A probabilistic approach to some problems in von Neumann algebras

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One of the most famous open questions concerning von Neumann algebras is to know whether free group factors with different numbers of generators are isomorphic or not

$$L(F^m) \simeq L(F^n) \qquad \text{if} \qquad n \neq m \quad ???$$

To try to attack such questions, Voiculescu introduced about twenty years ago free probability theory. Free probability theory is a probability theory for non-commutative variables equipped with a notion of freeness analogous to the classical notion of independence. This similarity permits to generalize many concepts from classical probability such as central limit theorems, Brownian motions etc and provides intuition to the domain. On the other hand, freeness is related to the usual notion of freeness on groups and is therefore meaningfull in standard operator algebras theory. Last but not least, independent large Gaussian random matrices were shown to be asymptotically free by D. Voiculescu [18]. Since then, large random matrices became a source of examples of interesting laws of non-commutative variables. In these proceedings, we shall describe how such a philosophy has been developped to try to answer the isomorphism problem and related issues. Even though this problem has not yet been settled we want to emphasize that such a strategy has already been fruitful (c.f. [21], [12], [13]). We hope to convince analysts and probabilists that these issues are very closely related with standard problems in analysis and probability.

We shall follow the following plan

(1) **Description of free probability framework.** Relation with large random matrices.

(2)**The isomorphism problem in free probability terms.**

(3)**Trying to disprove it by an entropy approach.** Entropy theory, large deviations techniques. Recent developments and discussion.

(4) **Conclusion.**

For completness, we provide in the appendix the proof of Gelfand-Naimark-Segal construction and show how non-commutative laws prescribe von Neumann algebras up to isomorphisms.

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1 Free probability versus *classical probability*

In this section, we provide a short introduction to free probability, comparing it with standard probability (which elements are written in *italic*).

1.1 The setting

A non-commutative (or W^*)- probability space is a couple $(\mathcal{A}, \tau)$ such that

- $\mathcal{A}$ is a von Neumann algebra, i.e a weakly closed sub-C^*-algebra of the space $B(H)$ of bounded linear operators on some Hilbert space H.
- τ is a state on $\mathcal{A}$, that is a complex-valued linear form on $\mathcal{A}$ such that

$$\overline{\tau(A)} = \tau(A^*), \quad \tau(AA^*) \geq 0, \quad \tau(I) = 1, \qquad \forall A \in \mathcal{A}.$$

We shall consider in the following **tracial states**, which are states satisfying the additional hypothesis that

$$\tau(AB) = \tau(BA), \qquad \forall \quad A, B \in \mathcal{A}.$$

Examples 1: a) Let $n \in \mathbb{N}$, $\mathcal{A} = M_n(\mathbb{C}) = B(\mathbb{C}^n)$. For any $v \in \mathbb{C}^n$ such that $\|v\|_{\mathbb{C}^n} = 1$, we set

$$\tau_v(M) =< v, Mv >_{\mathbb{C}^n} \quad \forall M \in M_n(\mathbb{C})$$

Then, τ_v is a state. There is a unique tracial state on $M_n(\mathbb{C})$, which is the normalized trace

$$\mathrm{tr}(M) = \frac{1}{n}\sum_{i=1}^{n} M_{ii}.$$

b) *Let $(X, \Sigma, d\mu)$ be a classical probability space. Then $\mathcal{A} = L^\infty(X, \Sigma, d\mu)$ equipped with*

$$\tau : f \to \int f d\mu$$

is a (non-)commutative probability space. Here, $L^\infty(X, \Sigma, d\mu)$ is seen as the space of bounded linear operator on the Hilbert space $H = L^2(X, \Sigma, d\mu)/\equiv$ equipped with the scalar product $< f, g >_\mu = \int f(x)g(x)d\mu(x)$ by the embedding given by the multiplication operator $M(f)g = fg$. Here, H is obtained by separating $L^2(X, \Sigma, d\mu)$ by the equivalence relation $f \equiv h \Leftrightarrow \mu(|f-g|^2) = 0$ so that $< ., . >_\mu$ furnishes it with a Hilbert structure.

c) Let G be a discrete group, and $(e_h)_{h\in G}$ be a basis of $\ell^2(G)$. Let $\lambda(h)e_g = e_{hg}$. Then, we take $\mathcal{A}$ to be the von Neumann algebra generated by the linear span of $\lambda(G)$. The (tracial) state is the linear form given, once restricted to $\lambda(G)$, by

$$\tau(\lambda(g)) = 1_{g=e}$$

Here, e denotes the neutral element.

We refer to [25] for further examples and details.

1.2 The law of m self-adjoint non commutative variables

Let $(\mathcal{A},\tau)$ be a non-commutative probability space. If $(X_1,\cdots,X_m)\in\mathcal{A}$, $X_i=X_i^*$, their joint law is given by the restriction of τ to the algebra generated by $(X_1,\cdots,X_m)$:

$$\tau_{X_1,\cdots,X_m}(P)=\tau(P(X_1,\cdots,X_m))\qquad \forall P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle$$

where $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ denotes the set of polynomial functions in m non-commutative variables. Such a definition can of course be extended to the case of non self-adjoint variables by taking polynomial functions of $(X_i,X_i^*)_{1\le i\le m}$ but we shall not consider this generalization here.

Classical setting *This definition is a generalization of the observation that, in the commutative setting, the law $\mu_{f_1,\cdots,f_m}$ of m bounded real-valued random variables $(f_i)_{1\le i\le m}\in L^\infty(X,\Sigma,\mu)$ is determined by their joint moments, i.e if $\mathbb{C}[X_1,\cdots,X_m]$ denotes the set of polynomial functions in m commutative variables, the law $\mu_{f_1,\cdots,f_m}$ is determined by*

$$\mu_{f_1,\cdots,f_m}(P)=\int P(f_1(\omega),\cdots,f_m(\omega))d\mu(\omega),\quad \forall P\in\mathbb{C}[X_1,\cdots,X_m].$$

As a consequence, the space $\mathcal{M}^m$ of laws of m self-adjoint non commutative variables can be seen as the set of linear forms on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ which are

a) **non negative** :

$$\tau(PP^*)\ge 0$$

for all $P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle$,

b) **with mass one** :

$$\tau(I)=1.$$

We shall assume that they are **tracial** ;

$$\tau(PQ)=\tau(QP)\quad \forall P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle.$$

This abstract point of view is actually equivalent to the previous one in the sense that by the Gelfand-Neumark-Segal (GNS) construction, being given $\mu\in\mathcal{M}^m$, we can construct a W^*-probability space $(\mathcal{A},\tau)$ and operators $(\mathbf{X}_1,\cdots,\mathbf{X}_m)$ such that

$$\mu=\tau_{\mathbf{X}_1,\ldots,\mathbf{X}_m}. \tag{1.1}$$

We recall this construction in Appendix 5.1 ; roughly speaking it shows that, as in the commutative setting, $\mathcal{A}$ can be thought as $L^\infty(\mu)$ in the sense that it is embedded into the space $B(L^2(\mu))$ of bounded linear operators on the space of functions with finite second moment. We shall also denote $W^*(X_1,\cdots,X_n)$ the von Neumann algebra $\mathcal{A}$.

If $R\in\mathbb{R}$, the subset $\mathcal{M}^m_R$ of $\mathcal{M}^m$ of variables uniformly bounded by R,

$$\mathcal{M}^m_R=\{\tau\in\mathcal{M}^m;\tau(X_i^{2n})\le R^{2n}\quad\forall n\in\mathbb{N}\}$$

is Polish when equipped with its weak-$*$ topology

$$\lim_{n\to\infty}\tau_n=\tau\Leftrightarrow\lim_{n\to\infty}\tau_n(P)=\tau(P)\qquad\forall P\in\mathbb{C}\langle X_1,\ldots X_m\rangle.$$

Classical setting. *Note that $\mathcal{M}_R^1$ is exactly the space $\mathcal{P}([-R,R])$ of probability measures on $[-R,R]$. The set $\mathcal{P}([-R,R]^m)$ of probability measures on $[-R,R]^m$ is more generally described as the set $\mathbb{C}[X_1,\cdots,X_m]^*$ of linear forms on $\mathbb{C}[X_1,\cdots,X_m]$ which are positive and with mass one, and it is a Polish space when equipped with its weak-$*$ topology.*

The assumption that the variables are bounded (i.e $R<\infty$) can be relaxed in the classical setting by considering bounded continuous test functions, $\mathcal{P}(\mathbb{R}^m)\subset\mathcal{C}_b(\mathbb{R}^m)^*$. This approach can be generalized to $\mathcal{M}^m$ by considering bounded non-commutative test functions (see [6]).

Example 2: Let $A_1^N,\cdots,A_m^N\in\mathcal{H}_N^m$, with spectral radius $||A_i^N||_\infty$ bounded by R for $1\le i\le m$, and consider

$$\hat{\mu}^N_{A_1^N,\cdots,A_m^N}(P)=\mathrm{tr}\left(P(A_1^N,\cdots,A_m^N)\right),\quad\forall P\in\mathbb{C}\langle X_1,\cdots X_m\rangle.$$

Then, $\hat{\mu}^N_{A_1^N,\cdots,A_m^N}\in\mathcal{M}_R^m$. If $(A_1^N,\cdots,A_m^N)_{N\in\mathbb{N}}$ is a sequence such that

$$\lim_{N\to\infty}\hat{\mu}^N_{A_1^N,\cdots,A_m^N}(P)=\tau(P),\quad\forall P\in\mathbb{C}\langle X_1,\cdots X_m\rangle,$$

then $\tau\in\mathcal{M}_R^m$ since $\mathcal{M}_R^m$ is Polish.

There is a well known question of A. Connes related with the last example : Can all $\tau\in\mathcal{M}^m$ be constructed as a limit of $\hat{\mu}^N_{A_1^N,\cdots,A_m^N}$ for a sequence $A_1^N,\cdots,A_m^N\in\mathcal{H}_N^m$, $N\in\mathbb{N}$?

Classical setting*In the case $m=1$, $\mathcal{M}_R^1=\mathcal{P}([-R,R])$ and the question amounts to ask whether for all $\mu\in\mathcal{P}([-R,R])$, there exists a sequence $(\lambda_1^N,\cdots,\lambda_N^N)_{N\in\mathbb{N}}$ such that*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\delta_{\lambda_i^N}=\mu.$$

This is well known to be true according to Birkhoff's theorem, but is still an open question for $m\ge 2$ in the non-commutative setting.

1.3 Notion of freeness

$X=(X_1,\cdots,X_m)$ are said to be free with $Y=(Y_1,\cdots,Y_n)$ iff for any $P_1,\cdots,P_q\in\mathbb{C}\langle X_1,\cdots,X_m\rangle$ and $Q_1,\cdots,Q_q\in\mathbb{C}\langle X_1,\cdots,X_n\rangle$ such that $\mu_X(P_i)=0$ and $\mu_Y(Q_i)=0\quad\forall\,1\le i\le q$,

$$\mu_{X,Y}(P_1(X)Q_1(Y)P_2(X)Q_2(Y)\cdots P_q(X)Q_q(Y))=0.\tag{1.2}$$

Freeness, as independence, uniquely defines the joint law from the marginals μ_X and μ_Y since one easily checks that $\mu_{X,Y}(P)$ is uniquely determined for any $P\in\mathbb{C}\langle X_1,\cdots,X_m,Y_1,\cdots,Y_m\rangle$ by induction over the degree of P.

Classical setting *In comparison, if X,Y are two bounded random variables with law τ, X is independent of Y under τ iff for all $P,Q\in\mathbb{C}[X_1,\cdots,X_m]\times\mathbb{C}[Y_1,\cdots,Y_m]$*

$$\mu_X(P)=0,\mu_Y(Q)=0\Rightarrow\tau(P(X)Q(Y))=0.$$

Note here that if X,Y are centered random variables which are commutative and independent under τ, $\tau(XYXY)=\tau(X^2)\tau(Y^2)>0$ whereas if they are free $\tau(XYXY)=0$.

Examples 3:

a) In the case of a discrete group considered in Example 1.c) with 2-free generators g_1, g_2 (in the usual sense that for any polynomials such that $P_i^j(g_j) \neq e$, $P_1^1(g_1)P_1^2(g_2)P_2^1(g_1)\cdots \neq e$), $(\lambda(g_1), \lambda(g_2))$ are also free in the sense that the law prescribed by $\tau_{g_1,g_2}(\lambda(g)) = 1_{g=e}$ for any element g of the group generated by g_1 and g_2 satisfies (1.2).

b) Voiculescu [18] : Take $X_1^N, X_2^N \in \mathcal{H}_N$ to be a sequence of uniformly bounded matrices with spectral distribution converging as N go to infinity toward μ_1 and μ_2 respectively. Then, if U follows Haar measure on $U(N)$,

$$\lim_{N\to\infty} \mathrm{tr}(P(X_1^N, UX_2^N U^*)) = \lim_{N\to\infty} \hat{\mu}^N_{X_1^N, UX_2^N U^*}(P) = \tau_{\mu_1,\mu_2}(P) \quad \forall P$$

$\tau_{\mu_1,\mu_2} \in \mathcal{M}^2$ is the distribution of two free variables with marginal distribution given by μ_1 and μ_2.

If X_2^N is distributed according to the Gaussian law (GUE) (that is a Gaussian Wigner matrix)

$$\mu_N(dX) = \frac{1}{Z_N} 1_{X\in\mathcal{H}_N} e^{-\frac{N^2}{2}\mathrm{tr}(X^2)} dX,$$

then for any unitary matrix U,

$$\mu_N(dX) = \mu_N(UdXU^*).$$

Hence, since by Wigner [27], $\hat{\mu}^N_{X_2^N}$ converges towards the semi-circular law

$$\sigma(dx) = (2\pi)^{-1}\sqrt{4-x^2}dx,$$

$\hat{\mu}^N_{X_1^N, X_2^N} \Rightarrow \tau_{\mu_1,\sigma}$.

1.4 Some notions borrowed from classical probability

The role played by Gaussian laws with respect to independence is played by semi-circular laws when freeness is considered. Indeed, if $(X_1, \cdots, X_n, \cdots)$ are free centered variables ($\tau(X_i) = 0$) with covariance one ($\tau(X_i^2) = 1$), $n^{-\frac{1}{2}}\sum_{i=1}^n X_i$ converges in distribution to a semi-circular distribution (c.f. [18]).

Classical Setting *When the $(X_1, \cdots, X_n, \cdots)$ are independent centered variables with covariance one, the well known central limit theorem asserts that $n^{-\frac{1}{2}}\sum_{i=1}^n X_i$ converges in distribution to a standard Gaussian variable.*

One can define a free Brownian motion $(S_t, t \geq 0)$ as a process starting from the origin and such that for all $t \geq s$, $(t-s)^{-\frac{1}{2}}(S_t - S_s)$ is free from $\sigma(S_u, u \leq s)$ and with semi-circular distribution. Free stochastic differential (Itô's) calculus can be constructed (c.f [2]). Namely, if $K_.$ is a function of non-commutative variables such that for $t \in \mathbb{R}$, K_t depends only on the algebra $\sigma(X_u, u \leq t)$ generated by $(X_u, u \leq t)$ and is uniformly Lipschitz with respect to the operator norm, then there exists a unique solution to the differential operator valued equations given by

$$dX_t = dS_t + K_t(X)dt, \tag{1.3}$$

as can be seen by using a standard Picard iteration argument.

2 The isomorphism problem

The fundamental observation (which belongs to free probability folklore) is that the law of the variables $X_1, \cdots, X_m$ determines the von Neumann algebra they generate. More precisely,

Lemma 2.1 *If $X_1, \cdots, X_m$ (resp. $Y_1, \cdots, Y_m$) are non-commutative variables with law τ_X and τ_Y,*

$$\tau_X = \tau_Y \Rightarrow W^*(X_1, \cdots, X_m) \simeq W^*(Y_1, \cdots, Y_m)$$

where $\mathcal{A} \simeq \mathcal{B}$ means that the two algebras are isomorphic.

The proof of this lemma is recalled in appendix 5.2.

Now, $W^*(X_1, \cdots, X_m) \simeq W^*(Y_1, \cdots, Y_n)$ iff there exists $F_1(X), \cdots, F_n(X)$ (resp. $G_1(Y), \cdots, G_m(Y)$) in $W^*(X_1, \cdots, X_m)^n$ (resp. in $W^*(Y_1, \cdots, Y_n)^m$) and unitary operators $U : L^2(W^*(X_1, \cdots, X_m)) \to L^2(W^*(Y_1, \cdots, Y_n))$ (resp. $V : L^2(W^*(Y_1, \cdots, Y_n)) \to L^2(W^*(X_1, \cdots, X_m))$) so that $Y_i = UF_i(X)U^*$ for $1 \le i \le n$ (resp. $X_i = VG_i(Y)V^*$ for $1 \le i \le m$). Hence, let us say that τ_X is equivalent to τ_Y, which we denote by $\tau_X \equiv \tau_Y$, iff τ_X and τ_Y are the pushforward of each other, that is that there exists $F \in W^*(X_1, \cdots, X_m)^n, G \in W^*(Y_1, \cdots, Y_n)^m$ such that

$$\tau_Y(P) = F_\# \tau_X(P) = \tau_X(P \circ F) \quad \tau_X(P) = G_\# \tau_Y(P) = \tau_Y(P \circ G) \quad \forall P.$$

Then, Lemma 2.1 shows that

$$W^*(X_1, \cdots, X_m) \simeq W^*(Y_1, \cdots, Y_m) \Leftrightarrow \tau_X \equiv \tau_Y. \tag{2.1}$$

The isomorphism problem : Let σ_m be the law of m free semi-circular variables $S_1, .., S_m$. By (2.1), $L(F^m) \simeq W^*(S_1, ., S_m)$. The isomorphism problem can therefore be recast into

$$W^*(S_1, ., S_m) \simeq W^*(S_1, ., S_n) \Leftrightarrow \sigma_m \equiv \sigma_n \Rightarrow m = n?$$

Classical setting :*it is well known that a probability measure on $\mathbb{R}^m$ is equivalent to a probability measure on $\mathbb{R}^n$ provided they have no atoms.*

3 Entropy approach

Voiculescu [20] introduced a quantity $\delta : \mathcal{M}^m \to [0, m]$, analogue to Minkowski dimension, such that for all $m \in \mathbb{N}$

$$\delta(\sigma_m) = m.$$

It is currently warmly discussed whether δ is an invariant of the von Neumann algebra, that is whether for all $\mu \in \mathcal{M}^m$, $\mu \equiv \sigma_m$ implies $\delta(\sigma_m) = \delta(\mu)$. If this is the case, then one has proved that

$$L(F^m) \not\simeq L(F^n) \text{ if } m \neq n.$$

To define δ, Voiculescu [20] built an Entropy theory based on microstates free entropy χ which we now define.

Let $\tau \in \mathcal{M}^m$ and define a micro-state $\Gamma_R(\tau, \epsilon, k)$ by

$$\Gamma_R(\tau,\epsilon,k) = \{A_1,\cdots,A_m \in \mathcal{H}_N : |\mathrm{tr}(A_{i_1}\cdots A_{i_p}) - \tau(X_{i_1}\cdots X_{i_p})| < \epsilon$$

$$\forall\, p \le k,\ \forall\, 1 \le i_j \le m, ||A_j||_\infty \le R \quad \forall\, 1 \le j \le m\}.$$

Then we set

$$\chi(\tau) := \lim_{\substack{\epsilon\downarrow 0\\ k\uparrow\infty, R\uparrow\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}(\Gamma_R(\tau,\epsilon,k)).$$

The original definition of Voiculescu uses the Lebesgue measure instead of the Gaussian measure but it is not hard to see (c.f [7]) that these two definitions are equivalent up to a Gaussian term $2^{-1}\sum \mu(X_i^2)$.

The classical analogue to χ is Boltzmann-Shannon entropy :

$$S(\mu) = \lim_{\substack{\epsilon\downarrow 0\\ k\uparrow\infty, R\uparrow\infty}} \limsup_{N\to\infty} \frac{1}{N} \log \widetilde{\mu}_N^{\otimes m}(\Gamma_R(\tau,\epsilon,k))$$

where $\widetilde{\mu}_N$ is the law of diagonal matrices with i.i.d standard Gaussian entries. In fact, for diagonal matrices

$$\tau(X_{i_1}\cdots X_{i_p}) = < \frac{1}{N}\sum_{i=1}^N \delta_{X_{ii}^1,\cdots,X_{ii}^m}, x_{i_1}\cdots x_{i_p} >$$

so that $\Gamma_R(\tau,\epsilon,k)$ is a small neighborhood of the empirical measure of the entries. Moreover, when the random variables are bounded, it is well known that the weak- topology generated by polynomial functions is equivalent to the topology generated by bounded continuous functions and hence we arrive to the more common definition of Boltzmann-Shannon entropy*

$$S(\mu) = \lim_{\epsilon\downarrow 0}\limsup_{N\to\infty} \frac{1}{N} \log \widetilde{\mu}_N^{\otimes m}\left(d\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_{ii}^1,\cdots,X_{ii}^m}, \mu\right) < \epsilon \right)$$

where d is a distance compatible with respect to the weak-topology such as Dudley's distance

$$d(\mu,\nu) = \sup\left\{ \left| \int f d\mu - \int f d\nu \right| ; |f(x)| \textit{ and } \left| \frac{f(x)-f(y)}{x-y} \right| \le 1, \forall x \ne y \right\}.$$

By Sanov's theorem (c.f [9], theorem 6.2.10), if γ is the standard Gaussian law $\gamma(dx) = (2\pi)^{-1} e^{-\frac{1}{2}x^2} dx$,

$$\begin{aligned} S(\mu) &:= \lim_{\epsilon\downarrow 0}\limsup_{N\to\infty} \frac{1}{N} \log \widetilde{\mu}_N^{\otimes m}\left(d\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_{ii}^1,\cdots,X_{ii}^m}, \mu\right) < \epsilon \right) \\ &= \lim_{\epsilon\downarrow 0}\liminf_{N\to\infty} \frac{1}{N} \log \widetilde{\mu}_N^{\otimes m}\left(d\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_{ii}^1,\cdots,X_{ii}^m}, \mu\right) < \epsilon \right) \\ &= S^*(\mu) \end{aligned}$$

where

$$S^*(\mu) := \begin{cases} -\infty \ \textit{if } \mu \not\ll \gamma^{\otimes m} \\ -\int \log \frac{d\mu}{d\gamma^{\otimes m}} d\mu \ \textit{otherwise.} \end{cases}$$

The natural question is to seek for a generalization of Sanov's theorem in the non-commutative setting, that is to show that in the definition of χ one can replace the lim sup by a lim inf and then to find a formula for this limit which does not depend on the description via micro-states.

When $m = 1$, Voiculescu showed that indeed one can replace in the definition of χ the lim sup by a lim inf and also that for all $\mu \in \mathcal{P}(\mathbb{R})$

$$\chi(\mu) = \chi^*(\mu) := \int \log|x-y| d\mu(x) d\mu(y) - \frac{1}{2}\int x^2 d\mu(x) + \frac{3}{4}.$$

When $m \geq 2$, Biane, Capitaine and myself [3] proved by using large deviations techniques that, for all $\tau \in \mathcal{M}^m$,

$$\chi^{**}(\tau) \leq \chi(\tau) \leq \chi^*(\tau).$$

χ^* had been previously defined by Voiculescu [22] by means of Free Fisher Information and called non-microstates free entropy since it does not depend on the micro-state and random matrices definition. It is the analogue of the relative entropy S^*. The question whether one can replace the lim sup by a lim inf is still wide open since as we shall see the equality between χ^* and χ^{**} is still unclear and actually related to very deep questions such as Connes's.

$\chi^*(\tau)$ and $\chi^{**}(\tau)$ can be seen as the cost to construct τ from free semi-circular increments :

$$\chi^*(\tau) := -\inf\left\{\frac{1}{2}\int_0^1 \phi(|K_t(X)|^2) dt\right\} \tag{3.1}$$

where the infimum is taken over all ϕ which are laws of continuous non-commutative processes $(X_t^1, \cdots, X_t^m)_{t\in[0,1]}$ which start at null operators $(X_0^1, \cdots, X_0^m) = (0, \cdots, 0)$ and end at time one at operators $(X_1^1, \cdots, X_1^m)$ with law τ and which satisfies in a weak sense

$$dX_t^i = dS_t^i + K_t^i(X) dt, \quad 1 \leq i \leq m \tag{3.2}$$

where K_t belongs to the von Neumann algebra generated by $(X_u, u \leq t)$ and S is a m-dimensional free Brownian motion (i.e $(S^1, \cdots, S^m)$ are m free Brownian motions). More precisely, assume to simplify that K_t is uniformly bounded so that the solution to (3.2) is uniformly bounded (note here that S_t is uniformly bounded since the semi-circular law is compactly supported). Let for $s \in [0,1]$, $\widetilde{\phi}_s$ be the law of $(X_{u\wedge s} + S_{u-s\vee 0}, u \in [0,1])$. Then (3.2) is satisfied in a weak sense iff for all $t \in [0,1]$, all polynomial functions $P = Q(X_{t_1}, \cdots, X_{t_n})$ on cylinders

$$\widetilde{\phi}^t(P) - \sigma(P) = \int_0^t \phi\left(\widetilde{\phi}^s(\nabla_s P|\mathcal{B}_s) K_s\right) ds \tag{3.3}$$

where σ is the law of a free Brownian motion and $\widetilde{\phi}^s(.|\mathcal{B}_s)$ denotes the orthogonal projection in $L^2(\phi)$ on the σ-algebra $\mathcal{B}_s = \sigma(X_u^i, 1 \leq i \leq m, u \leq s)$. ∇_s is the Malliavin operator

$$\nabla_s^l(x_{t_1}^{i_1} \dots x_{t_n}^{i_n}) = \sum_{p=1}^n 1_{i_p = l} x_{t_{p+1}}^{i_{p+1}} \dots x_{t_n}^{i_n} x_{t_1}^{i_1} \dots x_{t_{p-1}}^{i_{p-1}} 1_{[0,t_p]}(s)$$

It was shown in [3] that for nice $K_.$, (3.3) is a strong solution of (3.2).

χ^{**} is defined similarly but the infimum is restricted to processes for which the drift K is sufficiently smooth. It was shown in [3] that the infimum in the definition (3.1) of χ^* is taken at the distribution of a free Brownian bridge $(tX + (1-t)\int_0^t (1-u)^{-1} dS_u, t \in [0,1])$, where $X = (X_1, \cdots, X_m)$ has law τ and $S = (S_1, \cdots, S_m)$ is a m-dimensional free Brownian motion, free with X. Plugging this fact into the definition (3.1) of χ^* yields the initial definition of Voiculescu.

Classical setting : *it can be seen that S^* can be defined similarly by replacing $(S_t, t \geq 0)$ by a standard Brownian motion.*

The so-called unification problem (c.f Voiculescu [26]) is to prove that the lim sup can be replaced by a lim inf in the definition of χ and $\chi(\tau) = \chi^*(\tau)$ at least for τ such that $\chi(\tau) > -\infty$.

It seems that this problem is related with a better understanding of analysis of non-commutative functions and related to Connes's question. In fact, to show that $\chi^{**}(\tau) = \chi^*(\tau) = \chi(\tau)$, we would like to show that processes with smooth fields are dense, i.e that for any law of non-commutative processes ϕ, there exists a sequence (ϕ^ϵ) associated with a smooth field K^ϵ by (3.2) such that

$$\lim_{\epsilon \to 0} \phi^\epsilon = \phi$$

and

$$\lim_{\epsilon \to 0} \int_0^1 \phi^\epsilon(|K_t^\epsilon(X)|^2) dt = \int_0^1 \phi(|K_t(X)|^2) dt.$$

But because K^ϵ is smooth, $dX_t^\epsilon = dS_t + K_t^\epsilon(X^\epsilon)dt$ as a unique solution $X_t^\epsilon = F_t^\epsilon(S_s, s \leq t)$ with a smooth function F^ϵ as can be checked again by Picard argument.

Moreover, if H^N is an m-N-dimensional Hermitian Brownian (that is a $N \times N$ Hermitian matrix with Brownian motion entries) the asymptotic freeness of independent Wigner's matrices together with Wigner's convergence [27] imply that

$$\hat{\mu}^N_{H_t^N, t \geq 0} \Rightarrow \tau_{S_t, t \geq 0}.$$

Consequently, since F^ϵ are smooth,

$$\phi^\epsilon = \lim_{N \to \infty} \hat{\mu}^N_{A_1^N, \cdots, A_m^N} \quad \text{with } A_t^N = F_t^\epsilon(H_s^N, s \leq t) \in \mathcal{H}_N^m, t \in [0,1].$$

Thus, ϕ^ϵ can be approximated by non-commutative distribution of finite matrices and hence ϕ.

When $m = 1$, such a program was realized by taking for ϕ^ϵ the law obtained by convoluting ϕ by small Cauchy laws (c.f Zeitouni and myself [14]) but the generalization of this strategy to $m \geq 2$ fails on crucial analytic questions which are not yet understood in the non-commutative context.

The entropy dimension was defined by Voiculescu [20] for $\tau = \tau_{X_1, \cdots, X_m} \in \mathcal{M}^m$, if $S_1, \cdots, S_m$ are free semicircular variables, free with X, by

$$\delta(\tau) = m + \limsup_{\epsilon \downarrow 0} \frac{\chi(\tau_{X_1 + \epsilon S_1, \cdots, X_m + \epsilon S_m})}{|\log \epsilon|}.$$

It satisfies the following property

Property 3.1 *(a) $\delta(\tau_{X_1, \cdots, X_m}) = \sum_{i=1}^m \delta(\tau_{X_i})$ if $X_1, \cdots X_m$ are free (c.f. [20]).*

(b) $\chi(\tau) > -\infty$ implies $\delta(\tau) = m$ (c.f. [20]).

(c) If $m = 1$, $\mu \in \mathcal{P}(\mathbb{R})$, Voiculescu [20] proved that

$$\delta(\mu) = 1 - \sum_{t \in \mathbb{R}} \mu(\{t\})^2.$$

(d) By [3],

$$\delta^{**}(\tau) \leq \delta(\tau) \leq \delta^*(\tau)$$

where δ^* and δ^{**} are defined as δ but with χ^* (resp. χ^{**}) instead of χ.

Note that by (c), we see that when $m = 1$, δ counts the number of atoms which existence is crucial in isomorphism questions in the commutative setting.

Recent work of Connes, Shlyakhtenko [8] tried to define another invariant for von Neumann algebras.

They generalized the notion of L^2-homology and L^2-Betti numbers for a tracial von Neumann algebra, motivated by the measure-equivalence invariance of the group theoretical L^2-Betti numbers proved by Gaboriau [11]. They define an L^2-Betti number Δ.

They can show it is related to δ^*, and thus to δ by [3], by

$$\delta(\tau) \leq \delta^*(\tau) \leq \Delta(\tau).$$

Mineyev, Shlyakhtenko [15] proved that in the case of a finitely generated group

$$\delta^*(\tau) = \beta_1(G) - \beta_0(G) + 1$$

with the group L^2 Betti-numbers β.

Yet the question of the invariance of δ, δ^*, Δ is still open.

Another attempt was done by Haagerup et al. to try to prove that δ is NOT an invariant.

A good candidate for a counterexample was a priori the so-called *DT* operator T which is obtained as the limit of upper triangular matrices with i.i.d Gaussian variables above the diagonal.

The idea is that the circular operator, limit of square matrices with i.i.d Gaussian entries is such that $C = T + \widetilde{T}^*$ where $T, \widetilde{T}$ are free. It is known to generate a two dimensional free group factor and $\delta(C) = 2$. Thus, since T is generated with half as much random variables, it could be hoped that $\delta(T) < 2$. On the other hand, it was shown by Dykema and Haagerup [10] that T is isomorphic to $L(F^2)$ so that invariance of δ would be disproved if $\delta(T)$ was strictly smaller than 2.

However, it was recently shown by Aagaard [1] that

$$\delta^*(T) = 2$$

so that DT operators do not provide a counterexample for δ^*, nor for δ if one beleives the unification problem to hold true.

4 Conclusion

Free probability allows to express in probability terms many problems from non-commutative algebras, and hence gives to probabilists a chance to use their skill in this topic. However, open questions are often in the end analytic questions: Connes question and $\chi = \chi^*$ problem could be settled if we would understand better the regularizing properties of free convolution. The important use of Gaussian random matrices in this domain also connects it with combinatorics and physics, since tracial states which satisfy Connes approximating property can be seen as limit of matrix models, which have been used in these last domains to enumerate maps (see the review [28]).

5 Appendix

5.1 About the GNS construction

This construction can be summarized as follows (c.f [16],[17]). Consider the bilinear form on $\mathbb{C}\langle X_1, \ldots X_m\rangle^2$ given by

$$< P, Q >_\mu = \mu(PQ^*).$$

We then construct a Hilbert space $(H_\mu, <.,.>_\mu)$ as follows. We consider the left ideal

$$L_\mu = \{F \in L^2(\mu) : ||F||_\mu = 0\}$$

and the quotient space $h_\mu := \mathbb{C}\langle X_1, \ldots X_m\rangle / L_\mu$. We let η_μ be the inclusion map from $\mathbb{C}\langle X_1, \ldots X_m\rangle$ into h_μ. $<.,.>_\mu$ determines a pre-Hilbert structure on h_μ and therefore the completion H_μ of h_μ by the norm $||.||_\mu =<.,.>_\mu^{\frac{1}{2}}$ is a Hilbert space. The non-commutative polynomials $\mathbb{C}\langle X_1, \ldots X_m\rangle$ act by left multiplication on H_μ. In fact, if we denote for $P, Q \in \mathbb{C}\langle X_1, \ldots X_m\rangle$ $\pi_\mu(P)\eta_\mu(Q) = \eta_\mu(PQ)$, then $\pi_\mu(P)$ extends uniquely as a bounded linear operator on H_μ since

$$||\pi_\mu(P)(\eta_\mu(Q)||_\mu^2 = \mu(PQQ^*P) \le ||QQ^*||_\infty \mu(PP^*) = ||QQ^*||_\infty ||\pi_\mu(P)||_\mu^2.$$

Moreover, one checks that $\pi_\mu(\mathbb{C}\langle X_1, \ldots X_m\rangle)$ is an involutive algebra equipped with the operator norm

$$|||\pi_\mu(P)||| = \sup_{Q \in H_\mu} ||\eta_\mu(Q)||_\mu^{-1} ||\eta_\mu(PQ)||_\mu.$$

The involution is simply given by

$$(X_{i_1} \cdots X_{i_n})^* = X_{i_n} \cdots X_{i_1}.$$

We denote $\mathcal{A}_\mu$ the von Neumann obtained by completing $\pi_\mu(\mathbb{C}\langle X_1, \ldots X_m\rangle)$ by the weak topology on H_μ. $\mathcal{A}_\mu$ is equipped with the tracial state

$$\tau_\mu(\pi_\mu(P)) =< \pi_\mu(P)\eta_\mu(I), \eta_\mu(I) >_\mu = \mu(P).$$

We then easily check that $(\mathcal{A}_\mu, \tau_\mu)$ verify (1.1). In the sense that $\mathcal{A}_\mu \subset B(H_\mu)$ where H_μ is roughly speaking the space of square integrable functions, we can think of $\mathcal{A}_\mu$ as the set of bounded measurable functions $L^\infty(\mu)$.

5.2 Proof of Lemma 2.1

This fact can be deduced from proposition 3.3.7 of [16] and the previous proof of GNS construction when one notices that (π_μ, H_μ) can be seen to be a cyclic representation of $\mathbb{C}\langle X_1, \ldots X_m\rangle$ (c.f [16], section 3.3).

Let us however summarize it. The proof uses uniqueness of GNS representations. It can be recast in the general framework of two non-commutative probability space (M, τ_M) and (N, τ_N) ; if $\tau_M = \tau_N$ then we want to show that $N \simeq M$. Indeed, if we regard $M \subset B(H_{\tau_M})$ and $N \subset B(H_{\tau_M})$ as acting via the GNS representation then one defines a unitary operator

$$U : H_{\tau_M} \to H_{\tau_N}, \; U(\eta_{\tau_M}(P(X_1, \ldots, X_n))) = \eta_{\tau_N}(P(Y_1, \ldots, Y_n)),$$

for every polynomial P. The fact that $\tau_M(P(X_1, \ldots, X_n)) = \tau_N(P(Y_1, \ldots, Y_n))$ ensures that U is well defined and isometric on a dense subspace of H_{τ_M} and maps this dense subspace onto a dense subspace of H_{τ_N}. Hence we may extend it (uniquely) to a unitary $U : H_{\tau_M} \to H_{\tau_N}$. Finally, one checks that $UX_iU^* = Y_i$ and therefore $UMU^* = N$.

$\blacksquare$

Acknowledgments I wish to thank N. Brown and D. Shlyakhtenko for many useful conversations on the topic I discussed in these notes.

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