USES OF FREE PROBABILITY IN RANDOM MATRIX THEORY

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This article gives a short introduction to free probability theory and emphasizes its role as a natural framework to study random matrices with size going to infinity. We motivate the uses of free probability by a few applications, taken from joint works with Jones, Khrishnapur, Maurel-Segala, Shlyakhtenko and Zeitouni. The first concerns the study of non-normal matrices; we show that the empirical measure of the eigenvalues of non-normal random matrices whose law is invariant by conjugation converges and that the limit can be described by the so-called R-diagonal operators, which are random variables taken from free probability. This in particular generalizes the single ring theorem of Feinberg and Zee [1]. The second application deals with the topological expansion of Brézin, Itzykson, Parisi and Zuber [2]. We show that such expansion can be turned into an asymptotic expansion and that the limit can be constructed by 'free' Langevin dynamics. In particular, the convergence holds also in non-perturbative but convex situations.

Keywords: Random matrices, map enumeration.

1. Introduction

Large random matrices were first studied in theoretical physics by Wigner, in connection with quantum mechanics, as a model for the energy level of large nuclei. In an independent development in the early 1970s, Hugh Montgomery showed that, assuming the Riemann Hypothesis, and modulo some technicalities, the pair correlation function for the spacings between the (normalized) zeros of the Riemann zeta function on the critical line behaves like the pair correlation function for the (normalized) eigenvalues of a random matrix. At about the same time, G. 't Hooft and Brézin, Itzykson, Parisi and Zuber, showed that Gaussian random matrix integrals are generating functions for the enumeration of graphs embedded into surfaces which are sorted by their genus (the so-called topological expansion). Since that time an extraordinary variety of mathematical, physical and engineering systems have been related with Random Matrix Theory; it has emerged as an interdisciplinary scientific activity par excellence.

In this review, we shall concentrate on the global asymptotics of the spectrum of large random matrices, for instance reflected by the asymptotics of the (normalized) trace of powers of these matrices. We will argue that such asymptotics can be described in great generality in the framework of free probability and therefore that this theory is extremely useful to study large random matrices. We will illustrate this point by two problems from theoretical physics. The first concerns the study of the spectrum of non-normal large random matrices which appears in several areas of physics [3–10]. It was shown by Feinberg and Zee [1] that the spectral distribution of certain ensembles of random non-normal converges to a radially symmetric deterministic measure whose support is a single ring. This result came as a big surprise because the support of the singular values of these ensembles can be as diconnected as wished. This question is related with the law of non-commutative variables called 'R-diagonal operators' in free probability. Relying on a study of such operators by Haagerup and Larsen [11], we can generalize and prove rigorously Feinberg and Zee's result. Another range of applications of free probability concerns the first order of the topological expansion of Brézin, Itzykson, Parisi and Zuber which relates the asymptotics of matrix integrals with the enumeration of planar graphs. Indeed, again the limit can be described in terms of free variables, which in turns allows us to prove that the topological expansion can be turned into an asymptotic expansion, that it holds in non-perturbative but convex situations and finally, under such convexity hypothesis, that it has some properties, such as the connectivity of the support of the limiting matrices.

Free probability emerged in the eighties when Voiculescu realized that certain questions appearing in operator algebra theory could be phrased in probabilistic terms. Of course, such a probability theory should be concerned with non-commutative random variables, as is non-commutative probability theory. However, free probability theory differs from the latter by the notion of freeness. Similarly to the classical notion of independence, freeness is defined by certain relations between traces of words. These two components are the basis for a probability theory for noncommutative variables where many concepts taken from probability theory such as the notions of laws, convolution, convergence in law, independence, central limit theorem, Brownian motion, entropy, and more can be naturally defined. For instance, the law of one self-adjoint variable is simply given by the traces of its powers (which generalizes the definition through moments of compactly supported probability measures on the real line), and the joint law of several self-adjoint non-commutative variables is defined by the collection of traces of words in these variables. The joint law of free variables is defined by the law of each of them and a condition on the joint moments which defines them uniquely from the moments of the marginals. Convergence in law just means that the trace of any word in the noncommutative variables converges towards the right limit.

About ten years later, Voiculescu showed that free probability is the right framework to consider the asymptotics of random matrices as their size go to infinity. More specifically, he proved that an m-tuple of random matrices whose eigenvectors are genuinely independent, namely with a covariance matrix following independent unitary Haar distributed random matrices, converges to an m-tuple of free random variables. From that point, many concepts from standard probability theory could be brought to free probability, once suitably transposed through random matrices. Hence, in some manner, many concepts in free probability are inspired from random matrices. But free probability also developed on its own, or in relation with operator algebra theory, and provide now the natural framework to study the asymptotics of random matrices. We shall in the next section describe more precisely the basics of free probability.

2. Free probability theory

2.1. *Basics of free probability*

Free probability is a non-commutative probability theory with a notion of freeness very analogous to independence in classical probability theory. Variables in such a theory include finite size random matrices but also 'matrices with size going to infinity'. This notion represents the weak limit of matrices sequences. Namely m $N \times N$ Hermitian matrices $(M_1^N, \cdots, M_m^N)$ converge to $(M_1, \cdots, M_m)$ iff for all polynomial P

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\left(P(M_1^N, \cdots, M_m^N)\right) =: \tau\left(P(M_1, \ldots . M_m)\right).$$

The limit τ, when it exists, is just a linear form on the set of polynomials. It is analogous to a classical expectation. Indeed, because it is obtained as the limit of traces of matrices, it satisfies, for all polynomials P, Q in m non-commutative variables,

$$\tau(PQ) = \tau(QP) \quad \tau(PP^*) \geq 0 \quad \tau(I) = 1.$$

The second and third properties generalize the properties of positivity and mass of the classical expectation. τ is called a tracial state. As in classical probability, one likes to think about expectation of random variables and of laws of random variables. In fact, τ can be seen as the law of non-commutative variables in the sense that, if the matrices $(M_i^N, N \geq 0, 1 \leq i \leq m)$ are uniformly bounded for the operator norm by some constant R, one can construct a Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$, a vector $\zeta \in H$ and bounded linear operators $(M_1, \ldots, M_m)$ on H so that for all polynomial P,

$$\tau(P) = \langle P(M_1, \ldots, M_m)\zeta, \zeta \rangle. \tag{1}$$

This construction of H and $(M_1, \ldots, M_m)$ is called the Gelfand-Naimark-Seigal construction. Note that in the case where $m = 1$, Riesz's theorem asserts that τ is a classical probability measure on $[-R, R]$. One way to construct M_1 as a bounded operator on a Hilbert space is to take $H = L^2(\tau)$, once quotiented by the left ideal $\{h \geq 0, \tau(h) = 0\}$, and to put M_1 to be the left multiplication by x, so that for all $h \in L^2(\tau)$, $M_1 h(x) = xh(x)$.

In the sequel, τ will be a linear form on the set of polynomials and we will assume that we have constructed a Hilbert space H so that (1) holds and our random variables all live in $B(H)$.

Definition 2.1. $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n) \in (B(H), \tau)$ are free iff for all polynomials $P_1, \ldots, P_\ell$ and $Q_1, \ldots, Q_\ell$ so that $\tau(P_i(\mathbf{X})) = 0$ and $\tau(Q_i(\mathbf{Y})) =$

0 for all i, we have

$$\tau\left(P_1(\mathbf{X})Q_1(\mathbf{Y})\cdots P_\ell(\mathbf{X})Q_\ell(\mathbf{Y})\right)=0.$$

Note that the joint law $(\tau(P(\mathbf{X},\mathbf{Y})),P)$ is uniquely determined by the collection of the marginals $\tau(P(\mathbf{X}))$ and $\tau(Q(\mathbf{Y}))$, where Q,P runs over the set of polynomials in $\mathbf{X}$ or $\mathbf{Y}$.

The notion of freeness we just defined is related with the classical notion of freeness in groups. Indeed, let G be a group with generators $g_1,\ldots,g_m$, a neutral element e and define

$$\tau(g)=1_{g=e},\qquad \text{for } g\in G.$$

Then we claim that $g_1,\ldots,g_m$ are free under τ iff they are free in the group G. Indeed, $g_1,\ldots,g_m$ are free in G iff it is not possible that a non trivial word in $g_1,\ldots,g_m$ is trivial. But a non trivial word is just a word which can be reduced into a product $P_1(g_{i_1})\cdots P_k(g_{i_k})$ of words in the g_i, $i_{k+1}\neq i_k$. Because each of the word P_i in g_i is not trivial, $\tau(P_i)=0$ and then we see that the condition that $P_1(g_{i_1})\cdots P_k(g_{i_k})$ is not the neutral element exactly means that $\tau(P_1(g_{i_1})\cdots P_k(g_{i_k}))=0$ which is the freeness condition.

Next we show that freeness also appears when one considers random matrices with 'independent' basis of eigenvectors, asymptotically when the size of the matrices goes to infinity. Let $(U_1^N,\ldots,U_m^N)$ be $N\times N$ independent matrices following the Haar measure on the unitary group and $A_1^N,\ldots,A_m^N$ deterministic Hermitian matrices with spectral measures $\hat\mu_k^N=\frac{1}{N}\sum_{j=1}^N\delta_{\lambda_i^N(A_k^N)}$ which converge to μ_i for $1\le i\le m$ in moments, that is

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}(A_i^N)^k=\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^N(\lambda_j(A_i^N))^k=\int x^k d\mu_i(x).$$

Then, Voiculescu [12]

Theorem 2.1 (Voiculescu 91'). *Let* $X_i^N=U_i^NA_i^N(U_i^N)^*$. *For any polynomial* P *in* m *non-commutative variables,*

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\left(P(X_1^N,\ldots,X_m)\right)=\tau\left(P(X_1,\ldots,X_m)\right)\quad a.s\,.$$

with $(X_1,\ldots,X_m)$ *free and with marginal distribution* $(\mu_i)_{1\le i\le m}$.

When (random) matrices converge and their limit are free, we say that they are asymptotically free. In fact, Theorem 2.1 applies to many standard matrix models, such as for instance the classical Gaussian ensembles. Indeed, the ensemble of Hermitian $N\times N$ matrices with independent Gaussian entries with covariance N^{-1} taken from the GUE is by definition invariant under unitary conjugation; hence a matrix taken from the GUE can be written as $X^N=U^NA^N(U^N)^*$ where A^N is the diagonal matrix with entries given by the eigenvalues of X^N and U^N an independent matrix following the Haar measure on the unitary group. By Wigner's theorem [13]

Theorem 2.2 (Wigner 58'). *For all integer number d*

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\left((X^N)^d\right)=\begin{cases}0 & \text{if } d \text{ is odd,}\\ C_{\frac{d}{2}} & \text{otherwise,}\end{cases}\quad a.s.$$

where C_p denote the Catalan numbers. Moreover, the right hand side is also equal to $\int x^d d\sigma(x)$ with σ the semicircle law given by

$$d\sigma(x)=(2\pi)^{-1}1_{|x|\le 2}\sqrt{4-x^2}dx.$$

In particular, if one considers m independent matrices taken from the GUE, they will be asymptotically free. More precisely

Corollary 2.1. *m independent matrices taken from the GUE $(X_1^N,\dots,X_m^N)$ converge to m free semicircle variables.*

This corollary in fact generalizes to independent matrices with independent centered entries with finite moments and covariance N^{-1}, see [14, Theorem 5.4.2]. From Theorem 2.2 and its corollary, it is no surprise that freeness can be used to describe many asymptotics of random matrix problems. Reciprocally, it is clear that probability theory can be brought to the non-commutative setting by applying probability concepts to random matrices and then taking the large N limit. We next show how a few classical notions of probability theory generalize in free probability.

2.2. *Free Brownian motion*

Following Corollary 2.1, we are going to define the free Brownian motion as the limit of the Hermitian Brownian motion. $\{H_t^N, t\ge 0\}$, the Hermitian Brownian motion, is a $N\times N$ Hermitian matrix valued process whose entries are i.i.d. complex Brownian motions.

$$H_t^N(k\ell)=\frac{B_t(k\ell)+i\tilde{B}_t(k\ell)}{\sqrt{2N}}\quad k<\ell,\ H_t^N(kk)=\frac{B_t(kk)}{\sqrt{N}}\,.$$

It is not hard to check that for all $N\times N$ Hermitian matrix A such that $N^{-1}\mathrm{Tr}(A^2)=1$, $\{\mathrm{Tr}(AH_t^N), t\ge 0\}$ follows a real Brownian motion.

From Corollary 2.1, and the scaling property of Brownian motion, we deduce

Corollary 2.2. *For all $t_1,\dots,t_p\in\mathbb{R}^+$, the following limit exists*

$$\lim_{N\to\infty}\mathbb{E}[\frac{1}{N}\mathrm{Tr}(H_{t_1}^N\cdots H_{t_p}^N)]=:\tau(S_{t_1}\cdots S_{t_p})$$

$(S_t, t\ge 0)$ is a continuous process with values in $B(H)$ with free increments distributed according to the appropriately rescaled semicircle law.

This result is a direct consequence of Corollary 2.1 since the increments of the Hermitian Brownian motion are independent matrices distributed according to the GUE. One of the great application of the classical Brownian motion is based on its relation with the heat equation which governs its density, which allowed to get a new insight on the latter. Such a connection can be performed thanks to Itô's calculus which shows that the small variation of a function evaluated at the Brownian motion after a small time increment is given by the sum of the derivative of this function (multiplied by the increment of the Brownian motion) and the second derivative of this function (multiplied by the time increment). The last term reflects the lack of smoothness of the Brownian motion and provides the heat equation motion. A similar calculus holds with the free Brownian motion, but this time the additional term is given by a second order non-commutative differential, which differs from the Laplacian. For instance, if one evaluates the trace $v_k(t) := \tau((S_t)^k)$ of monomials in the free Brownian, it satisfies the Smoluchowski's type equation

$$\partial_t v_k(t) = \frac{k}{2}\sum_{p=0}^{k-2} v_p(t) v_{k-2-p}(t)$$

The differential calculus of the free Brownian motion can also be generalized to consider differential equation driven by the free Brownian motion.

Let $K : B(H)^m \to B(H)^m$ be a bounded Lipschitz function on $B(H)^m$

$$\max_{1\le i\le m} \|K^i(X) - K^i(Y)\|_{B(H)} \le \|K\|_L \max_{1\le i\le m} \|X_i - Y_i\|_{B(H)} \wedge 1\,.$$

Assume also that $K_i(X_1,\dots,X_m)$ is self-adjoint for any m-tuple $(X_1,\dots,X_m)$ of self-adjoint elements of $B(H)$. Then, for any $X_0 \in B(H)^m$, there exists a unique solution $X_.$ to

$$X_t^i = X_0^i + S_t^i + \int_0^t K^i(X_s)ds$$

with $(S^1,\dots,S^m)$ m free free Brownian motions. Note again that X_t can be seen to be the limit of the $N\times N$ Hermitian matrices so that X_0 is the limit of the $N\times N$ Hermitian matrices X_0^N and $X_.^N$ is the unique strong solution to the Langevin dynamics

$$X_t^{N,i} = X_0^{N,i} + H_t^{N,i} + \int_0^t K^i(X_s^N)ds$$

with i.i.d Hermitian Brownian motions $\{H^{N,i}, 1\le i\le m\}$.

In the case where $m=1$, K is just a bounded Lipschitz function on $\mathbb{R}$ and if we let τ_t be the spectral distribution of X_t (recall that τ_t is the probability measure on $\mathbb{R}$ so that $\tau_t(x^k) = \tau(X_t^k)$), Itô's calculus now gives, for any bounded twice continuously differentiable function f,

$$\partial_t \int f(x)d\tau_t(x) = \frac{1}{2}\int\int \frac{f'(x)-f'(y)}{x-y} d\tau_t(x)d\tau_t(y) + \int f'(y)K(y)d\tau_t(y).$$

2.3. *Free convolution*

Functional analysis can also be developped in free probability theory. For instance one can wonder what is the distribution of $A+B$, A and B being free variables with a prescribed distribution. This is, by Theorem 2.1, the limit of the spectral measure of $A_N + U_N B_N U_N^*$ when N goes to infinity. A similar question can be asked about the distribution of the product AB, A and B being free variables. Because AB is not self-adjoint, the moments of AB do not give the spectral measure of AB but in fact those of $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ when $A \geq 0$. Indeed, $\mu((A^{\frac{1}{2}}BA^{\frac{1}{2}})^n)$ equals $\mu((AB)^n)$ since μ is tracial and the spectral measure of $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ is given by its moments since it is self-adjoint. Hence, the law of AB describes the asymptotics of the spectral measure of $A_N^{\frac{1}{2}}U_N B_N U_N^* A_N^{\frac{1}{2}}$ as N goes to infinity and the spectral measure of A_N (resp. B_N) converges to μ_A (resp. μ_B) and U_N independent unitary.

The law of $A+B$ (resp. AB) is denoted $\mu_A \boxplus \mu_B$ (resp. $\mu_A \boxtimes \mu_B$). These probability measures are described by the R-transform and the S-transform respectively, which play the same role as a log-Fourier transform for the standard convolution. For example, we put

$$S_C(z) := \frac{1+z}{z} m_C^{-1}(z)$$

if $m_C(z) = \sum_{n\geq 1} \mu_C(x^n) z^n$. S_C is well defined at list when $\mu_C(x) \neq 0$ since then m_C^{-1} exists at list in a neighborhood of the origin by the implicit function theorem. Moreover, the knowledge of S_C on a set with accumulation points defines uniquely m_C and therefore the law μ_C. It is then known [14, Lemma 5.3.30] that, at list for small z's,

$$S_{A^{\frac{1}{2}}BA^{\frac{1}{2}}}(z) = S_A(z) S_B(z)\,.$$

Free probability theory offers many other interesting developments [14–17].

3. Single ring theorem

This section deals with one application of free probability results to the analysis of large non-normal random matrices. We consider the ensemble of random, non-Normal $N \times N$ matrices with law

$$dP_N(X_N) = \frac{1}{Z_N} e^{-N\mathrm{Tr}(V(X_N X_N^*))} dX_N$$

where $dX_N = \prod_{1\leq i,j\leq N} d\mathbb{R}e(X_N(ij)) d\Im(X_N(ij))$ is the Lebesgue measure on the set of $N \times N$ matrices with complex entries. V is a polynomial going to infinity at infinity so that Z_N is finite for each N. We consider the eigenvalues $\{\lambda_i^N\}_{1\leq i\leq N}$ of X_N and their empirical measure $L_N^V = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N}$. The following theorem was proved, albeit not entirely rigorously, by Feinberg and Zee [1].

Theorem 3.1 (Feinberg-Zee 97'). *Assume V is a polynomial. Then: L_N^V converges to a deterministic, rotationally invariant μ_V whose support consists of a* *single ring: there exists constants* $0 \le a < b < \infty$ *so that*

$$\mathrm{supp}(\mu_V) = \{re^{i\theta} : a \le r \le b, \theta \in [0, 2\pi[\}.$$

If $V(x) = gx^2 + mx$*, phase transition occurs when the support changes from a disc to an annulus.*

This result is rather surprising since one could imagine a priori that if the potential presents sufficiently deep attracting wells, the eigenvalues of X_N should concentrate in these wells so that the support of the limiting spectral measure should be disconnected. This is true when one considers the singular values of X_N, that is the eigenvalues of $\sqrt{X_N X_N^*}$. Let $\sigma_1^N \ge \sigma_2^N \ge \cdots \ge \sigma_N^N \ge 0$ denote the singular values of X_N. Their joint distribution is

$$\frac{1}{Z_N} \prod_{i<j} [(\sigma_i^N - \sigma_j^N)(\sigma_i^N + \sigma_j^N)]^2 e^{-N \sum_{i=1}^N V((\sigma_i^N)^2)} \prod_{i=1}^N \sigma_i^N d\sigma_i^N$$

One easily deduces (see e.g. [14, Section 2.6]) the convergence of the empirical measure of the singular values to the probability measure σ_V which minimizes

$$\mu \to \int V(x^2) d\mu(x) - \int \log |x^2 - y^2| d\mu(x) d\mu(y)$$

on the set of probability measures on the half line $\mathbb{R}^+$. If V has k sufficiently deep wells, it is easy to show that the support of this minimizer will consist of at list k intervals!

The surprising connectivity of the support of non-normal operators whose singular values have a very disconnected support was already observed in free probability by Haagerup and Larsen [11]. Indeed, note that under the Feinberg-Zee model, $X_N = U_N D_N V_N$ with D_N a real diagonal matrix with converging spectral distribution ($D_N = \mathrm{diag}(\sigma^N)$) and independent U_N, V_N, unitary matrices following Haar distribution. It is natural to wonder what can be said about the convergence of the empirical measure $L_{X_N} = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N}$ of the eigenvalues of $X_N = U_N D_N V_N$ when D_N is a real diagonal matrix with converging spectral distribution and independent Haar distributed matrices U_N, V_N, independent of D_N.

A similar question was considered in the framework of free probability. Indeed, Nica and Speicher [17] defined the so-called R-diagonal operators, which are operators which can be decomposed as $R = UD$ with D Hermitian and U unitary, U and D being free. Note that the distribution of the eigenvalues of UDV and UD is the same if U, V are two free unitary variables. The distribution of such operators was studied by Haagerup and Larsen [11].

Theorem 3.2 (Haagerup-Larsen 00'). *Take* $X_D := UDV$ *or* UD, U, V, D *free,* D *self-adjoint with law* μ_D, U, V *unitaries. The Brown measure (continuous analogue of the spectral measure) of* X_D *is rotation invariant and radially supported* *on an annulus*

$$\mu_{X_D}(B(0,f(t))) = t$$

where $f(t) = 1/\sqrt{S_{D^2}(t-1)}$.

$$\mathrm{supp}(\mu_{X_D}) = \{re^{i\theta}, r \in [(\mu_D(x^{-2}))^{-\frac{1}{2}}, (\mu_D(x^2))^{\frac{1}{2}}], \theta \in [0, 2\pi[\}$$

The Brown measure of a non-normal operator X is given as follows. For $z \in \mathbb{C}\backslash\mathbb{R}$, let ν^z be the spectral measure of $(z-X)(z-X)^*$. Since $(z-X)(z-X)^*$ is self-adjoint, ν^z can easily be computed from its moments for instance. Then, the Brown measure of X is the compactly supported probability measure μ_X on $\mathbb{C}$ so that for any smooth compactly supported function ψ

$$\int \psi(z) d\mu_X(z) = \frac{1}{4\pi} \int_{\mathbb{C}} \Delta\psi(z) \int \log|x| d\nu^z(x) dz.$$

One way to see that the Brown measure is related to our problem is the Green formula which reads, if $(\lambda_i^N)_{1\le i\le N}$ are the eigenvalues of X_N,

$$\begin{aligned}\sum_{i=1}^N \psi(\lambda_i^N) &= \frac{1}{2\pi}\int_{\mathbb{C}} \Delta\psi(z) \log|\prod_{i=1}^N (z-\lambda_i^N)| dz \qquad (2)\\ &= \frac{1}{4\pi}\int_{\mathbb{C}} \Delta\psi(z) \left(\sum_{i=1}^N \log|z-\lambda_i^N|^2\right) dz \\ &= \frac{N}{4\pi}\int_{\mathbb{C}} \Delta\psi(z) \int \log|x| dL_{(z-X_N)(z-X_N)^*}(x) dz\end{aligned}$$

where $L_{(z-X_N)(z-X_N)^*}(x)$ is the spectral measure of the self-adjoint operator $(z-X_N)(z-X_N)^*$. Taking $X_N = U_N D_N V_N$, we know by Theorem 2.1 that the spectral measures of the Hermitian matrices $(z-X_N)(z-X_N)^*$ converge to the law of $(z-UDV)(z-UDV)^*$. Hence, we expect $N^{-1}\sum_{i=1}^N \delta_{\lambda_i^N}$ to converge to the Brown measure of UDV, U, D free unitary variables, free with D. This is the content of the following theorem [18]

Theorem 3.3 (G., Krishnapur, Zeitouni '09). *Take* $X_N = U_N D_N V_N$. *Assume some technical conditions on* D_N *and that* $L_{D_N} = \frac{1}{N}\sum \delta_{D_N(ii)}$ *converges to a probability measure* μ_D *on* $\mathbb{R}^+$ *so that for some* $\kappa, \kappa' > 0$, *all* $z \in \mathbb{C}^+$ *so that* $\Im z \ge N^{-\kappa'}$,

$$\left| \frac{1}{N}\sum_{i=1}^N \frac{1}{z - D_N(ii)} - \int \frac{1}{z-x} d\mu_D(x) \right| \le \frac{1}{N^\kappa |\Im z|}.$$

Then $L_{X_N} = \frac{1}{N}\sum \delta_{\lambda_i^N}$ *converges weakly to* μ_{X_D} *in probability, i.e for any bounded continuous function* f *on* $\mathbb{C}$, $N^{-1}\sum_{i=1}^N f(\lambda_i^N)$ *converges to* $\int f(z) d\mu_{X_D}(z)$ *in probability.*

As a corollary of Theorem 3.3, we prove the the Feinberg-Zee "single ring theorem".

Corollary 3.1. *Let V denote a polynomial with positive leading coefficient. Let the n-by-n complex matrix X_n be distributed according to the law*

$$dP_n(X) = \frac{1}{Z_n} \exp(-n\mathrm{Tr}\ V(XX^*))dX\ ,$$

where Z_n is a normalization constant and dX the Lebesgue measure on n-by-n complex matrices. Then, the spectral measure L_{X_n} of X_n with law P_n converges weakly in probability to μ_{X_D} as in Theorem 3.3 with $\Theta = \sigma_V$.

The proof of theorem 3.3 is not straightforward since Theorem 2.1 only guarantees the weak convergence of $L_{(z-X_N)(z-X_N)^*}(x)$ whereas $\log|x|$ is not bounded. Hence, a complete proof requires to control the small eigenvalues of $(z-X_N)(z-X_N)^*$. Such a problem was also encountered in the case of a $N \times N$ matrix X_N with i.i.d entries. When the entries are Gaussian, Ginibre [19] proved that the spectral measure of X_N converges in probability to the so-called circular law, that is the uniform measure on the disk with radius one. Extending this result to non Gaussian entries happened to be a difficult task. The idea to use the Green formula (2) is due to Girko [20], who however did not address the difficulties related with the unboundedness of the logarithm. The circular law was proved under some conditions by Bai [21] and finally, in full generality, by Gotze and Tikhomirov [22] and Tao and Vu [23], by dealing with this question.

4. Enumeration of maps

The uses of matrix integrals as generating functions for the enumeration of graphs in physics and the so-called topological expansion are diverse. Let us give a few examples: the enumeration of triangulations following Brézin, Itzykson, Parisi and Zuber, the enumeration of meanders (Di Francesco et al), the study of loop configurations and the O(n) model (Eynard, Kostov ...), the application to knots theory (Zuber, Zinn Justin ...), the relation with algebraic geometry and topological string theory (the famous Dijkgraaf-Vafa conjecture states that Gromov-Witten invariants generating functions should be matrix integrals), Harer and Zagier (1986) in their article on the Euler characteristic of the moduli space of curves, and the famous work of Kontsevich. It became a cornerstone in free probability when Voiculescu (1984) found out that the combinatorics of moments of several independent matrices are, when their size goes to infinity, the same as the combinatorics of free variables. Since then, random matrices and their combinatorics appeared as a central tool in free probability (see e.g. the recent book of Nica and Speicher [17]). In this section, we shall precise some elements of the relation between these two fields. Let us first recall the key result of Brézin, Itzykson, Parisi and Zuber [2].

Theorem 4.1 (Brézin, Itzykson, Parisi and Zuber 78'). *Let* $V(X_1,\ldots,X_m) = \sum \beta_i q_i(X_1,\ldots,X_m) + \frac{1}{2}\sum X_i^2$ *with words* $(q_i)_{1\le i\le n}$ *and*

$$d\mu_V^N(X_1,\ldots,X_m) = (Z_V^N)^{-1} e^{-N\mathrm{Tr}(V(X_1,\ldots,X_m))} dX_1\cdots dX_m.$$

For any monomial P*, we have as a formal expansion in* β_i *and* N

$$\int [\frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_m))] d\mu_V^N = \sum_{g\ge 0} \frac{1}{N^{2g}} \tau_{\beta,g}(P) \tag{3}$$

with, for some integer numbers $M_g((1,P),(k_i,q_i))$*,*

$$\tau_{\beta,g}(P) = \sum_{k_1,\ldots,k_n\ge 0} \prod_{i=1}^{p} \frac{(-\beta_i)^{k_i}}{k_i!} M_g((1,P),(k_i,q_i))$$

The integer number $M_g((1,P),(k_i,q_i))$ is the number of certain graphs that we now describe.

First, let us define what is a polygon of type q for a word q in m letters $(X_i, 1\le i\le m)$. It is a polygon drawn on the sphere with one marked side and colored sides constructed as follows. We associate to each index $i, 1\le i\le m$, a color (called 'color i') and to each letter $X_i, 1\le i\le m$ a side with color i. A polygon of type $q = X_{i_1}\cdots X_{i_k}$ is a polygon embedded in the sphere by drawing the sides corresponding to the letters of q successively; following the orientation of the sphere, we first draw a side with color i_1, then of color i_2 until the end where the loose end of the side of color i_k is glued with the loose end of the side with color i_1. $M_g((k_i,q_i)_{1\le i\le p})$ is the number of coverings (also called maps) of a surface with genus g by k_i polygons of type q_i, $1\le i\le p$ where only sides of the same color can be glued together. The counting is done for labelled sides. Such enumeration question is highly non trivial, in particular when polygons are colored. They are related at criticality with statistical models on $\mathbb{Z}^2$ by the Knizhnik-Polyakov-Zamolodchikov relation [24].

In their seminal article [2], Brézin, Itzykson, Parisi and Zuber used their result to count some planar maps, that is coverings of the sphere, by estimating the related matrix integrals. This assumes that the large N limit can be taken in (3). This point was justified recently in a series of papers [25–30].

Theorem 4.2. *Let* $V = \sum \beta_i q_i(X_1,\ldots,X_m) + \frac{1}{2}\sum X_i^2$ *with words* $(q_i)_{1\le i\le n}$ *and*

$$d\mu_V^N(X_1,\ldots,X_m) = (Z_V^N)^{-1} e^{-N\mathrm{Tr}(V(X_1,\ldots,X_m))} dX_1\cdots dX_m.$$

For any $\ell > 0$*, if the* β_i*'s are small enough and* V *strictly convex, then for any word* P*,*

$$\int [\frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_m))] d\mu_V^N = \sum_{g=0}^{\ell} \frac{1}{N^{2g}} \tau_{\beta,g}(P) + o(\frac{1}{N^{2\ell}}) \tag{4}$$

with, for interesting integer numbers $M_g((1,P),(k_i,q_i))$,

$$\tau_{\beta,g}(P) = \sum_{k_1,\ldots,k_n\geq 0} \prod_{i=1}^{p} \frac{(-\beta_i)^{k_i}}{k_i!} M_g((1,P),(k_i,q_i))$$

Here, V is strictly convex iff there exists $c>0$ so that for any $p\times p$ Hermitian matrices $X_1,\ldots,X_m$, any $p\in\mathbb{N}$, $\mathrm{Tr}V(X_1,\ldots,X_m)$ is real and

$$\mathrm{Hess}(\mathrm{Tr}V(X_1,\ldots,X_m)) \geq cI.$$

(4) implies in particular that the free energy $\frac{1}{N^2}\log Z_V^N$ converges as N goes to infinity under the hypotheses of the theorem. We prove below, by using free probability ideas, that such a convergence extends to the case where V is strictly convex or even "locally strictly convex" provided we add a cutoff. If V is just some polynomial going to infinity at infinity so that the free energy is well defined, it is still an open problem to establish this convergence. The idea to extend the convergence to this convex situation is to use dynamics and type of Monte-Carlo approximation argument. More precisely, in strictly convex situations, it is well known that Langevin dynamics converge as time goes to infinity to their unique equilibrium measure, whatever is their initial distribution. This is true at the level of large matrices but also at the free probability limit. The idea is thus to approximate Gibbs measures by their dynamics, use that they converge to the free dynamics as dimension goes to infinity, and then that the free dynamics are close to their equilibrium state when time is large. In other words, we want to show that the diagram below is approximately commutative

$$\begin{array}{ccc}
X_t^N = H_t^N - \frac{1}{2}\int_0^t DV(X_s^N)ds & \to t\to\infty\to & \mu_V^N \\
\downarrow & & \downarrow \\
N\to\infty & & N\to\infty \\
\downarrow & & \downarrow \\
X_t = S_t - \frac{1}{2}\int_0^t DV(X_s)ds & \to t\to\infty\to & \tau_{\beta,0}
\end{array}$$

Of course, this approach can only be valid because the above approximations hold uniformly, with constants depending only on the convexity of the potential.

We say that V is locally strictly convex iff $\exists c>0$

$$\mathrm{Hess}(\mathrm{Tr}V(X_1,\ldots,X_m)) \geq cI$$

on the set of self-adjoint operators $X_1,\ldots,X_m$ (in a C^* algebra) so that $\|X_i\|_\infty \leq L(c)$ for some $L(c)$. We let

$$d\mu_V^{N,L}(X_1,\ldots,X_m) = (Z_V^{N,L})^{-1}\mathbf{1}_{\|X_i\|_\infty\leq L} e^{-N\mathrm{Tr}(V(X_1,\ldots,X_m))} dX_1\cdots dX_m.$$

We then can prove [31]

Theorem 4.3. *If $V = \frac{1}{2}\sum X_i^2 + \sum \beta_i q_i$ is "locally strictly convex", for L large enough (but finite if V is not globally strictly convex),*

$$\lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_m))d\mu_V^{N,L} = \tau_{\beta,0}(P)$$

with $\tau_{\beta,0}(P)$ the analytic extension of the generating function for the enumeration of coverings of the sphere.

The key step to prove this theorem is to show the convergence of the free dynamics to the tracial state $\tau_{\beta,0}$, which in turns also gives some non trivial properties of the latter.

Theorem 4.4. *Let $V = \frac{1}{2}\sum X_i^2 + \sum \beta_i q_i$ be "locally strictly convex", and X_t be the solution of*

$$X_t^i = S_t^i - \frac{1}{2}\int_0^t D_i V(X_s)ds.$$

Then

- *$\tau(P(X_t))$ converges, as $t \to \infty$, to $\tau_{\beta,0}(P(X))$.*
- *For small β_i's,*

$$\tau_{\beta,0}(P) = \sum_{k_1,\ldots,k_n\geq 0} \prod_{i=1}^{p} \frac{(-\beta_i)^{k_i}}{k_i!} M_0((1,P),(k_i,q_i))$$

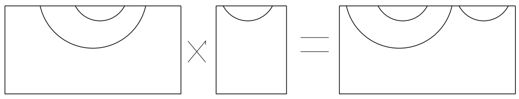
  and teh above series converges absolutely. $\tau_{\beta,0}(P)$ extends as an analytic function on the domain of local strict convexity of the potential.
- *X_i has a connected spectrum under $\tau_{\beta,0}$, and in fact more precisely any polynomial $P(X_1,\ldots,X_m)$ of $X_1,\ldots,X_m$ under $\tau_{\beta,0}$ is an operator with connected spectrum.*

The last point is an amazing application of operator algebra theory. Indeed, we show that $\tau_{\beta,0}$ is the law of the solution of the free diffusion as time goes to infinity and in fact that the convergence holds for the operator norm. This in turn guarantees that this limit belongs to the C^*-algebra constructed with the free increments of the free Brownian motion. The result follows since it is well known [32] that such C^*-algebra does not contain a projection.

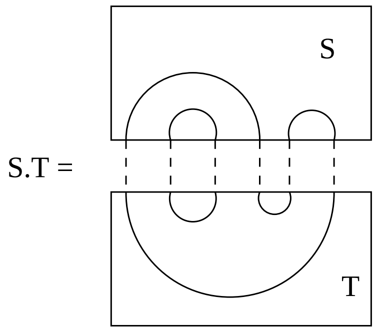
The fact that $\tau_{\beta,0}$ can be constructed as a limit of matrix models or free dynamics can also be used

- to compute $\tau_{\beta,0}(P)$ (see the enumeration of triangulations [2]),
- to show that some generating functions of combinatorial numbers are tracial states.

We next consider the second application. Even though this point can also be proved by combinatorial arguments, constructing matrix models for these enumeration questions often appear to be a short cut to prove that generating functions of interesting numbers are indeed tracial state. The goal of this application is to construct II_1 factors, and more precisely towers of factors with prescribed index. Recall that factors are von Neumann algebras (that is weakly closed algebras of bounded operators on a Hilbert space equipped with an involution and a neutral element) with a trivial center. They are of said to be of type II_1 if they are equipped with a tracial state. A tower of factors is a sequence of factors $(M_n)_{n\geq 0}$ which are embedded in each other ($M_n \subset M_{n+1}$). The index $[M_n; M_{n+1}]$ measures somehow the 'size' of M_{n+1} with respect to M_n as follows. It can be seen that M_{n+1} is generated by M_n and a projection e_{n+1} and then for all $x \in M_n$, $\mathrm{tr}(xe_{n+1}) = \lambda \mathrm{tr}(x)$ with $\lambda = 1/[M_n; M_{n+1}]$. It was shown by Jones [33] that the index can only take the values $\{4\cos^2 \pi/n, n = 3, 4, \cdots\} \cup [4, \infty)$. With Jones and Shlyakhtenko we have constructed a tower of subfactors for any possible values of the index, based on planar algebra structure (and in fact mainly Temperley-Lieb algebras). Temperley-Lieb elements are boxes containing non-intersecting strings. We can endow this set with the multiplication given by simply drawing the drawings next to each other



We also endow this algebra by the involution which is given by taking the symmetric picture of the element. We denote $S.R$ the drawing obtained, for two Temperley-Lieb diagrams S, R with the same number of boundary points, by drawing these two diagrams in front of each other and gluing the boundary points pairwise by straight lines.



We then obtain a collection of non intersecting loops (two in the above picture). We then consider the trace given by

$$\tau(S) = \sum_{R \in \mathrm{TL}} \delta^{\#\text{ loops in } S.R}$$

where the exponent in δ is the number of loops in the drawing defined by $S.R$ and we sum over all Temperley-Lieb diagrams R with the same number of boundary points than S. The next result proves [34] that if we take the weak completion of the resulting algebra we obtain a factor. Moreover, we can construct a tower by considering the kth multiplication to be given by gluing by embedded arches the k nearest neighbouring boundary points of the two elements and defining similarly a trace by summing over Temperley-Lieb elements except for the k boundary points at the two extremes which are glued by embedded arches.

Theorem 4.5 (G-Jones-Shlyakhtenko 07'). *Take*
$\delta \in I := \{2\cos(\frac{\pi}{n})\}_{n\geq 4} \cup]2, \infty[$
- *τ is a tracial state.*
- *The corresponding von Neumann algebra is a factor. A tower of factors with index δ^2 can be built.*

The fact that τ is a tracial state was first proved by using matrix models [34] but then a combinatorial proof was given [35]. The matrix models approach follows the idea of the planar algebra of a graph [36]. In fact, the idea to get the construction for integer values of δ is to use the embedding from Temperley-Lieb diagrams into the set of polynomials in δ variables as follows. Suppose that we are given a box B with $2k$ boundary points. Assume also that there are k non-crossing curves inside B which connect pairs of boundary points together, hence yielding a Temperley-Lieb element. Let π be the associated non-crossing pairing of $\{1, \ldots, 2k\}$ and denote $p \approx \ell$ if (p, ℓ) is a block of π. We associate to B the non commutative polynomial

$$P_B(X_1, \ldots, X_n) := \sum_{\substack{1\leq i_1,\ldots,i_{2k}\leq n \\ i_\ell = i_p \text{if} \ell \approx \text{p}}} X_{i_1} \cdots X_{i_{2k}}.$$

Taking the $(X_i, 1 \leq i \leq n)$ to be independent GUE matrices and letting the size going to infinity, we know that the expectation of the renormalized trace of polynomials in $(X_1, \ldots, X_n)$ converge to the number of non crossing pairings of the letters that can be build above this polynomial so that only all pairing contain only the same letter. By symmetry, it is not hard to see that when summing over all these graphs, each loop will come δ times, hence yielding the trace τ. For more general δ's, one has to sum over the vertices of a graph whose adjacency matrix has eigenvalue δ [34, 36].

This approach can be generalized as follows. Let $S_1, \ldots, S_n$ be Temperley-Lieb elements. Let $\beta_1, \ldots, \beta_n$ be small real numbers and for any Temperley-Lieb element S, define

$$\mathrm{Tr}_\beta(S) = \sum_{n_i \geq 0} \sum \prod_{1\leq i\leq n} \frac{\beta_i^{n_i}}{n_i!} \delta^{\sharp \text{loops}}$$

where we sum over all connected planar diagrams build over n_i diagrams S_i and one diagram S by matching the boundary points of these diagrams and we count the number of loops of the full picture. Then, we can prove

Theorem 4.6 (G-Jones-Shlyakhtenko 09').
Take $\delta \in I := \{2\cos(\frac{\pi}{n})\}_{n\geq 4} \cup]2, \infty[$. Tr_β is a tracial state, as a limit of matrix (or free probability) models.

The construction is made by considering, instead of independent Gaussian random matrices, random matrices interacting via a potential chosen appropriately.

5. Conclusion

In this review, we tried to advertise free probability theory to the physicists community. Indeed, it is particularly convenient to describe the asymptotics of random matrices with genuinely independent random eigenvectors which are given by free operators. Hence, many of such limits have already been studied in free probability, cf. the so-called R-diagonal operators which describe the limit of non-normal matrices. Moreover, because free probability has developed many powerful tools from classical probability, it can give new ideas to study random matrices, cf. Monte-Carlo type of ideas to generate a Gibbs measure by Langevin dynamics, which in turn allows to study several matrix models in non-perturbative situations. The relation between free probability theory and operator algebra theory, equipped with the classical notion of freeness, is also very important to analyze the asymptotics of random matrices, cf. the connectivity of the support of the spectral measure of random matrices interacting via a convex potential. Vice versa, the relation between random matrices and combinatorics developed by the so-called topological expansion allows to get more insight in operator algebra, cf. the construction of the tower of factors. Of course, there are much more applications and developments around these themes and we refer the readers to review articles and books [14–17]. To conclude, we would like however to point out that the range of applications of free probability mainly concerns random matrices whose eigenvectors are approximately uniformly distributed on the sphere (which correspond to Haar distributed eigenvectors). This is well known to be the case for instance for Wigner matrices with independent entries with high enough moments, which have “delocalized” eigenvectors. When the entries have no second moment, the asymptotic distribution is different [37–39], the eigenvectors more localized [40] and it is not clear how to interpret the limit in the free probability context.

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