Aging in particle systems

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1 Introduction

Statistical mechanics is devoted to the study of the thermodynamical properties of physical systems. Classical litterature on this topic often concerns their static or equilibrium properties. However, most systems in nature are not in equilibrium (see [25] for a discussion on this subject) and such a study can at best be a good approximation to reality. Even more, equilibrium can be completely irrelevant for some systems which can only be observed out of equilibrium. One can distinguish at least two classes of such systems. The first describes systems which are naturally out of equilibrium because they are submitted to a gradient of temperature, of potential etc... The second concerns systems which relax to equilibrium so slowly that the equilibrium will never be reached during the experiment or the simulation. For instance, glasses, jelly, toothpaste are example of media which, even though they seem in our everyday life much alike solids in equilibrium, still evolve on very long time scales. These systems are called glasses ; they appear when some parameter (such as temperature, pressure, etc) is changed in such a way that their relaxation time to equilibrium diverges. Such systems are very diverse and we shall later be more specifically interested in spin glasses. A canonical example of spin glasses is a metal with dilute magnetic impurities, which were shown to exhibit a rather peculiar behaviour by De Nobel and Chantenier. Such a medium can be modelled by a system of particles in random interaction or with a random external field (the randomness coming from the randomness of the distribution of the impurities in a given sample). These models are called disordered and we shall detail them later in this survey. There are many other materials that exhibit a glass phase ; let us quote some physic litterature on the glass phase of supraconductors [30], granular materials [8, 9] etc

One of the relevant properties which has been investigated recently for out of equilibrium dynamics is aging. A system is said to age if the older it gets, the longer it will take to forget its past. The age of the system is the time spent since the system reached its glass phase, which is often obtained

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by freezing it below the critical temperature. The experiment exhibiting aging is usually as follows. One considers a medium at time $t = 0$ at high temperature and freeze it at a temperature below the critical temperature T_c. One then measures a parameter $q(t_w, t_w + t)$ where t_w is the age of the system (i.e the time spent since the system was frozen in its glass phase) and $t + t_w$ the measurement time. The parameter $q(s,t)$ is often the covariance $E(X_t X_s) - E(X_t)E(X_s)$ of the observable X or the probability $P(X_t = X_s)$. Then, a system is said to age when $q(t_w, t_w + h(t_w))$ converges to a non zero constant as t_w goes to infinity, for some non trivial increasing function h. One usually observes the following. At large temperature, the system quickly equilibrates and the order parameter should rapidly become stationary ; $q(s,t) \approx q(s-t)$ for t, s reasonably large. At lower temperature, one observes usually data as represented in figure 1 ; the experimental covariances are not functions of $t - t_w$ only, but also depends on the age t_w of the system, and are therefore a more complicated function of t and t_w that one can investigate.

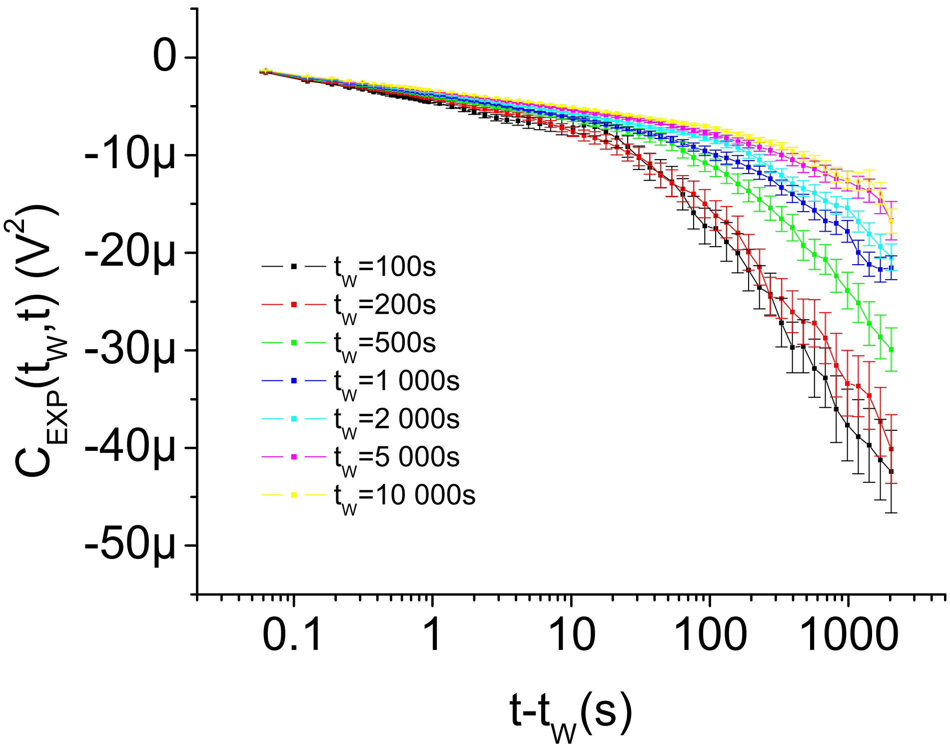

Figure 1: Experimental covariances $C_{\mathrm{EXP}}(t_w, t) - C_0 = < X(t_w)X(t) > - < X(t_w)^2 >$ in the insulating spin glass $CdCr_{1.7}In_{0.3}S_4$, mesured by D. Herisson and M.Ocio [26, 27].

For instance, it was observed in [27] that the covariance becomes approximately a function of

$\zeta = (1-\mu)$

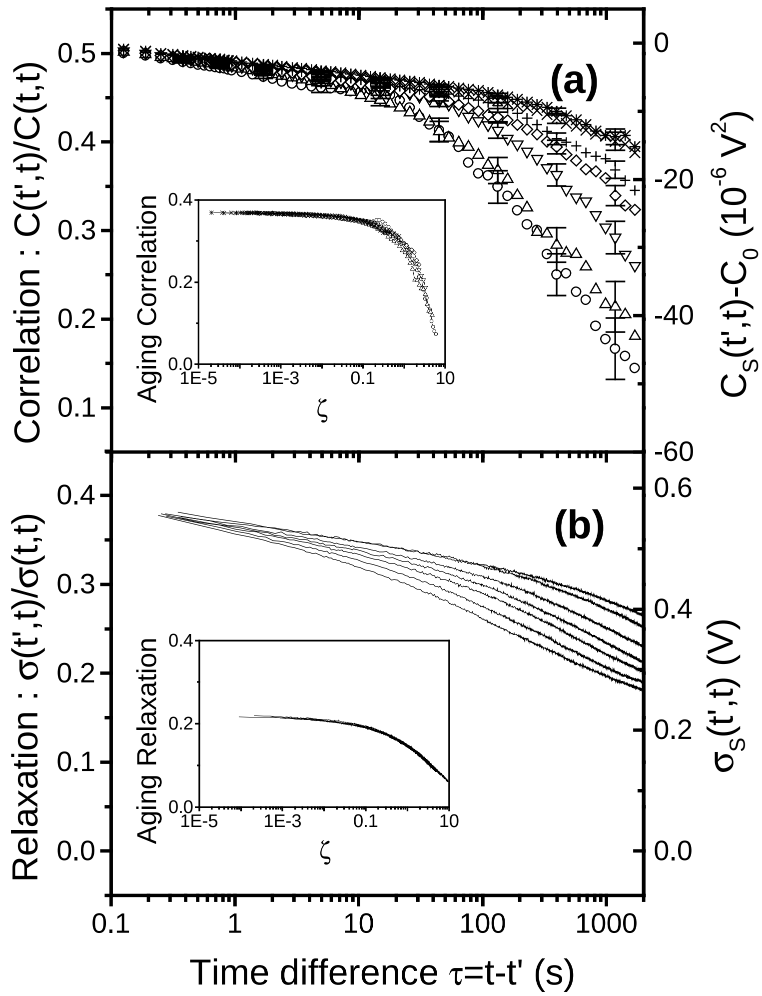

Figure 2: Experimental covariance in the insulating spin glass $CdCr_{1.7}In_{0.3}S_4$ at low temperature [26, 27].

Let us notice that the figures above are already taken in such a time scale that they do not show what happens for short times t. A more detailed study usually shows that at least two phenomena are going on ; on a short scale, when $t - t_w$ goes to infinity while t_w stays small enough, the system reaches a state where q is approximately given by a constant q_{EA} (whose value is represented by the initial flat part in the covariance diagrams above) and stays in this state quite a long time so that the systems seems to be in equilibrium and the dynamics looks stationary. However, on a longer scale, the system will undergo dramatic changes which will drive the parameter q to zero. The existence of different time scales related to slower and quicker processes is also a description of aging.

The mathematical understanding of aging has been undertaken only very recently and is still very limited. For the time being, aging phenomenon could be analysed for very few disordered models. The two main phenomenologies that have been isolated as a source of aging can be illustrated by two toys models: the so-called Bouchaud's trap model and the spherical Sherrington-Kirkpatrick model. Since these two models were introduced to understand the dynamics for the Sherrington-Kirkpatrick model of spin glass, we shall first describe this model. It is given by the quadratic Hamiltonian

$$H_J(\mathbf{x}) = \sum_{1 \le i < j \le N} J_{ij} x_i x_j$$

where $\mathbf{x} = (x_i, 1 \le i \le N)$ represent the particles or spins, which belong to a set M. M can be either discrete, for instance $M = \{-1, +1\}$ in the Ising model, or continuous, for instance $M = I\!R$ or M is a compact Riemaniann manifold such as a sphere in $I\!R^d$. The J_{ij}'s are centered independent random variables with variance N^{-1}, often assumed to be Gaussian for simplicity. If μ is a probability measure on M, a Gibbs (or equilibrium) measure for the Sherrington-Kirkpatrick model at temperature $T = \beta^{-1}$ is given by

$$\mu_N(d\mathbf{x}) = \frac{1}{Z_N} e^{\beta H_J(\mathbf{x})} \prod_{i=1}^{N} d\mu(x_i) \text{ with } Z_N = \int e^{\beta H_J(\mathbf{x})} \prod_{i=1}^{N} d\mu(x_i).$$

In the case $M = I\!R$, the associated Langevin dynamics (see section 2) were considered by Sompoliski and A. Zippelius (see [32, 29] and then by G. Ben Arous and myself [7, 21]). It is proved that the empirical measure $N^{-1} \sum_{i=1}^{N} \delta_{x^i_{[0,T]}}$ on path space converges as N goes to infinity for every time $T > 0$. Its limit is not Markovian (eventhough at finite N, its law is Markovian, it losts this property at the limit by self-averaging, the average of Markov laws being not necessarily Markovian) and given by a nonlinear equation. This limiting law is so complicated that the behaviour of its covariance could not be analysed so far, neither in the mathematics or in the physics litterature. A similar work was achieved by M. Grunwald for Ising spins and standard Glauber dynamics [20]. However, it is expected that the Langevin dynamics for Sherrington-Kirkpatrick dynamics ages and actually with infinitely many time scales.

Since this already simple model of spin glass was already to difficult to analyse, toys models were introduced to try to understand why aging could appear. Their study allowed to point out two major situations to generate aging.

(a) The first is a flat energy space; the particle system has a single ground state characterized by the lowest possible energy E_0, but there are many other states with energy E_n which is very close to E_0, more precisely $E_n - E_0$ is of the order of N^{-1} if N is the number of particles. Then, the dynamics will be likely to visit all these states in a finite time (independent of N) before finding the ground state. This process will create a long time memory of the history and aging. Hence, aging is here caused by the flatness of the bottom of attracting valley in the energy landscape, and the consequent difficulty for the system to find its most favorable state within this valley. It will in fact find it typically in a time depending on the age of the system, time after which it will begin to forget its past.

This phenomenon describes the spherical Sherrington-Kirkpatrick model, but should also describe the spherical p-spins model of Sherrington-Kirkpatrick. It is believed also that it should explain aging of the dynamics of the original Sherrington-Kirkpatrick model.

(b) The trap model; in this case, the evolution of the particle system is represented by a Markov process in a random energy landscape. The process will spend most time into deep valleys of lowest energy where it will be "trapped" and its evolution will be mostly driven by the seek of deeper valleys. The time spent in these valleys is random and aging will appear when the mean time spent in these valleys diverges.

This model was originally introduced by Bouchaud to understand aging in the Random Energy Model (REM) introduced by Derrida as a simplification of the Sherrington-Kirkpatrick model of spin glass. It was shown by G. Ben Arous, A. Bovier and V. Gayrard [2, 3] that this picture is indeed relevant. It also describes aging in the Sinaï model [17].

Note that in both cases, the main point is that the system has infinitely many favorable states which it can reach in finite time; this can be opposed to usual stationnary systems where ground states are separated by an energy barrier which diverges with the size of the system, forbidding the infinite system to visit several of them in a finite time.

2 Spherical model of spin glass

If $U : \mathbb{R} \to \mathbb{R}$ is some potential going to infinity fast enough at infinity, the Langevin dynamics at temperature $T = \beta^{-1}$ for the Sherrington-Kirkpatrick model are defined by the stochastic differential system

$$dx_t^i = -\frac{\beta}{2}\partial_{x^i} H_J^N(\underline{x}_t)dt - U'(x_t^i) + dB_t^i$$

with prescribed initial data. Here, $(B^i, 1 \le i \le N)$ are i.i.d Brownian motions.

One way to simplify considerably this system is to consider instead a smooth spherical constraint

$$dx_t^i = -\frac{\beta}{2}\partial_{x^i} H_J^N(\underline{x}_t)dt + U'(\frac{1}{N}\sum_{j=1}^{N}(x_t^j)^2)x_t^i + dB_t^i \tag{2.1}$$

with a function U on $\mathbb{R}^+$ such that

$$\limsup_{x\to\infty} \frac{U(x)}{x} = +\infty$$

in order to insure the almost sure boundedness of the empirical covariance under the dynamics (2.1). A hard spherical constraint was considered in [14] where a similar study was undertaken.

The great simplification offered by the spherical model is that the empirical covariance

$$K_N(s,t) = \frac{1}{N}\sum_{i=1}^N x_s^i x_t^i, \tag{2.2}$$

satisfies, at the large N limit, an autonomous equation. Indeed, one computes

$$K_N(t,s) = \frac{1}{N}\mathrm{tr}\left((e^{-\int_0^t \mathbf{V_N}(u)du}x_0 + \int_0^t e^{-\int_v^t \mathbf{V_N}(u)du}dB_v)(e^{-\int_0^s \mathbf{V_N}(u)du}x_0 + \int_0^s e^{-\int_v^s \mathbf{V_N}(u)du}dB_v)\right)$$

with $\mathbf{V_N}(u)$ the $N\times N$ matrix given by $\mathbf{V_N}(u) := U'(K_N(u,u))\mathbf{I} - \beta\mathbf{J}$ if $\mathbf{J}$ is the symmetric matrix with entries $\{J_{ij}, 1\le i\le j\le N\}$ above the diagonal. From this formula, it is easily seen that the long time behaviour of the covariance will be driven by the largest eigenvalues of the matrix $\mathbf{J}$. The eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ of the Wigner matrix $\mathbf{J}$ are well known ; λ_1 converges almost surely towards 2, but the difference of the next eigenvalues with λ_1 are of order N^{-1} so that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i} = \sigma \qquad \text{a.s.}$$

with σ the semi-circle law $\sigma(dx) = C\sqrt{4-x^2}dx$, which is absolutely continuous w.r.t Lebesgue measure, in particular in the neighborhood of 2. From this asymptotic, one deduces that if the $(x_0^i, 1\le i\le N)$ are independent equidistributed variables with law μ_0 (which corresponds to an infinite temperature initial condition), K_N converges almost surely towards K, solution of the renewal equation

$$K(t,s) = R_\beta(t)R_\beta(s)\mathcal{L}(\beta(t+s))\int x^2 d\mu_0(x) + \int_0^s \frac{R_\beta(t)R_\beta(s)}{R_\beta(v)^2}\mathcal{L}(\beta(t+s-2v))dv \tag{2.3}$$

with, for $\theta > 0$ and $t \ge 0$,

$$\begin{aligned}\mathcal{L}(\theta) &= \int e^{\theta\lambda}d\sigma(\lambda)\\ R_\beta(t) &= e^{-\int_0^t U'(K(s,s))ds}\end{aligned}$$

One can analyze this equation when

$$U(x) = \frac{c}{2}x^2$$

for some $c>0$. We find that for the solution of (2.3), if we let

$$\beta_c = \frac{c}{2\lambda^*}\int \frac{1}{\lambda^*-\lambda}d\sigma(\lambda). \tag{2.4}$$

and assume that

$$\sigma(\lambda^*-\lambda\le\theta) \simeq_{\theta\to 0} b_1\theta^q \tag{2.5}$$

for some $q>1$ and $b_1>0$ (and hence $\beta_c<\infty$), then, the unique solution K to (2.3) satisfies

1) For $\beta < \beta_c$, there exists $\delta_\beta > 0$ and $c_\beta \in I\!R^+$ so that for all $t, s \in I\!R^+$,

$$|K(t,s)| \le c_\beta e^{-\delta_\beta |t-s|}. \tag{2.6}$$

2) For $\beta = \beta_c$, $q \neq 2$, $t \gg s \gg 1$, we have the polynomial decay

$$K(t,s) \sim \begin{cases} (t-s)^{1-q} & \text{for } t/s \text{ bounded} \\ \frac{s^{1-\psi_q/2}}{t^{q-\psi_q/2}} & \text{otherwise ,} \end{cases} \tag{2.7}$$

where $\psi_q = \max(2-q, 0)$.

3) When $\beta > \beta_c$ we get that

$$K(t,s) \sim (s/t)^{q/2} , \tag{2.8}$$

so $K(t,s) \to 0$ if and only if $t/s \to \infty$.

Note that in the case where σ is the semi-circle appearing in the asymptotics of the spectral measure of $\mathbf{J}$, $q = \frac{3}{2}$. Hence, we see that aging appears for $\beta > \beta_c$ when the particles are initially independent. When starting from the top eigenvector, this phenomenon disappears (the system stays in the basin of attraction of the top eigenvector) ; in fact, for any fixed $\beta > \beta_c$, regardless of the way in which $t-s$ and s approach infinity,

$$K(t,s) \approx (\frac{2\beta^2}{c}\lambda^* - \int \frac{1}{\lambda^* - \lambda} d\sigma(\lambda))/(2\beta) > 0$$

There is thus no aging regime for this initial condition, which underlines the fact that aging phenomenon is very dependent on the initial conditions.

Note here that two factors were crucial to prove aging ; the flatness of the energy landscape near the ground state but also the fact that the interaction between the particles results with a nonlinear equation for the covariance (indeed, without this nonlinearity, it could be checked that the covariance would be asymptotically stationnary [22]). In fact, the randomness of the matrix $\mathbf{J}$ is not necessary, provided its eigenvalues distribution (which could be deterministic) is sufficiently flat next to the maximum eigenvalue.

On a technical point of view, it was crucial that the covariance C satisfies an autonomous equation. It was pointed out by L. Cugliandolo and J. Kurchan [15] that an autonomous system of equations could be obtained for the covariance and the so-called response function for p spherical models, leading to an analysis of aging phenomenon for these systems. In particular, they believe that in some cases, these models lead to more than two different time scales. I recently derived rigorously with G. Ben Arous and A. Dembo the same system of equations, but we have not yet achieved its long time analysis.

3 Bouchaud's trap model; an energy trap model

Bouchaud's random walk is a simple model of a random walk trapped by random wells. It was proposed as an approximation of the evolution of a more complex system in an energy lanscape with favorable valleys, located at sites given by a discrete set V, and with energies $\{E_x, x \in V\}$.

Let $G = (V, B)$ be a graph described by its set of vertices V and its bonds B. Two vertices are said to be neighbours if they are related by a bond.

Bouchaud's simplest random walk X is a Markov process who jumps from a site x to its neighbours $y : (x, y) \in B$ with a rate

$$w_{x,y} = e^{-\beta E_x}$$

and $w_{x,y} = 0$ if (x, y) are not neighbours. The $\{E_x, x \in V\}$ are independent random variables with exponential law ;

$$P(E_x > t) = e^{-t}, \qquad \forall t > 0.$$

Let $P^{\mathbf{E}}$ denote the quenched law of the Markov chain X (i.e given a realization of the energies $\mathbf{E} = \{E_x, x \in V\}$) and P its annealed law (i.e is the average over the randomness of the energies of the P^E's ; $P =< P^E >$).

The natural order parameters to consider here are either the two times probability

$$R^E(t_w, t_w + t) = P^E(X(t_w) = X(t_w + t))$$

or its annealed version

$$R(t_w, t_w + t) = P(X(t_w) = X(t_w + t)),$$

or can be the probability that the process did not jump between time t_w and time $t_w + t$;

$$\Pi^E(t_w, t_w + t) = P^E(X(t_w) = X(t_w + s); s < t), \quad \Pi(t_w, t_w + t) = P(X(t_w) = X(t_w + s); s < t).$$

Aging for such a model was first studied in the mathematic litterature by Fontes, Isopi and Newman [18] in the case where $V = \mathbb{Z}$. They proved that, when $\beta > 1$,

$$\lim_{t_w \to \infty} R(t_w, (1 + \theta)t_w) = f(\theta)$$

with a well-defined function f, showing an aging regime in the scale of the age of the system. On the other hand, it was shown (see [4]) that Π satisfies

$$\lim_{t_w \to \infty} \Pi(t_w, t_w + \theta t_w^\gamma) = q(\theta)$$

with a well defined function q and $\gamma = (1 + \beta)^{-1}$.

Combining these two results shows that the process will be able to quit a deep trap in a time of order t_w^γ but will not find a deeper trap before a time of order t_w.

4 Sinaï model

Bouchaud's trap model on $\mathbb{Z}$ describes also the long time behaviour of Sinaï 's random walk in random environment ; it is described as follows. Let $\mathbf{p} = (p_i, i \in \mathbb{Z}) \in [0, 1]^{\mathbb{Z}}$ be independent equidistributed variables with law μ. Sinai's Markov chain $X^{\mathbf{p}}$ is then given by

$$P\left(X_{n+1}^{\mathbf{p}} = i + 1 | X_n^{\mathbf{p}} = i\right) = 1 - P\left(X_{n+1}^{\mathbf{p}} = i - 1 | X_n^{\mathbf{p}} = i\right) = p_i, \;\; P\left(X_0^{\mathbf{p}} = 0\right) = 1.$$

Let $\rho_i := \frac{1-p_i}{p_i}$ and assume that $\mathbb{E}[\log \rho_0] = \int \log(x^{-1}-1)d\mu(x)$ is well defined. It is well known that if $\mathbb{E}[\log \rho_0] \neq 0$, the Markov chain is transient and will go to infinity when time goes to infinity. When $\mathbb{E}[\log \rho_0] = 0$, Sinaï [31] proved that the Markov chain $X^{\mathbf{P}}$, correctly renormalized, converges almost surely towards the deepest valley designed by the random environment that it could visit. More precisely, if we let

$$W^n(t) = \frac{1}{\log\, n} \sum_{i=0}^{\lfloor (\log n)^2 t \rfloor} \log \rho_i \cdot (\text{sign } t),$$

W^n will converge towards a Brownian motion W on $\mathbb{R}$. Then, the random walk $X^{\mathbf{P}}$, once divided by $(\log n)^2$, will converge towards the nearest point to the origin which corresponds to a well of depth greater or equal to one designed by W as shown in figure 3.

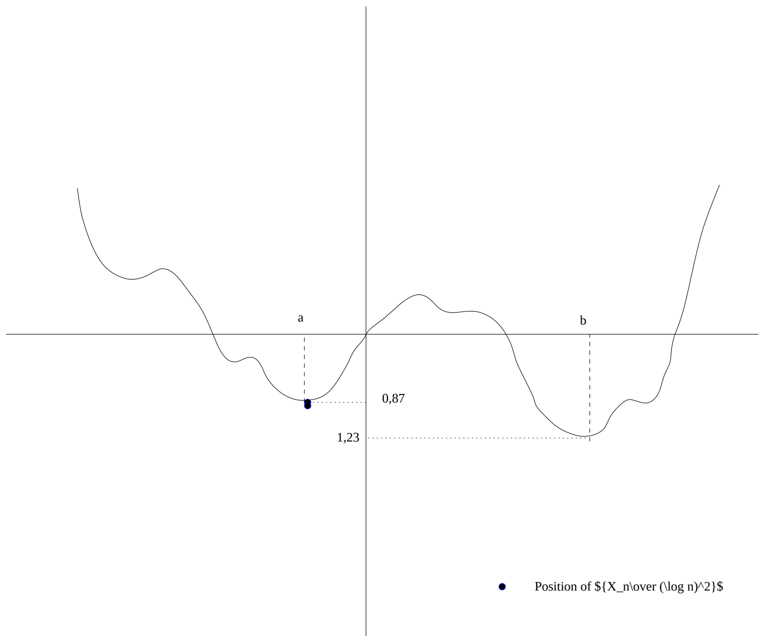

Figure 3: If $x_a > 1$, then $(\log n)^{-2} X^{\mathbf{P}}(n)$ converges to a, if $x_a < 1, x_b > 1$, it converges towards b etc.

The aging phenomenon will then also occur since when time is going on, the random walk will have found better and better attractors and will therefore like to stay longer there ; it was indeed shown (see [28, 19, 17]) that for any $h > 1$,

$$\lim_{\eta \to 0} \lim_{n \to \infty} P\left(\frac{|X_{n^h} - X_n|}{(\log n)^2} < \eta \right) = \frac{1}{h^2} \left[\frac{5}{3} - \frac{2}{3} e^{-(h-1)} \right] \tag{4.1}$$

5 Bouchaud's trap model on a complete graph

Generalisation of Bouchaud's model can be found in [4, 5], where jump rates depending not only on the site where the walk stands but also from the energy of the site where it wants to jump, higher dimension models are considered as well.

Bouchaud's random walk on a complete graph is also of interest since it is related with Derrida's random energy model. If G is the complete graph on M points, we denote $\Pi_M(t_w, t_w+t)$ the annealed probability that the walk stays in a given well during time t, then it was shown (see [10, 2, 3]) that Π_M converges as M goes to infinity. Moreover, its limit Π satisfies

$$\lim_{t_w\to\infty} \Pi(t_w, t_w+\theta t_w) = H(\theta)$$

with

$$H(\theta) = (\pi \mathrm{cosec}(\pi/\beta))^{-1} \int_\theta^\infty (1+x)^{-1} x^{-\beta} dx.$$

6 Aging for the Random Energy Model

Let us finally describe the Random Energy Model (**REM**) introduced by Derrida ; noticing that for any given $\mathbf{x}$, the Hamiltonian $H_J(\mathbf{x})$ for the Sherrington-Kirkpatrick model is a centered Gaussian variable and thinking that the $\mathbf{x}$ are Ising spins taking values $+1$ or -1, Derrida considered the Gibbs measure on $\{-1,+1\}^N$ given, for $\sigma \in \{-1,+1\}^N$, by

$$\mu_{\beta,N}(\sigma) = \frac{1}{Z_{\beta,N}} e^{\beta\sqrt{N}E_\sigma} \quad \text{with} \quad Z_{\beta,N} = \sum_{\sigma\in\{-1,+1\}^N} e^{\beta\sqrt{N}E_\sigma}.$$

Here, $\{E_\sigma, \sigma \in \{-1,+1\}^N\}$ are independent centered Gaussian variables with variance one, the independence hypothesis resulting with a great simplification with respect to the original Sherrington-Kirkpatrick model. A standard Glauber dynamic for this model is given by the transition kernel $p(\sigma,\eta)$ on $\{-1,+1\}^N$ which is null if σ and η differ at more than one site, given by $N^{-1}e^{-\beta\sqrt{N}E_\sigma^+}$ if σ and η only differ by a spin-flip, and $1-e^{-\beta\sqrt{N}E_\sigma^+}$ if $\sigma=\eta$. Then, it was shown in [2, 3] that the motion of these dynamics when seen only on the deepest traps created by the energies $\{E_\sigma, \sigma\in\{-1,+1\}^N\}$ will be described by Bouchaud's random walk on a complete graph of large number of vertices. In fact, with a well chosen threshold $u_N(E) \approx \sqrt{2N\log(2)} + \frac{E}{\sqrt{2N\log(2)}}$ and a natural scaling $c_{N,E} \approx e^{\beta\sqrt{N}u_N(E)}$, they proved that

$$\lim_{t_w\to\infty} \lim_{E\to-\infty} \lim_{N\to\infty} P\left(\left|\frac{\Pi_N(c_{N,E}t_w, c_{N,E}(t_w+t))}{H(tt_w^{-1})} - 1\right| > \epsilon \right) = 0$$

for any $\epsilon > 0$.

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