Combinatorial aspects of matrix models

Alice Guionnet*, Edouard Maurel-Segala†

9 septembre 2005

Abstract

We show that under reasonably general assumptions, the first order asymptotics of the free energy of matrix models are generating functions for colored planar maps. This is based on the fact that solutions of the Schwinger-Dyson equations are, by nature, generating functions for enumerating planar maps, a remark which bypasses the use of Gaussian calculus.

Keywords : Random matrices, non-commutative measure, map enumeration.

Mathematics Subject of Classification: 15A52, 46L50, 05C30.

1 Introduction

It has long been used in combinatorics and physics that moments of Gaussian matrices have a valuable combinatorial interpretation. The first result in this direction is due to Wigner [31] who proved that the trace of even moments of a $N \times N$ Hermitian matrix A with i.i.d centered entries with covariance N^{-1} converge as N goes to infinity towards the Catalan numbers which enumerate non crossing partitions. If one restricts to Gaussian entries, that is matrices following the law μ_N of the **GUE** which is the probability measure on the set $\mathcal{H}_N$ of $N \times N$ Hermitian matrices with density

$$\mu_N(dA) = \frac{1}{Z_N} 1_{A \in \mathcal{H}_N} e^{-\frac{N}{2}\mathrm{tr}(A^2)} \prod_{1 \le i \le j \le N} d\Re e(A_{ij}) \prod_{1 \le i < j \le N} d\Im m(A_{ij}),$$

it occurs that the corrections to this convergence count graphs which can be embedded on surface of higher genus, a fact which was used by Harer and Zagier [19]. This enumerative property was fully developed after 't Hooft, who considered generating functions of such moments. For instance, c.f Zvonkin [32], we have the formal expansion

$$F_N(tx^4) = \frac{1}{N^2} \log \int e^{-Nt\mathrm{tr}(A^4)} d\mu_N(A) = \sum_{g \ge 0} N^{-2g} \sum_{k \ge 1} \frac{(-t)^k}{k!} C(k,g)$$

with

$$C(k,g) = \mathrm{Card}\{ \text{ maps with genus } g \text{ with } k \text{ stars of valence } 4\}$$

*Ecole Normale Supérieure de Lyon, Unité de Mathématiques pures et appliquées, UMR 5669, 46 Allée d'Italie, 69364 Lyon Cedex 07, France. E-mail: aguionne@umpa.ens-lyon.fr.

†Ecole Normale Supérieure de Lyon, Unité de Mathématiques pures et appliquées, UMR 5669, 46 Allée d'Italie, 69364 Lyon Cedex 07, France. E-mail: emaurel@umpa.ens-lyon.fr

Here, maps are connected oriented diagrams which can be embedded into a surface of genus g in such a way that edges do not cross and the faces of the graph (which are defined by following the boundary of the graph) are homeomorphic to a disc. The counting is done up to equivalent classes, i.e. up to homeomorphism. Let us stress that the above equality is only formal and should be understood in the sense that all the derivatives at the origin on both sides of the equality match, it means that, for all $k \in \mathbb{N}$,

$$(-1)^k \partial_t^k F_N(tx^4)|_{t=0} = \sum_{g \geq 0} \frac{1}{N^{2g}} C(k,g)$$

which can be proved thanks to Wick's formula (note above that the sum is in fact finite).

Such expansions can be generalized to arbitrary polynomial functions (to enumerate maps with vertices of different degrees) and to several-matrices integrals which allow to enumerate colored maps. More precisely, let V be a polynomial of m non-commutative variables, $V = \sum_{i=1}^n t_i(q_i + q_i^*)$ with some monomials q_i and real parameters t_i (and with q_i^* being the adjoint of q_i as defined in section 2). Then, the free energy expands formally into

$$F_N(V) = \frac{1}{N^2} \log \int e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))} d\mu_N(A_1)\cdots d\mu_N(A_m) = \sum_{g\geq 0} \frac{1}{N^{2g}} F_g(t_1,\cdots,t_n)$$

where for $g \in \mathbb{N}$, F_g is a generating function for the enumeration of colored maps of genus g related to the monomials $(q_i)_{1\leq i \leq n}$.

The interest in such formal expansions lies in the hope to be able to estimate the free energy $F_N(V)$ when N goes to infinity by probability techniques, henceforth finding formulae for the generating functions $(F_g)_{g\geq 0}$. Such a strategy can only be validated if the expansion is not only formal, i.e. that for reasonable (eventually small but non zero) parameters $(t_1,\cdots,t_n)$, for all $k \in \mathbb{N}$ and for N large enough,

$$F_N(V) = \sum_{g=0}^{k} \frac{1}{N^{2g}} F_g(t_1,\cdots,t_n) + o(\frac{1}{N^{2k}}).$$

This means that one can invert the limits of t small and N large in the expansion.

Our aim is to look beyond this formal approach and try to justify this inversion of limits.

In the case of one matrix integrals, this problem is quite well understood at any level of the expansion and for any reasonable potentials V (see [1] and [12] for instance).

Several matrix models are much harder. In the physics literature, the focus is mostly on a few specific integrals; we refer the interested reader to the reviews [11, 15]. In the mathematical literature, fewer matrix integrals could be analyzed and only their first order asymptotics could be derived (see Mehta et al. [24, 23] and Guionnet et al. [16, 14]). Even for these last integrals, the relation of their first order asymptotics with the related enumeration problem was not yet established rigorously. In combinatorics, another road was opened by Bousquet-Melou and Schaeffer [7], following the ideas of Tutte [27], to enumerate colored planar maps; instead of studying matrix models, they used directly bijection between maps and well labeled trees.

In this paper, we shall focus on the first order asymptotics of matrix models. We shall provide a rigorous mathematical ground to relate them with the enumeration of planar maps.

To establish such a relation, we shall study an even more interesting quantity than the free energy, namely, the limiting empirical distribution of matrices; for $A_1,\cdots,A_m \in \mathcal{H}_N^m$, it is defined as the linear form on the set $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ of polynomials of m non-commutative variables so that

$$\hat{\mu}^N_{A_1,\cdots,A_m}(P) = \frac{1}{N}\mathrm{tr}(P(A_1,\cdots,A_m)) \quad \forall P \in \mathbb{C}\langle X_1,\cdots,X_m\rangle.$$

Let μ^N_V be the probability measure on $\mathcal{H}^m_N$ given by

$$\mu^N_V(dA_1,\cdots,dA_m) = e^{-N^2F_N(V)}e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))}\prod_{i=1}^m d\mu_N(A_i)$$

with $F_N(V)$ as above. In the sequel, we take a potential $V = V_{\bar{t}} = \sum_{i=1}^n t_i(q_i + q_i^*)$. Then we proceed in two steps to relate the first order asymptotic of $\hat{\mu}^N_{A_1,\cdots,A_m}$ under μ^N_V to the enumeration of planar maps.

• First, we study the solution τ in the algebraic dual $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ of $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ of the so-called Schwinger-Dyson equations **SD[V]**:

$$\tau_V((X_i + \mathcal{D}_iV)P) = \tau_V \otimes \tau_V(D_iP)$$

for all $P \in \mathbb{C}\langle X_1,\cdots,X_m\rangle$ and $i \in \{1,\cdots,m\}$. Here, D_i and $\mathcal{D}_i$ are respectively the non-commutative derivative and the cyclic derivative with respect to the i^{th} variable (see paragraph 2.2). We give sufficient conditions on V so that solutions to this equation exist and are unique.

Moreover, we relate solutions to **SD[V]** with generating functions of planar maps. To do that, let us associate to $(X_i)_{1\le i\le m}$ m branches of different colors, and to a monomial $q(\mathbf{X}) = X_{i_1}\cdots X_{i_p}$ a star with p colored branches by ordering clockwise the branches corresponding to $X_{i_1},\cdots,X_{i_p}$. Such a star is said to be of type q. Note that it has a distinguished branch, the first one, X_{i_1}, and its branches are oriented by the above clockwise order (one should imagine the star to be fat, each branch made of two parallel segments which have opposite orientation, the whole orientation being given by the clockwise order). This defines a bijection between non-commutative monomial and oriented stars with colored branches and one distinguished branch. Alternatively, a star can be seen as an oriented circle with colored dots and one marked dot. A planar map is a connected graph embedded into the sphere with colored stars, each branch of each star being glued with exactly one branch of the same color and the edges obtained in this way do not cross each other (see a more precise description of the planar maps we enumerate in subsection 2.5).

We can now relate Schwinger Dyson's equation and maps enumeration:

Theorem 1.1 *Let $R > 2$, then there exists an open neighborhood $U \subset \mathbb{R}^n$ of the origin (a ball of positive radius) such that:*

- *For $\bar{t} \in U$, there exists a unique $\tau_{\bar{t}} \in \mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ which is a solution to $\mathbf{SD}[V_{\bar{t}}]$ and such that for all p, for all $i_1,\cdots,i_p$ in $\{1,\cdots,m\}$, $|\tau_{\bar{t}}(X_{i_1}\cdots X_{i_1})| \le R^p$*
- *For all P monomial in $\mathbb{C}\langle X_1,\cdots,X_m\rangle$, $\bar{t} \to \tau_{\bar{t}}(P)$ is analytic on U and for all $k_1,\cdots,k_n$ integers, $(-1)^{\Sigma k_i}\partial_{t_1}^{k_1}\cdots\partial_{t_n}^{k_n}\tau_{\bar{t}}(P)|_{\bar{t}=0}$ is the number of maps with k_i stars of type q_i or q_i^* and one of type P.*

Hence, $(\tau_{V_{\bar{t}}})_{|t|\le\epsilon}$ are generating states for the enumeration of colored planar maps and Schwinger Dyson's equations can be viewed as the generating differential equations to enumerate colored planar maps. This is due to the fact that the action of the derivatives D_i and $\mathcal{D}_i$ on monomials, under the above bijection between stars and monomials, produces natural operations on planar maps.

• Then, we shall see (see section 3) that, under some appropriate assumptions on V, $\hat{\mu}^N_{A_1,\cdots,A_m}$ converges almost surely under $\mu^N_V(dA_1,\cdots,dA_m)$ towards a solution τ_V to the Schwinger-Dyson equations $\mathbf{SD}[V]$.

First, we show under rather general assumptions that the limit points of $\hat{\mu}^N_{A_1,\cdots,A_m}$ will solve a weak form of the Schwinger-Dyson equation (see section 3.1) which turns into its strong form if the limit points are compactly supported, i.e have all the moments of monomial functions of degree d bounded by R^d for some finite constant R. For small t_i's, this proves that $\hat{\mu}^N_{A_1,\cdots,A_m}$ converges almost surely towards the solution of $\mathbf{SD[V]}$ if we know that the limit points of $\hat{\mu}^N_{A_1,\cdots,A_m}$ satisfy such a bound. We then give sufficient conditions to obtain such an a priori estimate.

We consider convex potential V (see section 3.2) for which we have:

Theorem 1.2 *Let U be the set of t_i's for which $V = V_{\bar{t}}$ is convex, then there exists $\epsilon > 0$ such that for $(t_i)_{1\le i\le n} \in U \cap B(0,\epsilon)$, $\hat{\mu}^N_{A_1,\cdots,A_m}$ converges in $L^1(\mu^N_{V_{\bar{t}}})$ and almost surely to the unique solution to $\mathbf{SD}[V_{\bar{t}}]$ as described in theorem 1.1*

For general potential V, we obtain a similar result provided we add a cut-off (see section 3.3).

Coming back to the free energy of matrix models, we conclude (see Theorem 3.3) that when the empirical distribution of matrices converges towards the solution to Schwinger-Dyson's equations, the free energy is also a generating function of the associated planar maps.

As a consequence, we can apply these results to the study of Voiculescu's microstates entropy (see section 4) and show that the microstates entropy can be estimated at the solutions to $\mathbf{SD[V]}$ when the t_i's are small enough.

Finally, we compare diverse approaches to the enumeration of planar maps by either using matrix models or combinatorics techniques.

The results of this paper are clearly known, at least at a subconscious level, by physicists, but we could not find any proper reference on the subject. However, we want to emphasize that the use of Schwinger Dyson's equations is well spread in physics. This paper is rather elementary but provides a mathematical framework to the study of matrix models and related map enumeration. We hope it will demystify this interesting field of physics to mathematicians, or at least to probabilists. The generalization of the techniques developed in this paper to higher order expansions is the subject of a forthcoming article.

2 Schwinger Dyson's equations and combinatorics

2.1 Tracial states

Let $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ be the set of polynomial functions in m self-adjoint non-commutative variables. We endow $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ with the involution given for all $z\in\mathbb{C}$, all $i_1,\cdots,i_p\in\{1,\cdots,m\}$ and all $p\in\mathbb{N}$, by

$$(zX_{i_1}\cdots X_{i_p})^* = \bar{z}X_{i_p}\cdots X_{i_1}.$$

We will say that P in $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ is self-adjoint if $P^*=P$.

For any $R>0$, completing $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ for the norm

$$||P||_R = \sup_{\mathcal{A}\ C^*-\text{algebra}} \sup_{\substack{a_1,\ldots,a_m\in\mathcal{A},\\ a_i=a_i^*,\|a_i\|_{\mathcal{A}}\le R}} ||P(a_1,\cdots,a_m)||_{\mathcal{A}}$$

produces a C^*-algebra $\mathbb{C}\langle X_1,\cdots,X_m\rangle_R = (\mathbb{C}\langle X_1,\cdots,X_m\rangle, ||.||_R, *)$.

We let $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ be the set of real linear forms on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ (i.e linear forms such that $\tau(a^*) = \overline{\tau(a)}$), and denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R^*$ the subset of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ of continuous forms with respect to the norm $||.||_R$, i.e the topological dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R$.

We let $\mathcal{M}^m$ be the set of laws of m bounded self-adjoint non-commutative variables, that is the subset of elements τ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ such that

$$\tau(PP^*) \geq 0, \quad \tau(PQ) = \tau(QP) \quad \forall P, Q \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \quad \tau(I) = 1. \tag{1}$$

It is not hard to see that for any $R < \infty$, $\mathcal{M}_R^m = \mathbb{C}\langle X_1, \cdots, X_m\rangle_R^* \cap \mathcal{M}^m$ is a compact metric space for the weak*-topology by Banach-Alaoglu theorem. Elements of $\mathcal{M}^m = \cup_{R\geq 0}\mathcal{M}_R^m$ are said to be compactly supported, by analogy with the case $m = 1$ where they are indeed compactly supported probability measures. A family $(\tau_t)_{t\in I}$ of elements of $\mathcal{M}_R^m$ for some $R < \infty$ is said to be uniformly compactly supported.

To deal with variables which do not have all their moments, we eventually can change the set of test functions and, following [8], consider instead of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ the complex vector space $\mathcal{C}_{st}^m(\mathbb{C})$ generated by the Stieljes functionals

$$ST^m(\mathbb{C}) = \{ \prod_{1\leq i\leq p}^{\rightarrow} (z_i - \sum_{k=1}^m \alpha_i^k \mathbf{X}_k)^{-1}; \quad z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{R}, p \in \mathbb{N}\} \tag{2}$$

where $\prod^{\rightarrow}$ is the non-commutative product. We can give to $ST^m(\mathbb{C})$ an involution and a norm

$$||F||_\infty = \sup_{\mathcal{A}C^*-\text{algebra}} \sup_{a_i = a_i^* \in \mathcal{A}} ||F(a_1, \cdots, a_m)||_\infty$$

where the supremum is taken eventually on unbounded operators affiliated with $\mathcal{A}$, which turns it into a C^*-algebra. We will denote $\mathcal{C}_{st}^m(\mathbb{R}) = \{G = F + F^*, F \in \mathcal{C}_{st}^m(\mathbb{C})\}$. We will let $\mathcal{M}_{ST}^m$ be the set of linear forms on $\mathcal{C}_{st}^m(\mathbb{C})$ which satisfy (1) (but with functions of $\mathcal{C}_{st}^m(\mathbb{C})$ instead of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$). If one equips $\mathcal{M}_{ST}^m$ with its weak topology, then $\mathcal{M}_{ST}^m$ is a compact metric space (see [8]).

2.2 Non-commutative derivatives

We let $D_i : \mathbb{C}\langle X_1, \cdots, X_m\rangle \to \mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ be the non-commutative derivative with respect to X_i given by the Leibnitz rule

$$D_i(PQ) = D_iP \times (1 \otimes Q) + (P \otimes 1) \times D_iQ$$

for any $P, Q \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ and the condition

$$D_iX_j = 1_{i=j} 1 \otimes 1.$$

In other words, if P is a non-commutative monomial

$$D_iP = \sum_{P = P_1 X_i P_2} P_1 \otimes P_2$$

where the sum runs over over all possible decomposition of P as $P_1 X_i P_2$. This definition can be extended to $\mathcal{C}_{st}^m(\mathbb{C})$ by keeping the above Leibnitz rule (but with P, Q in $\mathcal{C}_{st}^m(\mathbb{C})$) and

$$D_i(z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1} = \alpha_i (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1} \otimes (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1}.$$

We also define the cyclic derivative $\mathcal{D}_i$ as follows. Let $m : \mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle \to \mathbb{C}\langle X_1, \cdots, X_m\rangle$ (resp. $\mathcal{C}^m_{st}(\mathbb{C}) \otimes \mathcal{C}^m_{st}(\mathbb{C}) \to \mathcal{C}^m_{st}(\mathbb{C})$) be defined by $m(P \otimes Q) = QP$. Then, we set

$$\mathcal{D}_i = m \circ D_i.$$

If P is a non-commutative monomial, we have

$$\mathcal{D}_i P = \sum_{P = P_1 X_i P_2} P_2 P_1.$$

2.3 Schwinger-Dyson's equation

Let V be self-adjoint and consider the following equation on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$; we say that $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies the Schwinger-Dyson equation with potential V, denoted in short **SD[V]**, if and only if for all $i \in \{1, \cdots, m\}$ and $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\tau(I) = 1, \quad \tau \otimes \tau(D_i P) = \tau((\mathcal{D}_i V + X_i)P) \qquad \mathbf{SD[V]}$$

These equations are called Schwinger-Dyson's equations in physics, but in free probability, one would rather say that the conjugate variable (or alternatively the non-commutative Hilbert transform) $D_i^* 1$ under τ is equal to $X_i + \mathcal{D}_i V$ for all $i \in \{1, \cdots, m\}$.

2.4 Uniqueness of the solutions to Schwinger-Dyson's equations for small parameters

In this paper, we shall restrict ourselves to non-oscillatory integrals, that is to the case where $\mathrm{tr}(V(X_1, \cdots, X_m))$ is real for any m-uple of Hermitian matrices. In other words,

$$\mathrm{tr}(V(X_1, \cdots, X_m)) = \mathrm{tr}(V^*(X_1, \cdots, X_m)) = \mathrm{tr}(2^{-1}(V + V^*)(X_1, \cdots, X_m))$$

for any m-uple of Hermitian matrices. Thus, we shall assume that

$$V(X_1, \cdots, X_m) = V_{\bar{t}}(X_1, \cdots, X_m) = \sum_{i=1}^n t_i (q_i(X_1, \cdots, X_m) + q_i^*(X_1, \cdots, X_m))$$

where the q_i's are monomial functions of m non-commutative indeterminates and $\bar{t} = (t_1, \cdots, t_n)$ are real parameters.

In this paragraph, we shall consider solutions to $\mathbf{SD}[V_{\bar{t}}]$ which satisfy a compactness condition that we shall discuss in the following subsections. Let $R \in \mathbb{R}^+$ (We will always assume $R \geq 1$ without loss of generality).

(H(R))*An element $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies* **(H(R))** *if and only if for all $k \in \mathbb{N}$,*

$$\max_{1 \leq i_1, \cdots, i_k \leq m} |\tau(X_{i_1} \cdots X_{i_k})| \leq R^k.$$

In the sequel, we denote D the degree of V, that is the maximal degree of the q_i's; $q_i(X) = X_{j_1^i} \cdots X_{j_{d_i}^i}$ with, for $1 \leq i \leq n$, $\deg(q_i) =: d_i \leq D$ and equality holds for some i.

The main result of this paragraph is

Theorem 2.1 *For all $\bar{t} \in \mathbb{R}^n$, there exists $A(\bar{t}) = A(|t|) \in \mathbb{R}^+$ with $|t| = \max_{1 \leq i \leq n} |t_i|$, $A(|t|)$ goes to infinity when $|t|$ goes to zero, so that for $R \leq A(|t|)$, there exists at most one solution $\tau_{\bar{t}}$ to* $\mathbf{SD}[V_{\bar{t}}]$ *which satisfies* **(H(R))**.

Remark: Note here that it could be believed at first sight that the solutions to **SD[V]** are not unique since they depend on the trace of high moments $\tau(q_j P)$. However, our compactness assumption **(H(R))** gives uniqueness because it forces the solution to be in a small neighborhood of the law $\tau_0 = \sigma^m$ of m free semi-circular variables, so that perturbation analysis applies. We shall see in Theorem 2.3 that this solution is actually the one which is related with the enumeration of maps.

Proof.

Let us assume we have two solutions τ and τ'. Then, by the equation **SD[V]**, for any monomial function P of degree $l-1$, for $i \in \{1, \cdots, m\}$,

$$(\tau - \tau')(X_i P) = ((\tau - \tau') \otimes \tau)(D_i P) + (\tau' \otimes (\tau - \tau'))(D_i P) - (\tau - \tau')(\mathcal{D}_i V P)$$

Hence, if we let for $l \in \mathbb{N}$

$$\Delta_l(\tau, \tau') = \sup_{\text{monomial } P \text{ of degree l}} |\tau(P) - \tau'(P)|$$

we get, since if P is of degree $l-1$,

$$D_i P = \sum_{k=0}^{l-2} p_k^1 \otimes p_{l-2-k}^2$$

where p_k^i, $i = 1, 2$ are monomial of degree k or the null monomial, and $\mathcal{D}_i V$ is a finite sum of monomials of degree smaller than $D-1$,

$$\begin{aligned}\Delta_l(\tau, \tau') &= \max_{P \text{ of degree } l-1} \max_{1 \le i \le m} \{|\tau(X_i P) - \tau'(X_i P)|\} \\ &\le 2 \sum_{k=0}^{l-2} \Delta_k(\tau, \tau') R^{l-2-k} + C|t| \sum_{p=0}^{D-1} \Delta_{l+p-1}(\tau, \tau')\end{aligned}$$

with a finite constant C (which depends on n only). For $\gamma > 0$, we set

$$d_\gamma(\tau, \tau') = \sum_{l \ge 0} \gamma^l \Delta_l(\tau, \tau').$$

Note that under **(H(R))**, this sum is finite for $\gamma < (R)^{-1}$. Summing the two sides of the above inequality times γ^l we arrive at

$$d_\gamma(\tau, \tau') \le 2\gamma^2(1 - \gamma R)^{-1} d_\gamma(\tau, \tau') + C|t| \sum_{p=0}^{D-1} \gamma^{-p+1} d_\gamma(\tau, \tau').$$

We finally conclude that if $(R, |t|)$ are small enough so that we can choose $\gamma \in (0, R^{-1})$ so that

$$2\gamma^2(1 - \gamma R)^{-1} + C|t| \sum_{p=0}^{D-1} \gamma^{-p+1} < 1$$

then $d_\gamma(\tau, \tau') = 0$ and so $\tau = \tau'$ and we have at most one solution. Taking $\gamma = (2R)^{-1}$ shows that this is possible provided

$$\frac{1}{4R^2} + C|t| \sum_{p=0}^{D-1} (2R)^{p-1} < 1$$

so that when $|t|$ goes to zero, we see that we need R to be smaller than $A(|t|)$ of order $|t|^{-\frac{1}{D-2}}$.

□

2.5 Combinatorics

In this paragraph we describe the combinatorial objects we are considering. Let us associate a colored star to any monomial. We associate to each $i \in \{1, \cdots, m\}$ a different color. Then, we define a bijection between oriented branch-colored stars with a distinguished branch and non-commutative monomials as follows. For any $i \in \{1, \cdots, m\}$, we associate to X_i a branch of color i. We shall say that a star is of type $q(X_1, \cdots, X_m) = X_{i_1} \cdots X_{i_l}$ if it is a star with l branches which we color clockwise ; the first branch will be of color i_1, the second of color i_2 ... etc ... until the l^{th} branch is colored with color i_l. Note that this star possesses a distinguished branch, the one corresponding to X_{i_1}, and an orientation, corresponding to the clockwise order. By convention, the star of type $q = 1$ is simply a point.

FIG. 1 – *The star of type* $q(X) = X_1^2 X_2^2 X_1^4 X_2^2$

A planar map is a connected graph embedded into the sphere with colored stars, each branch is glued with exactly one branch of the same color and the edges obtained in this way do not cross each other. Hence, branches are thought as half edges. Maps are only considered up to an homeomorphism of the sphere. Now we will be interested in enumerating maps with a fixed set of stars, we define for q_i a family of non necessarily distinct monomials and k_i a family of integers:

$$\mathcal{M}_0((q_1, k_1), (q_2, k_2), \ldots, (q_n, k_n)) = \sharp\{\text{planar maps build with } k_i \text{ stars of type } q_i\}.$$

We denote in short $\mathcal{M}_0(P, (q_1, k_1), \ldots, (q_n, k_n)) = \mathcal{M}_0((P, 1), (q_1, k_1), \ldots, (q_n, k_n))$. In that set, each star is labeled and has a marked branch (which corresponds to its first variable) so that for example $\mathcal{M}_0((X^4, 2)) = 36$. Now what we are really interested in is enumerating maps with a fixed number of stars of type q or q^* so that we define:

$$\mathcal{M}((q_1,k_1),\ldots,(q_n,k_n)) = \sum_{\substack{1\le p_i\le k_i,\\ 1\le i\le n}} \prod_{i=1}^{n} C_{k_i}^{p_i} \mathcal{M}_0((q_1,p_1),(q_1^*,k_1-p_1),\ldots,(q_n,k_n),(q_n^*,k_n-p_n))$$

and $\mathcal{M}(P,(q_1,k_1),\ldots,(q_n,k_n)) = \mathcal{M}((P,1),(q_1,k_1),\ldots,(q_n,k_n))$.

This quantity enumerates the number of ways to build a map on stars of fixed types up to the symmetry induced on stars by the operand $*$.

Due to the fact that everything is labeled, we enumerate lots of very similar objects. A way to avoid this problem is to look at the maps as they are enumerated by combinatoricians (see [7]). The idea is to forget every label and to add a root which is defined as a star and a branch of this star. We will say that a map is rooted at a monomial of type P if its root is of type P with the marked branch the first one in the above construction of a star from a monomial. We can define for P a monomial, k_i a family of integers and q_i a family of (this time) distinct monomials.

$$\begin{aligned}\mathcal{D}_0(P,(q_1,k_1),(q_2,k_2),\ldots,(q_n,k_n)) \quad = \sharp\{ \quad & \text{rooted planar maps with } k_i \text{ stars of type } q_i \\ & \text{and one of type } P \text{ which is the root}\}\end{aligned}$$

and

$$\mathcal{D}(P,(q_1,k_1),\ldots,(q_n,k_n)) = \sum_{\substack{1\le p_i\le k_i,\\ 1\le i\le n}} \mathcal{D}(P,(q_1,p_1),(q_1^*,k_1-p_1),\ldots,(q_n,k_n),(q_n^*,k_n-p_n))$$

To go from these rooted maps to the previous one we only have to label each star and be careful about the symmetry of the stars in order to specify a branch by star. More precisely, let us define the degree of symmetry $s(q)$ of a monomial q as follows. Let $\omega : \mathbb{C}\langle X_1,\cdots,X_m\rangle \to \mathbb{C}\langle X_1,\cdots,X_m\rangle$ be the linear function so that for all $i_k \in \{1,\cdots,m\}$, $1\le k\le p$

$$\omega(X_{i_1}X_{i_2}\cdots X_{i_p}) = X_{i_2}\cdots X_{i_p}X_{i_1}$$

and, with $\omega^p = \omega\circ\omega^{p-1}$, define

$$s(q) = \sharp\{0\le p\le \deg\,(q)-1|\omega^p(q)=q\}.$$

We easily see that for all monomial P, distinct monomials q_i (but eventually, one of them may be equal to P), and integers k_i:

$$\mathcal{D}(P,(q_1,k_1),(q_2,k_2),\ldots,(q_n,k_n)) = \frac{\mathcal{M}(P,(q_1,k_1),(q_2,k_2),\ldots,(q_n,k_n))}{\Pi_{i=1}^n k_i! s_i^{k_i}} \tag{3}$$

2.6 Graphical interpretation of Schwinger-Dyson's equations

We shall now make an assumption on the solutions of Schwinger-Dyson's equation $\mathbf{SD}[V_{\overline{t}}]$ when the parameters belong to an open convex neighborhood of the origin, namely

(H) *There exists a convex neighborhood $U\in\mathbb{R}^n$, a finite real number R and a family $\{\tau_{\overline{t}}, t\in U\}$ of linear forms on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ so that for all $\overline{t}$ in U, $\tau_{\overline{t}}$ is a solution of* $\mathbf{SD}[V_{\overline{t}}]$ *which satisfies* $H(R)$.

Note that up to take a smaller set U, we can assume that the conclusions of Theorem 2.1 are valid, i.e $R \leq A(|t|)$ for all $\bar{t} \in U$ since $A(|t|)$ blows up as $|t|$ goes to zero.

The central result of this article is then

Theorem 2.2 *Assume that (H) is satisfied. Then*

1. *For any $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\bar{t} \in U \to \tau_{\bar{t}}(P)$ is $\mathcal{C}^\infty$ at the origin in the sense that for all $\bar{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$ there exists $\epsilon(k_1+k_2+\cdots+k_n) > 0$ so that $\partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_{\bar{t}}(P)$ exists on $U_\epsilon = U \cap B(0,\epsilon)$ with $B(0,\epsilon) = \{\bar{t} \in \mathbb{R}^n : |t| \leq \epsilon\}$. We let $\tau^{\bar{k}}(P) = (-1)^{k_1+\cdots+k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_{\bar{t}}(P)|_{\bar{t}=0}$. Then, we have for all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ and all $i \in \{1, \cdots, m\}$,*

$$\tau^{\bar{k}}(X_i P) = \sum_{\substack{0 \leq p_j \leq k_j \\ 1 \leq j \leq n}} \prod_{j=1}^n C_{k_j}^{p_j} \tau^{\bar{p}} \otimes \tau^{\bar{k}-\bar{p}}(D_i P) + \sum_{1 \leq j \leq n} k_j \tau^{\bar{k}-1_j}(\mathcal{D}_i q_j + \mathcal{D}_i q_j^* P) \tag{4}$$

   where $1_j(i) = 1_{i=j}$ and $\tau^{\bar{k}}(1) = 1_{\bar{k}=0}$.

2. *For any monomial $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, any $k_1, \cdots, k_n \in \mathbb{N}$,*

$$\tau^{\bar{k}}(P) = \mathcal{M}(P, (q_1, k_1), \ldots, (q_n, k_n)).$$

Proof.

• The smoothness of $\bar{t} \to \tau_{\bar{t}}$ comes as in the proof of Theorem 2.1 from Schwinger-Dyson's equations and induction on the degree of the test polynomial function. Denote $V = V_{\bar{t}}$, $\tau = \tau_{\bar{t}}$ and take $\bar{t} = (t_1, \cdots, t_n), \bar{t}' = (t_1', t_2', .., t_n') \in U$. By **SD[V]**,

$$(\tau_{\bar{t}} - \tau_{\bar{t}'})[(X_i + \mathcal{D}_i V_{\bar{t}})P] = (\tau_{\bar{t}} - \tau_{\bar{t}'}) \otimes \tau_{\bar{t}}(D_i P) + \tau_{\bar{t}'} \otimes (\tau_{\bar{t}} - \tau_{\bar{t}'})(D_i P) + \tau_{\bar{t}'}[(\mathcal{D}_i V_{\bar{t}'} - \mathcal{D}_i V_{\bar{t}})P]$$

By our finite moment assumption, we deduce that if P is a monomial function of degree $l-1$, for any $i \in \{1, \cdots, m\}$,

$$|\tau_{\bar{t}}[(X_i + \mathcal{D}_i V_{\bar{t}})P] - \tau_{\bar{t}'}[(X_i + \mathcal{D}_i V_{\bar{t}})P]|$$

$$\leq 2 \sum_{k=0}^{l-2} \max_{Q \text{ monomial of degree } \leq k} |\tau_{\bar{t}}[Q] - \tau_{\bar{t}'}[Q]| R^{l-2-k} + \sum_{1 \leq i \leq n} |t_i - t_i'| R^{l+D-1}.$$

Thus we deduce that for any $p \in \mathbb{N}$,

$$\begin{aligned} \Delta_l(\tau_{\bar{t}}, \tau_{\bar{t}'}) &= \max_i \max_{P \text{ monomial of degree } p-1} |\tau_{\bar{t}}(X_i P) - \tau_{\bar{t}'}(X_i P)| \\ &\leq 2 \sum_{k=0}^{l-2} \Delta_k(\tau_{\bar{t}}, \tau_{\bar{t}'}) R^{l-2-k} + \sum_{i=1}^n |t_i| \Delta_{l+d_i-1}(\tau_{\bar{t}}, \tau_{\bar{t}'}) + \sum_{1 \leq i \leq n} |t_i - t_i'| R^{l+D-1}. \end{aligned}$$

Now, let $\gamma \in (0, R^{-1})$ and let's sum both sides of this inequality multiplied by γ^l to obtain, with $d_\gamma(\tau_{\bar{t}}, \tau_{\bar{t}'}) = \sum_{l \geq 0} \gamma^l \Delta_l(\tau_{\bar{t}}, \tau_{\bar{t}'})$,

$$d_\gamma(\tau_{\bar{t}}, \tau_{\bar{t}'}) \leq 2(1-\gamma R)^{-1} \gamma^2 d_\gamma(\tau_{\bar{t}}, \tau_{\bar{t}'})$$

$$+ \sum_{i=1}^n |t_i| \gamma^{-d_i+1} d_\gamma(\tau_{\bar{t}}, \tau_{\bar{t}'}) + (1-\gamma R)^{-1} \sum_{1 \leq i \leq n} |t_i - t_i'| R^{D-1}.$$

Since by definition $\Delta_l(\tau_{\overline{t}}, \tau_{\overline{t}'}) \le 2R^l$, $d_\gamma(\tau_{\overline{t}}, \tau_{\overline{t}'})$ is finite for $\gamma R < 1$ we arrive at

$$(1 - 2\gamma^2(1 - R\gamma)^{-1} - \sum_{1\le i\le n} |t_i|\gamma^{-D+2}) d_\gamma(\tau_{\overline{t}}, \tau_{\overline{t}'}) \le (1 - R\gamma)^{-1} \sum_{1\le i\le n} |t_i - t_i'| R^{D-1}.$$

Now, for $|t|$ small enough, we can find $\gamma = \gamma(|t|) > 0$ so that

$$1 - 2\gamma^2(1 - R\gamma)^{-1} - \sum_{1\le i\le n} |t_i|\gamma^{-D+2} > 0$$

and so

$$\sum_{l\ge 0} \gamma^l \Delta_l(\tau_{\overline{t}}, \tau_{\overline{t}'}) \le C(\overline{t}) \sum_{1\le i\le n} |t_i - t_i'|$$

which implies that for all $l \in \mathbb{N}$

$$\Delta_l(\tau_{\overline{t}}, \tau_{\overline{t}'}) \le C(\overline{t})\gamma^{-l} \sum_{1\le i\le n} |t_i - t_i'|$$

so that for any monomial function P, $\overline{t} \to \tau_{\overline{t}}(P)$ is Lipschitz in $U_\epsilon := U \cap B(0, \epsilon)$ for ϵ small enough. Moreover, we have proved that there exists $\eta_0(\epsilon) = \gamma^{-1} < \infty$, so that

$$\Delta_l(\tau_{\overline{t}}, \tau_{\overline{t}'}) \le C_0(\epsilon)\eta_0(\epsilon)^l |\overline{t} - \overline{t'}| \text{ with } |\overline{t} - \overline{t'}| = \max_{1\le i\le n} |t_i - t_i'|. \tag{5}$$

Consequently, $\tau_{\overline{t}}$ is almost surely differentiable in U_ϵ and the derivative satisfies

$$\partial_{t_k} \tau_{\overline{t}}[(X_i + \mathcal{D}_i V_{\overline{t}})P] + \tau_{\overline{t}}[\mathcal{D}_i q_k P] = \partial_{t_k} \tau_{\overline{t}} \otimes \tau_{\overline{t}}(D_i P) + \tau_{\overline{t}} \otimes \partial_{t_k} \tau_{\overline{t}}(D_i P) \tag{6}$$

for almost all $\overline{t} \in U_\epsilon$. Since $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ is countable, these equalities hold simultaneously for all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$ almost surely, let U_ϵ' be this subset of U_ϵ of full probability.

(5) implies that

$$\max_{1\le k\le m} \max_{P \text{ monomial of degree } l} |\partial_{t_k} \tau_{\overline{t}}(P)| \le C_0(\epsilon)\eta_0(\epsilon)^l$$

for all $\overline{t} \in U_\epsilon'$. This bound in turn shows that we can redo the argument as above to see that for $|\overline{t}|$ small enough, $\overline{t} \to \partial_{t_k}\tau_{\overline{t}}(P)$ is Lipschitz. Indeed, if we set

$$\Delta_1(l) = \Delta_l^1(\partial\tau_{\overline{t}}, \partial\tau_{\overline{t}}) = \max_{1\le k\le m} \max_{P \text{ monomial of degree } l} |\partial_{t_k}\tau_{\overline{t}}(P) - \partial_{t_k}\tau_{\overline{t}'}(P)|$$

we get, for $\overline{t}', \overline{t} \in U_\epsilon'$,

$$\Delta_1(l) \le 2\sum_{k=0}^{l-2} \Delta_1(k) R^{l-2-k} + C_0(\epsilon)|\overline{t} - \overline{t'}| l \eta_0(\epsilon)^l + \sum_{i=1}^{n} |t_i| \Delta_1(l + d_i - 1)$$

so that we get that by summation, for $\gamma < \min(R^{-1}, \eta_0(\epsilon)^{-1})$,

$$(1 - 2(1 - R\gamma)^{-1}\gamma^2 - \sum_{i=1}^{n} |t_i|\gamma^{-d_i+1}) \sum_{l\ge 0} \Delta_1(l)\gamma^l \le \gamma^2 C_0(\epsilon)(1 - \gamma\eta_0(\epsilon))^{-2} |\overline{t} - \overline{t'}|.$$

Hence, again, we can choose $\eta_1(\epsilon) < \infty$ big enough so that there exists $C_1(\epsilon) < \infty$ so that if ϵ is small enough

$$\Delta_1(l) \le C_1(\epsilon)\eta_1(\epsilon)^l |\overline{t} - \overline{t'}|.$$

In particular, this shows that we can extend $\overline{t} \in U'_\epsilon \to \partial_{t_k}\tau_{\overline{t}}(P)$ for all monomial functions P continuously in U_ϵ and so the equality (6) holds everywhere. Now, we can proceed by induction to see that $\overline{t} \to \tau_{\overline{t}}(P)$ is $\mathcal{C}^\infty$ differentiable in a neighborhood of the origin. More precisely, for any $\overline{k} = (k_1, \cdots, k_n)$ there exists $\epsilon = \epsilon(k_1 + k_2 + \cdots + k_n) > 0$ so that on U_ϵ,

$$\tau_{\overline{t}}^{\overline{k}}(P) = (-1)^{k_1+\cdots+k_n}\partial_{t_1}^{k_1}\cdots\partial_{t_m}^{k_n}\tau_{\overline{t}}(P)$$

exists and furthermore satisfies the equation

$$\tau_{\overline{t}}^{\overline{k}}((X_i + \mathcal{D}_i V_{\overline{t}})P) = \sum_{\substack{0\le p_i \le k_i\\ 1\le i\le n}} \prod_{i=1}^{n} C_{k_i}^{p_i} \tau_{\overline{t}}^{\overline{p}} \otimes \tau_{\overline{t}}^{\overline{k}-\overline{p}}(D_i P) + \sum_{1\le j\le m} k_j \tau_{\overline{t}}^{\overline{k}-1_j}((\mathcal{D}_i q_j + \mathcal{D}_i q_j^*)P)$$

Applying this result at the origin, we obtain the announced result.

• We now show the combinatorial interpretation of (4). It is based on the observation that the $\{\tau^{\overline{k}}(P), \overline{k} \in \mathbb{N}^n\}$ and the $\{\mathcal{M}(P, (k_1, q_1)\cdots(k_n, q_n), \overline{k} \in \mathbb{N}^n\}$ satisfy the same inductive relations.

Let us first interpret graphically $\tau^0 = \tau_0$. τ_0 satisfies by definition **SD[0]** which is well known to have a unique solution given by the law of m free semi-circular variables (see Voiculescu [28]). Then, $\tau_0(X_{i_1}\cdots X_{i_k})$ can be computed for instance using cumulants techniques as developed by R. Speicher [26]; it counts the number of planar maps which can be constructed from the star associated to $X_{i_1}\cdots X_{i_k}$ by gluing together the edges of the star of the same color. A way to prove that is to remark first that if we have a star with two branches of the same color, there is only $1 = \tau^0(1)$ ways to glue them. We then proceed by induction over the degree of the monomial function. We let $\mathcal{M}(P)$ be the number of planar maps with labeled stars with a star of type P and shall show that it satisfies the same induction relation than $\tau^0(P)$. Let $i \in \{1, \cdots, m\}$ and $P = X_i Q$. To compute $\mathcal{M}(X_i Q)$, we break the edge between the distinguished branch X_i and the other branch of Q with which it was glued, then erasing these two branches. Since the maps are planar, this decomposes the planar map into two planar maps (see figure 2) corresponding respectively to the stars Q_1, Q_2 for any possible choices of Q_1, Q_2 so that $Q = Q_1 X_i Q_2$. Hence

$$\mathcal{M}(X_i Q) = \sum_{Q = Q_1 X_i Q_2} \mathcal{M}(Q_1)\mathcal{M}(Q_2).$$

Thus, if $\mathcal{M}(R) = \tau_0(R)$ for all monomial of degree strictly smaller than P,

$$\mathcal{M}(X_i Q) = \sum_{Q = Q_1 X_i Q_2} \tau_0(Q_1)\tau_0(Q_2) = \tau_0 \otimes \tau_0(D_i Q)$$

which completes the argument since the right hand side is exactly $\tau_0(X_i Q)$.

We now consider the general case; let us assume that for $|\overline{k}| \le M$, the graphical interpretation has been obtained for all monomial and that for $|\overline{k}| = M + 1$, it has been proved for monomial of degree smaller or equal to L. By the preceding, we can take $M \ge 1$ and $L \ge 1$ since for all $\overline{k} \ne 0$, $\tau^{\overline{k}}(1) = 0$. Again, we shall show that $\mathcal{M}(P, (q_1, k_1), \cdots, (q_n, k_n))$ satisfies the same induction relation than $\tau^{\overline{k}}(P)$.

FIG. 2 – *The decomposition* $P(X) = X_1X_2^2X_1^4X_2^2$ *into* $X_1X_2^2X_1 \otimes X_1^2X_2^2$

Let us consider a star of type X_iP (rooted at the branch X_i, with its inner orientation) with P a monomial of degree less than L and $|\overline{k}| = \sum k_i = M+1$. Now, in order to compute $\mathcal{M}(X_iP, (q_1, k_1), \cdots, (q_n, k_n))$, we break the edge between the distinguished branch X_i (which has color i) and the other branch with which it was glued.

The first possibility is that it was glued with an edge of the star P. Then, since the maps are planar, this decomposes the map in two planar maps. If this branch was given by the X_i so that $P = P_1X_iP_2$, one of this planar map contain the star of type P_1 and the other the star of type P_2, which have also a distinguished branch and are oriented. If one of this planar map is glued with k_j stars of type q_j or q_j^* , $0 \leq k_j \leq n$, the other map is glued with the remaining stars, that is $k_j - p_j$ stars of type q_i or q_i^*. There are $\prod_{j=1}^n C_{k_j}^{p_j}$ ways to choose p_j among k_j stars of type q_j or q_j^* for $1 \leq j \leq n$ (recall here that stars are labeled). Since we do that for all (P_1, P_2) so that P have the above decomposition, we obtain the planar maps corresponding actually to the stars associated with the monomials of D_iP. Note that the case where one of the monomial in D_iP is the monomial 1 shows up when $P = X_iQ$ or QX_i for some monomial Q and the weight corresponds then to the case where we glue the first branch X_i in X_iP with its left or right neighbor. In this case, none of these two branches can be glued with another star, and there is only one possibility to glue these two branches otherwise, which corresponds to the weight $\tau^{\overline{k}}(1) = 1_{\overline{k}=0}$.

Hence, the number of planar maps corresponding to this configuration is given by

$$\sum_{\substack{0\leq p_j\leq k_j\\ 1\leq j\leq n}} \sum_{P=P_1X_iP_2} \prod_{1\leq j\leq n} C_{k_j}^{p_j} \mathcal{M}(P_1, (q_1, p_1), \cdots, (q_n, p_n))\mathcal{M}(P_2, (q_1, k_1 - p_1), \cdots, (q_n, k_n - p_n))$$

$$= \sum_{\substack{0\leq p_j\leq k_j\\ 1\leq j\leq n}} \prod_{1\leq j\leq n} C_{k_j}^{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k}-\overline{p}}(D_iP)$$

where we finally used our induction hypothesis.

The other possibility is that this edge is glued with a star of type q_j^ϵ for some $j \in \{1, \cdots, n\}, \epsilon \in \{., *\}$. In this case, erasing the edge means that we destroy a star of type q_j^ϵ and replace the stars of type $X_i P$ and q glued together with a single star P glued with the star q_j^ϵ in place of X_i with an edge of color i removed; if $q_j^\epsilon = Q_1 X_i Q_2$, we replace the two stars of type $X_i P$ and q_j^ϵ by a single one of type $Q_2 Q_1 P$ (see figure 3). Since we do that with all the possible edges of color i in q_j^ϵ, we find that we can glue all monomials appearing in $\mathcal{D}_i q_j^\epsilon$, and so the corresponding weight is given by $\tau^{\overline{k}-1_j}(\mathcal{D}_i q_j^\epsilon P)$ times k_j, the number of ways to choose one star among k_j of type q_j^ϵ.

FIG. 3 – *The merging of* $q(X) = X_1^2 X_2^2 = X_1 X_1 X_2^2$ *and* $X_1 P$ *into* $X_1 X_2^2 P$

Hence, by induction, we proved that the number of planar maps with k_j stars of type q_j or q_j^* and one of type $X_i P$ is given by

$$\mathcal{M}((X_i P, 1), (q_1, k_1), ..., (q_n, k_n)) = \sum_{\substack{0 \le p_j \le k_j \\ 1 \le j \le m}} \prod_{j=1}^{n} C_{k_j}^{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k}-\overline{p}}(D_i P) + \sum_{1 \le j \le m} k_j \tau^{\overline{k}-1_j}(\mathcal{D}_i(q_j + q_j^*)P) \tag{7}$$

$$= \tau^{\overline{k}}(X_i P) \tag{8}$$

for all $i \in \{1, \cdots, m\}$. This shows that the graphical interpretation holds for all L and $|\overline{k}| \le M+1$. We can start the induction since we know that $\tau^{\overline{k}}(1) = 1_{\overline{k}=0}$. This completes the proof.

□

Remarks:

1. This graphical interpretation can be sometimes simplified for particular V. For example, consider $V_{t,u,c} = tA^4 + uB^4 + cAB$ which appears in the Ising model. First, one may notice that this is not in the form of the theorem but we can replace $\mathrm{tr}(V)$ by

$$\mathrm{tr}(\frac{t}{2}(A^4 + A^4) + \frac{u}{2}(B^4 + B^4) + \frac{c}{2}(AB + BA))$$

so that the theorem can be applied. Now, one may see that there will some redundancy in the enumeration of maps given by this potential as for example vertices of type AB are isomorphic to vertices of type BA. But there will be some simplification with the factors $\frac{1}{2}$ so that finally the theorem will give, if $\tau_{t,u,c}$ is the a solution to $\mathbf{SD}[\mathbf{V_{t,u,c}}]$ given by the theorem, that for all monomial P,

$$(-1)^{m+n+r}\partial_t^m\partial_u^n\partial_c^r\tau_{t,u,c}(P)|_{t=u=c=0} = \mathcal{M}_0(P,(A^4,m),(B^4,n),(AB,r))$$

2. Note that this graphical approach can be generalized to matrix models with more complex potentials involving tensor products. For example, one can consider a potential V which is a sum of monomials and of tensor products of monomials:

$$V_{\bar{t}} = \sum_i t_i(q_i^1\otimes\cdots\otimes q_i^d + (q_i^1)^*\otimes\cdots\otimes(q_i^d)^*)$$

and the associated measure with density with respect to $\mu_N^{\otimes m}$ given by $Z_N^{-1}e^{-N^{2-d}(\mathrm{tr})^{\otimes d}V_{\bar{t}}}$. Then one can write the generalized Swinger Dyson's equation:

$$\tau\otimes\tau(D_iP) = \tau(X_iP) + \sum_{k,j} t_k\tau^{\otimes d_k}(q_k^1\otimes\cdots\otimes\mathcal{D}_iq_k^jP\otimes\cdots\otimes q_k^d$$
$$+(q_k^1)^*\otimes\cdots\otimes\mathcal{D}_i(q_k^j)^*P\otimes\cdots\otimes(q_k^d)^*)$$

The previous results remain valid up to a graphical interpretation of the new term. For example $q^1\otimes\cdots\otimes q^k$ will be a bunch of k loops, the first one containing the branches of the star of q^1, in the clockwise order, the first of which is the marked one, the second one the branches of q^2 ... The additional constraint being that vertices which will be placed in a loop can not be linked to any vertices in an other loop.

2.7 Existence of an analytic solution to Schwinger-Dyson's equation

The aim of this section is to prove that for all monomials $(q_j)_{1\le j\le n}$, there exists a convex neighborhood of the origin (actually an open ball) and a finite constant R so that hypothesis **(H)** of section 2.6 is satisfied. Moreover, we show it depends analytically on $\bar{t}$ in a neighborhood of the origin. Let $V_{\bar{t}}$ be as before.

Theorem 2.3 *There exists an open neighborhood $U\subset\mathbb{R}^n$ of the origin (a ball of positive radius) such that for $\bar{t}\in U$, there exists $\tau_{\bar{t}}\in\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ satisfying $\mathbf{SD}[V_{\bar{t}}]$ such that:*

- *$\bar{t}\to\tau_{\bar{t}}$ is analytic on U, i.e. there exists $\tau^{\bar{k}},\bar{k}\in\mathbb{N}^n$ in $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ such that for all P in $\mathbb{C}\langle X_1,\cdots,X_m\rangle$, t in U,*

$$\tau_{\bar{t}}(P) = \sum_{\bar{k}\in\mathbb{N}^n}\prod_{1\le i\le n}\frac{(-t_i)^{k_i}}{k_i!}\tau^{\bar{k}}(P)$$

and the serie converges absolutely on U.
- $\tau^{\bar{k}}(P) = (-1)^{\Sigma k_i}\partial_{t_1}^{k_1}\cdots\partial_{t_n}^{k_n}\tau_{\bar{t}}(P)|_{\bar{t}=0} = \mathcal{M}(P,(q_1,k_1),\cdots,(q_n,k_n))$
- *There exists $R<\infty$ so that for all $\bar{t}\in U$, all $i_i\cdots i_l\in\{1,\cdots,m\}^l$, all $l\in\mathbb{N}$,*

$$|\tau_{\bar{t}}(X_{i_1}\cdots X_{i_l})|\le R^l.$$

Remark: It will be useful to use sometimes for $\tau_{\overline{t}}$ an alternative expression related to the enumeration of rooted maps. Using (3), one can obtain inside the domain of convergence, for all monomial P:

$$\tau_{\overline{t}}(P) = \sum_{\overline{k}\in\mathbb{N}^n} \prod_{1\le i\le n} (-s(q_i)t_i)^{k_i} \mathcal{D}(P,(q_1,k_1),\cdots,(q_n,k_n)).$$

Proof.

If we have such a solution, it satisfies assumption **(H)** and by (4) if $\overline{k}! = \prod k_i!$,

$$\frac{\tau^{\overline{k}}(X_iP)}{\overline{k}!} = \sum_{\substack{0\le p_j\le k_j\\ 1\le j\le n}} \sum_{P=P_1X_iP_2} \frac{\tau^{\overline{p}}(P_1)}{\overline{p}!}\frac{\tau^{\overline{k}-\overline{p}}(P_2)}{(\overline{k}-p)!} + \sum_{\substack{1\le j\le m\\ k_j\neq 0}} \frac{\tau^{\overline{k}-1_j}(\mathcal{D}_i(q_j+q_j^*)P)}{(\overline{k}-1_j)!}$$

where the second sum runs over all monomials P_1, P_2 so that P decomposes into $P_1X_iP_2$. We can use this formula to define the $\tau^{\overline{k}}$ by induction, the graphical interpretation is directly satisfied.

We must control the growth of the $\tau^{\overline{k}}(P)$'s. Our induction hypothesis will be that for $\overline{k}$ so that $\sum_i k_i \le M-1$ and all monomial P, as well as for $\sum k_i = M$ and monomials P of degree smaller than L,

$$\left|\frac{\tau^{\overline{k}}(P)}{\overline{k}!}\right| \le A^{\sum k_i} B^{degP} \prod_i C_{k_i} C_{degP}$$

where the C_k are the Catalan's numbers which satisfy

$$C_{k+1} = \sum_{p=0}^{k} C_pC_{k-p}, \quad C_0 = 1, \quad \frac{C_{k+l}}{C_l} \le 4^k \quad \forall l,k\in\mathbb{N}. \tag{9}$$

Here, $\deg P$ denotes the degree of the monomial P and we can assume $B\ge 2$ without loss of generality. Our induction is trivially true for $\overline{k}=0$ and all L since $\tau^{\overline{0}} = \sigma^m$ is the law of m free semi-circular variables which are uniformly bounded by 2 so that

$$|\tau^{\overline{0}}(P)| \le 2^{degP}$$

Moreover, it is satisfied for all $\overline{k}$ and $L=0$ since then $\tau^{\overline{k}}(1) = 1_{\overline{k}=0}$. Let us assume that it is true for all $\overline{k}$ such that $\sum k_i \le M-1$ and all monomials, and for $\overline{k}$ such that $\sum k_i = M$ and monomials P of degree less than L for some $L\ge 0$. Then

$$\begin{aligned}
\left|\frac{\tau^{\overline{k}}(X_iP)}{k!}\right| &\le \sum_{\substack{0\le p_j\le k_j\\ 1\le j\le n}} \sum_{P=P_1X_iP_2} A^{\sum k_i} B^{degP-1} \prod_{i=1}^{n} C_{p_j}C_{k_j-p_j}C_{degP_1}C_{degP_2}\\
&\quad +2\sum_{1\le l\le n} A^{\sum k_j -1}\prod_j C_{k_j} B^{degP+deqq_l-1} C_{degP+deqq_l-1}\\
&\le A^{\sum k_i} B^{degP+1}\prod_i C_{k_i} C_{degP+1}\left(\frac{4^n}{B^2} + 2\frac{\sum_{1\le j\le n} B^{degq_j-2}4^{degq_j-2}}{A}\right)
\end{aligned}$$

where we used (9) in the last line. It is now sufficient to choose A and B such that

$$\frac{4^n}{B^2} + 2\frac{\sum_{1\le j\le n} B^{degq_j-2}4^{degq_j-2}}{A} \le 1$$

(for instance $B = 2^{n+1}$ and $A = 4nB^{D-2}4^{D-2}$) to verify the induction hypothesis works for polynomials of all degrees (all L's).

Then

$$\tau_{\overline{t}}(P) = \sum_{k \in \mathbb{N}^n} \prod \frac{(-t_i)^{k_i}}{k_i!} \tau^{\overline{k}}(P)$$

is well defined for $|t| < (4A)^{-1}$. Moreover, for all monomial P,

$$|\tau_{\overline{t}}(P)| \leq \sum_{k \in \mathbb{N}^n} \prod_{i=1}^{n} (4t_i A)^{k_i} (4B)^{degP} \leq \prod_{i=1}^{n} (1 - 4At_i)^{-1} (4B)^{degP}.$$

so that for small t, $\tau_{\overline{t}}$ has an uniformly bounded support.

□

Hence, we see that the enumeration of planar maps could be reduced to the study of Schwinger-Dyson's equations **SD[V]**. For instance, the asymptotics of such enumeration can be obtained by studying the optimal domain in which the solutions are analytic. Matrix models can be useful to study also the solution, e.g. we shall deduce from this approach that the solutions to **SD[V]** are tracial states (the positivity condition being unclear a priori).

3 Existence of tracial states solutions to Schwinger-Dyson's equations from matrix models

We let $V = V_{\overline{t}}$ be a polynomial function as before, and consider

$$Z_V^N = \int e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))} \mu_N(dA_1) \cdots \mu_N(dA_m)$$

and μ_V^N the associated Gibbs measure

$$\mu_V^N(dA_1, \cdots, dA_m) = (Z_V^N)^{-1} e^{-N\mathrm{tr}(V(A_1,\cdots,A_m))} \mu_N(dA_1) \cdots \mu_N(dA_m).$$

This is well defined provided that we assume that the monomials of highest degree in $V_{\overline{t}}$ are sufficiently large to make Z_V^N finite. We shall assume for instance that

$$V_{\overline{t}}(\mathbf{X}) = \sum_{1 \leq i \leq n} t_i (q_i(\mathbf{X}) + q_i^*(\mathbf{X})) + \sum_{n+1 \leq i \leq n+m} t_i X_{i-n}^D \qquad (10)$$

with D even and monomial functions q_i of degree less or equal than $D-1$ and $t_i > 0$ for $i \in \{n+1, \cdots, n+m\}$. We shall see in the last paragraph of this section that such assumption can be removed provided a cut-off is added.

The empirical distribution of m matrices $\mathbf{A} = (A_1, \cdots, A_m) \in \mathcal{H}_N^m$ is defined as the element of $\mathcal{M}_{ST}^m$ such that

$$\hat{\mu}_{\mathbf{A}}^N(F) := \hat{\mu}_{A_1,\cdots,A_m}^N(F) = \frac{1}{N} \mathrm{tr}(F(A_1, \cdots, A_m))$$

for all $F \in \mathcal{C}_{st}^m(\mathbb{C})$. Note that the empirical distribution could be defined as well as an element of $\mathcal{M}^m$ but since the random matrices $(A_1, \cdots, A_m)$ under μ_V^N have a priori no uniformly bounded spectral radius, the topology of weak convergence would not be suitable then.

We shall see that if we know that a limit point of $\hat{\mu}^N_{\mathbf{A}}$ under μ^N_V are compactly supported, then it satisfies **SD[V]**. In a second part, we shall give examples of potential V for which this assumption is satisfied. Finally, we discuss localized matrix integrals and show that bounded solutions to **SD[V]** for small potentials can always be constructed by localized matrix integrals.

3.1 Limit points of empirical distribution of matrices following matrix models satisfy the SD[V] equations

We claim that

Theorem 3.1 *Assume* (10). *Then*

1. *There exists $M < \infty$ so that*

$$\limsup_{N\to\infty} \hat{\mu}^N_{A_1,\cdots,A_m}(X_i^D) \le M$$

   μ^N_V *almost surely for all $i \in \{1,\cdots,m\}$.*

2. *The limit points of $\hat{\mu}^N_{A_1,\cdots,A_m}$ for the $\mathcal{C}^m_{st}(\mathbb{C})$-topology satisfy the 'weak' Schwinger-Dyson equation*

$$\tau\otimes\tau(D_i F) = \tau((\mathcal{D}_i V + X_i)F) \quad \textbf{(WSD)[V]}$$

   for all $F \in \mathcal{C}^m_{st}(\mathbb{C})$.

Note here that $(\mathcal{D}_i V + X_i)F$ does not belong to $\mathcal{C}^m_{st}(\mathbb{C})$ so that it is not clear what **(WSD)[V]** means a priori. We define it by the following; there exists a sequence $V^\delta \in \mathcal{C}^m_{st}(\mathbb{C})$ so that

$$\lim_{\delta\to 0}\max_{1\le i\le m}\sup_{\tau(X_i^D)\le M} \tau(|\mathcal{D}_i V^\delta - \mathcal{D}_i V - X_i|) = 0$$

from which, since any $F \in \mathcal{C}^m_{st}(\mathbb{C})$ is uniformly bounded,

$$\lim_{\delta\to 0}\max_{1\le i\le m}\sup_{\tau(X_i^D)\le M} |\tau(F\mathcal{D}_i V^\delta) - \tau(F(\mathcal{D}_i V + X_i))| = 0$$

is well defined.

Proof.

- The first point is trivial since by Jensen's inequality,

$$Z^V_N \ge \exp\{-N^2\int \frac{1}{N}\mathrm{tr}(V(\mathbf{A}))\prod_{1\le i\le m} d\mu_N(A_i)\} \ge \exp\{cN^2\}$$

for some $c > -\infty$, where the last inequality comes from the fact that (see [28])

$$\lim_{N\to\infty}\int \frac{1}{N}\mathrm{tr}(V(\mathbf{A}))\prod_{1\le i\le m} d\mu_N(A_i) = \sigma^m(V) < \infty$$

where σ^m is the law of m free semi-circular variables.

Now, observe that by Hölder's inequality,

$$|\hat{\mu}^N_{\mathbf{A}}(q_i)| \le \max_{1\le i\le m} \hat{\mu}^N_{\mathbf{A}}(|X_i|^{D-1}+1)$$

so that we deduce

$$\hat{\mu}_{\mathbf{A}}^N(V) \geq \sum_{i=1}^m \left(t_{i+n}\hat{\mu}_{\mathbf{A}}^N(A_i^D) - c(\bar{t})\hat{\mu}_{\mathbf{A}}^N(|A_i|^{D-1}) - c(\bar{t})\right)$$

with a finite constant $c(\bar{t})$. Since $t_{i+n} > 0$, we conclude that $\hat{\mu}_{\mathbf{A}}^N(V) \geq m|t|M/2$ when $\max_{1\leq i\leq m} \hat{\mu}_{\mathbf{A}}^N(A_i^D) \geq M$ for M large enough. Thus

$$\mu_V^N\left(\max_{1\leq i\leq m} \hat{\mu}_{\mathbf{A}}^N(A_i^D) \geq M\right) \leq e^{-2^{-1}N^2Mm|t|}e^{-cN^2} \tag{11}$$

goes to zero exponentially fast when $M > \frac{2c}{m|t|}$. The claim follows by Borel-Cantelli's lemma.

• We proceed as in [9], following a common idea in physics, which is to make, in Z_V^N, the change of variables $X_i \to X_i + N^{-1}F(\mathbf{X})$ for a given $i \in \{1, \cdots, m\}$ and $F \in \mathcal{C}_{st}^m(\mathbb{R})$. Noticing that the Jacobian for this change of variable is

$$|J| = e^{N\hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N(D_iF)+O(1)}$$

we get that

$$\int e^{(N\hat{\mu}_{\mathbf{A}}^N\otimes\hat{\mu}_{\mathbf{A}}^N(D_iF)-N^2\hat{\mu}_{\mathbf{A}}^N(N^{-1}X_iF(\mathbf{X})+V(X_i+N^{-1}F(\mathbf{X}))-V(X_i))}\mu_V^N(d\mathbf{A}) = O(1)$$

from which we deduce that

$$\int_{\max_{1\leq i\leq m}\hat{\mu}_{\mathbf{A}}^N(A_i^D)\leq M} e^{(N\hat{\mu}_{\mathbf{A}}^N\otimes\hat{\mu}_{\mathbf{A}}^N(D_iF)-N^2\hat{\mu}_{\mathbf{A}}^N(N^{-1}X_iF(\mathbf{X})+V(X_i+N^{-1}F(\mathbf{X}))-V(X_i))}\mu_V^N(d\mathbf{A}) = O(1).$$

Hence, we conclude by Chebychev inequality and (11) that for M big enough, any $\delta > 0$, there exists $\eta > 0$, so that if we denote

$$E_N = \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N(D_iF) - N\hat{\mu}_{\mathbf{A}}^N(N^{-1}X_iF(\mathbf{X}) + V(X_i + N^{-1}F(\mathbf{X})) - V(X_i))$$

then

$$\mu_V^N\left(\{\max_{1\leq i\leq m} \hat{\mu}_{\mathbf{A}}^N(X_i^D) \leq M\} \cap \{|E_N| \leq \delta\}\right) \geq 1 - e^{-\eta N}.$$

Moreover,

$$\hat{\mu}_{\mathbf{A}}^N(V(X_i + N^{-1}F(\mathbf{X})) - V(X_i))) = N^{-1}\hat{\mu}_{\mathbf{A}}^N(\mathcal{D}_iVF) + R_N$$

with a rest R_N of order $N^{-2}\max_{1\leq i\leq m}\hat{\mu}_{\mathbf{A}}^N(X_i^{D-2})$ which we can neglect on $\max_{1\leq i\leq m}\hat{\mu}_{\mathbf{A}}^N(A_i^D) \leq M$. This shows, by Borel-Cantelli's Lemma, that for all $F \in \mathcal{C}_{st}^m(\mathbb{R})$,

$$\hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N(D_iF) - \hat{\mu}_{\mathbf{A}}^N(X_iF + \mathcal{D}_iVF)$$

goes to zero almost surely. This result extends to $F \in \mathcal{C}_{st}^m(\mathbb{C})$ since it can always be decomposed into the sum of two elements of $\mathcal{C}_{st}^m(\mathbb{R})$. Moreover, if we let $A_i^\epsilon = A_i(1+\epsilon A_i^2)^{-1} = A_i(\sqrt{-1}+\sqrt{\epsilon}A_i)^{-1}(-\sqrt{-1}+\sqrt{\epsilon}A_i)^{-1} \in \mathcal{C}_{st}^m(\mathbb{C})$, then again by Hölder's inequality $\tau(|\mathcal{D}_iV(A_i)-\mathcal{D}_iV(A_i^\epsilon)|)$ goes to zero uniformly on $\max_{1\leq i\leq m}\tau(A_i^D) \leq M$. This shows that $\mu \to \mu((\mathcal{D}_iV + X_i)F)$ is continuous for the weak $\mathcal{C}_{st}^m(\mathbb{C})$-topology on $\{\mu(A_i^D) \leq M\}$ for any $F \in \mathcal{C}_{st}^m(\mathbb{C})$. Therefore, since $\mathcal{M}_{ST}^m$ is compact, we conclude that any limit point of $\hat{\mu}_{\mathbf{A}}^N$ satisfies

$$\tau \otimes \tau(D_iF) = \tau((X_i + \mathcal{D}_iV)F)$$

□

We therefore have the

Corollary 3.2 *Assume that there exists a limit point τ_V of $\hat{\mu}^N_{\mathbf{A}}$ under μ^N_V which is compactly supported. Then, it satisfies Schwinger-Dyson's equation* **SD[V]**.

Proof.

The proof is straightforward since if τ_V is compactly supported it is equivalent to say that τ_V satisfies **WSD[V]** or **SD[V]** since $\mathcal{C}^m_{st}(\mathbb{C})$ is dense in the set of polynomial functions (approximate the A_i's by the A_i^δ's defined in the previous proof).

□

Let us also give the final argument to deduce convergence of the free energy from the previous considerations.

Theorem 3.3 *1. Assume that $\hat{\mu}^N_{\mathbf{A}}$ converges in $\mathcal{M}^m_{ST}$ almost surely or in expectation under $\mu^N_{V_{\bar{t}}}$ towards $\tau_{\bar{t}}$ solution to* $\mathbf{SD}[V_{\bar{t}}]$ *for $\bar{t}$ in a convex neighborhood U of the origin. Assume furthermore that $\max_p \mu^N_{V_{t_1,\cdots,t_{k-1},s,t_{k+1},\cdots,t_n}}(\hat{\mu}^N_{\mathbf{A}}(|X_p|^l))$ is uniformly bounded for $(t_1,\cdots,t_{k-1},s,t_{k+1},\cdots,t_n)\in U$ and N large enough for some l strictly greater than the degree of $V_{\bar{t}}$. Then,*

$$F^N_{V_{\bar{t}},k} = N^{-2}\log\left(\frac{Z^N_{V_{\bar{t}}}}{Z^N_{V_{(t_1,\cdots,t_{k-1},0,t_{k+1}\cdots t_n)}}}\right)$$

converges as N goes to infinity towards a limit $F_{V_{\bar{t}},k}$. Moreover,

$$F_{V_{\bar{t}},k} = -\int_0^{t_k}\tau_{(t_1,\cdots,t_{k-1},s,t_{k+1},\cdots,t_n)}(q_k+q_k^*)ds.$$

If furthermore $\tau_{\bar{t}}$ is uniformly compactly supported in U, we deduce that $\bar{t}\to F_{V_{\bar{t}}}$ is $\mathcal{C}^\infty$ in a neighborhood of the origin and $(-1)^{\sum p_i}\partial^{p_1}_{t_1}\cdots\partial^{p_n}_{t_n}F_{V_{\bar{t}},k}|_{\bar{t}=0}$ is the number of planar maps with p_i stars of type q_i or q_i^* when $p_k\geq 1$.

2. Assume that for $\bar{t}$ in a open convex neighborhood U of the origin the limit points of $\hat{\mu}^N_{\mathbf{A}}$ under $\mu^N_{V_{\bar{t}}}$ are uniformly compactly supported. Assume further that $\max_p|\mu^N_{V_{\bar{t}}}(\hat{\mu}^N_{\mathbf{A}}(|X_p|^l))|$ is uniformly bounded (independently of $\bar{t}\in U$) for N large enough and some l strictly larger than the degree of $V_{\bar{t}}$. Then, $\hat{\mu}^N_{\mathbf{A}}$ converges $\mu^N_{V_{\bar{t}}}$-almost surely towards $\tau_{\bar{t}}$ described in Theorem 2.3 for $\bar{t}\in U\cap B(0,\epsilon)$ for some $\epsilon>0$ small enough and for $\bar{t}\in U\cap B(0,\epsilon)$,

$$F^N_{V_{\bar{t}}} = N^{-2}\log(Z^N_{V_{\bar{t}}})$$

converges as N goes to infinity towards

$$F_{V_{\bar{t}}} = \sum_{\bar{k}\in\mathbb{N}^n\backslash(0,..,0)}\prod_{1\leq i\leq n}\frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}((q_1,k_1),\cdots,(q_n,k_n)).$$

Note above that the last serie as a positive radius of convergence according to Theorems 2.2 and 2.3. This emphasizes that the possible divergence of $F^N_{V_{\bar{t}}}$ does not survive the large N limit.

Proof.

• By differentiating $N^{-2}\log Z^N_{V_{\bar{t}}}$ with respect to t_k we obtain that

$$\partial_{t_k}N^{-2}\log Z^N_{V_{\bar{t}}} = -\mu^N_{V_{\bar{t}}}(\hat{\mu}^N_{\mathbf{A}}(q_k+q_k^*)).$$

But, under assumption, $(\hat{\mu}^N_{\mathbf{A}}(q_k+q_k^*))_{N\in\mathbb{N}}$ converges almost surely and is uniformly integrable so that $\mu^N_{V_{\bar{t}}}(\hat{\mu}^N_{\mathbf{A}}(q_k+q_k^*))$ is a uniformly bounded sequence which converges as N goes to infinity towards $\tau_{\bar{t}}(q_k+q_k^*)$ for $\bar{t}\in U$. Integrating with respect to t_k yields the convergence with $F_{V_{\bar{t}}}$ as above by dominated convergence theorem. The last part of the first point theorem is a direct consequence of Theorem 2.2.

• By Corollary 3.2 and Theorem 2.1, we see that our hypothesis implies that for $\bar{t}\in U\cap B(0,\epsilon)$ for some $\epsilon>0$, the limit points of $\hat{\mu}^N_{\mathbf{A}}$ are unique and given by $\tau_{\bar{t}}$. Hence, $\hat{\mu}^N_{\mathbf{A}}$ converges in $\mathcal{M}^m_{ST}$ almost surely towards $\tau_{\bar{t}}$. Since we assumed our family uniformly integrable, we deduce that $\mu^N_{V_{\bar{t}}}(\hat{\mu}^N_{\mathbf{A}}(q_l+q_l^*))$ converges as N goes to infinity towards $\tau_{\bar{t}}(q_l+q_l^*)$ for all $l\in\{1,\cdots,n\}$ and we see as above that for all $i\in\{1,\cdots,n-1\}$,

$$\frac{1}{N^2}\log\left(\frac{Z^N_{V_{(0,..,0,t_i..t_n)}}}{Z^N_{V_{(0,..,0,t_{i+1}..t_n)}}}\right)$$

converges as N goes to infinity towards a limit

$$F_{V_{\bar{t}},i}=-\int_0^{t_i}\tau_{(0,\cdots,0,s,t_{i+1},\cdots,t_n)}(q_i+q_i^*)ds.$$

Hence, since we know that $F_{V_0}=0$,

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N^2}\log Z^N_{V_{\bar{t}}} &= \sum_{i=1}^n\lim_{N\to\infty}\frac{1}{N^2}\log\left(\frac{Z^N_{V_{(0,..,0,t_i..t_n)}}}{Z^N_{V_{(0,..,0,t_{i+1}..t_n)}}}\right)\\
&= -\sum_{i=1}^n\int_0^{t_i}\tau_{(0,\cdots,0,s,t_{i+1},\cdots,t_n)}(q_i+q_i^*)ds\\
&= \sum_{i=1}^n\sum_{k_i,..,k_n\in\mathbb{N}^{n-i}}\prod_{i+1\le j\le n}\frac{(-t_j)^{k_j}}{k_j!}\frac{(-t_i)^{k_i+1}}{(k_i+1)!}\tau^{(0,..,0,k_i,\cdots,k_n)}(q_i+q_i^*)
\end{aligned}$$

where we used in the last line Theorem 2.3. Noting that $\tau^{(0,..,0,k_i,\cdots,k_n)}(q_i+q_i^*)$ is the number of planar maps with k_j stars of type q_j or q_j^* for $j\ge i+1$ and k_i+1 stars of type q_i or q_i^* , we conclude the proof.

□

We shall in the next section provide a generic example where the assumption of the second point of Theorem 3.3 is satisfied (in fact, a slightly different version since we do not prove that the almost sure limit points of $\hat{\mu}^N_{\mathbf{A}}$ satisfy our compactness assumption, but their average do, which still guarantees the result).

3.2 Convex interaction models

Let us assume that we consider a matrix model with potential V such that

$$\phi^N_{V,a}:(A_k(ij))\in(\mathbb{R}^{N^2})^m\to\mathrm{tr}(V(A_1,\cdots,A_m))-\frac{a}{2}\sum_{k=1}^m\mathrm{tr}(A_k^2)\tag{12}$$

is convex in all dimensions for some $a < 1$, i.e the Hessian of $\phi^N_{V,a}$ is non negative for all $N \in \mathbb{N}$. An example is V of the form

$$V(A_1, \cdots, A_m) = \sum_{i=1}^{n} t_i (\sum_{k=1}^{m} \alpha^i_k A_k)^{2p_i}$$

with non-negative t_i's, integers p_i's and real α's. Indeed, by Klein's lemma (c.f. [18]), since $x \to (\sum \alpha_k x_k)^{2p_i}$ is convex,

$$\mathbf{A} \to \mathrm{tr}(\sum \alpha_i A_i)^{2p_i}$$

is also convex (Here $\mathbf{A}$, by an abuse of notations, denotes the entries of the m-uple of matrices $\mathbf{A} = (A_1, \cdots, A_m)$).

Then, we shall prove that

Theorem 3.4 *Let V be a self-adjoint polynomial function which satisfies (12). Then*

- *There exists $R_V < \infty$ so that*

$$\limsup_{N\to\infty} \mu^N_V(\hat{\mu}^N_{\mathbf{A}}(A_i^{2n})) \le (R_V)^n$$

for all $n \in \mathbb{N}$ and $i \in \{1, \cdots, m\}$. Here, R_V is uniformly bounded by some R_M when the quantities $(a, V(0,0,..,0), (\mathcal{D}_i V(0,0,..,0))_{1\le i\le m})$ are bounded by M.

- *$\mu^N_V[\hat{\mu}^N_{\mathbf{A}}]$ is tight and its limit points satisfy* **SD[V]**.
- *Take $V = V_{\bar{t}} = \sum_{i=1}^n t_i q_i$ and let U_a be the set of t_i's for which $V_{\bar{t}}$ satisfies (12) for a given $a < 1$. For $\epsilon > 0$ small enough, when $(t_i)_{1\le i\le n} \in U_a \cap B(0,\epsilon)$, $\hat{\mu}^N_{\mathbf{A}}$ converges in $L^1(\mu^N_V)$ and almost surely to the unique solution to* **SD[V]**.
- *Assume that U_a contains $\cup_{1\le i\le n}\{(0,..,0,t_i,..,t_n), 0 \le t_i \le \delta\}$ for δ small enough. Then, for $\epsilon > 0$ small enough, for $\bar{t} \in U \cap B(0,\epsilon)$,*

$$F^N_{V_{\bar{t}}} = N^{-2} \log(Z^N_{V_{\bar{t}}})$$

converges as N goes to infinity towards

$$F_{V_{\bar{t}}} = \sum_{\bar{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\le i\le n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}((q_1,k_1),\cdots,(q_n,k_n)).$$

Remark : Observe that our hypothesis is verified for all quadratic interaction models such as the Ising model, the q-Potts model ... etc ... as soon as the self potential of each matrix is convex.

Proof.

We can assume without loss of generality that $a = 0$ since otherwise we just make a shift on the covariance of the matrices under μ_N. The idea is to use Brascamp-Lieb inequality (c.f Harge [20] for recent improvements) which shows that since

$$f(\mathbf{A}) = e^{-N\mathrm{tr}V(A_1,\cdots,A_m)}$$

is log-concave, for all convex function g on $(\mathbb{R})^{mN^2}$

$$\mu^N_V(g(\mathbf{A}-\mathbf{M})) = \int g(\mathbf{A}-\mathbf{M}) \frac{f(\mathbf{A})\prod d\mu_N(A_i)}{\int f(\mathbf{A})\prod d\mu_N(A_i)} \le \int g(\mathbf{A}) \prod d\mu_N(A_i) \tag{13}$$

with

$$\mathbf{M} = \int \mathbf{A} d\mu_V^N.$$

Here $\mathbf{A}$ denotes the set of entries of the matrices $(A_1, \cdots, A_m)$. Let us apply (13) with $g(\mathbf{A}) = \mathrm{tr}(A_k^{2p})$ which is convex by Klein's lemma. Hence,

$$\mu_V^N \left(\mathrm{tr}((A_k - \mathbb{E}[A_k])^{2p})\right) \leq \mu_N(\mathrm{tr}(A^{2p})) \tag{14}$$

where $\mathbb{E}[A_k](ij) = \mu_V^N(A_k(ij))$ for $1 \leq i, j \leq N$. By Soshnikov, Theorem 2 p.17 in [25], there exists a finite constant C so that for all $p \leq \sqrt{N}$,

$$\mu_N(\mathrm{tr}(A^{2p})) \leq CN4^p.$$

In particular,

$$\limsup_{N\to\infty} \mu_V^N[\frac{1}{N}\mathrm{tr}((A_k - \mathbb{E}[A_k])^{2p})] \leq 4^p. \tag{15}$$

Also, by Chebychev's inequality we find that if $||A||_\infty$ denotes the spectral radius of A, for all $k \in \{1, \cdots, m\}$

$$\mu_V^N(||A_k - E[A_k]||_\infty \geq 3) \leq \mu_V^N(\mathrm{tr}((A_k - \mathbb{E}[A_k])^{2p} \geq 3^{2p}) \leq CN(\frac{2}{3})^{2p}$$

for all $p \leq \sqrt{N}$. Taking $p = \sqrt{N}$, we deduce by Borel Cantelli's lemma that

$$\limsup_{N\to\infty} ||A_k - E[A_k]||_\infty \leq 3 \quad \text{a.s.} \tag{16}$$

We now control $\mathbb{E}[A_k]$ uniformly. Since the law of A_k is invariant by the action of the unitary group, we deduce that for all unitary matrix U,

$$\mathbb{E}[A_k] = \mathbb{E}[UA_kU^*] = U\mathbb{E}[A_k]U^* \Rightarrow \mathbb{E}[A_k] = \mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))I. \tag{17}$$

We now bound $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))$ independently of N. Since V is convex, there are real numbers $(\gamma_i)_{1\leq i\leq m}$ and $c > -\infty$, $\gamma_i = \mathcal{D}_i V(0, \cdots, 0)$ and $c = V(0, \cdots, 0)$ so that for all $N \in \mathbb{N}$ and all matrices $(A_1, \cdots, A_m) \in \mathcal{H}_N^m$,

$$\mathrm{tr}(V(A_1, \cdots, A_m)) \geq \mathrm{tr}(\sum_{i=1}^m \gamma_i A_i + c).$$

By Jensen's inequality, we know that $Z_N^V \geq e^{-dN^2}$ for some $d < \infty$ and so Chebychev's inequality implies that for all $y > 0$, all $\lambda > 0$,

$$\begin{aligned}
\mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geq y\right) &\leq e^{(d-c)N^2 - \lambda y N^2}[\int e^{-N\sum_{i=1}^m \gamma_i \mathrm{tr}(A_i) + N\lambda \mathrm{tr}(A_k)} \prod_{i=1}^m d\mu_N(A_i) \\
&\quad + \int e^{-N\sum_{i=1}^m \gamma_i \mathrm{tr}(A_i) - N\lambda \mathrm{tr}(A_k)} \prod_{i=1}^m d\mu_N(A_i)] \\
&\leq 2e^{(d-c)N^2 - \lambda y N^2} e^{\frac{N^2}{2}\sum_{i\neq k}\gamma_i^2 + \frac{N^2}{2}(\gamma_k+\lambda)^2}
\end{aligned}$$

Optimizing with respect to λ shows that there exists $A < \infty$ so that

$$\mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geq y\right) \leq e^{AN^2 - \frac{N^2}{4}y^2}$$

and so

$$\mu_V^N(|\hat{\mu}_{\mathbf{A}}^N(X_k)|) = \int \mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geq y\right) dy \leq 4\sqrt{A} + \int_{y \geq 4\sqrt{A}} e^{-\frac{N^2}{4}(y^2 - 4A)} dy \leq 8\sqrt{A}$$

where we assumed N large enough in the last line. Hence, we have proved that

$$\limsup_N |\mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))| < 8\sqrt{A}. \tag{18}$$

Plugging this result in (15) and (17) we obtain for all $p \geq 1$:

$$\begin{aligned}
\limsup_{N\to\infty} \mu_V^N[\hat{\mu}_{\mathbf{A}}^N((A_k)^{2p})] &\leq 2^{2p-1} \limsup_{N\to\infty} \mu_V^N[\frac{1}{N}\mathrm{tr}((A_k - \mu_V^N[A_k])^{2p})] \\
&\quad + 2^{2p-1} \limsup_{N\to\infty} (\mu_V^N(\frac{1}{N}\mathrm{tr}[A_k])^{2p}) \\
&\leq 2^{2p-1}4^p + 2^{2p-1}(8\sqrt{A})^{2p} \leq R_V^{2p}
\end{aligned}$$

with $R_V = 4(1 + 8\sqrt{A})$. To prove the convergence of $\mu_N^V[\hat{\mu}_{\mathbf{A}}^N]$, remember that $\mu_N^V[\hat{\mu}_{\mathbf{A}}^N]$ is tight for the $\mathcal{C}_{st}^m(\mathbb{C})$-topology. To study its limit point, recall $\int xe^{-x^2/2}f(x)dx = \int f'(x)e^{-x^2/2}dx$ so that, for $P \in \mathcal{C}_{st}^m(\mathbb{C})$,

$$\begin{aligned}
\int \frac{1}{N}\mathrm{tr}(A_k P) d\mu_N^V(\mathbf{A}) &= \frac{1}{2N^2}\sum_{ij} \int \partial_{A_k(ij)}(Pe^{-N\mathrm{tr}(V)})_{ji} \prod d\mu_N(A_i) \\
&= \frac{1}{2N^2}\sum_{ij}\int \Bigg(\sum_{P = QX_kR} 2Q_{ii}R_{jj} \\
&\quad - N\sum_{l=1}^{n} \sum_{q_l = QX_kR} t_l \sum_{h=1}^{N} 2P_{ji}Q_{hj}R_{ih} \Bigg) d\mu_V^N(\mathbf{A}) \\
&= \int \left(\frac{1}{N^2}(\mathrm{tr}\otimes\mathrm{tr})(D_k P) - \frac{1}{N}\mathrm{tr}(\mathcal{D}_k V P) \right) d\mu_V^N(\mathbf{A})
\end{aligned}$$

which yields

$$\int \left(\hat{\mu}_{\mathbf{A}}^N((X_k + \mathcal{D}_k V)P) - \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N(D_k P) \right) d\mu_N^V(\mathbf{A}) = 0$$

Now, by convexity of V we have concentration of $\hat{\mu}_{\mathbf{A}}^N$ under μ_N^V (since log-Sobolev inequality is satisfied uniformly, according to Bakry-Emery criterion, and that Herbst's argument therefore applies, see [2], sections 6 and 7): for all Lipschitz function f on the entries

$$\mu_V^N\left(\mathbf{A} : |f(\mathbf{A}) - \mu_V^N(f)| \geq \delta\right) \leq e^{-\frac{\delta^2}{2\|f\|_{\mathcal{L}}^2}}$$

where $||f||_{\mathcal{L}}$ is the Lipschitz constant of f. Since for $P \in \mathcal{C}^m_{st}(\mathbb{C})$, $\mathbf{A} \to \hat{\mu}^N_{\mathbf{A}}(P)$ is Lipschitz with constant of order N^{-1} (see [17]), we conclude that since $D_i P \in \mathcal{C}^m_{st}(\mathbb{C}) \otimes \mathcal{C}^m_{st}(\mathbb{C})$, for all $P \in \mathcal{C}^m_{st}(\mathbb{C})$,

$$\lim_{N\to\infty} \left| \int \hat{\mu}^N_{\mathbf{A}} \otimes \hat{\mu}^N_{\mathbf{A}}(D_k P) d\mu^V_N(\mathbf{A}) - \mu^V_N[\hat{\mu}^N_{\mathbf{A}}] \otimes \mu^V_N[\hat{\mu}^N_{\mathbf{A}}](D_k P) \right| = 0.$$

Thus

$$\limsup_{N\to\infty} \left(\mu^N_V(\hat{\mu}^N_{\mathbf{A}}((X_k + \mathcal{D}_k V)P)) - \mu^N_V[\hat{\mu}^N_{\mathbf{A}}] \otimes \mu^N_V[\hat{\mu}^N_{\mathbf{A}}](D_i P) \right) = 0$$

If τ is a limit point of $\mu^N_V[\hat{\mu}^N_{\mathbf{A}}]$ for the weak $\mathcal{C}^m_{st}(\mathbb{C})$-topology, we can use the previous moment estimates to show that even though $X_k + \mathcal{D}_k V$ is a polynomial function, $\mu^N_V(\hat{\mu}^N_{\mathbf{A}}((X_k + \mathcal{D}_k V)P))$ converges along subsequences towards $\tau((X_k + \mathcal{D}_k V)P))$, and of course $\mu^N_V[\hat{\mu}^N_{\mathbf{A}}] \otimes \mu^N_V[\hat{\mu}^N_{\mathbf{A}}](D_k P)$ converges towards $\tau \otimes \tau(D_k P)$. Hence, we get that the limit points of $\mu^N_V[\hat{\mu}^N_{\mathbf{A}}]$ satisfy the **WSD[V]**. By the previous moment estimate, this limit points are compactly supported, hence they satisfy **SD[V]**. Similarly, by Corollary 3.2, $\hat{\mu}^N_{\mathbf{A}}$ is almost surely tight and its limit points satisfy **SD[V]** according to (16) and (18).

When $V = V_{\bar{t}}$, observe that $R_{\bar{t}}$ is uniformly bounded when $|t| \leq M$ since $V_{\bar{t}}(0, \cdots, 0)$ and $(\mathcal{D}_i V_{\bar{t}}(0, \cdots, 0))_{1\leq i\leq m}$ depends continuously on $\bar{t}$. Thus, the first point of the theorem shows that the limit points of $\mu^N_{V_{\bar{t}}}[\hat{\mu}^N_{\mathbf{A}}]$ are uniformly compactly supported. Hence, since also we have seen that they satisfy $\mathbf{SD}[V_{\bar{t}}]$, for $\bar{t}$ small enough, $\hat{\mu}^N_{\mathbf{A}}$ converges in expectation (and therefore almost surely by concentration), to the unique solution to $\mathbf{SD}[V_{\bar{t}}]$. The last point is now a direct consequence of Theorem 3.3.

□

Hence, we see here that convex potentials have uniformly compactly supported limit distributions so that we can apply the whole machinery. We strongly believe that this property extends to much more general potentials. However, we shall see in the next section that we can localize the integral to make sure that all limit points are uniformly compactly supported and still keep the enumerative property, hence bypassing the issue of compactness.

3.3 The uses of diverging integrals

In the domain of matrix models, diverging integrals are often considered. For instance, if one wants to consider triangulations, one would like to study the integral

$$Z_N(tx^3) = \int e^{tN\mathrm{tr}(M^3)} d\mu_N(M)$$

which is clearly infinite if t is real. The same kind of problem arises in many other models (c.f. the dually weighted graph model [21]). However, we shall see below that at least as far as planar maps are concerned, we can localize the integrals to make sense of it, while keeping its enumerative property. Namely, let $V_{\bar{t}} = \sum t_i q_i$ and let us consider the localized matrix integrals given, for $L < \infty$, by

$$Z^{N,L}_{V_{\bar{t}}} = \int_{||\mathbf{A}||_\infty \leq L} e^{-N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))} \prod d\mu_N(A_i)$$

and the associated Gibbs measure

$$\mu^{N,L}_{V_{\bar{t}}}(d\mathbf{A}) = (Z^{N,L}_{V_{\bar{t}}})^{-1} 1_{||\mathbf{A}||_\infty \leq L} e^{-N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))} \prod d\mu_N(A_i).$$

Here, $||\mathbf{A}||_\infty = \max_{1\le i\le m} ||A_i||_\infty$ and $||A_i||_\infty$ denotes the spectral radius of the matrix A_i.

We shall prove that

Theorem 3.5 *There exists $\epsilon_0 > 0$ so that for $\epsilon < \epsilon_0$, there exists $L_0(\epsilon)$ and $L(\epsilon)$, $L(\epsilon)$ going to infinity and $L_0(\epsilon)$ going to 2 as ϵ goes to zero, so that for $\bar{t} \in B(0,\epsilon)$, for all $L \in [L_0(\epsilon), L(\epsilon)]$,*

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z_{V_{\bar{t}}}^{N,L} = \sum_{\bar{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\le i\le n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}((q_1,k_1),\cdots,(q_n,k_n))$$

Moreover, under $\mu_{V_{\bar{t}}}^{N,L}$, $\hat{\mu}_{\mathbf{A}}^N$ converges almost surely towards $\tau_{\bar{t}}$ described in Theorem 2.3.

This shows that, up to localization, the first order asymptotics of matrix models give the right enumeration for any polynomials. The diverging integrals often considered in physics should be therefore thought to be conveniently localized to keep their combinatorial virtue, and are then as good as others. In view of Lemma 3.6, this localization procedure should not damage the rest of the large N expansion neither. Note that when $m = 1$, the localization amounts to restrict the integral in the domain of strict convexity of the potential, henceforth again avoiding all issues of escaping eigenvalues.

Proof.

The proof is very close to that of Theorem 3.1 except that we have to be careful to make perturbations which do not change the constraint $||\mathbf{A}||_\infty \le L$. Let $i \in \{1,\cdots,m\}$ and consider the perturbation $A_i \to A_i + N^{-1}h(A_i)$ and $A_j \to A_j$ for $j \neq i$ with a compactly supported function h which vanishes on $[-R,R]^c$. Then for $L > R$, for sufficiently large N, and $||A_i||_\infty \le L$, $||A_i + N^{-1}h(A_i)||_\infty \le L$ so that we see that the limit points of $\hat{\mu}_{\mathbf{A}}^N$ under the localized Gibbs measure $\mu_V^{N,L}$ satisfy for $i \in \{1,\cdots,m\}$,

$$\mu\otimes\mu(D_i h(X_i)) = \mu((\mathcal{D}_i V + X_i)h(X_i)). \tag{19}$$

These limit points are also laws of operators bounded by L, but we shall see that in fact this bound can be improved to become independent of L for good L's. We fix a limit point μ below; μ is a tracial state. We proceed as before by taking $h(x) = P(\phi_R(x))$ with a polynomial test function P and ϕ_R a smooth cutoff function with uniformly bounded derivative (say by 2). Then,

$$\mu\otimes\mu(D_i h(X_i)) = \mu\otimes\mu(D_i P(\phi_R(X_i)) \times \phi_R'(X_i)\otimes 1)$$

so that if $P(x) = x^{2k+1}$,

$$|\mu\otimes\mu(D_i h(X_i))| \le 2\sum_{p=0}^{2k} \mu(|\phi_R(X_i)|^p)\mu(|\phi_R(X_i)|^{2k-p}).$$

Hence, we get, for $i \in \{1,\cdots,m\}$,

$$\mu(X_i\phi_R(X_i)^{2k+1}) \le 2\sum_{p=0}^{2k} \mu(|\phi_R(X_i)|^p)\mu(|\phi_R(X_i)|^{2k-p}) + |t|\sum_{j=0}^{n} \mu(|\mathcal{D}_i q_j(\mathbf{X})||\phi_R(X_i)|^{2k+1})$$

Note that we can bound above the last term by Hölder's inequality so that

$$\sum_{j=0}^{n} \mu(|\mathcal{D}_i q_j(\mathbf{X})||\phi_R(X_i)|^{2k+1}) \le C \max_{1\le j\le m} \{\mu(|X_j|^{2k+D-1}), \mu(|X_j|^{2k})\}$$

where we assumed $|\phi_R(x)| \le |x|$. Taking $\phi_R(x) = x$ when $|x| \le R/2$ and $\phi_R(x) = 0$ when $|x| \ge R$, ϕ_R linear in between, we can now use monotone convergence theorem (letting R going to infinity) to obtain

$$\max_{1\le j\le m} \mu(X_j^{2(k+1)}) \le 2\sum_{p=0}^{2k} \max_{1\le j\le m} \mu(|X_j|^p) \max_{1\le j\le m} \mu(|X_j|^{2k-p}) + C|t| \max_{1\le j\le m} (\mu(X_j^{2k}) + \mu(|X_j|^{D+2k-1}))$$

Noting that $\mu(|X_j|^{D+2k-1}) \le L^{D-1}\mu(|X_j|^{2k})$ since under μ the operators are uniformly bounded by L, we can improve this uniform bound as follows. We make the induction hypothesis that

$$B_k := \max_{1\le j\le m} \mu(|X_j|^k) \le C_k A^k$$

for $k \le 2r$, with C_k the Catalan numbers and some $R > 1$. Then, by Hölder's inequality,

$$(B_{2r+1})^{\frac{2r+2}{2r+1}} \le B_{2r+2} \le 2C_{2r+1}A^{2r} + C|t|(1+L^{D-1})A^{2r}C_{2r}.$$

We can assume without loss of generality that $B_{2r+1} \ge 1$ so that we get

$$B_{2r+2} \vee B_{2r+1} \le C_{2r+1}A^{2r+1}\left(2A^{-1} + C|t|L^D\right)$$

and so we can continue our induction if L is not too large (L at most of order $(2C|t|)^{-\frac{1}{D}}$) so that there is $A > 0$ so that $2A^{-1} + C|t|L^D \le 1$. Hence all limit points are non-commutative laws of operators with norms uniformly bounded by $4A \ll L$. Thus if $R > 4A$ we conclude by (19) that all limit points satisfy $\mathbf{SD}[V_{\bar t}]$. Moreover, Theorem 2.1 and 2.3 apply to show that there is a unique solution to this equation which are laws of operators bounded by R (provided R, and so L, is neither too big ($L \le O(|t|^{-D^{-1}})$, for uniqueness), nor too small ($L \ge L_0(|t|) \ge L_0(0) = 2$ for existence). Note here that $L_0(|t|)$ is the smallest real number such that $|\tau_{\bar t}(P)| \le L_0(|t|)^{deg(P)}$ for all monomial P, which converges to 2 as $\bar t$ goes to zero by Theorem 2.3. Hence, $\hat\mu^N_{\mathbf{A}}$ has a unique limit point, $\tau_{\bar t}$, and thus converges towards it.

The formula of the free energy is then derived as in Theorem 3.3 since L is fixed independently of $\bar t$ small enough.

□

Let us remark that if we define, following Voiculescu [29], a microstates $\Gamma(\mu, n, N, \epsilon)$ for $\mu \in \mathcal{M}_1^{(m)}$, $n \in \mathbb{N}$, $N \in \mathbb{N}$, $\epsilon > 0$, as the set of matrices $A_1, .., A_m$ of $\mathcal{H}_N^m$ such that

$$|\mu(\mathbf{X}_{i_1}..\mathbf{X}_{i_p}) - \frac{1}{N}\mathrm{tr}(\mathbf{A}_{i_1}..\mathbf{A}_{i_p})| < \epsilon \tag{20}$$

for any $1 \le p \le n$, $i_1, .., i_p \in \{1, .., m\}^p$, then we have

Lemma 3.6 *For all $\delta > 0$ small enough, for $L \in [L_0(\delta), L(\delta)]$, $V = V_{\bar t}$ and $|t| \le \delta$,*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \int_{||\mathbf{A}||_\infty \le L} e^{-N tr(V(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m)$$

$$= \lim_{\epsilon\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_V,n,N,\epsilon)\cap ||\mathbf{A}||_\infty \le L} e^{-N tr(V(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m)$$

$$= \lim_{\epsilon\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_V,n,N,\epsilon)} e^{-N tr(V(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m)$$

Proof.

The first equality is a direct consequence of the previous theorem since it is equivalent to the fact that $\mu_{V_{\bar{t}}}^{N,L}(\Gamma(\tau_V, n, N, \epsilon))$ goes to one. The second comes from the fact that for n greater than the degree of V,

$$\lim_{\epsilon\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_V,n,N,\epsilon)\cap ||\mathbf{A}||_\infty \le L} e^{-N\mathrm{tr}(V(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m)$$
$$= -\tau_V(V) + \lim_{\epsilon\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_V, n, N, \epsilon) \cap ||\mathbf{A}||_\infty \le L\right)$$
$$= -\tau_V(V) + \lim_{\epsilon\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_V, n, N, \epsilon)\right)$$

where we used in the last equality the result of [3], which hold when τ_V is the law of bounded operators with norms strictly smaller than L (see the last remark in [3]).

□

Therefore, the localization should not affect the full expansion of the integral since second order asymptotics are usually obtained first by a localization on microstates in order to use precise Laplace method's.

As a corollary, we also deduce that for all $V_{\bar{t}}$ with $\bar{t}$ small enough, the limits of empirical distributions of matrices given by localized matrix models provide solutions of $\mathbf{SD}[V_{\bar{t}}]$. Since these limits have to be tracial states, we deduce that when there is a unique solution it has to be a tracial state. Thus,

Corollary 3.7 *The compactly supported solutions of* $\mathbf{SD}[V_{\bar{t}}]$ *are tracial states when* $\bar{t}$ *is sufficiently small.*

Note that if $(P_i)_{1\le i\le m}$ is the conjugate variable of a tracial state, Voiculescu [30] have shown that $P_i = \mathcal{D}_i P$ for $1 \le i \le m$ and some polynomial P. This fact should be compared with our graphical interpretation which works only because P_i is a cyclic derivative.

4 Applications to free entropy

Let us recall that Voiculescu's microstates entropy is defined, for $\tau \in \cup_R \mathcal{M}_R^m$, by

$$\chi(\tau) = \lim_{\substack{\epsilon\to 0, n\to\infty \\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau, n, N, \epsilon) \cap ||\mathbf{A}||_\infty \le L\right)$$

with $\Gamma(\tau, n, \epsilon, N)$ the microstates defined in (20). Note that the original definition of Voiculescu is not with respect to the Gaussian measure, but with respect to the Lebesgue measure. However, both definitions only differ by a quadratic term (see [9]). It is an (important) open problem whether in general one can replace the limsup by a liminf in the definition of χ. However, from the previous considerations, we can see the following

Theorem 4.1 *Let* $n \in \mathbb{N}$ *and* $(q_i)_{1\le i\le n}$ *be monomials in* m *non-commutative variables* $\mathbf{X} = (X_1, \cdots, X_m)$. *Let* $V_{\bar{t}}(\mathbf{X}) = \sum_{i=1}^n t_i(q_i(\mathbf{X}) + q_i^*(\mathbf{X}))$. *By Theorem 2.3, we know that there exists*

$\epsilon > 0$ so that for $|t| < \epsilon$, there exists a unique compactly supported solution $\tau_{\bar{t}}$ to $\mathbf{SD}[V_{\bar{t}}]$. Then, also for $|t| \le \epsilon$,

$$\chi(\tau_{\bar{t}}) = \lim_{\substack{\epsilon\to 0, n\to\infty \\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_{\bar{t}}, n, N, \epsilon) \cap \{||\mathbf{A}||_\infty \le L\}\right).$$

Moreover,

$$\chi(\tau_{\bar{t}}) = -\sum_{k\in\mathbb{N}^n\setminus(0,\cdots,0)} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \left(\sum_{j=1}^n k_j - 1\right) \mathcal{M}((q_1,k_1),\cdots,(q_n,k_n)).$$

Remark: In particular, we see as could be expected that $\chi(\tau_{\bar{t}}) > -\infty$ and so all the solutions to Schwinger-Dyson's equations for small parameters are laws of von Neumann algebras which are isomorphic to the free group factor.

Proof.

In fact,

$$\begin{aligned}
\chi(\tau_{\bar{t}}) &= \lim_{\substack{\epsilon\to 0, n\to\infty \\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\bar{t}},n,N,\epsilon)\cap\{||\mathbf{A}||_\infty\le L\}} e^{N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))-N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A}) \\
&= \tau_{\bar{t}}(V_{\bar{t}}) + \lim_{\substack{\epsilon\to 0, n\to\infty \\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\bar{t}},n,N,\epsilon)\cap\{||\mathbf{A}||_\infty\le L\}} e^{-N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A}) \\
&\le \tau_{\bar{t}}(V_{\bar{t}}) + F_{\bar{t}}
\end{aligned}$$

where the last inequality holds with

$$F_{\bar{t}} = \lim_{L\to\infty} \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{||\mathbf{A}||_\infty \le L'} e^{-N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A})$$

for L' chosen as in Lemma 3.6. On the other hand,

$$\begin{aligned}
& \lim_{\substack{\epsilon\to 0, n\to\infty \\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_{\bar{t}}, n, N, \epsilon) \cap ||\mathbf{A}||_\infty \le L\right) \\
&= \tau_{\bar{t}}(V_{\bar{t}}) + \lim_{\substack{\epsilon\to 0, n\to\infty \\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\bar{t}},n,N,\epsilon)\cap\{||\mathbf{A}||_\infty\le L} e^{-N\mathrm{tr}(V_{\bar{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A}) \\
&= \tau_{\bar{t}}(V_{\bar{t}}) + F_{\bar{t}} + \lim_{\epsilon\to 0, n\to\infty} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_{V_{\bar{t}}}^{N,L'}\left(\Gamma(\tau_{\bar{t}}, n, N, \epsilon)\right) \\
&= \tau_{\bar{t}}(V_{\bar{t}}) + F_{\bar{t}}
\end{aligned}$$

where we used in the last term Theorem 3.5 which implies

$$\lim_{N\to\infty} \mu_{V_{\bar{t}}}^{N,L}\left(\Gamma(\tau_{\bar{t}}, n, N, \epsilon)\right) = 1$$

for all $\epsilon > 0, n \in \mathbb{N}$. Thus, we see that χ is equal to its liminf definition and moreover

$$\chi(\tau_{\bar{t}}) = \tau_{\bar{t}}(V_{\bar{t}}) + F_{\bar{t}}.$$

Now, by Theorems 3.5 and 2.3,

$$F_{\overline{t}} = \sum_{\overline{k}\in\mathbb{N}^n\backslash(0,..,0)} \prod_{1\le i\le n} \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}((q_1,k_1),\cdots,(q_n,k_n))$$

whereas

$$\tau_{\overline{t}}(V_{\overline{t}}) = \sum_{i=1}^n t_i \sum_{\substack{k_j\in\mathbb{N},\\ 1\le j\le n}} \prod_{1\le j\le n} \frac{(-t_j)^{k_j}}{k_j!}\mathcal{M}((q_1,k_1),\cdots,(q_{i-1},k_{i-1}),(q_i,k_i+1),(q_{i+1},k_{i+1}),\cdots,(q_n,k_n))$$

from which the formula for $\chi(\tau_{\overline{t}})$ is easily derived.

□

5 Applications to the combinatorics of planar maps

For the sake of completeness, we summarize in this last section, the results of a few papers devoted to the enumeration of planar maps, either by a combinatorial approach or by a matrix model approach.

5.1 The one matrix case

We now consider the simpler case $m = 1$ where we only have one matrix. Let $V_{\overline{t}}(A) = \sum_{i=1}^{2D} t_i A^i$ with $t_{2D} > 0$ a polynomial potential with an even leading power. Then it has been proven in [4] Theorem 5.2 that the empirical measure satisfies a large deviation principle:

Theorem 5.1 *Let*

$$J(\mu) = \int \left(\frac{x^2}{2} + V_{\overline{t}}(x)\right) d\mu(x) - \int\int \log|x-y| d\mu(y) d\mu(x)$$

and

$$I(\mu) = J(\mu) - \inf_{\nu\in P(\mathbb{R})} J(\nu)$$

then the sequence of empirical measure $\hat{\mu}^N$ satisfies a large deviation principle in the scale N^2 with good rate function I. Moreover, the minimum of I is reached at a unique probability measure $\mu_{\overline{t}}$ so that

$$\frac{x^2}{2} + V_{\overline{t}}(x) - 2\int \log|y-x| d\mu_{\overline{t}}(y) = C_{\overline{t}}, \quad \mu_{\overline{t}} a.s.$$

with a finite constant $C_{\overline{t}}$, and where the left hand side dominates the right hand side on the whole real line.

One can notice that differentiating in x the last equation, we recover the Schwinger Dyson's equation. It is not sufficient in general to determine the solution uniquely; one need the inequality on the whole real line to fix the support of the solution.

These questions have also been investigated with the method of orthogonal polynomials which give a rather sharp description of the limit measure and emphasizes a structure similar to the semi-circular law. More precisely Theorem 3.1 in [12] gives:

Theorem 5.2 *There exists $t > 0$ and $\gamma > 0$ such that if for all i, $|t_i| < t$ and $t_{2D} > \gamma \sum_{i<2D} t_i$ then $\mu_{\bar{t}}$ is absolutely continuous with density $\Psi_{\bar{t}}$ of the form:*

$$\Psi_{\bar{t}}(x) = \frac{1}{2\pi} 1_{[a,b]}(x)\sqrt{(x-a)(x-b)}h(x)$$

with

$$h(z) = \int_{C(z,R)} \frac{V'_{\bar{t}}(s)}{\sqrt{(s-a)(s-b)}} \frac{ds}{s-z}$$

where R is such that $a, b \in C(z, R)$. Besides, the boundaries a and b can be find by the equations:

$$\int_a^b \frac{V'_{\bar{t}}(s)}{\sqrt{(s-a)(b-s)}} ds = 0$$

$$\int_a^b \frac{sV'_{\bar{t}}(s)}{\sqrt{(s-a)(b-s)}} ds = 2\pi$$

We now look at combinatorics of the Schwinger-Dyson's equation with one variable, for $V_{\bar{t}}(x) = \sum_{i=1}^{2D} t_i x^i$. Remember that from Theorem 2.3, $\mu_{\bar{t}}$ can be seen as the generating function of graphs counted by the numbers of stars of valence i:

$$\mu_{\bar{t}}(x^p) = \sum_{k_1,\cdots,k_{2D}\in\mathbb{N}} \prod_{i=1}^{2D} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_0((P,1), \{(X^i, k_i)\}_{1\leq i\leq 2D}).$$

Hence, Theorem 5.2 allows to estimate the numbers of one color planar maps. A more direct combinatorial approach can be developed by considering for instance the dual of those graphs. The dual of a graph is simply obtained by replacing each face by a star and each edge by a transverse edge which link the two stars which come from the face adjacent to the edge. In that operation each star is replaced by a face of the same valence. As we work on the sphere we can decide that the face which comes from the star X^p is the external face.

So $\mu_{\bar{t}}(X^{p+1})$ is also the generating function of connected planar graphs with an external face of valence $p+1$ and enumerated by the number of faces of a given valence. Those objects are classical ones in combinatorics and we can follow [27] to find an equation on these generating functions. The idea is to try to cut the first edge of the external face, then two cases may occur: either the graph is disconnected and we obtain two graphs or it isn't disconnected and the external face has grown. This two cases corresponds in the dual graph to the fact that the first branch of the root is a loop or not which is exactly what we use to build our combinatorial interpretation so that we can retrieve the Schwinger Dyson's equation from this fact. Just by using the equation given by this decomposition and some algebraic tools, combinatoricians have solved some models. For example [5] gives an equation on the generating function $M(u, v)$ of maps whose internal faces have degree living in a fixed set $\mathcal{D} \subset \mathbb{N}$ and enumerated by their number of edges and the degree of the external face. To translate this in our framework, one can consider for a finite $\mathcal{D}$ with an even maximal element,

$$V_{\bar{t}}(X) = \sum_{d\in\mathcal{D}} t_d X^d$$

Then under this potential, for small t, the limit measure $\mu_{\bar{t}}$ will satisfy our combinatorial interpretation. Then

$$M(u, v) = \sum_{p\in\mathbb{N}} \mu_{(-u^{\frac{d}{2}})_{d\in\mathcal{D}}}(X^p)v^p$$

Now Theorem 1 of [5] states:

Theorem 5.3 *For a serie $F(z) = \sum_i a_i z^i$ we will note $[z^i]F(z)$ the i^{th} coefficient a_i. Then there exists a unique power serie R satisfying*

$$R = 1 - 4R_1 v - 4R_2 v^2$$

with

$$R_1 = \frac{u}{2}\sum_{i\in D}[v^{i-1}](R^{\frac{i}{2}}) \text{ and } R_2 = \frac{u}{2}\sum_{i\in D}[v^i](R^{\frac{i}{2}}) + u - 3R_1^2.$$

The number m_n of maps with n edges such that every degree of internal face lies in D is then

$$m_n = [u^n]\frac{(R_2(u) + R_1(u)^2)(R_2(u) + 9R_1(u)^2)}{(n+1)u^2}$$

The techniques to prove these results are most often purely algebraic. The main difference in nature than we could meet between the approaches by matrix models or by combinatorics to these enumerations is that the first provides for free additional structure; it shows that these enumerations can be expressed in terms of a probability measure $\mu_{\bar{t}}$. This point generalizes to any number of colors where the enumeration can be expressed in terms of tracial states. One may hope that this positivity condition could help in solving these combinatorial problems.

5.2 Ising model on random graphs

This model is defined by $m = 2$ and

$$V(A, B) = V_{Ising}(A, B) = -cAB + V_1(A) + V_2(B).$$

In the sequel, we denote in short A for X_1 and B for X_2. It is clear that for $|c| < 1$, V is a convex potential as defined in (12) if V_1, V_2 are convex (write $-2AB = (A-B)^2 - A^2 - B^2$ or $2AB = (A+B)^2 - A^2 - B^2$ to see that up to a quadratic term $2^{-1}|c|A^2 + 2^{-1}|c|B^2$, V is convex) Hence we deduce from Theorem 3.4 that

Corollary 5.4 *For $c \in \mathbb{R}$ and $V_i(x) = \sum_{j=1}^D t_j^i x^{2j}$, $i = 1, 2$, set $V_{\bar{t},c}(A, B) = -cAB + V_1(A) + V_2(B)$. Let, for $\delta > 0$, $U_\delta = \cap_{i,j}\{0 \le t_j^i \le \delta\} \cap \{|c| < 1 - \delta\}$. Then, for $\delta > 0$ small enough and $(\bar{t}, c) \in U_\delta$, $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N)$ converges towards the solution $\mu_{\bar{t},c}$ of $\mathbf{SD}[V_{\bar{t},c}]$ as N goes to infinity. Moreover*

$$\mu_{\bar{t},c}(P) = \sum_{\substack{\bar{k}\in\mathbb{N}^{2D}\\ r\in\mathbb{N}}} \prod_{i,j} \frac{(-t_j^i)^{k_j^i}}{k_j^i!}\frac{c^r}{r!}\mathcal{M}_0((P,1), (A^{2j}, k_j^1)_{1\le j\le D}, (B^{2j}, k_j^2)_{1\le j\le D}, (AB, r)),$$

and

$$F(\bar{t}, c) - F(\bar{t}, 0) = \sum_{\substack{\bar{k}\in\mathbb{N}^{2D}\\ r\ge 1}} \prod_{i,j} \frac{(-t_j^i)^{k_j^i}}{k_j^i!}\frac{c^r}{r!}\mathcal{M}_0((A^{2j}, k_j^1)_{1\le j\le D}, (B^{2j}, k_j^2)_{1\le j\le D}, (AB, r)).$$

Remark: Note that we took potentials V_1 and V_2 as polynomials with even powers to guarantee our convexity relation but this condition could easily been relaxed by taking more sophisticated domains than U_δ in which the polynomials would remain convex.

Proof.

This result is a consequence of Theorem 2.2, 3.4 and 3.3. Note here that the control on $\mu_V^N(N^{-1}\mathrm{tr}(AB))$ assumed in Theorem 3.3 is satisfied due to Theorem 3.4 which provides a uniform bound when $|c| < \xi$ for $\xi < 1$.

□

According to the graphical interpretation, the limiting measure is linked to planar maps with stars whose type are the monomial of V_1, V_2 and stars of type AB. Those maps are very close from Ising configuration on planar graphs except that two stars of type AB can be linked together. For integers $(k_j^i)_{i\in\{A,B\},1\le i\le D}$, define

$$\begin{aligned}\mathcal{I}(\{k_j^i\}, r, P) \quad = \sharp\{ \quad & \text{planar maps with } k_j^i \text{ stars of color } i \text{ and degree } 2j, \\ & \text{one star of type } P \text{ (if } P \neq 0) \text{ and } r \text{ stars of type } AB \\ & \text{such that there's no link between any of the } r\ AB\text{-stars. } \}\end{aligned}$$

and its rooted counterpart:

$$\begin{aligned}\mathcal{J}(\{k_j^i\}, r, P) \quad = \sharp\{ \quad & \text{rooted planar maps with } k_j^i \text{ stars of color } i \text{ and degree } 2j, \\ & \text{one star of type } P \text{ wich is the root and } r \text{ stars of type } AB \\ & \text{such that there's no link between any of the } r\ AB\text{-stars. } \}\end{aligned}$$

There's a relation between these quantities similar to (3):

$$\mathcal{I}(\{k_j^i\}, r, P) = \mathcal{J}(\{k_j^i\}, r, P) r! \Pi_{i,j} k_j^i! (2j)^{k_j^i} \tag{21}$$

We can now relate these numbers to our limit measure:

Proposition 5.5 *Let $\mu_{\bar{t},c}$ be as in 5.4, then on its radius of convergence,*

$$\mu_{\bar{t},c}(P) = \left(\frac{1}{1-c^2}\right)^{\frac{deg\ P}{2}} \sum_{\substack{k_j^i \in \mathbb{N}^{2D} \\ r \in \mathbb{N}}} \prod_{i,j} \frac{1}{k_j^i!} \left(\frac{-t_j^i}{(1-c^2)^j}\right)^{k_j^i} \frac{c^r}{r!} \mathcal{I}(\{k_j^i\}, r, P)$$

and

$$F(\bar{t}, c) - F(\bar{t}, 0) = \frac{1}{1-c^2} \sum_{k_j^i \in \mathbb{N}, i \in \{1,2\}, j \in \{1,D\}, r \ge 1} \prod_{i,j} \frac{1}{k_j^i!} \left(\frac{-t_j^i}{(1-c^2)^j}\right)^{k_j^i} \frac{c^r}{r!} \mathcal{I}(\{k_j^i\}, r, 0)$$

Proof.

First we define a projection π from rooted maps to rooted Ising graph such that if M is a map $\pi(M)$ is obtained by deleting pairs of AB stars which are glued. We now apply Corollary 5.4, and translate its result in term of rooted diagrams using (3):

$$\mu_{\overline{t},c}(P) = \sum_{\substack{\overline{k}\in\mathbb{N}^{2D} \\ r\in\mathbb{N}}} \prod_{i,j} (-2jt_j^i)^{k_j^i} c^r \mathcal{D}_0(P, (A^{2j}, k_j^1)_{1\le j\le D}, (B^{2j}, k_j^2)_{1\le j\le D}, (AB, r))$$

All the maps M appearing in that sum are such that $\pi(M)$ is an Ising graph rooted at a star of type P. For a fixed Ising graph G we must find the contribution in that sum of $\pi^{<-1>}(G)$. But we can construct every graph in that set by adding pairs of stars AB on the edges of G. The numbers of edges of G is $e_G = \frac{degP}{2} + \sum_{i,j} 2jk_j^i$ so that to get the whole contribution of $\pi^{<-1>}(G)$ we have to multiply the contribution of G by

$$\sum_{a_1,\cdots,a_{e_G}\in\mathbb{N}} c^{2\sum a_i} = \left(\frac{1}{1-c^2}\right)^{\frac{degP}{2}+\sum_{i,j} 2jk_j^i}.$$

In that sum, a_i stands for the number of pairs of AB stars added on the i^{th} edge. Summing on every graphs, we obtain:

$$\mu_{\overline{t},c}(P) = \left(\frac{1}{1-c^2}\right)^{\frac{\deg P}{2}} \sum_{\substack{\overline{k}\in\mathbb{N}^{2D} \\ r\in\mathbb{N}}} \prod_{i,j} \left(\frac{-2jt_j^i}{(1-c^2)^{2j}}\right)^{k_j^i} c^r \mathcal{J}(\{k_j^i\}, r, P)$$

and the result follows by using (21).

The second point can be proven by proceeding in the same way.

□

In the rest of this section, we compare a few different approaches to solve the enumeration problem of the Ising model. In short, let us emphasize that, for the time being, combinatorial and orthogonal polynomials approaches give the more complete and explicit results. However, these techniques are still limited to very few models. The Schwinger-Dyson's equation or the large deviation approaches can be developed for a much wider range of models (such as q-Potts, induced QCD etc). However, it seems to us that these arguments still need some mathematical efforts to provide as transparent and powerful results (namely for the first a mathematical study of the so-called master-loop equations, and for the second a clear understanding of the relations between complex Burgers equations and the master-loop equations). A striking difference between the combinatorial and the matrix model approaches seems to reside in the fact that matrix models provide for free information on the structure of the generating function of the number of planar maps, for instance as the Stieljes transform of a probability measure with connected support.

5.2.1 Orthogonal polynomial approach

Here we take $V_1 = V_2 = (g/4)x^4$. By using orthogonal polynomials techniques, it was proved by Mehta [23] that the corresponding free energy $F_{g,c}$ satisfies

$$F_{g,c} - F_{0,c} = \int_0^1 (1-x)[\log f(x) - \log \frac{cx}{2(1-c^2)}]dx$$

with $f(x) = f_{g,c}$ solution to the algebraic equation

$$f(x)\{(1 - 6\frac{g}{c}f(x))^{-2} - c^2\} + 12g^2 f^3(x) - \frac{1}{2}cx = 0$$

and the root to be taken equals $2^{-1}cx(1-c^2)^{-1}$ when $g=0$.

Starting from there, a simpler expression as been derived in [6] (equation (16), (17) with $h = z/g$):

$$
\begin{aligned}
F_{z,c} &= \frac{1}{2}\ln h(z) + \frac{h^2(z)}{2}\left(\frac{z-1}{2(3z-1)^3} + c^2\frac{z+1}{3z-1} + \frac{c^4}{2}(3z^4-3z^2+1)\right) \\
&\quad -h(z)\left(\frac{1}{3z-1} + c^2(1-z^2)\right) + \frac{1}{2}\ln(1-z^2) + \frac{3}{4}
\end{aligned}
$$

with

$$
h(z) = \frac{(1-3z)^2}{1-c^2(1-3z)^2(1-3z^2)} \tag{22}
$$

Hence, by the preceding, Mehta's result gives a formula for the generating function of $\mathcal{J}$ in the quadrangulation case. However, it does not a priori gives the limiting spectral measures of the matrices. Moreover, this strategy could be only developed completely and rigorously for the Ising model and the matrix coupled in chain model [10].

5.3 Direct combinatorial approach

We can also relate this result to the work of Bousquet-Melou and Schaeffer [7]. Their approach is purely combinatorial; they use bijection with well labeled trees (whose generating functions are well understood) to obtain algebraic equations for the generating functions of the Ising model. Let $I(X,Y,u)$ be the generating function of the Ising model on quasi-tetravalent graphs, (i.e. tetravalent graphs except for the root which is bivalent and black) where X (resp. Y) counts the black (resp. white) tetravalent stars and u the bicolored edges:

$$
I(X,Y,u) = \sum_{m,n,r\in\mathbb{N}} X^m Y^n u^r \sharp\{\text{ quasi -tetravalent maps with } m \text{ tetravalent black stars, } n \text{ tetravalent white stars and } r \text{ bi-colored edges}\}.
$$

If $P(x,y,u)$ is the solution to the algebraic equation:

$$
P = 1 + 3xyP^3 + \frac{P(1+3xP)(1+3yP)}{u^2(1-9xyP^2)^2} \tag{23}
$$

Then, by [7], Proposition 1 p.4, I can be written in function of $P(x,y,u)$ with $x = X(u-\frac{1}{u})^2$ and $y = Y(u-\frac{1}{u})^2$ as

$$
I(X,Y,u) = (1-u^{-2})\left(xP^3 + \frac{P(1-3xP-2xP^2-6xyP^3)}{1-9xyP^2} - \frac{yu^{-2}P^3(1+3xP)^3}{(1-9xyP^2)^3}\right).
$$

On the other hand, according to Proposition 5.5, if $V = tA^4 + uB^4 - cAB$ and $\mu_{t,u,c}$ is the associated limit measure then on its domain of convergence,

$$
I(X,Y,u) = (1-u^2)\mu_{X(1-u^2)^{-2},Y(1-u^2)^{-2},u}(A^2).
$$

If we make the following change of variable in (23):

$$
x = y = \frac{-z}{3c^2h(z/3)}, P = -c^2h(z/3), u = c
$$

then we find (22). Hence, a combinatorial approach can be developed to solve the problem of the enumeration of planar maps of the Ising model, a strategy which requires some combinatorial insight. The next approach we present, developed in particular by Staudacher, Kazakov and Eynard, is a direct analysis of the **SD[V]** equations. It is a purely analytical and rather robust strategy.

5.4 Direct study of the SD$[V_{Ising}]$ equations

Here, the analysis is based on Theorem 3.4 which asserts that if V_1, V_2 are convex, for small parameters, $\hat{\mu}^N_{A,B}$ converges almost surely towards the solution $\mu_{\bar{t},c}$ of **SD**$[V_{\bar{t},c}]$ which is a generating function for the enumeration of maps. Hereafter we take $c = 1$ up to a rescaling $\bar{x} = \sqrt{c}x$, $\bar{y} = \sqrt{c}x$, $V_1(x) = \bar{V}_1(\bar{x})$, $\mu_{\bar{t}}(P(A, X_2)) = \mu_{\bar{t},1}(P(\sqrt{c}^{-1}A, \sqrt{c}^{-1}X_2))$. Following Eynard [13], we shall analyze the solutions of the Schwinger Dyson's equation. Observe that the following considerations hold for any range of parameters, not only small parameters. For large parameters, we do not know that the Schwinger Dyson's equation has a unique solution but we still know that any limit point of the empirical measure of the random matrices still satisfies it. In the next section, we shall see that for the Ising model and any range of parameters, there is a unique such limit point, and it will therefore enjoy the properties described below. We here summarize the main result, as found in Eynard [13]. Let $\mu_{\bar{t}}$ be a solution of **SD**$[V_{Ising}]$

$$\mu_{\bar{t}}((W_1'(A) - B)P) = \mu_{\bar{t}} \otimes \mu_{\bar{t}}(D_A P),$$
$$\mu_{\bar{t}}((W_2'(B) - A)P) = \mu_{\bar{t}} \otimes \mu_{\bar{t}}(D_B P),$$

with D_A (resp. D_B) the non-commutative derivative with respect to A (resp. B) μ_A (resp. μ_B) and $W_i(z) = z^2/2 + V_i(z)$. Now, let μ_A (resp. μ_B) be the spectral measure of the matrix A (resp. B) then we shall obtain an algebraic equation for $H\mu_A(x)$ (resp. $H\mu_B(x)$) the Stieljes transform of the limiting measure μ_A (resp. μ_B) given, for $x \in \mathbb{C}\backslash\mathbb{R}$ by:

$$H\mu_A(x) = \mu_{\bar{t}}(\frac{1}{x - A}) = \int \frac{1}{x - y} d\mu_A(y)$$

Property 5.6 *Let for $x, y \in \mathbb{C}\backslash\mathbb{R}$, $Y(x) = H\mu_A(x) - W_1'(x)$ and $X(y) = H\mu_B(x) - W_2'(x)$. Then, there exists a polynomial function*

$$E(x, y) = \sum_{i,j=1}^{d-1} a_{ij}(\bar{t}) x^i y^j$$

so that for all $x, y \in \mathbb{C}\backslash\mathbb{R}$

$$E(X(y), y) = 0 \qquad E(x, Y(x)) = 0.$$

In particular, μ_A and μ_B are absolutely continuous with respect to Lebesgue measure, with Hilbert transform $H\mu_A$ and $H\mu_B$ so that $Y(x) = H\mu_A(x) - W_1'(x)$ satisfies the same algebraic equation with $x \in \mathbb{R}$.

Proof.

Note that since we know that $\mu_{\bar{t}}$ is compactly supported, we can take in **SD**$[V_{Ising}]$ Stieljes functions instead of polynomials P since the latest are dense by Weirstrass theorem.

We take $P = P(A) = (x - A)^{-1}$ in the second equation in $\mathbf{SD}[V_{Ising}]$ to obtain:

$$\mu_{\bar{t}}\left(\frac{W_2'(B)}{x - A}\right) = -1 + xH\mu_A(x)$$

Then we use this in the first equation written with

$$P(A, B) = \frac{1}{(x - A)} \frac{(W_2'(y) - W_2'(B))}{(y - B)}$$

to get after some calculation

$$U(x, y)(y - Y(x)) = (Y(x) - W_1'(x))(x - W_2'(y)) + 1 - Q(x, y) \tag{24}$$

where

$$U(x, y) = \mu_{\bar{t}}\left(\frac{1}{(x - A)} \frac{W_2'(y) - W_2'(B)}{(y - B)}\right),$$

and

$$Q(x, y) = \mu_{\bar{t}}\left(\frac{W_1'(x) - W_1'(A)}{(x - A)} \frac{W_2'(y) - W_2'(B)}{(y - B)}\right).$$

To obtain our algebraic equation, we simply define

$$E(x, y) = (Y(x) - W_1'(x))(x - W_2'(y)) + 1 - Q(x, y)$$

and we obtain the famous "Master-loop equation"

$$E(x, Y(x)) = 0$$

by taking $y = Y(x)$ in (24). In a symmetric way, we can show that if $X(y) = H\mu_B(x) - W_2'(y)$ then we also have $E(X(y), y) = 0$. Note that E is a polynomial function. Hence, this shows that $Y(x)$, $X(y)$ and so the generating functions $H\mu_A(x)$ and $H\mu_B(y)$ are solution to an algebraic equation. However, this equation still contains a certain numbers of unknown; $\{\mu_{\bar{t}}(A^p B^q), p \leq \deg(V_1) - 2, q \leq \deg(V_2) - 2\}$. It is argued in physics that when $\bar{t}$ is small, the supports of μ_A and μ_B should be connected and therefore $(x, Y(x))$ and $(X(y), y)$ should then be genus zero curves. Then, these unknowns should be determined by the asymptotic behavior of $X(y)$ and $Y(x)$ at infinity

$$X(y) \simeq W_2'(y) - \frac{1}{y}(1 + o(1)), \quad Y(x) \simeq W_1'(x) - \frac{1}{x}(1 + o(1)).$$

Note in passing that, as solutions of an algebraic equation, $H\mu_A$ and $H\mu_B(x)$ extends continuously (but in general not differentially) to the real line (eventually as an extended complex number). As a consequence, μ_A and μ_B have densities with respect to the Lebesgue measure, as the limits of the imaginary part of the Stieljes transform on the real line.

□

5.5 Large deviations approach

A large deviation approach was developed in [16], see also Matytsin [22]. Again, we take $c = 1$ up to rescaling and denote $W_i(x) = x^2/2 + V_i(x)$ for $i = 1, 2$. The main advantage of this strategy is to be valid in the whole range of the parameters. Otherwise, it should provide the same type of information than in the previous paragraph. Namely,

Property 5.7 *For any polynomials V_1, V_2 going to infinity faster than x^2, $\hat{\mu}^N_{A,B}$ converges almost surely towards $\mu_{\overline{t},1} = \mu_{\overline{t}}$ which is uniquely defined by the Schwinger-Dyson's equations*

$$\mu_{\overline{t}} \otimes \mu_{\overline{t}}(D_A P) = \mu_{\overline{t}}((W_1'(A) - B)P), \qquad \mu_{\overline{t}} \otimes \mu_{\overline{t}}(D_B P) = \mu_{\overline{t}}((W_2'(B) - A)P) \tag{25}$$

and by the fact that $\mu_{\overline{t}}|_A$ and $\mu_{\overline{t}}|_B$ (which are the limits of $\hat{\mu}^N_A$ and $\hat{\mu}^N_B$ respectively) are the unique minimizers of

$$S^{V_1,V_2}(\mu) = \mu_A(W_1) + \mu_B(W_2) - 2^{-1} \int\int \log|x-y| d\mu^A(x) d\mu^A(y)$$

$$-2^{-1} \int\int \log|x-y| d\mu^B(x) d\mu^B(y) + \frac{1}{2} \inf_{\rho,m} \{ \int_0^1 \int \frac{m_t(x)^2}{\rho_t(x)} dx dt + \frac{\pi^2}{3} \int_0^1 \int \rho_t(x)^3 dx dt \}$$

where the inf is taken over m, ρ so that $\mu_t(dx) = \rho_t(x)dx \in \mathcal{P}(\mathbb{R})$, $\mu_0(x \in .) = \mu_A(x \in .)$, $\mu_1(x \in .) = \mu_B(x \in .)$, and

$$\partial_t \rho_t(x) + \partial_x m_t(x) = 0.$$

The infimum in $(\rho_., m_.)$ is taken along the solution to a complex Burgers equation; let $\Omega = \{x \in \mathbb{R}, t \in (0,1) : \rho_t(x) > 0\}$ and define on Ω $u_t(x) = \rho_t(x)^{-1} m_t(x)$ and $f_t(x) = u_t(x) + i\pi \rho_t(x)$. Then on Ω,

$$\partial_t f_t(x) + f_t(x) \partial_x f_t(x) = 0.$$

Moreover, with $\mu_A = \mu_{\overline{t}}|_A$ and $\mu_B = \mu_{\overline{t}}|_B$, for μ_A-almost all x

$$W_1'(x) - u_0(x) = H\mu_A(x), \quad \mu_A \text{ a.s.}, \quad W_2'(x) + u_1(x) = H\mu_B(x), \quad \mu_B \text{ a.s.} \tag{26}$$

In comparison with the previous statements, we note that the above results hold for all c and V_1, V_2, and not only for small parameters.

Proof.

Most of the proof is contained in [16] where the convergence of $\hat{\mu}^N_A$, $\hat{\mu}^N_B$ towards the unique minimizers of S^{V_1,V_2} was proved (see Theorem 3.3 in [16]), as well as the fact that the limit is compactly supported and that $\mu_{\overline{t}}$ satisfies (25) but for $P \in \mathcal{C}^m_{st}(\mathbb{R})$ (see section 3.2.1, p. 555 and 558, in [16]). It clearly extends to polynomial functions since $\mu_{\overline{t}}$ is compactly supported as its marginals are. The only point we stress here is that this imply that $\mu_{\overline{t}}$ is also uniquely determined. Indeed, by proceeding by induction over the degree in B of a monomial function P, we see that

$$\tau(BP) = -\tau \otimes \tau(D_A P) + \tau(W_1'(A)P)$$

defines uniquely all the moments $\tau(P(A,B))$ from those of $\tau(Q(A))$. Note here that this is specific to the interaction under consideration; in general the solutions of **SD[V]** is not determined by their restriction to one variable.

□

Using for instance the fact that if we let $g_t(x) = tf_t(x) + x$, the Wronskian of (f, g) is null, we find that on each connected component of Ω, there exists an analytic function F so that

$$tf_t(x) + x = F(f_t(x)).$$

In a small parameter region, it should easily be arguable that Ω is connected, as it is when the parameters are null (where the solution at time t can be seen to be a semi-circular variable with variance $1 - t + t^2$). According to the previous section, we know that f_t extends continuously to $t = 0$ and $t = 1$ since μ_A and μ_B have densities which yields

$$x = F(f_0(x)) \quad f_1(y) + y = F(f_1(y)) \tag{27}$$

for all x in the support of μ_A and all y in the support of μ_B. Noting that $f_0(x) = W_1'(x) - \overline{H\mu_A}(x) = -\overline{Y(x)}$, $f_1(x) = -W_2'(x) + H\mu_B(x) = -X(x)$ it is tempting to hope that (27) yields the same result that Property 5.6, namely that $(Y(x), x)$ and $(y, X(y))$ satisfy the same algebraic equation. Our knowledge of this field is much too limited to unable us to get this conclusion.

Acknowledgments: We are extremely grateful to B. Eynard and G. Schaeffer for many comments which helped us to compare the different mathematical approaches to the enumeration of planar maps. We also thank A. Okounkov for many useful discussions.

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