]\ dr + \int_s^t \Big[\sigma(r, X_r^{(n)}) - \sigma(r, X_r^{(n-1)})\Big]\ dB_r.$$

Set

$$\alpha_t^{(n)} = \mathbb{E}\left(\sup_{s\le u\le t} |X_u^{(n+1)} - X_u^{(n)}|^p\right)$$

Then,

$$\begin{aligned}\alpha_t^{(n)} \quad &\le \quad 2^p \left[\mathbb{E}\left(\sup_{s\le u\le t}\left|\int_s^u \left(b(r,X_r^{(n)}) - b(r,X_r^{(n-1)})\right)\, dr\right|^p\right)\right.\\ &+ \quad \left.\mathbb{E}\left(\sup_{s\le u\le t}\left|\int_s^u \left(\sigma(r,X_r^{(n)}) - \sigma(r,X_r^{(n-1)})\right)\, dB_r\right|^p\right)\right].\end{aligned}$$

Note

$$\Sigma_1(n) = \mathbb{E}\left(\sup_{s\le u\le t}\left|\int_s^u \left(b(r,X_r^{(n)}) - b(r,X_r^{(n-1)})\right)\, dr\right|^p\right)$$

and

$$\Sigma_2(n) = \mathbb{E}\left(\sup_{s\le u\le t}\left|\int_s^u \left(\sigma(r,X_r^{(n)}) - \sigma(r,X_r^{(n-1)})\right)\, dB_r\right|^p\right).$$

By Burkholder's inequality, we can bound $\Sigma_2(n)$ by

$$\Sigma_2(n) \le C_p\ \mathbb{E}\left[\left(\int_s^t \left(\sigma(r,X_r^{(n)}) - \sigma(r,X_r^{(n-1)})\right)^2\, dr\right)^{p/2}\right].$$

We are going to use the following result:
Let f be a positive function, then

$$\left(\int_s^t f(r)\, dr\right)^{p/2} \le (t-s)^{p/2-1}\int_s^t f(r)^{p/2}\, dr.$$

Then

$$\begin{aligned}\Sigma_2(n) &\le C_p\ (t-s)^{p/2-1}\ \int_s^t \mathbb{E}\left[\left|\sigma(r,X_r^{(n)}) - \sigma(r,X_r^{(n-1)})\right|^p\right]\, dr.\\ &\le C_p\ L^p (t-s)^{p/2-1}\ \int_s^t \mathbb{E}\left[\left|X_r^{(n)} - X_r^{(n-1)}\right|^p\right]\, dr\end{aligned}$$

by using hypothesis (H_1). Consequently, there exists a constant k_2 such that

$$\Sigma_2(n) \le k_2\int_s^t \mathbb{E}\left[\sup_{s\le u\le r}\left|X_u^{(n)} - X_u^{(n-1)}\right|^p\right]\, dr.$$

Using the same type of argument we show that there exists a constant k_1 such that

$$\Sigma_1(n) \le k_1\int_s^t \mathbb{E}\left[\sup_{s\le u\le r}\left|X_u^{(n)} - X_u^{(n-1)}\right|^p\right]\, dr.$$

We deduce that there exists a constant K such that

$$\alpha_t^{(n)} \le K\int_s^t \alpha_r^{(n-1)}\, dr.$$

By induction

$$\alpha_t^{(n)} \leq K^n \alpha_t^{(0)} \int_s^t \, dt_1 \int_s^{t_1} \, dt_2 \ldots \int_s^{t_{n-1}} \, dt_r$$
$$\leq \frac{K^n}{n!} \alpha_t^{(0)} (t-s)^n \leq \frac{K^n}{n!} \alpha_t^{(0)} T^n$$

But

$$\begin{aligned}
\alpha_t^{(0)} &= \mathbb{E}\left[\sup_{s \leq u \leq t} \left| \int_s^u b(r,x) \, dr + \int_s^u \sigma(r,x) \, dB_r \right|^p \right] \\
&\leq 2^p \left(\mathbb{E}\left[\sup_{s \leq u \leq t} \left| \int_s^u b(r,x) \, dr \right|^p \right] + \mathbb{E}\left[\sup_{s \leq u \leq t} \left| \int_s^u \sigma(r,x) \, dB_r \right|^p \right] \right) \\
&\leq 2^p \left((t-s)^{p-1} \mathbb{E}\Big(\int_s^t |b(r,x)|^p \, dr \Big) + (t-s)^{p/2-1} \mathbb{E}\Big(\int_s^t |\sigma(r,x)|^p \, dr \Big) \right) \\
&\leq KA^p \mathbb{E}\Big((1+|x|)^p \Big) = C < \infty
\end{aligned}$$

by assumption. Hence,

$$\alpha_t^{(n)} \leq C \, \frac{(KT)^n}{n!}.$$

We deduce that

$$\sum_{n=0}^{\infty} \left(\mathbb{E}\left[\sup_{s \leq u \leq t} |X_u^{(n+1)} - X_u^{(n)}|^p \right] \right)^{1/p} \leq C^{1/p} \sum_{n=0}^{\infty} \left(\frac{(KT)^n}{n!} \right)^{1/p} < \infty$$

Hence, $(X_u^{(n)})$ is a Cauchy sequence in L^p. Consequently, the limit exists and belong to L^p.

ii) Uniqueness of the solution :
Let X and X' be two solutions of the SDE (8.2) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})$. Then, we show that X and X' are not distinguishable in the sense that

$$\mathbb{P}[\exists u \in]s,t]; X_u \neq X'_u] = 0,$$

by using Gronwall's lemma :
Let g be a Borelian fonction defined on $[0,T]$ with values in $\mathbb{R}^+$ such that

$$\sup_{t \leq T} g(t) < \infty.$$

If

$$g(t) \leq A + B \int_0^t g(s) \, ds,$$

then for all $t \in [0,T]$,

$$g(t) \leq A \exp(Bt).$$

In particular if $A = 0$, then $g(t) = 0$.
Let $T_n = \inf\{t;\ |X_t| = n \ \text{ or } \ |X'_t| = n\}$. Then there exists a constant $K > 0$ such that for all $t \in [0, T_n]$,

$$\mathbb{E}(|X_t - X'_t|^2) \leq K\ \mathbb{E}(\int_s^t |X_r - X'_r|^2\, dr).$$

The process X is therefore a modification on $[0, T_n]$ of X', hence on $\mathbb{R}^+$ ($T_n \uparrow +\infty$ $\mathbb{P}-$a.s.), which implies indistinguishability of X and X' by continuity.

□

The previous result shows that the solutions exists path wise and is measurable with respect to the filtration of the Brownian motion. This is what is called a *strong* solution. A weaker form of existence and uniqueness is given in terms of laws; there is uniqueness in law when two solutions started from the same initial data have the same law. It is possible to have existence of a weak solution and uniqueness in law without path wise uniqueness. For instance if B is a Brownian motion with $B_0 = x$ and we set

$$W_t = \int_0^t \text{sign}(B_s) dB_s$$

where the sign of x is -1 if $x \leq 0$ and $+1$ otherwise. by Lévy's characterization, W is a Brownian motion started from zero and we have

$$B_t = x + \int_0^t \text{sgn}(B_s) dW_s$$

Any solution of the latter differential equation is a Brownian motion by Lévy's characterization. However, if $x = 0$, B and $-B$ are two path wise distinct solutions as

$$-B_t = \int_0^t \text{sgn}(-B_s) dB_s + 2\int_0^t 1_{B_s=0} dB_s$$

where the last term is a martingale with vanishing quadratic variations, hence is almost surely equal to zero. However, existence in law and pathwise uniqueness yields uniqueness in law (by Yamada-Wantanabe Thm)

Proof of Gronwall's lemma.
Iterarting the condition on g we get for all $n \geq 1$,

$$g(t) \leq A + A(Bt) + A\frac{(Bt)^2}{2} + \ldots + A\frac{(Bt)^n}{n!} + B^{n+1}\int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_n} ds_{n+1} g(s_{n+1}).$$

If g is bounded by a constant C, the last term is bounded by $C(Bt)^{n+1}/(n+1)!$, hence goes to zero as n goes to infinity □

8.1. Strong Markov property and diffusion processes. In an ordinary differential equation, the future of the trajectory of a particle is entirely determined by its present position. The stochastic analogue for stochastic differential equations is true as well: solutions to SDE's have the strong Markov property, i.e., the distribution of their future depends only on their present position. (In fact, SDE solutions should be viewed as the prototypical example of a strong Markov process.)

THEOREM 8.2. *(Strong markov property). Assume that σ and b are two Lipschitz functions. Then for all $x \in \mathbb{R}^d$, if X^x denotes a weak solution started from x if F is any measurable nonnegative functional on $C(\mathbb{R}^+, \mathbb{R}^d)$ then almost surely, for any stopping time T*

$$\mathbb{E}[F(X^x_{T+t}, t \geq 0)|\mathcal{F}_T] = \mathbb{E}[F(X^y_t, t \geq 0)]_{y=X_T}$$

on the event $T < \infty$.

PROOF. We can assume without loss of generality that T is finite up to replace it by $T \wedge n$. As X is solution we have

$$X^x_{T+t} - X^x_t = \int_T^{T+t} \sigma(X^x_s) dB_s + \int_T^{T+t} b(X^x_s) ds$$

We thus only need to show that for any pre visible locally bounded process and continuous local martingales X we have, if $X^T_t = X_{T+t} - X_T$,

$$\int_T^{T+t} H_s dX_s = \int_0^t H_{s+T} dX^T_s \tag{8.4}$$

But this is clear if $H_s(x) = 1_{s \in [t_1, t_2]} 1_{x \in A}$, extends to simple processes by linearity and then by Itô isometry extends to all $H \in L^2$. The general result is deduced by localization. Therefore we deduce that $Y_t = X_{T+t}$ satisfies

$$Y_t = Y_0 + \int_0^t \sigma(Y_s) dB^T_s + \int_0^t b(Y_s) ds$$

and therefore by strong existence and uniqueness the result follows. □

We next provide a brief introduction to the theory of diffusion processes, which are Markov processes characterized by martingale properties. We first construct these processes with SDE's and then move on to describe some fundamental connection with PDE's. In the next section we show how diffusions arise as scaling limits of Markov chains. Define for $f \in C^2(\mathbb{R}^d)$ the infinitesimal generator L given by

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i} \tag{8.5}$$

We assume that $(a_{ij}(x))_{1 \leq i,j \leq d}$ is a symmetric nonnegative matrix for all $x \in \mathbb{R}^d$.

DEFINITION 8.3. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. We say that a processs X is an L-diffusion if for all $f \in C_b^2(\mathbb{R}^d)$, the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$$

is a local martingale.

We next check that any solution of the SDE (8.1) is an L-diffusion for some choices of a and b, simply as a direct consequence of Itô's formula.

THEOREM 8.4. *Let X be a solution to the SDE* (8.1) *for σ and b measurable functions. Then, X is a L-diffusion for $a = \sigma\sigma^T$ and b.*

Any non negative matrix a has a square root σ. If a is positive definite, a is Lipschitz iff σ is. Hence, if we assume that $a \geq cId$ with $c > 0$ independent of x, a and b Lipschitz, then there exists an L-diffusion (the solution of the SDE)

8.2. Applications to PDEs.

8.2.1. *The Cauchy problem.* Solutions toSDE's can be used to construct solutions of PDEs, more precisely of the Cauchy problem.

THEOREM 8.5. *Let $g : \mathbb{R}^d \rangle \mathbb{R}$ be a bounded C^2 function and let X be an L-diffusion with $a = \sigma\sigma^T$ and σ, b uniformly Lipschitz. Then if we define:*

$$u(t, x) = \mathbb{E}_x[g(X_t)]$$

is the unique solution in $C^{2,1}(\mathbb{R}_+ \times \mathbb{R}^d)$ to the problem:

$$\partial_t u(t, x) = Lu(t, x) \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad u(0, x) = g(x) \quad \forall x \in \mathbb{R}^d$$

PROOF. It is obvious by applying Itô's formula that

$$\partial_t \mathbb{E}_x[g(X_t)] = \partial_t \mathbb{E}_x[g(x) + \int_0^t Lg(X_s)ds] = \mathbb{E}_x[Lg(X_t)]$$

If t goes to zero we deduce that

$$\partial_t \mathbb{E}_x[g(X_t)]|_{t=0} = Lg(x)$$

by continuity of Lg. To deduce the result for all $t > 0$ we need to show that the operator L "commutes" with $\mathbb{E}_x$. This is due to the strong Markov property, which implies that the law P_t of X_t is such that $P_{t+\epsilon} = P_\epsilon * P_t$ and therefore for all $t > 0$

$$\partial_t \mathbb{E}_x[g(X_t)] = \lim_{\epsilon \rangle 0} \frac{1}{\epsilon}(P_{t+\epsilon} - P_t)(g) = \lim_{\epsilon \rangle 0} \frac{1}{\epsilon}(P_\epsilon - I)(P_t(g)) = \partial_s P_s|_{s=0} * P_t g = L\mathbb{E}_x[g(X_t)]$$

Hence $u(t,x)=\mathbb{E}_x[g(X_t)]$ indeed satisfies Cauchy problem. To prove uniqueness let v be another solution and let $f(t,x)=v(T-t,x)$ for $t\le T$. Due to Itô's formula, $M_t=v(T-t,X_t), t\le T$ is a martingale and therefore

$$\mathbb{E}[v(0,X_T)]=\mathbb{E}_x[M_T]=\mathbb{E}_x[M_0]=v(T,x)$$

which completes the proof as $v(0,x)=g(x)$ by hypothesis.

8.2.2. *The Dirichlet problem.* We have already seen that the laws of solutions of SDE's are solutions for the heat equation. This relation can be used to construct solutions of other questions, such as the Dirichlet or Poisson problem. In the following we fix an open set D in $\mathbb{R}^d$, an infinitesimal generator L as in (8.5) as well as a function ϕ smooth on the boundary of D. The Dirichlet problem consists in finding a function u $C^2(D)$ such that

$$Lu(x)=0 \quad \forall x\in D \quad , \lim_{\substack{x\to y\\ x\in D}} u(x)=\phi(y) \quad \forall y\in\partial D\,.$$

The Poisson problem consists in requiring that $Lu+g=0$ for some continuous function g on D, and keep the same boundary constraint. The natural candidate to be a solution of Poisson problem in view of what we have already seen is

$$u(x)=\mathbb{E}_x[\int_0^{T_D} g(X_s)ds+\phi(X_{T_D})1_{T_D<\infty}]$$

where X_t is the solution of the SDE with generator L (here we assume a,b uniformly Lipschitz) and $T_D=\inf\{t>0: X_t\notin D\}$. Indeed, if u is a solution, we can apply Itô's calculus to find that

$$u(X_{t\wedge T_D})=u(x)-\int_0^{t\wedge T_D} g(X_s)ds+\int_0^{t\wedge T_D}\nabla u(X_s).dB_s$$

from which the result formally follows by taking expectation and letting t going to infinity, if the solution is indeed C^2. It turns out that a sufficient condition to make this argument rigorous when $g=0$ is that L is uniformly elliptic, that is $(a_{ij}(x))_{1\le i,j\le d}\ge cI$ for some $c>0$, we refer to Oksendal book, Thm 9.2.14 for a full proof.

8.2.3. *Feynmann-Kac formula.* Feynamnn Kac formula allows to construct a slightly different type of PDE.

THEOREM 8.6. *Let $f\in C_b^2(\mathbb{R}^d)$ and $V\in C_b^0(\mathbb{R}^d)$, V being uniformly bounded. We set*

$$u(t,x)=\mathbb{E}_x[f(B_t)\exp\left(\int_0^t V(B_s)ds\right)]\,.$$

Then, u is the unique solution $w\in C_b^{1,2}(\mathbb{R}_+\times\mathbb{R}^d)$ of

$$\partial_t u=\frac{1}{2}\nabla u+Vu \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, u(0,.)=f \quad x\in\mathbb{R}^d$$

The proof is again an easy application of Itô's formula.

8.2.4. *Stroock-Varadhan martingale problem.* Let $(\sigma_{i,j})_{1\le i,j\le d}$ and $(b_i)_{1\le i\le d}$ be measurable functions with values in $\mathbb{R}$ and set $a=\sigma\sigma^T$.

Definition 8.7. *We say that a process X with values in $\mathbb{R}^d$ together with a filtered probability space $(\Omega,\mathcal{G},(|Fa_t)_{t\ge 0}$ solves the martingale problem $M(a,b)$ iff for all $1\le i,j\le d$ the processes*

$$Y^i=(X_t^i-\int_0^t b_i(X_s)ds, t\ge 0) \text{ and } (Y_t^iY_t^j-\int_0^t a_{i,j}(X_s)ds, t\ge 0)$$

are local martingales.

Of course the second condition implies that the martingale bracket of X is as well given by the integral of a.

As an exercise, we leave the reader to check that if X solves $M(a,b)$, is adapted and continuous, iff for all function $f\in C^2(\mathbb{R}^d,\mathbb{R})$

$$M_t^f=f(X_t)-f(X_0)-\int_0^t Lf(X_s)ds$$

is a local martingale, with L the infinitesimal generator

$$Lf(x)=\frac{1}{2}\sum_{i,j}a_{i,j}(x)\partial_i\partial_j f(x)+\sum b_i(x)\partial_i f(x)$$

It is also left as an exercise that any solution of the SDE (8.1) solves the martingale problem $M(\sigma\sigma^T,b)$. Remarkably the converse is true.

Theorem 8.8. *Let $a=\sigma\sigma^T$ and let X and a filtered probability space $(\Omega,\mathcal{G},(|Fa_t)_{t\ge 0}$ solve the martingale problem $M(a,b)$. Then there exists an $(\mathcal{G}_t)_{t\ge 0}$ Brownian motion $(B_t,t\ge 0)$ in $\mathbb{R}^d$ defined on an enlarged probability space so that (X,B) solves the SDE*

$$X_t^i=X_0^i+\int_0^t b_i(X_s)ds+\sum_j\int_0^t \sigma_{ij}(X_s)dB_s^j$$

Proof. Assume that σ is invertible and set $Y_t^i=X_t^i-\int_0^t b_i(X_s)ds$. Define

$$B_t^i=\int_0^t\sum(\sigma^{-1})_{i,k}(X_s)dY_s^k$$

B is a local martingale with martingale bracket equal to identity. Hence, by Léve's characterization it is a Brownian motion. Moreover X satisfies the announced SDE as

$$\sum_j\int_0^t\sigma_{ij}(X_s)dB_s^j=\sum_{j,k}\int_0^t\sigma_{ij}(X_s)\sigma^{-1})_{j,k}(X_s)dY_s^k=Y_t^i$$

When σ is not everywhere invertible, let us first restrict ourselves to $d = 1$ for simplicity. Then we put

$$B_t = \int_0^t 1_{\sigma(X_s)\neq 0}\sigma^{-1}(X_s)dY_s + \int_0^t 1_{\sigma(X_s)=0}dW_s\,,$$

with W an additional Brownian motion (hence the necessity to enlarge the filtration), independent from Y. Then again by Lévy characterization, B is a Brownian motion whereas

$$\int_0^t \sigma(X_s)dB_s = \int_0^t 1_{\sigma(X_s)\neq 0}dY_s$$

To conclude, we use Lemma 7.14 which implies that

$$\int_0^t 1_{\sigma(X_s)\neq 0}dY_s = Y_t \quad a.s$$

as

$$\langle \int_0^\cdot 1_{\sigma(X_s)\neq 0}dY_s, \int_0^\cdot 1_{\sigma(X_s)\neq 0}dY_s\rangle_t = \langle Y, Y\rangle_t$$

To conclude in the general case we first observe that we can find a matrix-valued function O which is orthogonal, and a diagonal matrix d with non negative entries such that for all x

$$a(x) = O(x)d(x)O(x)^T$$

If $O(x) = Id$, then the question amounts to the one dimensional one. So we put

$$B_t = \int_0^t O(X_s)F(d(X_s))O(X_s)^T dY_s + \int_0^t O(X_s)G(d(X_s))O(X_s)^T dW_s\,.$$

with $F(x) = \sqrt{x}1_{x\neq 0}$ and $G(x) = 1_{x=0}$, and W a d dimensional Brownian motion, independent from Y Then, we also have $\sigma = O(x)\sqrt{d(x)}O(x)^T$ and therefore

$$\int_0^t \sigma(X_s)dB_t = \int_0^t O(X_s)\sqrt{d(X_s)}O(X_s)^T[O(X_s)F(d(X_s))O(X_s)^T dY_s$$

$$+O(X_s)G(d(X_s))O(X_s)^T dW_s]$$

as $\sqrt{d(X_s)}F(d(X_s)) = I$ whereas $\sqrt{d(X_s)}G(d(X_s)) = 0$, we conclude as above that

$$\int_0^t \sigma(X_s)dB_t = Y_t \qquad a.s$$

from which the conclusion follows.

□

9. Appendix

9.1. Monotone class Theorem. The monotone class theorem is a result from measure theory that we use in several instances in this course. Let E be a set and $P(E)$ the set of subsets of E. If $C \subset P(E)$, $\sigma(C)$ denotes the smallest sigma-algera generated by C (this is also the intersection of all the σ algebras containing C)

Definition 9.1. A subset M of $P(E)$ is said to be monotone class if

(1) $E \in M$,
(2) If $A, B \in M$ then $B \backslash A \in M$,
(3) if A_n is an increasing sequence in M, then $\cup_{n\in\mathbb{N}} A_n \in M$.

Any σ algebra is monotone class. Any intersection of monotone class is monotone class. If C is a set in $P(E)$ we can define the monotone class $M(C)$ generated by C by putting

$$M(C) = \cap_{M \text{monotone class}, C \subset M} M$$

Lemma 9.2. *If $C \subset P(E)$ is stable under finite intersection then $M(C)$ is the σ algebra $\sigma(C)$ generated by C.*

Proof. As $\sigma(C)$ is monotone class, we clearly have $M(C) \subset \sigma(C)$. We therefore only need to show that $M(C)$ is a σ-algebra. The only thing we need to show is that it is stable under finite intersections (indeed, by going to the complement, this will imply it is stable under finkte union, and then by countable union by taking an increasing limit). To that end set for $A \in E$

$$M_A = \{B \in M(C) : A \cap B \in M(C)\}$$

Take first $A \in C$. As C is stable under finite intersection, clearly $C \subset M_A$. Moreover M_A is monotone class as

$E \subset M_A$,

if $B, B' \in M_A$ and $B \subset B'$, $A \cap (B \backslash B') = (A \cap B') \backslash (A \cap B) \in M(C)$ and so $B \backslash B' \in M_A$

If $B_n \in M_A$ for all n, $B_n \subset B_{n+1}$, $A \cap (\cup B_n) = \cup (A \cap B_n) \in M(C)$ and therefore $\cup B_n \in M(C)$.

Hence, $M(C) \subset M_A$, or in other words

$$\forall A \in C, \forall B \in M(C), \quad A \cap B \in M(C) \tag{9.1}$$

To conclude we need to take $A \in M(C)$. By the above, $C \subset M_A$. By exactly the same arguments as above (but replacing the stability by intersection of C by (**??**)) we deduce that M_A is class monotone. Hence $M(C) \subset M_A$ and therefore $M(C)$ is stable under finite intersection.

□

We will apply this lemma as follows:

- Let $\mathcal{A}$ be a σ-algebra and (μ, ν) be two probability measures on $(E, \mathcal{A})$. Assume that there exists $C \subset A$ stable under finite intersection such that $\sigma(C) = \mathcal{A}$ and $\mu(B) = \nu(B)$ for all $B \in C$. Then $\mu = \nu$ (indeed $\{A \in \mathcal{A} : \mu(A) = \nu(A)\}$ is class monotone).
- Let $(X_i)_{i \in I}$ be a family of random variables on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and let $\mathcal{G}$ be a sub σ algebra. To show that the σ-algebras $\sigma(X_i, i \in I)$ and $\mathcal{G}$ are independent it is enough to establish that $(X_{i_1}, \ldots, X_{i_p})$ is independent of $\mathcal{G}$ for all $i_1, \ldots, i_p \in I$ (Observe that the set of events which only depends on finitely many $X_i, i \in I$ is stable under finite intersection and generates $\sigma(X_i, i \in I)$)
- Let $(X_i)_{i \in I}$ be a family of random variables on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and Z be a bounded real variable. To show that for some $j \in I$
$$\mathbb{E}[Z|X_i, i \in I] = \mathbb{E}[Z|X_j]$$
it is enough to show that for any choice of $i_1, \ldots i_p \in I$,
$$\mathbb{E}[Z|X_j, X_{i_1}, \ldots, X_{i_p}] = \mathbb{E}[Z|X_j]$$
(observe that the set of events A such that $\mathbb{E}[1_A Z] = \mathbb{E}[1_A \mathbb{E}[Z|X_j]]$ is monotone class)

9.2. Useful discrete martingale properties.

THEOREM 9.3. *Let $(X_n)_{n \in \mathbb{N}}$ be a submartingale such that*
$$\sup_{n \geq 0} \mathbb{E}[X_n^+] < \infty$$
Then, the limit $\lim_n X_n = X_\infty$ exists almost surely. Moreover, X_∞ belongs to L^1

Observe above that the convergence does not hold in L^1 in general (cf $M_n = \prod_{k \leq n} X_k$ with i.i.d Bernouilli X_k which equals 2 with probability 1/2 and 0 otherwise)

PROOF. Let $T_0 = 0$, $T_i = \inf\{p > \tilde{T}_{i-1} : \exists k > p : X_p \leq a \leq b \leq X_k\}$ and put $\tilde{T}_i$ the smallest such k. Then we set
$$u_n([a, b]) = \sum_{i \geq 1} 1_{T_i \leq n}$$
the number of times M_n crosses $[a, b]$ upward. If for all $a < b$ $u_n([a, b])$ converges towards a finite limit, then the sequence M_n converges or goes in absolute value to infinity. It is therefore enough to prove that $u_n([a, b])$ has finite expectation by Doob's uncrossing inequality
$$(b - a)\mathbb{E}[u_n([a, b])] \leq \mathbb{E}[(M_n - a)^+] \tag{9.2}$$

To prove this inequality, define the two sequences of stopping times $\tau_0 = 0$,

$$\sigma_{j+1} = \inf\{k > \tau_j : M_n \leq a\} \quad \tau_{j+1} = \min\{k > \sigma_{j+1} : M_n \geq b\}$$

Then, as $\tau_n \geq n$, it goes to infinity and we can write with $Y_n = (M_n - a)^+$,

$$Y_n = Y_0 + \sum_{i=1}^{\infty}(Y_{n\wedge\tau_i} - Y_{n\wedge\sigma_i}) + \sum_{i=0}^{\infty}(Y_{n\wedge\sigma_{i+1}} - Y_{n\wedge\tau_i})$$

But

$$\sum_{i=1}^{\infty}(Y_{n\wedge\tau_i} - Y_{n\wedge\sigma_i}) \geq (b-a)u_n([a,b])]$$

whereas as M_n is a submartingale, Y_n is also a submartingale so the for all i

$$\mathbb{E}[Y_{n\wedge\sigma_{i+1}} - Y_{n\wedge\tau_i}] \geq 0$$

from which (9.2) follows. □