Chapter 21

Wigner matrices

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Abstract

This is a brief survey of some of the important results in the study of the eigenvalues and the eigenvectors of Wigner random matrices, i.e. random hermitian (or real symmetric) matrices with i.i.d entries. We review briefly the known universality results, which show how much the behavior of the spectrum is insensitive to the distribution of the entries.

21.1 Introduction

In the fifties, Wigner introduced a very simple model of random matrices to approximate generic self-adjoint operators. It is given as follows. Consider a family of independent, zero mean, real or complex valued random variables $\{Z_{i,j}\}_{1\leq i<j}$, independent from a family $\{Y_i\}_{1\leq i}$ of i.i.d centered real-valued random variables. Consider the (real-symmetric or hermitian) $N \times N$ matrix X_N with entries

$$X_N(j,i) = \bar{X}_N(i,j) = \begin{cases} Z_{i,j}\,, & \text{if } i<j, \\ Y_i\,, & \text{if } i=j. \end{cases} \tag{21.1.1}$$

We call such a matrix a *Wigner matrix*, and if the random variables $Z_{i,j}$ and Y_i are Gaussian, we use the term *Gaussian Wigner matrix*. The case of Gaussian Wigner matrices in which $EY_1^2 = 2$ and $E|Z_{1,2}|^2 = 1$ is of particular importance, since their law is invariant under the action of the orthogonal, (resp. unitary) group if the entries are real (resp. complex), see e.g. Chapter 3 in this handbook. In the Gaussian case, the distribution of the matrix is invariant by the action of the natural symmetry group $SO(N)$ (respectively $SU(N)$), the eigenvalues of the matrix X_N are independent of the eigenvectors which are Haar distributed. The joint distribution $P_N^{(\beta)}$ of the eigenvalues $\lambda_1(X) \leq \cdots \leq \lambda_N(X)$ is given by

$$P_N^{(\beta)}(dx_1, \ldots, dx_N) = \bar{C}_N^{(\beta)} \mathbf{1}_{x_1 \leq \cdots \leq x_N} |\Delta(x)|^\beta \prod_{i=1}^N e^{-\beta x_i^2/4} dx_1 \cdots dx_N \,, \tag{21.1.2}$$

where

$$\bar{C}_N^{(\beta)} = \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta(x)|^\beta \prod_{i=1}^N e^{-\beta x_i^2/4} dx_i \right)^{-1} .$$

In this formula, $\beta = 1$ (resp. 2) if the entries are real (resp. complex). The distributions of the eigenvalues of these random matrices are usually called the Gaussian orthogonal ensembles (GOE) or the Gaussian Unitary Ensemble (GUE) respectively. These and more general invariant ensembles are the main focus of the other chapters of this handbook. The present chapter is devoted to the study of general non-invariant Wigner ensembles and of their universal properties, that is the properties they share with the invariant Gaussian ensembles, at the global and at the local levels, in the bulk and at the edge of the spectrum. This is a very wide domain of research, impossible to summarize with any depth in the format of this review article. The study of the properties of the spectrum of Wigner ensembles has a very long and rich history, and a few recent expository works cover the results and their proof in much greater depth than can be achieved here ([Bai99], [And09]). Compounding the difficulty is the fact that the field has seen recently a burst of very important results. Indeed, the question of the universality in the bulk at the local level for the spectrum of Wigner matrices, which had long been seen as one of the central open problems in Random Matrix Theory, has been solved recently by two groups of mathematicians (L.Erdos, J.Ramirez, B.Schlein, S.Peche, H.T. Yau and T.Tao and V.Vu). These new universality results open a very exciting period for the field of Random Matrix Theory (see the Bourbaki seminar [Gui10]). We will try to cover succinctly those new developments, without any claim for completeness given the rapid pace of recent progress. We will give some of the universality results as well as some results known for the Gaussian ensembles which have not yet been proved to be universal, i.e. the open questions left after the recent wave of progress. We will also give some of the known limits of universality, i.e. some of the cases where it is known that the results differ from the Gaussian ensembles. Before proceeding let us comment very briefly and informally about the methods of approach for general non-invariant Wigner ensembles. In fact there are very few such possible approaches, essentially three. Let us note first that the main tool used for invariant ensembles, i.e. the explicit computation of the distribution of the spectrum, is of course not possible for the non-invariant ensembles. The first general approach for non-invariant Wigner ensembles is the moment method, which consists in computing the asymptotic behavior of the moments of the spectral measure L_N, that is the empirical measure of the renormalized eigenvalues of X_N, or equivalently the normalized trace of powers of the random matrix

$$\int x^k dL_N(x) = \frac{1}{N^{\frac{k}{2}+1}} \mathrm{Tr}[X_N^k]$$

The second one, the resolvent method, consists in computing the asymptotic behavior of the normalized trace of the resolvent, i.e the Stieltjes transform of the spectral measure

$$\int \frac{1}{z-x} dL_N(x) = \frac{1}{N} \mathrm{Tr}[(zId - N^{-\frac{1}{2}} X_N)^{-1}].$$

For a long time these two methods were basically the only ones available. The survey [Bai99] shows how far one can go using these tools. Another approach to universality is based on explicit formulas and concerns a special case of non-invariant ensembles, the Gaussian divisible case. This is the case of random Wigner matrices, where the distribution of the entries is the convolution of an arbitrary distribution and of a centered Gaussian one. These matrices can be written as

$$X_N = \sqrt{\varepsilon} G_N + \sqrt{1-\varepsilon} V_N \tag{21.1.3}$$

with a matrix G_N taken from the GOE or the GUE, independent from a self-adjoint matrix V_N, and some $\varepsilon \in (0,1)$. Note that when V_N is a Wigner matrix, so is X_N. K.Johansson [Joh01] studied such matrices when X_N is taken from the GUE based on rather explicit formulas for the joint law of the eigenvalues of X_N. Another point of view is based on the fact that the spectrum of such matrices is described by the Dyson Brownian motion, see e.g. [And09, Theorem 4.3.2], at time $t = -\log \varepsilon$ which is a weak solution to the system

$$d\lambda_i^N(t) = \frac{\sqrt{2}}{\sqrt{\beta N}} dW_i(t) + \frac{1}{N} \sum_{j:j\neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} dt\,, \quad i = 1, \ldots, N\,, \tag{21.1.4}$$

with initial condition given by the eigenvalues of V_N and $W_i, 1 \leq i \leq N$ independent Brownian motions. This fourth perspective, based on Dyson's Brownian motion, has been considerably strengthened and proved to be very useful for far more general matrices, in the recent work of the group around Erdos, Schlein and Yau.

Universality can also be proved by using approximation arguments from the above models, see e.g. [R06, Tao09a, Tao09b].

21.2 Global properties

In this section, we describe the global properties of the spectrum of Wigner matrices. It turns out that, when the entries have a finite second moment, the eigenvalues of X_N are of order $\sqrt{N}$ with overwhelming probability, and the global properties of the spectrum of X_N shall be described by the spectral measure

$$L_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\frac{\lambda_i^N(X)}{\sqrt{N}}}$$

which is a probability measure on the real line. Note that in particular, for any $a < b$

$$L_N([a,b]) = \frac{1}{N}\sharp\{i : \lambda_i^N(X) \in \sqrt{N}[a,b]\}$$

is the proportion of normalized eigenvalues falling in the interval $[a,b]$.

The first result of RMT is of course Wigner's theorem which says that L_N converges towards the semi-circle law. We will give some of the known results about the fluctuations around this limit. We will also mention concentration results, and see that the law of L_N concentrates under fairly general hypothesis. One first result which seems out of reach for general Wigner ensembles concerns the large deviations of the spectral measure. Whereas in the case of the Gaussian ensembles, one knows a full large deviations principle, it seems impossible at this time to get such a result for any non-invariant ensemble. We will recall this large deviations principle as well as the moderate deviations principle for Gaussian ensembles. A moderate deviation has been given for some non-invariant Gaussian divisible ensembles.

Recently, it was observed [Erd08] that the convergence of the spectral measure towards the semi-circle law holds in a very local sense, that is can be obtained on intervals with width going to zero just more slowly than the typical spacing between eigenvalues. This result thus interpolates between global and local properties of the spectrum. Moreover, this local convergence is intimately related with the fact that the eigenvectors are *delocalized* in the sense that all of their entries are at most of order $(\log N)^4/\sqrt{N}$ with overwhelming probability.

Finally, in section 21.2.5, we will describe the global behavior of the spectrum when the entries do no have a finite second moment but have heavy tails, provided they belong to the domain of attraction of an α-stable law.

21.2.1 Convergence to the semi-circle law

The main result of this section goes back to Wigner [Wig55] and gives the weak convergence of L_N towards the semi-circle law σ given by

$$d\sigma(x) = \frac{1}{2\pi}\mathbf{1}_{[-2,2]}\sqrt{4-x^2}dx\,.$$

Theorem 21.2.1 *Assume $E[|Z_{1,2}|^2] = 1$ and $E[Y_1^2] < \infty$. Then, for any continuous function f with polynomial growth,*

$$\lim_{N\to\infty} \int f(x)dL_N(x) = \int f(x)d\sigma(x)\, a.s.$$

This theorem was originally proved for polynomial functions and the convergence in probability, under the condition that the entries have all their moments finite (see [Wig55] or [And09, Theorem 2.1.1]). However, almost sure convergence can be obtained by using Borel-Cantelli Lemma and fine estimates of the covariances [And09, Exercise 2.1.16]. Finally, the assumptions that all moments are finite can be removed by approximation using Hoffman-Wielandt inequality [And09, Lemma 2.1.19].

The next question is about speed of convergence of the spectral measure to the semi-circle law. It is not universal, and should depend on the law of the entries and the metric used to measure it. In [Bai93, Theorem 4.1], it was shown that

$$\sup_{x\in\mathbb{R}} |E[L_N((-\infty, x])] - \sigma([-2, x \vee -2])| = O(N^{-\frac{1}{4}})$$

under the assumption that the entries have a finite fourth moment. When the sixth moment is finite, it was shown [Bai09, Lemma 6.1] that the speed is at most of order $O(N^{-\frac{1}{2}})$. For complex Gaussian divisible ensembles, it was proved to be at most of order $O(N^{-\nu})$ for all $\nu < 2/3$ in [Got07, Theorem 1.2]. The conjecture is that the optimal speed is $O(N^{-1})$, as proved for the GUE case in [Got05]. A similar question may be asked for the expectation of the distance between L_N and σ. We refer to [Cha04] and references therein.

21.2.2 Fluctuations around the semi-circle law

The standard fluctuations results of the spectral measure for invariant Wigner ensembles states that for any smooth enough function f the law of

$$\delta_N(f) = N(\int f(x)dL_N(x) - \int f(x)d\sigma(x))$$

converges towards a Gaussian variable (cf [Joh98] for the Gaussian ensembles). Note here the intriguing speed $1/N$, to be compared with the classical $1/\sqrt{N}$ speed of the classical central limit theorem. Such a result for non-invariant ensembles was first proved in the slightly different Wishart matrix model case by Jonsson [Jon82] for polynomial functions f. We refer the reader to [And09, Theorem 2.1.31] for similar techniques for Wigner matrices under the hypothesis that the entries have all their moments finite and the centering is done with respect to the mean instead of the limit. This last result was extended in

[Lyt09] under only the finite fourth moment condition. An interesting point is that the covariance of the limiting Gaussian variable depends on the fourth moment of the distribution of the entries. The fluctuations result for $\delta_N(f)$ was generalized, under the assumption of a finite sixth moment, in [Bai09, Theorem 1.1] for functions f with four continuous derivatives. An interesting feature of [Bai09] is a generalization of the result of [Joh98] for the GOE which shows that the limiting Gaussian variable is not centered in general in the real case.

Such central limit theorems hold for many models of random matrices, see e.g. unitary matrices following the Haar measure [Dia94].

21.2.3 Deviations and Concentration properties

Under stronger conditions about the distribution of the entries, more can be said about the convergence of the spectral measure. One can prove concentration results. Indeed, it turns out that the evaluation of the spectral measure along a smooth function is a smooth function of the entries. Therefore, the theory of concentration of measure applies to such random variables. For instance, we have the following concentration of measure property. To begin with, recall that a probability measure P on $\mathbb{R}$ is said to satisfy the *logarithmic Sobolev inequality* (LSI) with constant c if, for any differentiable function f,

$$\int f^2 \log \frac{f^2}{\int f^2 dP} dP \le 2c \int |f'|^2 dP.$$

When the distribution of the entries satisfy log-Sobolev inequality, not only the spectral measure concentrates but also the eigenvalues themselves. More precisely we have the following statement.

Theorem 21.2.2 *Suppose that the laws of the independent entries $\{X_N(i,j)\}_{1\le i\le j\le N}$ all satisfy the (LSI) with constant c/N. Then, for any Lipschitz function f on $\mathbb{R}$, for any $\delta > 0$,*

$$P\left(|\int f(x)dL_N(x) - E[\int f(x)dL_N(x)]| \ge \delta\right) \le 2e^{-\frac{1}{4c|f|_{\mathcal{L}}^2}N^2\delta^2}. \quad (21.2.1)$$

Further, for any $k \in \{1, \ldots, N\}$,

$$P\left(|f(N^{-\frac{1}{2}}\lambda_k(X_N)) - Ef(N^{-\frac{1}{2}}\lambda_k(X_N))| \ge \delta\right) \le 2e^{-\frac{1}{4c|f|_{\mathcal{L}}^2}N\delta^2}. \quad (21.2.2)$$

These results can also be generalized to the case when the distribution of the entries satisfy a Poincaré inequality or are simply bounded (but then the test function f has to be convex). We refer to [Gui00] or [And09] for precise statements and generalizations. Interestingly, concentration inequalities hold under the mere assumption of independence (in fact only of the vectors $((X_N(ij))_{j\le i})_{1\le j\le N}$), but with a worst speed estimate [Bord10].

Theorem 21.2.3 *Assume f has finite variation norm*

$$\|f\|_{TV} := \sup_{\substack{k\in\mathbb{N} \\ x_0<x_1<x_2\cdots<x_k}} \sum_{\ell=1}^{k} |f(x_\ell) - f(x_{\ell-1})|.$$

Then, for any $\delta > 0$,

$$P\left(|\int f(x)dL_N(x) - E[\int f(x)dL_N(x)]| \geq \delta \|f\|_{TV} \right) \leq 2e^{-\frac{1}{2}N\delta^2}. \quad (21.2.3)$$

The proof follows from martingale inequalities.

The advantage of concentration inequalities is that they hold for any fixed N. However, they do not provide in general the optimal asymptotic speed. Such optimal constants are provided by moderate deviations principle, or by large deviations principle. Recall that a sequence of laws $(P_N, N \geq 0)$ on a Polish space Ξ satisfies a large deviation with good rate function $I : \Xi \to \mathbb{R}^+$ and speed a_N going to infinity with N if and only if the level sets $\{x : I(x) \leq M\}, M \geq 0$, of I are compact and for all closed set F

$$\limsup_{N\to\infty} a_N^{-1} \log P_N(F) \leq -\inf_F I$$

whereas for all open set O

$$\liminf_{N\to\infty} a_N^{-1} \log P_N(O) \geq -\inf_O I.$$

Large deviation results for the spectral measure of Wigner ensembles are still only known for the Gaussian ensembles since their proof is based on the explicit joint law of the eigenvalues $P_N^{(\beta)}$. This question was studied in [Ben97], in relation with Voiculescu's non-commutative entropy. The latter is defined as the real-valued function on the set $M_1(\mathbb{R})$ of probability measures on the real line given by

$$\Sigma(\mu) = \begin{cases} \int\int \log|x-y| d\mu(x)d\mu(y) & \text{if } \int \log|x| d\mu(x) < \infty\,, \\ -\infty & \text{else}\,. \end{cases} \quad (21.2.4)$$

Theorem 21.2.4 *Let*

$$I_\beta(\mu) = \begin{cases} \frac{\beta}{2}\int x^2 d\mu(x) - \frac{\beta}{2}\Sigma(\mu) - c_\beta & \text{if } \int x^2 d\mu(x) < \infty, \\ \infty & \text{else}\,, \end{cases} \quad (21.2.5)$$

with $c_\beta = \inf_{\nu\in M_1(\mathbb{R})}\{\frac{\beta}{2}\int x^2 d\nu(x) - \frac{\beta}{2}\Sigma(\nu)\}$.

Then, the law of L_N under $P_N^{(\beta)}$, as an element of $M_1(\mathbb{R})$ equipped with the weak topology, satisfies a large deviation principle in the scale N^2, with good rate function I_β.

A moderate deviations principle for the spectral measure of the GUE or GOE is also known, giving a sharp rate for the decrease of deviations from the semi-circle law in a smaller scale.

Theorem 21.2.5 *For any sequence $a_N \to 0$ so that $Na_N \to \infty$, the sequence $Na_N(L_N([x,\infty)) - \sigma([x,\infty)))$ in $L^1_c(\mathbb{R})$ equipped with the Stieljes'-topology and the corresponding cylinder σ-field satisfies the large deviation principle in the scale $(Na_N)^2$ and with good rate function*

$$J(F) = \sup\{\int h'(x)F(x)dx - \frac{1}{2}\int_0^1 (h')^2(\sqrt{s}x)\sigma(dx)\}\,.$$

Here the supremum is taken over the complex vector field generated by the Stieltjes functions $f(x) = (z-x)^{-1}, z \in \mathbb{C}$ and the Stieltjes'-topology is the weak topology with respect to derivatives of such functions.

This moderate deviation result does not have yet a fully universal version for general Wigner ensembles. It has however been generalized to Gaussian divisible matrices (21.1.3) with a deterministic self-adjoint matrix V_N with converging spectral measure [Dem01] and to Bernouilli random matrices [Dor09].

21.2.4 The local semi-circle law and delocalization of the eigenvectors

A crucial result has been obtained much more recently, by Erdös, Schlein and Yau [Erd08, Theorem 3.1] proving that the convergence to the semi-circle law holds also locally, namely on more or less any scale larger than the typical spacings between the normalized eigenvalues which is of order N^{-1}. More precisely, they consider the case where the distribution of the entries has sub-exponential tails, and in the case of complex entries when the real and the imaginary part is independent. They then showed the following theorem.

Theorem 21.2.6 *For an interval $I \subset]-2,2[$, let $\mathcal{N}_I$ be the number of eigenvalues of $X_N/\sqrt{N}$ which belong to I. Then, there exists positive constants c, C so that for all $\kappa \in (0,2)$, all $\delta \le c\kappa$, any $\eta > \frac{(\log N)^4}{N}$ sufficiently small, we have*

$$P\left(\sup_{|E|\le 2-\kappa}\left|\frac{\mathcal{N}_{[E-\eta,E+\eta]}}{2N\eta} - \rho_{sc}(E)\right| > \delta\right) \le Ce^{-c\delta^2\sqrt{N\eta}}\,. \tag{21.2.6}$$

Amazingly, this local convergence, concentration inequalities, independence and equi-distribution of the entries, entails the delocalization of the eigenvectors of X_N, namely it was shown in [Erd08] that

Corollary 21.2.1 *Under the same hypotheses, for any $\kappa > 0$ and $K > 0$, there exists positive finite constants C, c such that for all $N \geq 2$,*

$$P\left(\exists v \text{ so that } X_N v = \sqrt{N}\mu v,\ \|v\|_2 = 1,\ \mu \in [-2+\kappa, 2-\kappa] \text{ et } \|v\|_\infty \geq \frac{(\log N)^{\frac{9}{2}}}{N^{\frac{1}{2}}}\right)$$

is bounded above by $Ce^{-c(\log N)^2}$.

21.2.5 The limits of universality: the spectral measure of heavy tailed Wigner random matrices

This brief section shows that there are natural obvious limits to the universality properties of Random Matrix Theory. When the second moment of the distribution of the entries is infinite, the global behavior of the spectrum cannot be expected to be similar to the Gaussian invariant cases, an indeed it changes dramatically. Let us assume more precisely that the entries $(Z_{i,j})_{i \leq j}, (Y_i)_{i \geq 0}$ are independent and have the same distribution P on $\mathbb{R}$ belonging to the domain of attraction of an α-stable law. This means that there exists a slowly varying function L so that

$$P(|x| \geq u) = \frac{L(u)}{u^\alpha}$$

with $\alpha < 2$. We then let

$$a_N := \inf\{u : P(|x| \geq u) \leq \frac{1}{N}\}$$

which is of order $N^{\frac{1}{\alpha}} \gg \sqrt{N}$ for $\alpha < 2$. Note that a_N is the order of magnitude of the largest entries on a row (or column) of the matrix X_N or of the sum of the entries on a row (centered if $\alpha < 1$). Then, it was stated in [Bou94] (see Chapter 13 of this handbook) and proved in [Ben08] (see [Bel09] for the proof of the almost sure convergence, and [Bord09] for another approach) that the eigenvalues of X_N are of order a_N and that the spectral measure of X_N correctly renormalized converges towards a probability measure which is different from the semi-circle law. More precisely we have

Theorem 21.2.7 *Let $\alpha \in (0, 2)$ and put $L_N = N^{-1} \sum_{i=1}^N \delta_{a_N^{-1}\lambda_i}$. Then*

- *L_N converges weakly to a probability measure μ_α almost surely.*
- *μ_α is symmetric, has unbounded support, has a smooth density ρ_α which satisfies*

$$\rho_\alpha(x) \sim \frac{L_\alpha}{|x|^{\alpha+1}} \qquad |x| \to \infty.$$

21.3 Local properties in the bulk

The analysis of the local properties of the eigenvalues of X_N in the bulk of the spectrum goes back to Gaudin, Dyson and Mehta, among others, for the Gaussian ensembles. The asymptotic behavior of the probability that no eigenvalues belong to an interval of width N^{-1}, the asymptotic distribution of the typical spacing between two nearest eigenvalues, the asymptotic behavior of the k-points correlation functions are, for instance, well understood [Meh04]. The generalization of these results to large classes of non-invariant Wigner ensembles has been a major challenge for a long time. (as an anecdotal evidence of this, it could be noted that during the 2006 conference in Courant Institute in honor of Percy Deift's 60th birthday, a panel of five experts all quoted this as the main open question of Random Matrix Theory).

The first universality result in his direction was obtained by K.Johansson [Joh01] for the correlation functions of complex Gaussian divisible entries (21.1.3). Johansson's proof follows and expands on an idea of Brézin and Hikami [Bré97]. This universality result has been recently extended to real entries by using dynamics and Dyson Brownian motion (21.1.4) by Erdös, Ramirez, Schlein, Yau and Yin [Erd09a, Erd09b].

It was only very recently generalized to a general case of non-invariant ensembles (with an assumption of sub-exponential moments and the same four first moments as Gaussian variables) by Tao and Vu [Tao09a]. Combining these two sets of results, one can prove the universality of the local statistics of the eigenvalues in the bulk provided the entries have sub-exponential tails (and at least 3 points in their support in the real case). We shall detail these results in the next section, about spacing distributions and refer to the original papers for correlation functions.

21.3.1 Spacings in the bulk for Gaussian Wigner ensembles

Let us first concentrates on a sample of local results for the Hermitian case. Based on the fact that the law $P_N^{(2)}$ is determinantal, the following asymptotics are well known, see [Tra94], with ρ_{sc} the density of the semi-circle law.

Theorem 21.3.1 • *For any $x \in (-2, 2)$, any $k \in \mathbb{N}$, the joint law of k (unordered) rescaled eigenvalues $\eta_i = N^{\frac{1}{2}}\rho_{sc}(x)^{-1}(\lambda_i - N^{\frac{1}{2}}x)_{1\le i\le k}$ converges vaguely towards the measure which is absolutely continuous with respect to Lebesgue measure and with density*

$$\rho^{(k)}(\eta_1, \dots, \eta_k) = C_k \det\left((S(\eta_i, \eta_j))_{1\le i,j\le k}\right)$$

with S the Sine kernel

$$S(y_1, y_2) := \frac{\sin(y_1 - y_2)}{\pi(y_1 - y_2)}\,.$$

- *For any compact set B,*

$$\lim_{N\to\infty} P_N^{(2)}\left(\lambda_i \notin N^{\frac{1}{2}}x + N^{-\frac{1}{2}}\rho_{sc}(x)B, i=1,\ldots,N\right) = \Delta(B,S) \tag{21.3.1}$$

with Δ the Fredholm determinant

$$\Delta(B,S) := 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \cdots \int_A \det\left((S(x_i,x_j))_{i,j=1}^k\right) \prod_{i=1}^k dx_i \,.$$

This result implies, see e.g. [Meh04, Appendix A.8] (or a more complete proof in [And09, Theorem 4.2.49] together with [Dei99] which gives the uniform asymptotics of the Hermite kernel towards the Sine kernel on intervals with width going to infinity with N) that the empirical spacing distribution between two eigenvalues, near a given point in the bulk, converges. More precisely, we give the statement for spacings near the origin. Define the Gaudin distribution by

$$P_{Gaudin}([2t,+\infty)) = -C\partial_t \Delta(S, \mathbf{1}_{(-t,t)^c})$$

and consider a sequence l_N increasing to infinity, such that $l(N) = o(N)$.

Theorem 21.3.2 *The number of eigenvalues of $N^{-\frac{1}{2}}X_N$ at a distance less than l_N/N whose nearest neighbors spacing is smaller than $N^{-1}\pi s$, divided by l_N, converges in probability towards $P_{Gaudin}([0,s])$.*

These results generalize to the real case, even though the law $P_N^{(1)}$ is not determinantal anymore. For instance one has, see e.g. [And09, Section 3.1.2]

Theorem 21.3.3 *There exists an increasing function $F_1 : \mathbb{R}^+ \to \mathbb{R}^+$, $F_1(0) = 0, F_1(\infty) = 1$ so that*

$$1 - F_1(t) = \lim_{N\to\infty} P_N^{(1)}\left(\cap_{1\le i\le N}\{\lambda_i^N \in \frac{1}{\sqrt{N}}(-\frac{t}{2},\frac{t}{2})^c\}\right)$$

Explicit formulae for F_1 are known, see e.g. [And09, Theorem 3.1.6].

21.3.2 Universality for Gaussian divisible Wigner matrices

We will consider matrices given by (21.1.3) with V_N a Wigner matrix. In the complex case, we shall assume that the real and imaginary part of the entries of V_N are independent and that their common distribution have sub-exponential tails. Then, we have

Theorem 21.3.4 *For any $\varepsilon \in (0,1]$, the results of Theorem 21.3.1 extend to the eigenvalues of $X_N = \sqrt{\varepsilon}G_N + \sqrt{1-\varepsilon}V_N$.*

This result was proved by Johansson [Joh01] in the Hermitian case, based on the fact that the density of the eigenvalues of X_N is then explicitly given by the Harich-Chandra-Itzykson-Zuber integral. It was initially valid only in the middle of the bulk, i.e. very near the center of the semi-circle, but has been later extended to the whole bulk. He showed more recently [Joh10] that the existence of the second moment is sufficient to grant the result. In the real-symmetric case, such a formula does not exist and the result was only recently proved by Erdös, Schlein and Yau by noticing that the entries of X_N can be seen has the evolution of Brownian motions starting from the entries of V_N and taken at time $-\log\varepsilon$, see (21.1.4). Based on this fact, Erdös, Schlein and Yau developed techniques coming from hydrodynamics theory to show that a small time evolution of the Brownian motion is sufficient to guarantee that the correlation functions are close to equilibrium, given by the Gaussian matrix. In fact, they could show that this time can even be chosen going to zero with N. The universality result of Theorem 21.3.4 is weaker than those stated in Chapter 6 of this handbook where convergence of the density of the joint law of the eigenvalues is required; for the time being such an averaging is needed to apply Erdös, Schlein and Yau techniques.

21.3.3 Universality and the four moments theorem

The approach proposed by Tao and Vu to prove universality follows Lindenberg's replacement argument, which is classical for sums of i.i.d random variables. The idea is to show that the eigenvalues of X_N are smooth functions of the entries so that one can replace one by one the entries of a Wigner matrix by the entries of another Wigner matrix (for which we can control the local properties of the spectrum because they are Gaussian or Gaussian divisible) and control the difference of the expectations by $o(N^{-2})$ if the four first moments are the same. The statement of the result is more precisely the following. Let us consider two Wigner matrices X_N and X'_N whose entries have sub-exponential tails (in fact a sufficiently large number of moments should be enough) and, in the complex case, with independent imaginary and real part. We assume that the moments $C(\ell,p) = E[\Re(X_{ij})^\ell \Im(X_{ij})^p]$ are the same for X_N and X'_N for all $\ell + p \le 4$. We denote by $\lambda_1(M_N) \le \lambda_2(M_N) \cdots \le \lambda_{N-1}(M_N) \le \lambda_N(M_N)$ the ordered eigenvalues of a Hermitian $N \times N$ matrix M_N. Then, Theorem 15 of [Tao09a] states that

Theorem 21.3.5 *For any $c_0 > 0$ sufficiently small, any $\epsilon \in (0,1)$ and $k \ge 1$, any function $F : \mathbb{R}^k \to \mathbb{R}$ such that*

$$|\nabla^j F(x)| \le N^{c_0}$$

for all $0 \le j \le 5$ *and* $x \in \mathbb{R}^k$*, we have for any* $i_1, \ldots, i_k \in [\epsilon N, (1-\epsilon)N]$*,*

$$\left| E\left[F(\lambda_{i_j}(\sqrt{N}X_N), 1 \le j \le k)\right] - E\left[F(\lambda_{i_j}(\sqrt{N}X'_N), 1 \le j \le k)\right]\right| \le N^{-c_0}.$$

One can then take X'_N to be a Gaussian or a Gaussian divisible matrix to deduce from the previous results on such matrices that any Wigner matrix having the same moments of order smaller than four have asymptotically the same spacing distributions in the bulk. It is not hard to see [Tao09a, Corollary 30] that one can always match the first four moments with those of a Gaussian divisible matrix provided the law of the entries have at least three points in their support. In the Hermitian case, using a finer analysis based on the explicit formula for Harich-Chandra-Itzykson-Zuber integral, one can prove [Erd09b] that Theorem 21.3.4 holds with ε of order $N^{-\frac{3}{4}}$ in which case one can always approximate the first four moments of the entries with such a Gaussian divisible law to get rid of the "three points in the support" hypothesis [Erd09c].

21.3.4 The extreme gaps

We saw in the preceding question that the typical spacing between normalized eigenvalues in the bulk is of order N^{-1}, and that the distribution of a typical spacing is known and universal. What about the size and distribution of the smallest and of the largest spacings?

For the GUE ensemble, it has recently been proved ([Ben10] after initial results in [Vin]) that the smallest spacing between normalized eigenvalues in the bulk, say in the interval $(-2+\epsilon, 2-\epsilon)$ for a positive ϵ, is of order $N^{-4/3}$. The distribution of this smallest spacings is known. Moreover the point process of the smallest spacings is asymptotically Poissonian.

We first consider the smallest gaps, studying the point process

$$\chi^N = \sum_{i=1}^{N-1} \delta_{(N^{4/3}(\lambda_{i+1}-\lambda_i), \lambda_i)} 1_{|\lambda_i|<2-\epsilon},$$

for any arbitrarily small fixed $\epsilon > 0$. Then, Theorem 1.4 of [Ben10] states that

Theorem 21.3.6 *As* $N \to \infty$*, the process* χ^N *converges to a Poisson point* χ *process with intensity*

$$E[\chi(A \times I)] = \left(\frac{1}{48\pi^2}\int_A u^2 du\right)\left(\int_I (4-x^2)^2 dx\right)$$

for any bounded Borel sets $A \subset \mathbb{R}_+$ *and* $I \subset (-2+\epsilon, 2-\epsilon)$*.*

In fact this result obtained in [Ben10] is the same one would get if one assumed (wrongly) that the spacings are independent. The correlations between the spacings are not felt across the macroscopic distance between the smallest gaps.The following corollary about the smallest gaps is an easy consequence of the previous theorem. Introduce $t_1^{(n)} < \cdots < t_k^{(n)}$ the k smallest spacings in I, i.e. of the form $\lambda_{i+1} - \lambda_i$, $1 \leq i \leq n-1$, with $\lambda_i \in I$, $I = [a, b]$, $-2 < a < b < 2$. Let

$$\tau_k^{(n)} = \left(\int_I (4 - x^2)^2 dx / (144\pi^2) \right)^{1/3} t_k^{(n)}.$$

Corollary 21.3.1 *For any* $0 \leq x_1 < y_1 < \cdots < x_k < y_k$, *with the above notations*

$$P\left(x_\ell < n^{4/3}\, \tilde{\tau}_\ell^{(n)} < y_\ell, 1 \leq \ell \leq k \right) \underset{n\to\infty}{\longrightarrow} \left(e^{-x_k^3} - e^{-y_k^3} \right) \prod_{\ell=1}^{k-1} \left(y_\ell^3 - x_\ell^3 \right).$$

In particular, the k^{th} *smallest normalized space* $N^{4/3}\tau_k^N$ *converges in law to* τ_k, *with distribution*

$$P\left(\tau_k \in dx\right) = \frac{3}{(k-1)!}\, x^{3k-1} e^{-x^3} dx.$$

For the largest spacings, the situation is less understood. Nevertheless it is proven that the largest gaps normalized by $\frac{\sqrt{logN}}{N}$ converges to a constant.

The universality question related to the extreme gaps is still open. Can these results be generalized to non-invariant ensembles? The real-symmetric case is not understood even for the GOE.

The size $\frac{\sqrt{\log N}}{N}$ of the largest spacing is natural, it is the same as one would guess from the tail of the Gaudin distribution by making the ansatz that the spacings are independent. But it is also interestingly small. Indeed this size $\frac{\sqrt{\log N}}{N}$ is also the size of the standard deviation of the position of a given eigenvalue in the bulk [Gus]. Thus the maximal spacing is not bigger than a typical fluctuation of one eigenvalue! This is of course linked to the fact that two adjacent eigenvalues have perfectly correlated Gaussian fluctuations. Let us mention here that Gustavsson's result mentioned above has been proved to be universal [Tao09a].

21.4 Local properties at the edge

It has long been observed that extreme eigenvalues of random matrices tend to stick to the bulk, and hence converge to the boundary of the limiting spectral measure. We shall describe this phenomena, study the fluctuations of the extreme eigenvalues and their universality. For heavy tailed distribution of the entries, and as mentioned above in the case of the spectral measure, this universality breaks down for the behavior of the extreme eigenvalues. The extreme eigenvalues are then Poissonian and the associated eigenvectors are very localized.

We will also mention an interesting universality question related to the sensitivity of the extreme eigenvalues to the addition of a finite rank matrix.

21.4.1 Convergence of the extreme eigenvalues

The convergence of the extremal eigenvalues of a Wigner matrix towards the boundary of the support of the semi-circle law goes back to Füredi and Komlós [Fur81]. The following result has been proved by [Bai99]

Theorem 21.4.1 *Assume that the fourth moment of the distribution of the entries is finite. Then,* $\bar{\lambda}_N := \max_{1\le i\le N} \lambda_i^N$ *and* $\underline{\lambda_N} := \min_{1\le i\le N} \lambda_i^N$ *converge to* 2 *and* -2 *in probability.*

The proof relies on fine estimates of the moments of the averaged spectral measure, at powers going to infinity with N faster than logarithmically. It is not hard to see that the convergence in probability can be improved into an almost sure convergence. This result breaks down if the fourth moment is not finite.

21.4.2 Fluctuations of the extreme eigenvalues and the Tracy-Widom law

For the Gaussian ensembles, one can again rely on the explicit joint law of the eigenvalues to study precisely the fluctutations of the extreme eigenvalues. This is again simpler for the GUE, based on the determinantal structure of the law $P_N^{(2)}$. After the work of Mehta [Meh04], a complete mathematical analysis was given by the works of P. Forrester [For93] and C. Tracy and H. Widom [Tra94, Tra00]. The main result goes as follows.

Theorem 21.4.2 *For* $\beta = 1$ *or* 2*, for any* $t \in \mathbb{R}$*, the largest eigenvalue* $\bar{\lambda}_N$ *of* X_N *is such that*

$$F_\beta(t) := \lim_{N\to\infty} P_N^{(\beta)}\left(N^{\frac{1}{6}}(\bar{\lambda}_N - 2\sqrt{N}) \le t\right)$$

with F_β *the partition function for the Tracy-Widom law.*

This result can be generalized to describe the joint convergence of the k-th largest eigenvalues.

21.4.3 Universality

Based on very fine combinatorial arguments A. Soshnikov [Sos99] showed that the moments of the spectral measure up to order $N^{\frac{2}{3}}$ are the same than those of the Gaussian ensembles. This allowed him to show that

Theorem 21.4.3 *Assume that the entries* $(Z_{i,j})_{i\le j}$ *and* $(Y_i)_{i\ge 0}$ *are i.i.d with distribution which is symmetric and with sub-Gausian tail. Then,the results of Theorem 21.4.2 extend to* X_N.

By approximation arguments, it was shown in [R06] that it is sufficient to have the first 36 moments finite, whereas new combinatorial arguments allowed in [Kho09] to reduce the assumption to the twelve first moments finite. It is conjectured, and proved in the Gaussian divisible case [Joh10], that the optimal assumption is to have four moments finite. However, the hypothesis that the law is symmetric could not be completely removed using this approach, see [PS07] for an attempt in this direction.

21.4.4 Universality and the four moments theorem

T.Tao and V.Vu have generalized their four moments theorem to deal with the eigenvalues at the edge [Tao09b, Theorem 1.13].

Theorem 21.4.4 *Let* X_N *and* X'_N *be two Wigner matrices with entries with sub-exponential tails and moments which match up to order four. Assume in the case of complex entries that the real and the imaginary parts are independent. Then, there exists a small constant* $c_0 > 0$ *so that for any function* $F : \mathbb{R}^k \to \mathbb{R}$ *such that*

$$|\nabla^j F(x)| \le N^{c_0}$$

for all $0 \le j \le 5$ *and* $x \in \mathbb{R}^k$, *we have for any* $i_1, \ldots, i_k \in [1, N]$,

$$\left| E\left[F(\lambda_{i_j}(\sqrt{N}X_N), 1 \le j \le k)\right] - E\left[F(\lambda_{i_j}(\sqrt{N}X'_N), 1 \le j \le k)\right]\right| \le N^{-c_0}.$$

As a corollary, we have [Tao09b, Theorem 1.16].

Corollary 21.4.1 *Under the assumptions of the previous theorem, let* k *be a fixed integer number and* X_N *a Wigner matrix with centered entries with sub-exponential tails. Assume that they have the same covariance matrix as the GUE (resp. GOE) and that all third moments vanish. Then the joint law of* $((\lambda_N(X_N/\sqrt{N}) - 2)N^{\frac{2}{3}}, \ldots, (\lambda_{N-k}(X_N/\sqrt{N}) - 2)N^{\frac{2}{3}})$ *converges weakly to the same limit as for the GUE (resp. GOE).*

Note that the advantage of this approach is that it does not require the symmetry of the distribution of the entries as Soshnikov's result did.

21.4.5 Extreme eigenvalues of heavy tailed Wigner matrices

Let us now consider what happens when the entries have no finite moments of order four but are in the domain of attraction of an α-stable law as in section 21.2.5. Then, the behaviour of the extreme eigenvalues are dictated by the largest entries of the matrix, and therefore the point process of the extreme eigenvalues, once correctly normalized, converges to a point process, as in the extreme value theory of independent variables. More precisely, assume that the $(Z_{i,j}, i \leq j)$ and $(Y_i, i \geq 0)$ are i.i.d with law P and let

$$b_N = \inf\{x : P(u : |u| \geq x) \leq \frac{2}{N(N+1)}\}$$

which is of order $N^{\frac{2}{\alpha}}$. Then it was shown in [Auf09], generalizing a result from [Sos04], that

Theorem 21.4.5 *Take $\alpha \in (0,4)$ and for $\alpha \in [2,4)$ assume that the entries are centered. Then, the point process*

$$\mathcal{P}_N = \sum_{i \geq 0} \mathbf{1}_{\lambda_i \geq 0} \delta_{b_N^{-1} \lambda_i}$$

converges in distribution to the Poisson point process on $(0, \infty)$ with intensity $\rho(x) = \frac{\alpha}{x^{1+\alpha}}$.

This result simply states that the largest eigenvalues behave as the largest entries of the matrix.

21.4.6 Non universality and Extreme eigenvalues of finite rank perturbation of Wigner matrices

In [Baik05], the authors consider the effect on the largest eigenvalue of adding a finite rank matrix to a Gaussian sample covariance matrix. The phenomena they observed, namely a phase transition in the asymptotic behavior and fluctuations of the extreme eigenvalues, also happens when one considers Wigner matrices, see e.g. [Cap09a]. A finite rank perturbation can pull some eigenvalues away from the bulk.

We will dwell only on the simplest case, where the rank of the perturbation is one. But more is known, see e.g. [Fer07, Cap09a, Bai08]. Consider the deformed Wigner matrix

$$M_N = \frac{1}{\sqrt{N}} X_N + A_N$$

with X_N a Wigner matrix with independent entries with symmetric law μ satisfying Poincaré inequality and A_N a deterministic rank-one matrix. We will look at the two extreme cases.

First let $A_N = \frac{\theta}{N} J_N$, where $\theta > 0$ and J_N is the matrix whose entries are all ones, see e.g.[Fer07]. Obviously A_N is rank-one and its eigenvalues are 0 and θ. The only non trivial eigenvector of A_N is maximally delocalized. When the parameter θ is small enough ($\theta < 1$), the perturbation has no influence on the top of the spectrum, the top eigenvalue λ_1 of M_N "sticks to the bulk", i.e. λ_1 converges to 2, the edge of the bulk. If X_N is Gaussian it is proved that the fluctuations are also unaffected by the perturbation, i.e that $(\lambda_1 - 2)N^{2/3}$ converges to the Tracy-Widom distribution. This result is expected to be universal. When the parameter θ is large enough ($\theta > 1$), the top eigenvalue λ_1 is pulled away from the bulk, it converges to $\rho_\theta = \theta + \frac{1}{\theta} > 2$. Moreover in this case the fluctuations of the top eigenvalue are in the scale $\sqrt{N}$ and Gaussian, i.e $\sqrt{N}(\lambda_1 - \rho_\theta)$ converges to a Gaussian distribution. This result is universal.

But if one now chooses a rank-one perturbation with a very localized eigenvector, the situation is quite different. Let

$$A_N = diag(\frac{\theta}{N}, 0, \dots, 0)$$

Then again A_N is rank-one and its eigenvalues are 0 and θ. The only non trivial eigenvector of A_N is maximally localized. In this case again when θ is small enough ($\theta < 1$), the perturbation has no influence on the top of the spectrum, the top eigenvalue λ_1 of M_N "sticks to the bulk", i.e. λ_1 converges to 2, the edge of the bulk. Its fluctuations are Tracy-Widom in the invariant case and expected to also be in the non-invariant cases. When the parameter θ is large enough ($\theta > 1$), the top eigenvalue λ_1 is pulled away from the bulk, it converges to $\rho_\theta > 2$. But now the fluctuations are no longer Gaussian, $\sqrt{N}(\lambda_1 - \rho_\theta)$ converges in distribution to the convolution of a Gaussian law and of μ. This limiting law thus remembers the law of the entries and this result is definitely not universal. The reason is quite clear, and is due to the fact that, because of the localization of the top eigenvector of the perturbation A_N, the top eigenvalue of M_N remembers very much one entry of the matrix X_N.

This interesting non-universal behavior has been studied in more general cases in [Cap09b].

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