LOOP MODELS, RANDOM MATRICES AND PLANAR ALGEBRAS

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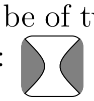
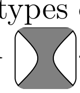
Abstract. We define matrix models that converge to the generating functions of a wide variety of loop models with fugacity taken in sets with an accumulation point. The latter can also be seen as moments of a non-commutative law on a subfactor planar algebra. We apply this construction to compute the generating functions of the Potts model on a random planar map.

1. Introduction

Loop models naturally appear in a wide variety of statistical models as the boundaries of random regions. For example, it is well-known that the Potts model on a planar map is equivalent to a loop model. This loop model can be viewed as living on a 4-valent planar map, where the vertices, represented by two nonintersecting strings, can be of two types depending on the shading of the regions delimited by the strings:  and . We are interested in summing over all planar maps (in a sense that will be made precise below) obtained by gluing these strings together while respecting the shading (i.e., making the planar map bicolored).

In this paper, we consider such loop models where vertices can be chosen in a much more general class. Let us consider the most basic case where the vertices are given by Temperley–Lieb elements. A Temperley–Lieb element consists of an outside box all of whose $2k$ boundary points are connected by non-crossing strings inside of the box, with a boundary point marked by a $*$ on one side of the boundary and a black-white shading so that two regions whose boundaries intersect are shaded differently.

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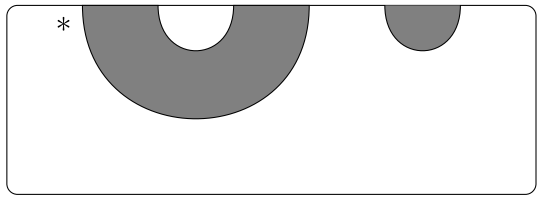


Clearly, as for the Potts model, given any family of Temperley–Lieb elements, one can try to enumerate the number of connected planar diagrams that can be built on them while respecting the shading.

In this paper we undertake to construct matrix models for such loop models. More precisely, consider the real-valued functional given by

$$\mathrm{tr}_t(S) = \sum_{n_1,\ldots,n_k=0}^{\infty} \prod_{i=1}^{k} \frac{t_i^{n_i}}{n_i!} \sum_{P \in P(n_1,\ldots,n_k,S)} \delta^{\#\text{ loops in } P} \tag{1}$$

where $P(n_1, \ldots, n_k, S)$ is the set of connected diagrams (tangles) built on

(1) n_i Temperley–Lieb elements S_i, $1 \le i \le k$,
(2) one Temperley–Lieb element tangle S,

by joining their boundary points so that

(1) the diagram can be embedded in the sphere so that the strings do not intersect,
(2) the shading is compatible in the sense that two boundary points can only be matched by a string if the shading is the same on each side of the string in both tangles.

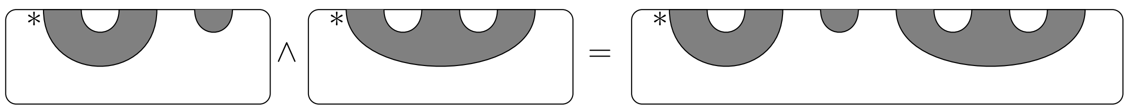
Note that since each element of Temperley–Lieb algebra comes with a marked boundary point, the sum runs over all possible matchings of these marked points irrespective of symmetries. The loop parameter δ is often called fugacity.

We shall prove the following result.

Theorem 1. *Let $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\ge 3} \cup [2, \infty[$ and let $S, S_1, \ldots, S_k$ be k fixed Temperley–Lieb elements. Then for $t_i, 1 \le i \le k$, small enough real numbers, tr_t can be built as a limit of matrix models.*

Matrix models for loop models were already constructed in the physics literature in a wide variety of cases. In fact, for integer δ, the construction of the matrix model follows from the seminal work of 't Hooft [28] and Brézin–Itzykson–Parisi–Zuber [5]. Indeed, these works have shown that the (formal) limits of traces of words of random matrices can be interpreted as generating functions for planar diagrams. This idea can be used for loop models by embedding Temperley–Lieb elements into the set of polynomials in δ non-commutative variables. To consider non-integer values of δ, we follow Jones' idea to generalize this embedding to non-commutative variables labeled by the edges of a bipartite graph, whose adjacency matrix possesses the eigenvalue δ. In subfactor theory, this embedding corresponds to an embedding of a subfactor planar algebra into a graph planar algebra. Note that similar constructions have been used for statistical lattice models [27] and even for matrix models [21, 22, 23] in a somewhat similar (but less general) construction than ours.

While this construction of tr_t allows to compute it in some cases, see section 3, it is also a way to prove that tr_t is a tracial state on Temperley–Lieb algebra in the following sense. Namely, let us denote by $\mathrm{TL}_{k,\pm}$ the set of Temperley–Lieb elements with k strings and $\pm = +$ (resp. $-$) if $*$ belongs to a white (resp. black) region. If $S \in \mathrm{TL}_{k,+}$ and $T \in \mathrm{TL}_{n,+}$, we can define the product $S \wedge T \in \mathrm{TL}_{k+n,+}$ given by the tangle which puts the element of $\mathrm{TL}_{k,+}$ entirely to the left of the element of $\mathrm{TL}_{n,+}$, see [12, Section 2].



An involution from $\mathrm{TL}_{n,+}$ into $\mathrm{TL}_{n,+}$ is defined by $S^* = \varphi(S)$ where φ is an orientation-reversing diffeomorphism. We denote by $Gr_0(\mathrm{TL})$ the $*$-algebra generated by the above multiplication and involution. We then have the following consequence of Theorem 1.

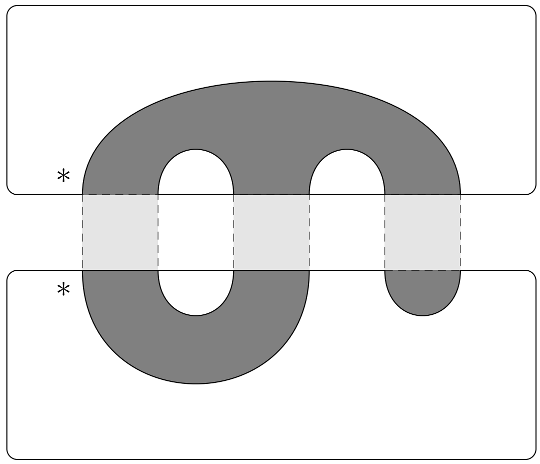
Corollary 2. *Let $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 3} \cup [2,\infty[$ and $S_1,\ldots,S_k$ be fixed Temperley–Lieb elements. Then for $t_i, 1\leq i\leq k$, small enough real numbers, tr_t is a tracial state on $Gr_0(TL)$, that is for any $S,T \in Gr_0(TL)$*

$$tr_t\,(S\wedge T) = tr_t\,(T\wedge S)\,, \qquad tr_t\,(SS^*) \geq 0\,.$$

In fact, it is possible to define tr_t on a wider class of algebras, namely the so-called subfactor planar algebras. These algebras have the particularity that their elements can be composed by means of planar diagrams (usually called in this context planar tangles) so as to give a picture made of loops. It is therefore the natural setting in which to pick the vertices of loop models. We shall define subfactor planar algebras in the next section but the reader can meanwhile think of them as the Temperley–Lieb algebra. In the case where $t = 0$, three of us studied tr_0 in such a generality. In [12], we defined the trace tr_0 on an arbitrary subfactor planar algebra. In the case where this subfactor planar algebra is just the Temperley–Lieb algebra with fugacity δ,

$$\mathrm{tr}_0(S) = \sum_{T\in \mathrm{TL}_{n,\epsilon}} \delta^{\#\text{loops in } \langle S,T\rangle}$$

where $\langle S,T\rangle$ is obtained by drawing the two Temperley–Lieb diagrams in front of each other and joining their boundary points by straight lines, thus obtaining a collection of loops.



One of the results of [12] is that

Theorem 3. *tr_0 is a tracial state on any subfactor planar algebra, as a limit of matrix models.*

The fact that tr_0 is a tracial *state* is not obvious but could also be derived by a combinatorial approach in [20]; such an approach is not yet developed for $\mathrm{tr}_t, t \neq 0$. We shall not study the properties of the von Neumann algebra associated to tr_t by the GNS construction as in [12, 13] when $t = 0$ in this paper and leave this question for future research.

Also in the case of a subfactor planar algebra, loops are assigned a fugacity δ and the definition (1) of tr_t can be extended, with $S, S_1, \ldots, S_k$ elements of this algebra. We can not prove that tr_t is a tracial state in such a generality but obtain the following result.

Theorem 4. *Let P be a finite-depth subfactor planar algebra and let $S_1, \ldots, S_k$ be elements of P. Then, for t small enough, tr_t is a tracial state on P, as a limit of matrix models.*

We refer the reader to [26, 9, 19] for the definition of finite-depth subfactor planar algebra. This class in any case includes any planar algebra with fugacity δ in the set $\{2\cos(\frac{\pi}{n})\}_{n\geq 3}$. However, our restriction is quite simple to understand. As we already pointed out above, our construction relies on an embedding of the subfactor planar algebra in a graph planar algebra. The finite depth condition implies that the graph attached to this subfactor planar algebra is finite [19], and as a result our random matrix construction requires only finitely many random matrices. On the contrary, we need to be able to define the joint law of infinitely many random matrices to construct the matrix model for infinite depth subfactor planar algebra, which we can do only when we can prove that the correlations of the matrices decay sufficiently fast. This is the case for Temperley–Lieb algebra with fugacity $\delta \geq 2$ and the graph A_∞, see section 2.5.2. Thus, Theorem 4 also holds for this infinite depth planar algebra.

We finally would like to mention that even though tr_t is not a state for other values of the fugacity, it is tracial for a much wider range of parameters by analytic continuation. Indeed, at least for small paramaters t, tr_t depends analytically on δ. This remark is important as it allows to extend equalities (and in particular formulas for tr_t or the traciality property) from a set with an accumulation point such as $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 3}$ to all δ's for the t_i's small enough. To simplify the notations, we state (in Appendix A, Lemma 22) the following lemma.

Lemma 5. *Let $S, S_1, \ldots, S_k$ be elements of a planar algebra. For a positive real number C, denote*

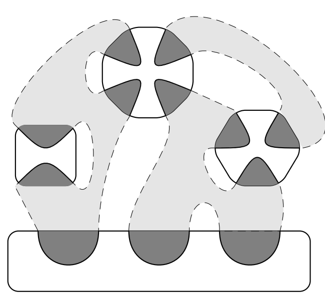
$$B_C = \{(\delta, t_1, \ldots, t_k) \in \mathbb{C}^{k+1}; \max_{1\leq i\leq k} |\delta t_i| < C\}.$$

Then, for C small enough, the function

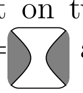
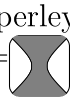
$$tr(S) : (\delta, t) \to tr_t(S) = \sum_{n_1,\ldots,n_k=0}^{\infty} \prod_{i=1}^{n_k} \frac{t_i^{n_i}}{n_i!} \sum_{P\in P(n_1,\ldots,n_k,S)} \delta^{\#\text{ loops in }P}$$

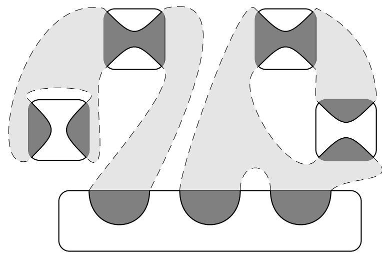
is well defined and analytic in B_C.

We next consider two classical loop models, and show that we can indeed use the matrix models we have constructed to compute partition functions of the loop models. Since the fugacity can take its value at least in the set $\{2\cos(\pi/n)\}_{n\geq 3}$, this allows to determine these partition functions for any fugacity by analyticity. The first model we consider one where the potential is constructed with Temperley–Lieb elements with non nested strings and black inside (that is depending only on a cup shaded black inside with the notations of [12])



In section 3.1, see Lemma 15, we identify the law of cup (the element made with only one string and black inside) under tr_t as a probability measure minimizing a certain entropy, whose Cauchy–Stieltjes functional can be computed.

The second model we consider is the one mentioned in the opening paragraph, built on two Temperley–Lieb elements with two strings and opposite shading $S_1 =$  and $S_2 =$ . We study the law of cup under tr_t:



We show in section 3.3 that the generating function of cup under tr_t is related with the Cauchy–Stieljes transform of an auxiliary model which shows up thanks to the Hubbard–Stratonovitch transformation, see Proposition 16. We then solve this auxiliary model based on the remark that it depends only on the eigenvalues of the matrices involved. Therefore, standard large deviations techiques [2] can be used and the asymptotics of this model are described by a variational formula, see Proposition 17. We finally study the optimizer of this variational formula and show that its Stieljes transform can, up to a reparametrization, be expressed as a ratio of theta functions, see Proposition 20. We summarize below our main results on the Potts model.

Theorem 6. *Let tr_t be the tracial state built on the two TL elements S_1 and S_2 with two strings and opposite shading. Assume that t_1, t_2 are small enough. Let B_n be the tangle with n non nested strings and black shading inside and put for small γ*

$$C(\gamma, A, B) = \sum_{n\geq 0} \gamma^n tr_t(B_n).$$

- *There exists an auxiliary probability measure ν_+ on the real line so that if we let $M(z) = \int \sum_{n\geq 0} z^n x^n d\nu_+(x)$, $\gamma(z) = \frac{\sqrt{8t_1}z}{1-z^2M(z)}$ is invertible on a neighborhood of the origin, with inverse $z(\gamma)$, and*

$$C(\gamma, A, B) = \frac{\alpha z(\gamma)}{\gamma} M(z(\gamma)).$$

- *There exists another auxiliary probability measure ν_- on the real line so that (ν_+, ν_-) is the unique minimizer of*

$$\sum_{\epsilon=\pm}\left(\frac{1}{2}\int x^2 d\nu_\epsilon(x) - \Sigma(\nu_\epsilon)\right) + \delta \iint \log|1+\alpha x + \beta y| d\nu_+(x)d\nu_-(y)$$

with Σ the free entropy $\Sigma(\mu) = \iint \log|x-y|d\mu(x)d\mu(y)$.
- *There exists $a_1 < a_2 < b_1 < b_2$ so that ν_+ (resp. ν_-) is supported on $[-a_2/\sqrt{8t_1}, -a_1/\sqrt{8t_1}]$ (resp. $[(b_1-1)/\sqrt{8t_2}, (b_2-1)/\sqrt{8t_2}]$). Moreover if we set*

$$u(z) := \frac{i}{2}\sqrt{(b_1-a_1)(b_2-a_2)} \int_{b_2}^{z} \frac{dz'}{\sqrt{(z'-a_1)(z'-a_2)(z'-b_1)(z'-b_2)}}$$

and let $z(u)$ be its inverse,

$$\omega_+(u) = \int \frac{1}{z(u) + \sqrt{8t_1}x} d\nu_+(x) = \frac{1}{\sqrt{8t_1}} M(\frac{z(u)}{\sqrt{8t_1}}),$$

then, if q is such that $\delta = q + q^{-1}$, $q = e^{i\pi\nu}$,

$$\omega_+(u) = \frac{1}{q - q^{-1}} [(\varphi_+(u) - \varphi_+(-u)) + R(z(u))]$$

with, if Θ is the theta function given by (34),

$$\varphi_+(u) = c_+ \frac{\Theta(u - u_\infty - 2\nu K)}{\Theta(u - u_\infty)} + c_- \frac{\Theta(u + u_\infty - 2\nu K)}{\Theta(u + u_\infty)}$$

and

$$R(z) = \frac{2q}{1 - q^2} \frac{z}{\sqrt{8t_1}} + \frac{q^2 + 1}{1 - q^2} \frac{(z - 1)}{\sqrt{8t_2}}.$$

The constants $a_1, a_2, b_1, b_2, u_\infty, C_+, c_-, K$ are defined by implicit equations. If $\delta = 2\cos(\frac{\pi}{n})$, ν is an integer and ω_+ satisfies an algebraic equation.

Whereas the matrix model for the first model is well-known, the matrix model for the second one has only been solved in some special cases in the literature, such as the $O(n)$ model [8, 25, 24, 10, 4] which corresponds to the case where the shading is neglected, that is $t_1 = t_2$. Moreover, matrix models are usually provided for δ integer, whereas our approach allows a general construction of the matrix model for all the values of δ above, a set which accumulates at 2. As mentioned earlier, our construction is closely related to Pasquier's [27], though the latter is in a slightly different context, namely that of statistical models on the square lattice (whereas there is no underlying lattice in our construction; it is in some sense random). See also [21, 22, 23] for another application of Pasquier's construction in the context of matrix models. Moreover, the enumeration problem corresponding to our second matrix model was recently considered in [3] in an equivalent formulation, namely the Potts model; we shall comment on the exact relation to our work in section 3.

2. From planar algebras to matrix models

In this section, we introduce the loop matrix models and prove Theorems 1 and 4. The matrix models depends on a bipartite graph which shows up in Jones's subfactor theory (see [19] and references therein). We first recall the definition of a subfactor planar algebra and the construction of the planar algebra from a bipartite graph. The example that the reader can keep in mind is the Temperley–Lieb algebra.

We then define a family of random matrices associated to a bipartite graph whose adjacency matrix has eigenvalue δ corresponding to some Perron-Frobenius vector μ.

We next consider the case of a finite graph and introduce the Gibbs measure associated to tr_t and prove Theorem 4 for planar subalgebras of graph planar algebras. Finally, we extend our construction to certain infinite graphs.

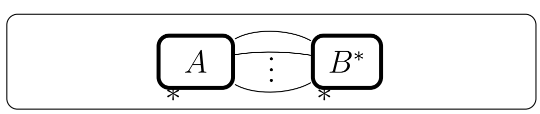
2.1. **Planar algebras.** We first recall some generalities on planar algebras. The reader is referred to [17] or [12] for a more extensive introduction.

Recall [17] that a *planar algebra* is a collection of vector spaces $\mathcal{P} = \{P_k^{\pm}\}$ endowed with an action of *planar tangles*.

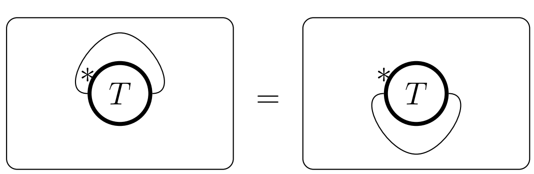
A planar tangle is a drawing consisting of an *output disk* D_0 and some number of *input disks* $D_1, \ldots, D_k$ in the interior of D ($k \geq 0$). Each disk has an even number of marked boundary points. On each disk, one of the boundary segments is marked and called the *initial segment*. The boundary points are joined by strings drawn in the interior of D_0 and outside all $D_1, \ldots, D_k$; in addition there may be some number of closed strings not connected to any of the D_i's. All of the strings are non-crossing. Lastly, some of the regions between the strings are supposed to be shaded, so that each string lies between a shaded and an unshaded region. Planar tangles can be composed by gluing the output disk of one tangle into an input disk of another tangle so as to match up the initial segments. In doing so, one must ensure that the numbers of boundary points and the shadings match.

The main axiom of a planar algebra is the existence, for each tangle T with disks $D_0, \ldots, D_k$ as above, so that D_j has $2b_j$ boundary points, of a multilinear map $M_T : P_{b_1}^{\sigma_1} \times \cdots \times P_{b_k}^{\sigma_n} \to P_{b_0}^{\sigma_0}$, where $\sigma_j = +$ if the initial segment of D_j is adjacent to a white region, and $\sigma_j = -$ otherwise. The maps M_T are supposed to be compatible with the operation of composition of tangles and invariant under isotopy. Moreover, the vector spaces $\{P_k^{\pm}\}$ are equipped with an involution compatible with M_T in the sense that $M_T(f^*) = M_{\varphi(T)}(\varphi \circ f)^*$ for any orientation reversing diffeomorphism φ.

A *subfactor* planar algebra is a planar algebra so that $\dim(P_0^{\sigma}) = 1$. As a consequence we can define a sesquilinear form on each $P_n^{\pm}$ by

$\langle A, B \rangle =$ 

where the outside region is shaded according to $\pm$. We also require that $\langle\, ,\, \rangle$ is positive definite and $M_{T_1} = M_{T_2}$ where T_1 and T_2 are the following two 0-tangles:



Once $P_{0,\pm}$ have been identified with the scalars there is a canonical scalar δ associated with a subfactor planar algebra with the property that the multilinear map associated to any tangle containing a closed string is equal to δ times the multilinear map of the same tangle with the closed string removed. By positivity of the scalar product, δ has to be positive and in fact it is well-known that the possible values of δ form the set $\{2\cos(\pi/n) : n = 3, 4, 5, \ldots\} \cup [2, \infty)$ [16].

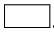
2.1.1. *Example 1: Temperley–Lieb algebra TL.* It is not hard to see that the Temperley–Lieb algebra is a planar algebra. Indeed, given a planar tangle T and some elements $B_1, \ldots, B_k \in \mathrm{TL}$ one can glue these elements into the input disks of T. Next, one can remove all closed strings by replacing each closed string by the factor δ. What results is another TL tangle, which is the result of applying the map M_T to $B_1, \ldots, B_k$. Clearly, M_T is invariant by isotopy and $P_0^{\pm}$ has dimension one. Finally, the canonical scalar product is positive definite according to [17]. One way to prove it is by verifying that the map of TL into a graph planar algebra is a planar algebra map. It thus takes the canonical bilinear form on TL to the canonical bilinear form on a graph planar algebra, where non-negativity can be verified directly. We will write $\mathrm{TL}(\delta)$ when we wish to emphasize the loop parameter (fugacity) δ.

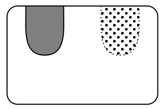
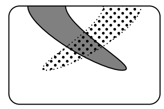
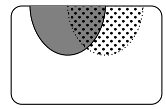
2.1.2. *Example 2: The planar algebra of two stitched Temperley–Lieb algebras.* Later in the paper, we will use (in addition to the Temperley–Lieb algebra) the stitched planar algebra $\mathcal{P} = \mathrm{TL}(\delta_1) \circledcirc \mathrm{TL}(\delta_2)$. This will be needed in order to realize the so-called $O(n, m)$ model in physics.

The n-th graded component of P_n of $\mathcal{P}$ is given by

$$P_n^{\pm} = \bigoplus_{\pi} (TL \circledcirc TL)_{\pi}^{\pm}$$

where the sum is over all partitions π of $\{1, 2, \ldots, 2n\}$ into two subsets of even sizes $2(p_\pi)$ and $2(q_\pi)$ and $(\mathrm{TL} \circledcirc \mathrm{TL})_{\pi}^{\pm} = \mathrm{TL}_{p_\pi}^{\pm} \otimes \mathrm{TL}_{q_\pi}^{\pm}$. Graphically, one can view P_n as the span of the collection of isotopy classes of tangles obtained by superimposing an arbitrary TL tangle colored red with an arbitrary TL tangle colored black, in such a way that the strings of the red and the black TL tangles are allowed to only intersect transversally, and so that the resulting tangle has a total of $2n$ boundary points (counted regardless of color). The partition π then corresponds to the colorings of the boundary points of the resulting diagram. We assume that the checkerboard coloring of the two superimposed TL diagrams are retained and are independently superimposed.

The isotopies need not preserve the intersections of the red and black strings, but must preserve the partition π. We also assume that one of the boundary regions is marked “first” (it could be of either color). Two different isotopy classes of diagrams, T and S, are presented below (black strings are indicated by solid lines and red by dotted lines, and the four possible shadings of regions are indicated by , , , ):

$T =$  $=$  $S =$ 

The planar algebra structure of P_n is defined as follows. Given a planar tangle T and diagrams $A_1, \dots, A_k$ in $\mathcal{P}$, the result $M_T(A_1, \dots, A_k)$ is obtained by gluing the diagrams $A_1, \dots, A_k$ into the input disks of T and suming over all possible ways of extending the colorings and shadings of the A_i's to the resulting tangle. The construction of $\mathcal{P}$ is a particular case of a more general construction $\mathcal{P}_1 \circledcirc \mathcal{P}_2$, which is possible for any pair of planar algebras $\mathcal{P}_1, \mathcal{P}_2$. This construction is presented in Appendix C.

2.2. **On the planar algebra of a graph.** As in [12], we shall use the construction of planar algebras from bipartite graphs, as introduced in [18]. The key ingredient here is the fact that every subfactor planar algebra (in particular, TL) embeds into a graph planar algebra. This makes it possible to "coordinatize" planar algebras.

We first fix notations.

Let $\Gamma = (V, E)$ be a bipartite graph with vertices $V = V_- \cup V_+$ so that any edge is either from V_+ to V_- or V_- to V_+. We denote by E_+ (resp. by E_-) the set of (oriented) edges starting in V_+ (resp. V_-). Thus $E = E_+ \cup E_-$. We let μ be a fixed Perron Frobenius eigenvector with eigenvalue δ for the adjacency matrix of Γ. The vector μ has positive entries. If $e \in E$, we denote by e^o the edge with opposite orientation. We denote by L the set of loops on Γ, L^+ (resp. L^-) the set of loops starting in V_+ (resp. V_-) (so $L = L^+ \cup L^-$). We denote by $L(v)$ the set of loops starting at $v \in V$. We finally let $P^\Gamma = \cup_{n,\pm} P^\Gamma_{n,\pm}$ where $P^\Gamma_{n,\pm}$ is the vector space of bounded functions on loops on Γ of length $2n$ starting and ending in V_+ for the plus sign and V_- for the minus sign. In the following, $s(e)$ (resp. $t(e)$) is the starting (resp. target) vertex of an edge e (note that several edges can have the same starting and ending points).

Example 7. Consider the graph with one vertex in V_+, one vertex in V_- and n edges between them. In this case, $\delta = n$ is an integer, and the eigenvector μ is identically equal to 1. The TL algebra embeds into the planar algebra of this graph for integer fugacity δ.

We next describe the action of planar tangles on P^Γ. Let T be a planar tangle with k input disks and let $L_1, \dots, L_k$ be loops on Γ. To define the planar algebra structure, we must exhibit $M_T(L_1, \dots, L_k)$ as an element of the planar algebra, i.e., we must prescribe its value on a loop L. The value

$$M_T(L_1, \dots, L_k)(L)$$

is computed as follows. First, label the marked points on the input disks of T with the edges comprising the loops $L_1, \ldots, L_k$, clockwise starting from the marked vertex (and the beginning of L_j). Next, label the marked points on the output disk of T with the edges of L, clockwise starting from the marked vertex. As a result, we obtain a *labeled tangle*, and we'll set $M_T(L_1, \ldots, L_k)(L)$ equal to the value of this labeled tangle which we compute as follows. First, we remove all closed loops in the tangle T (let us say, p loops total) and multiply $M_T(L_1, \ldots, L_k)(L)$ by δ^p. We'll again denote the tangle with removed interior loops by T.

Next, we isotope the tangle in such a way that each input disk becomes a rectangle whose top is horizontal, and so that all strings emanate from the top (in this way, the marked initial segment comprises the sides and bottom of the rectangle). Put

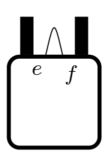
$$\sigma(e) = \sqrt{\frac{\mu(t(e))}{\mu(s(e))}}, e \in E.$$

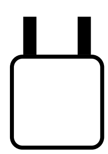
Then the value $M_T(L_1, \ldots, L_k)(L)$ is zero unless each string connects points which are labeled by opposite edges. Otherwise,

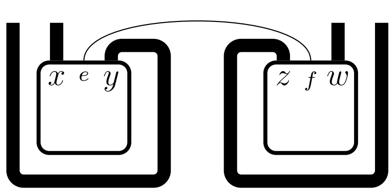
$$M_T(L_1, \ldots, L_k) = \prod_{\text{strings } s} \sigma(e_s)^{-\theta_s/\pi} \tag{2}$$

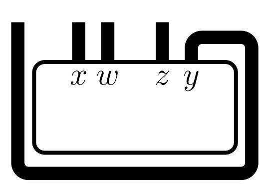
where e_s is the start of s and $\theta_s = \int_s d\theta$ is the total winding angle of the string s. Here $d\theta$ stands for the 1-form $ydx - xdy$ on the coordinate plane. (Note that the choice of which edge is selected as the start of a string s is irrelevant: if s' is the string s traversed backwards, then we get a non-zero value for the tangle iff $e_{s'} = e_s^o$; note that $\sigma(e^o) = \sigma(e)^{-1}$ and $\theta_{s'} = -\theta_s$.)

We note the identities (thick lines indicate an arbitrary number of parallel strings, e and f are two arbitrary edges, and x, y, z, w are arbitrary paths on graphs so that xey and zfw are loops, and all planar algebra elements are arranged so that the marked initial segment is at top-left):

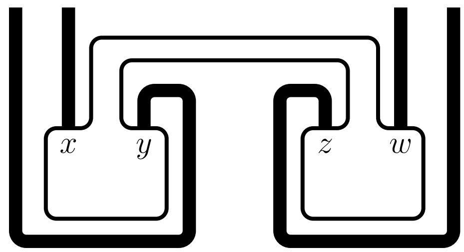


$= \delta_{e=f^o}\sigma(e)$




$= \delta_{e=f^o}\sigma(e)\sigma(z)^{-1}$


where $\sigma(z) = \prod \sigma(z_j)$ if $z = z_1 \ldots z_l$. (The last operation can be though of as replacing the string connecting e and f by a "tunnel" joining the two planar algebra elements:



followed by an isotopy "straightening" the resulting picture).

Using these operations, any labeled tangle can be simplified leaving only strings connecting the outer disk to inner disks. Each of these remaining strings can be removed by contributing a certain multiplicative factor according to (2).

2.2.1. *Example 1 continued: Temperley–Lieb algebra inside the planar algebra of a bipartite graph.* A particular case of the planar algebra axiom is the natural embedding from $\mathrm{TL}(k,\pm)$ into a linear span of loops, following [12]. Indeed, Temperley–Lieb tangles are tangles with no input disks, and thus produce elements in any planar algebra.

Suppose that we are given a box B with $2k$ boundary points (arranged so that all boundary points are at the top and $*$ is at position 0 from the top-left). Assume also that there are k non-crossing curves inside B which connect pairs of boundary points together. Let π be the associated non-crossing pairing. We let L_B be the set of loops in Γ so that $w \in L_B$ iff $w = e_1 \cdots e_{2k}$ with

- $e_n = e_\ell^o$ if $\{n, \ell\}$ is a block of the partition π (which is denoted $n \overset{\pi}{\sim} \ell$)
- $s(e_1) \in V_+$ (resp. in V_-) if $B \in \mathrm{TL}(k,+)$ (resp. $B \in \mathrm{TL}(k,-)$).

For $e \in E$, $\sigma(e) := \sqrt{\frac{\mu(t(e))}{\mu(s(e))}}$ and for $w \in L_B$, we define the weight

$$\sigma_B(w) = \sigma(e_{i_1}) \cdots \sigma(e_{i_n}) \text{ if } e_{i_k} = e_{j_k}^o \text{ whenever } i_k \overset{\pi}{\sim} j_k \text{ and } \ i_k < j_k,$$

We then associate to the Temperley–Lieb tangle B the element of $w_B \in P^\Gamma$ whose value on a loop L is zero unless $L \in L_B$ and for $L \in L_B$,

$$w_B(L) = M_B(L) = \sigma_B(L).$$

To be consistent with [12] we denote by $Gr_0 P^\Gamma$ the set of such linear combinations in the linear span of L. $B \to M_B$ is thus an algebra embedding from $Gr_0(\mathrm{TL})$ into $Gr_0 P^\Gamma$.

2.2.2. *Example 2 continued: The planar algebra of two stitched Temperley–Lieb algebras realized inside a graph planar algebra.* Let $\mathcal{P} = \mathrm{TL}(\delta_1) \circledcirc \mathrm{TL}(\delta_2)$ as in §2.1.2. We'll realize $\mathcal{P}$ inside a graph planar algebra. Let Γ_r, Γ_b be two graphs so that the associated planar algebras contain $\mathrm{TL}(\delta_r)$ and $\mathrm{TL}(\delta_b)$, respectively. We can thus choose Γ_x to be A_∞ if $\delta_x \geq 2$ and otherwise A_n for some n (related to δ_x). By appendix C, $\mathcal{P}$ embeds into the graph planar algebra of Γ.

Let $\Gamma = \Gamma_r \times \Gamma_b$. More precisely, the vertices of Γ are pairs (v_r, v_b) with $v_x \in \Gamma_x$, $x \in \{b, r\}$. The pair (v_r, v_b) is *even* iff either both v_r, v_b are even or both are odd; the pair (v_r, v_b) is odd otherwise. The edges of Γ are of two kinds: the *red* edges, consisting of pairs (e, v) with e an edge in Γ_r and v a vertex in Γ_b; this is an edge from $(s(e), v)$ to $(t(e), v)$; and *black* edges, consisting of pairs (f, w) with f an edge in Γ_b and w a vertex in Γ_r; this edge goes from $(w, s(f))$ to $(w, t(f))$. For an edge in Γ, $e = (f, w)$, put $u(e) = f$ (which is in Γ_r or Γ_b according to whether e is red or black). Note that Γ is a bi-partite graph, since each edge in Γ changes the parity of one of the components of a vertex (v_r, v_b). By Appendix C, Γ is the principal graph of $\mathcal{P}$.

Let μ be the Perron-Frobenius eigenvector for Γ, given at a vertex (v, w) by the product of the eigenvectors of Γ_r and Γ_b. For e and edge of Γ, put $\sigma(e) = (\mu(t(e))/\mu(s(e)))^{1/2}$. Let $c(x)$ be the color of the x-th boundary point of T.

Let $T \in P_n$ be a diagram, and let

$$R_T = \{(x, y) : \text{boundary points } x \text{ and } y \text{ are connected in } T\}.$$

The embedding of $\mathcal{P}$ into the graph planar algebra of Γ is given by sending T to the function $f_T \in \mathcal{P}^\Gamma$ given by:

$$f_T(e_1 \cdots e_{2n}) = \prod_{\substack{(x,y) \in R_T \\ x<y}} \sigma(e)\delta_{e \text{ has same color as } c(x)}\ \delta_{u(e_x)=u(e_y)^o},$$

for any loop $e_1 \cdots e_{2n}$ in Γ.

2.3. **Random matrices associated with a graph.** In the sequel, we fix a graph Γ with an eigenvalue δ and a Perron-Frobenius eigenvector μ as in the previous part. For $e \in E_+$, X_e^M is a $[M_{s(e)}] \times [M_{t(e)}]$ matrix with i.i.d entries with variance $(M_{s(e)} M_{t(e)})^{-\frac{1}{2}}$ for some integer numbers $(M_v, v \in V)$ so that

$$\lim_{M \to \infty} \frac{M_v}{M} = \mu(v).$$

We put $X_{e^o} = X_e^*$ and for a matrix $(A_{nm,e}, 1 \le n \le M(s(e)), 1 \le m \le M(t(e)), e \in E)$ we define the states:

$$\mathrm{tr}(A) = \sum_{v \in V} \frac{1}{M} \sum_{1 \le n \le M_v} A_{nn,\,vv} \quad \mathrm{Tr}_V(A) = \sum_{v \in V} \mu(v) \sum_{1 \le n \le M_v} A_{nn,\,vv}$$

and denote, for a word $w = e_1 \cdots e_k$, $X_w = X_{e_1} \cdots X_{e_k}$. In order that X_w is a square matrix, we shall assume that w is a loop. We denote

$$d_v X_w^N = \begin{cases} X_w^N \text{ if } w = e_1 \cdots e_n \text{ and } s(e_1) = v \\ 0 \text{ otherwise.} \end{cases}$$

The center-valued trace Tr_0 on $Gr_0 P^\Gamma$ is given by the equation, for $x = e_1 \cdots e_{2k}$

$$\mathrm{Tr}_0(x)(v) = 1_{s(e_1)=v} \langle x, T_k \rangle$$

with $T_k = \sum_{B\in \mathrm{TL}(k)} w_B$ the sum of all TL diagrams with k strings and where for two loops x, y on Γ, $\langle x, y\rangle = \delta_{x=y}$. Thus $\mathrm{Tr}_0(x)$ is a complex-valued function on the set of vertices of the graph.

We have the following theorem from [12, Proposition 2]

Theorem 8. *Let $v \in V$ and $w = e_1e_2\cdots e_k \in L(v)$. Then*

$$Tr_0(w)(v) = \lim_{M\to\infty} \frac{1}{\mu(v)} E[\mathrm{tr}(X_w^M)] \tag{3}$$

In the case where $x = w_T$ for some subfactor planar algebra element T, $Tr_0(w_T) = C(T)1$ is constant and $C(T) = tr_0(T)$.

Note that in [12], we had an additional dimension N so that $X_e^{M,N}$ converges to a matrix with free variables entries as N goes to infinity. This is however not needed since M goes to infinity.

Note that our random matrix model has the following interpretation. Consider the graph planar algebra P^Γ, and let x be an element labeled by some loop $e_1, \ldots, e_{2n}$. One can instead label the tangle by our random matrices $X_{e_1}, \ldots, X_{e_{2n}}$. In this case, the equations (2) governing the action of planar tangles on P^Γ can be summarized as follows: whenever a string connects two points labeled by random matrices X and Y, the string can be removed at the cost of a multiplicative factor given by the expected value $E(Tr_0(XY)(v))$, where v is the start of the edge corresponding to X.

We shall give a new proof of Theorem 8 based on the so-called Schwinger–Dyson equation.

Theorem 9. *For $t = (t_1, \cdots, t_k) \in \mathbb{R}^k$ small enough, there exists a probability measure μ_t^M on $(X_e, e \in E)$ so that for $v \in V$ and $w = e_1e_2\cdots e_k \in L(v)$, there exists a limit*

$$Tr_t(w)(v) = \lim_{M\to\infty} \frac{1}{\mu(v)} \int [\mathrm{tr}(X_w^M)] d\mu_t^M. \tag{4}$$

In the case where $x = w_T$ for some Temperley–Lieb diagram T (or, more generally, T comes from a subfactor planar algebra inside $\mathcal{P}^\Gamma$), $Tr_t(w_T) = C_t(T)1$ is constant and $C_t(T) = tr_t(T)$ is given by (1)*.*

In the next subsections we shall prove this theorem.

2.4. **The case of finite graphs.**

For $\delta = 2\cos(\frac{\pi}{n})$, $n \geq 3$ (or, more generally, if the planar algebra under consideration is finite-depth), the graph Γ can be chosen to be finite. This is the case for $\mathrm{TL}(\delta)$ if $\delta = 2\cos(\frac{\pi}{n+1})$; in this case the graph is A_n the graph with n vertices and $n-1$ edges. Another such example is the case of the graph with two vertices joined by n edges (see e.g. [18, Examples 4.1 and 4.4]). We study the finite-graph case first.

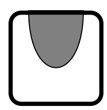
2.4.1. *Definition of the matrix models.* Let $\mathcal{P} \subset \mathcal{P}^\Gamma$ be a subfactor planar algebra realized inside a graph planar algebra. We consider the law μ_M of $|E_+|$ independent $M_{s(e)} \times M_{t(e)}$ matrices with i.i.d Gaussian entries with variance $(M_{s(e)}M_{t(e)})^{-\frac{1}{2}}$. We denote $\|M\|_\infty$ the spectral norm of a matrix M (that is the spectral radius of $\sqrt{MM^*}$), $X^M = (X_e^M)_{e\in E_+}$ the collection of these matrices and $\|X^M\|_\infty := \max_{e\in E_+} \|X_e^M\|_\infty$. We set, for given elements $B_1, \ldots, B_k \in \mathcal{P}$, real numbers $t_1, \ldots, t_k$ and some $K > 2$,

$$\mu_t^{M,K}(dX^M) := \frac{\mathbf{1}_{\|X^M\|_\infty \le K}}{Z_t^{M,K}} e^{\sum_{i=1}^k t_i M \mathrm{Tr}_V(X^M_{B_i})} \mu_M(dX^M)$$

where we denoted in short $X_B = X_{w_B} = \sum_{w\in L_B} \sigma_B(w) X_w$.

Example 10. Consider the TL algebra for $\delta \in \{2\cos(\frac{\pi}{n}), n \ge 3\}$. Then TL can be embedded into the graph planar algebra $\mathcal{P}^\Gamma$ where $\Gamma = A_n$. We can consider the following potentials:

- Let B be the element of TL given by the tangle with only one string:

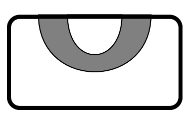


We can either see it as a cup with black inside and white outside or with the opposite shading, both leading to the same potential in the matrix model

$$\mathrm{Tr}_V\left(\sum_{v\in V_+}\sum_{e':s(e)=v} \sigma(e) X_e X_e^*\right) = M\mathrm{tr}\left(\sum_{\substack{v\in V_+\\ s(e)=v}} \sqrt{\mu(v)\mu(t(e))} X_e X_e^*\right)$$

$$= \mathrm{Tr}_V\left(\sum_{v\in V_-}\sum_{s(e)=v} \sigma(e) X_e X_e^*\right).$$

- Let B be the element of TL given by a tangle with two strings, two white regions and one black:



Then, it is given by

$$w_B = \sum_{e\in E_+}\sum_{f\in E_-} \sigma(e)\sigma(f) X_e X_f X_f^* X_e^*$$

with contribution to the potential

$$\mathrm{Tr}_V(\sum_{e\in E_+}\sum_{f\in E_-}\sigma(e)\sigma(f)X_eX_fX_f^*X_e^*) = \mathrm{Tr}_V\left((\sum_{e\in E_-}\sigma(e)X_eX_e^*)^2\right)$$

where it is understood that products which make no sense give no contribution. Inverting the shading amounts to exchanging E_+ and E_-.

The main result of this section is

Proposition 11. *Let $K > 2$. Then, there exists $t(K) > 0$ so that for $\max_{1\le i\le k}|t_i| \le t(K)$, for any loop $w \in L(v)$ there exists a limit*

$$Tr_t(X_w)(v) = \frac{1}{\mu(v)}\lim_{M\to\infty}\mu_t^{M,K}(tr(X_w)).$$

Moreover, $Tr_t(X_B)(v) = tr_t(B)$ for element $B \in \mathcal{P}^\Gamma$ and any vertex v.

The convergence of the matrix model is a small generalization of [14, Theorem 3.5] (where only Hermitian random matrices where considered), whereas the identification of the limit is based on the analysis of the so-called Schwinger–Dyson (or loop) equations. Note that in most papers in the physics literature, the cutoff $K < \infty$ is not taken, leading sometimes to diverging integrals. The advantage of adding this cutoff is that all integrals are well defined and moreover for small t_i's, the Gibbs measure $\mu_t^{M,K}$ has a strictly log-concave density, providing many interesting properties which allow to put on a firm mathematical ground the above convergence. In fact, we can remove the cutoff in case the density is strictly log-concave. K has to be chosen strictly greater than 2 (the edge of the support of the semi-circle law, which is greater or equal to that of the Pastur-Marchenko) so that the limit does not depend on it for t small enough. We start by recalling these properties and sketching the proof of [14, Theorem 3.5].

2.4.2. *Convexity assumption and consequences.* In the sequel, given an element of the planar algebra $W = \sum_{i=1}^k t_iB_i$, with $B_i \in \mathcal{P}^\Gamma$ delta functions along loops B_i, we shall denote by $\widetilde{W}$ the polynomial in the X_e's given by

$$\widetilde{W} = \sum_{i=1}^k t_i\sum_{v\in V}\mu(v)d_vX_{B_i}^M$$

so that $\mathrm{Tr}_V(\sum t_iX_{B_i}^M) = M\mathrm{tr}(\widetilde{W})$. As in [14], we shall assume that the map from the set of $|E_+|$ Hermitian matrices into $\mathbb{R}$ given by

$$(X_e^M)_{e\in E_+} \to M\mathrm{tr}(-\widetilde{W} + \frac{1}{2}\sum_{e\in E}\sqrt{\mu(s(e))\mu(t(e)}(X_e^M)(X_e^M)^*) \tag{5}$$

is strictly convex (with Hessian bounded below by c for some $c>0$ independent of M) on $\{\|X^M\|_\infty \le K\}$. This is always true for t sufficiently small, depending on K, with c going to $m=\min\{\sqrt{\mu(t(e))\mu(s(e))}, e\in E_+\}>0$ as t goes to zero.

As a consequence of strict convexity, we have concentration inequalities under $\mu_t^{M,K}$, see [11, Section 6.3] namely for any $w\in L$,

$$\mu_t^{M,K}\left(\left|\mathrm{tr}(X_w^M)-\mu_t^{M,K}\left(\mathrm{tr}(X_w^M)\right)\right|^2\right)\le \frac{C(w,K)}{(M)^2}. \tag{6}$$

We also have Brascamp-Lieb inequalities, see [11, Section 6.5], and so by comparison to the Gaussian law for which we know that the spectral radius is bounded with overwhelming probability, we can prove that there exists $\ell(c)$ (which only depends on c) so that

$$\mu_t^{M,K}\left(\max_{e\in E}\|X_e^M\|_\infty \ge \ell(c)\right)\le e^{-\delta(c)M}$$

for some $\delta(c)>0$. We assume we have chosen $K>\ell(c)$. In particular the family $\{\mu_t^{M,K}(\mathrm{tr}(X_w^M)), w\in L\}$ is tight. We will denote μ_t a limit point. We next show any limit point satisfies the so-called Schwinger–Dyson equation and that this equation as a unique solution when the t_i's are small.

2.4.3. *Schwinger–Dyson (or loop) equations.* Let us fix $e\in E$ and $P=X_w$ with w a path from $t(e)$ to $s(e)$. By using and integration by parts and concentration inequalities, we obtain (see [11, Section 8.1] or [14, Theorem 3.1]), that

$$\lim_{M\to\infty}\left(\mu_t^M(\mathrm{tr}((\sqrt{\mu(s(e))\mu(t(e))}X_e^M-D_{e^o}\tilde{W})P))-\mu_t^M(\mathrm{tr})\otimes\mu_t^M(\mathrm{tr})\left(\partial_{e^o}P\right)\right)=0,$$

where

$$\partial_e X_w=\sum_{w=w_1ew_2}X_{w_1}\otimes X_{w_2}\quad D_eX_w=\sum_{w=w_1ew_2}X_{w_2w_1}.$$

Let τ be a limit point of $\mu_t^M\mathrm{tr}$. Thus τ satisfies the Schwinger–Dyson equation: for every path from $t(e)$ to $s(e)$,

$$\tau\left((\sqrt{\mu(s(e))\mu(t(e))}X_e-D_{e^o}\widetilde{W})X_w\right)-\tau\otimes\tau\left(\partial_{e^o}X_w\right)=0 \tag{7}$$

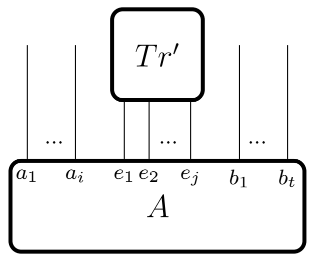
Using the map from $\mathcal{P}^\Gamma$ that sends $g_1,\dots,g_n$ to $X_{g_1}\dots X_{g_n}$ we can use τ to define a collection of linear maps from $\mathcal{P}^\Gamma$ with values in $\mathcal{P}^\Gamma$, which we will for now denote by $\mathrm{Tr}'_{i,j}$. By definition, if $P\in\mathcal{P}^\Gamma$ is the delta function on the loop $a_1,a_2,\dots,a_i,e_1,\dots,e_j,b_1,\dots,b_r$, then

$$\mathrm{Tr}'_{i,j}(P)=\delta_{s(b_1)=t(a_i)}\left\{\frac{1}{\mu(s(e_1))}\tau(X_{e_1}\dots X_{e_j})\right\}Q$$

where Q is the delta function supported on the loop $a_1\dots a_ib_1\dots b_r$.

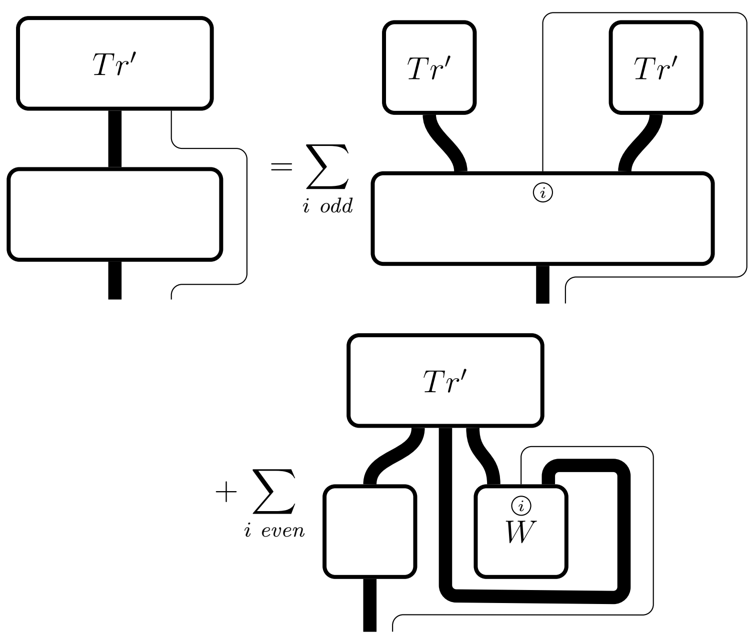
Then the Schwinger–Dyson equation is an equation on these linear maps. We will use the following graphical notation for the result of $\mathrm{Tr}_{i,j}$ applied to the delta function supported at the loop $a_1, \ldots, a_i, e_1, \cdots, e_j, b_1, \ldots, b_t, \in \mathcal{P}^\Gamma$ (we supress i, j since they can be read off from the numbers of the various strings):

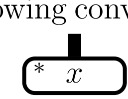
(8)

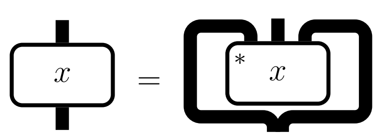


Lemma 12. *The Schwinger–Dyson equation* (7) *is equivalent to the following diagrammatic equation:*

(9)

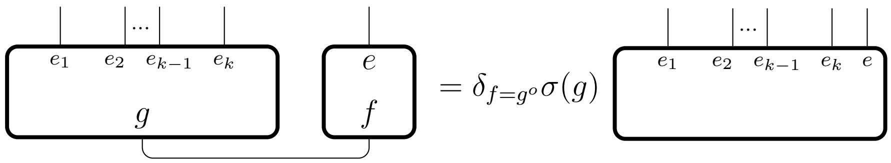


Here we use the following conventions. Thick lines indicate an arbitrary number of strings. Also, given  $\in \mathcal{P}^\Gamma$ we'll set

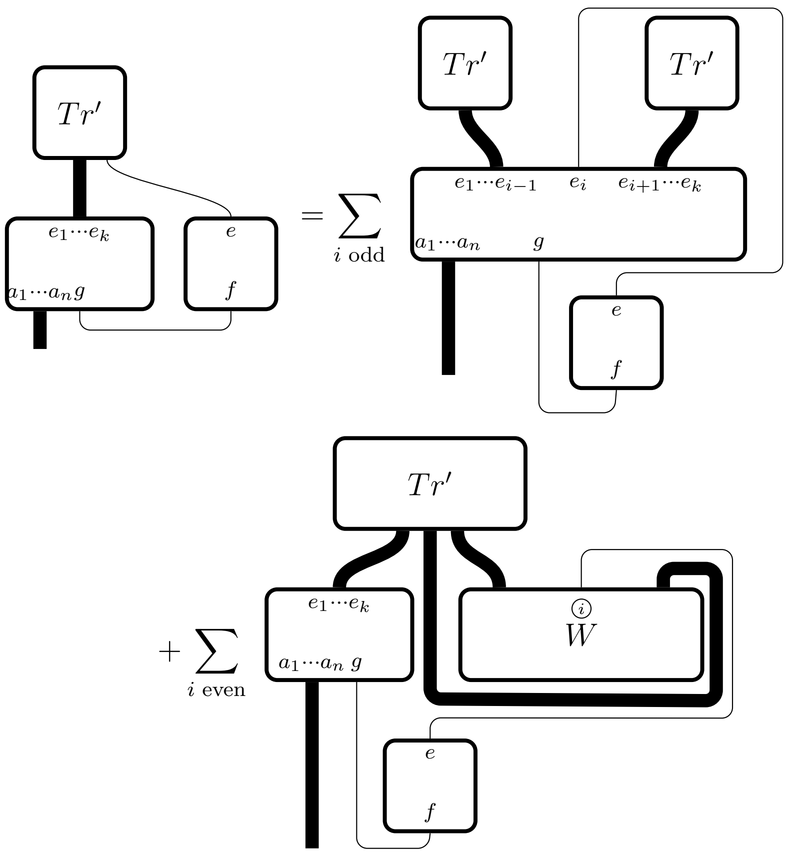


where by convention there is an arbitrary fixed number (same in all diagrams) of strings between the marked initial segment of x and the last string that is bent around x on the left.

Proof. Consider $e, f, g, e_1, \ldots, e_n \in E(\Gamma)$ so that the paths $e_1, e_2, \ldots, e_k, g$ and e, f form loops. Consider the following planar operation applied to $P = e_1, \ldots, e_k, g$ and e, f (more precisely, P is the element of $\mathcal{P}^\Gamma$ which is the delta function on the loop $e_1, \ldots, e_k, g$, etc.):

(10)


Let $e_1 \ldots e_k\ g\ a_n \ldots a_1$ and $e\ f$ be two loops in Γ. Consider the following equation:

(11)


It is easily seen (by noticing that g and f are arbitrary subject to $t(f) = s(g)$) that (11) is equivalent to (9).

Next, note that both sides of (11) are zero unless the region containing the point at infinity in all diagrams is labeled by $v = t(e)$ (so in particular $t(a_1) = s(a_n) = v$). Consider now the three diagrams comprising equation (11). Put

$$R = \delta_{f=g^o}\sigma(g)a_1 \ldots a_n.$$

The diagram in the upper-left corner is exactly

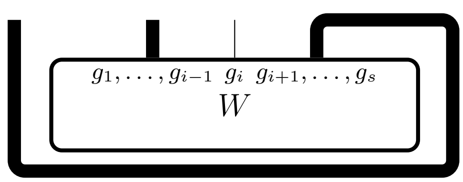
$$Q_1 = \frac{1}{\mu(t(e))} \tau(e_1 \dots e_k e) R.$$

Consider now the diagram in the upper-right corner. The term in the summation is zero unless $e_i = e^o$. In particular, the region to the right of the string emanating from e_i can be labeled by $s(e)$ and the region to the right of the string emanating from e_k can be labeled $t(e)$. By definition of Tr', we then get that this diagram is equal to

$$Q_2 = \frac{1}{\mu(t(e))} \frac{1}{\mu(s(e))} \tau \otimes \tau \left(\sum_{i \text{ odd}} e_1 \dots e_{i-1} \otimes e_{i+1} \dots e_k \right) \sigma(e)^{-1} R.$$

Note that we can actually replace the sum over odd i by the sum over all i, since $\tau(e_1 \dots e_{i-1}) = 0$ unless i is odd.

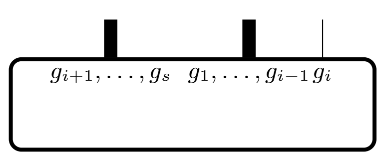
Assume first that W is the delta function supported on the loop $g_1 \dots g_s$. Let us now consider for i even the diagram

$D_i =$ 

Since all strings emanating from $g_{i+1}, \dots, g_s$ make a 360^o rotation in the drawing, according to (2)

$$D_i = \sigma(g_{i+1})^2 \cdots \sigma(g_s)^2 D_i'$$

where

$D_i' =$ 

To see this, we can compare the results of glueing all strings of D_i to some diagrams $B_1, \dots, B_k$ versus the results of glueing the corresponding strings of D_i' to the same diagrams. As we apply (2) to remove strings, the strings starting from D_i associated to g_j, $i+1 \le j \le s$ contribute an extra factor of $\sigma(g_j)^2$ as compared to their contribution if they were to start from D_i'.

Using this and noting that $\widetilde{W} = \mu(s(g_1))W = \mu(t(g_s))W$, the value of the bottom diagram in (11) is

$$\begin{aligned} Q_3 &= \frac{1}{\mu(s(e))} \sum_{i \text{ even}} \frac{1}{\mu(t(g_s))} \sigma(g_{i+1})^2 \dots \sigma(g_s)^2 \\ &\qquad \times \tau(e_1 \dots e_k g_{i+1} \dots g_s g_1 \dots g_{i-1}) \delta_{e^o = g_i} \sigma(e)^{-1} R \\ &= \frac{1}{\mu(s(e))} \sum_{i \text{ even}} \frac{1}{\mu(s(g_i))} \tau(e_1 \dots e_k g_{i+1} \dots g_s g_1 \dots g_{i-1}) \delta_{e^o = g_i} \sigma(e)^{-1} R \\ &= \frac{1}{\mu(s(e))} \frac{1}{\mu(t(e))} \tau(e_1 \dots e_k D_{e^o} W) \sigma(e)^{-1} R \end{aligned}$$

By linearlity, the same equation holds for arbitrary W.

Now, (11) is equivalent to saying that

$$Q_1 = Q_2 + Q_3.$$

Let us multiply each Q_i by the factor $\alpha = \mu(t(e))^{3/2}\mu(s(e))^{1/2}$. Then:

$$\alpha Q_1 = (\mu(t(e))\mu(s(e))^{1/2} \tau(e_1 \dots e_k e) R.$$

Similarly,

$$\begin{aligned} \alpha Q_2 &= \mu(t(e))^{3/2-1} \mu(s(e))^{1/2-1} \sigma(e)^{-1} \tau \otimes \tau \left(\sum_{i \text{ odd}} e_1 \dots e_{i-1} \otimes e_{i+1} \dots e_k \right) R \\ &= \tau \otimes \tau \left(\sum_{i \text{ odd}} e_1 \dots e_{i-1} \otimes e_{i+1} \dots e_k \right) R \end{aligned}$$

Finally,

$$\begin{aligned} \alpha Q_3 &= \mu(s(e))^{1/2-1} \mu(t(e))^{3/2-2} \sigma(e)^{-1} \left(\frac{1}{\mu(t(e)} \tau(e_1 \dots e_k D_{e^o} W) \right) R \\ &= \tau(e_1 \dots e_k D_{e^o} W) R. \end{aligned}$$

Since $e, f, a_1, \dots, a_n$ are arbitrary, we see that (11) is just a multiple of (7). $\square$

Lemma 13. *For any $c > 0$ and $K > \ell(c)$, there exists $t(c, K)$ so that when $|t_i| \le t(c, K)$ for all $i \in \{1, \dots, k\}$, there exists a unique solution to (7) so that $\mu(|X_e|^{2p}) \le K^p$ for all $e \in E_+$ and $p \in \mathbb{N}$.*

Proof. The proof follows the arguments of [14, Section 2.4] that we however repeat for further uses. Existence is already guaranteed. To prove uniqueness we assume we have two solutions μ_1 and μ_2 and let

$$\delta(n) = \sup_{w \in L, |w| \le n} |\mu_1(X_w) - \mu_2(X_w)|$$

with $|w|$ the number of letters in the word w. Then, (7) implies that, if we let $m = \min(\mu(s(e))\mu(t(e)))^{\frac{1}{2}} > 0$,

$$\delta(n+1) \le 2m^{-1} \sum_{p=1}^{n-1} \delta(p) K^{n-1-p} + A(t)\delta(n+D-1) \tag{12}$$

with D the degree of W and if $D_{e^0} W = \sum_i t_i \sum_{j=1}^{k_i^e} q_{ij}^e$ with some monomials q_{ij}^e with degree at most $D-1$, then

$$A(t) = \max_e (\mu(s(e))\mu(t(e)))^{-\frac{1}{2}} \sum_i |t_i| k_i^e$$

For $\gamma < 1/K$, the sum $\Delta(\gamma) = \sum_{p\ge 1} \gamma^p \delta(p)$ is finite and satisfies

$$\Delta(\gamma) \le \frac{\gamma^2}{1-\gamma K} \Delta(\gamma) + A(t)\gamma^{-D+2}\Delta(\gamma).$$

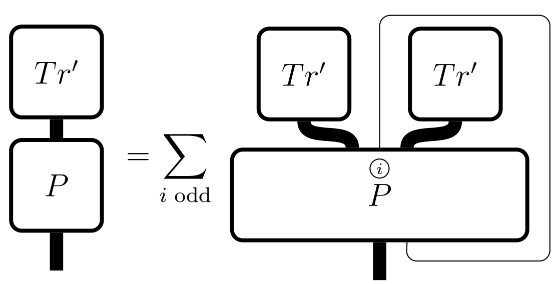
We then choose t small enough so that

$$\frac{\gamma^2}{1-\gamma L} + A(t)\gamma^{-D+2} < 1$$

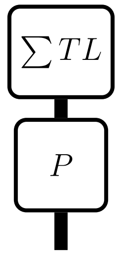
for some $\gamma \in]0, L^{-1}[$, which guarantees that $\Delta(\gamma) = 0$ and therefore $\delta(n) = 0$ for all $n \ge 0$. $\square$

In the next section we study Schwinger–Dyson equations for laws on P^Γ and P.

2.4.4. *Proof of Theorem 8 in the finite graph case.* In the case $t = 0$, we deduce Theorem 8. Equation (9) implies, for $t = 0$

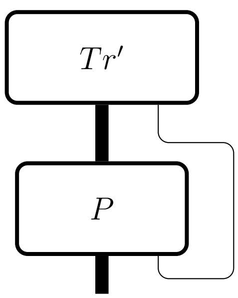


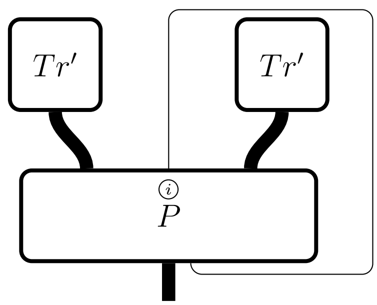
This equation clearly has a unique solution, since it defines the maps $\mathrm{Tr}'_{i,j}$ recursively in terms of $\mathrm{Tr}'_{i,j'}$ with $j < j'$. We claim that in fact $\mathrm{Tr}'_{ij} = (\mathrm{Tr}_0)_{ij}$. Recall that Tr_0 is given by

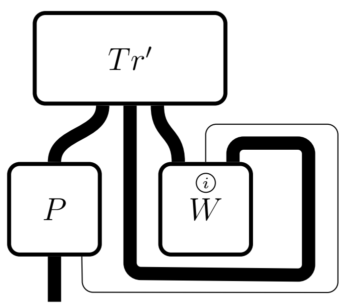
$Tr_0 =$ 

where $\sum TL$ stands for the sum of all TL diagrams. If we follow the rightmost top string of P, it will be connected to one of the other vertical strings of P (and, for parity reasons, it will be an odd string). From this we see that Tr_0 satisfies the same recursive relation as Tr'. $\square$

2.5. **Proof of Proposition 11.** Let us now consider the case $t \neq 0$. Then $\mathrm{Tr}'_{i,j}$ satisfy the equation

(13) 

$= \sum_{i \text{ odd}}$ 

(14) $+ \sum_{i \text{ even}}$ 

We claim that $\mathrm{Tr}' = \mathrm{Tr}_t$ (recall that $\mathrm{Tr}_t(P)$ is a function on the set of vertices on the graph described in Proposition 11). To see the equality, one can argue as in Lemma 13 that for sufficiently small t_i's, there is at most one solution to this equation. On the other hand, Tr_t clearly satisfies the same recursive equation: it expresses the fact that the rightmost top string of P must either come back to P, or be connected to a copy of the potential W. Thus $\mathrm{Tr}' = \mathrm{Tr}_t$.

$\square$

2.5.1. *Free energy.* We recap here how to get from the convergence of the tracial state that of the free energy

$$F_t^M = \frac{1}{M^2} \log Z_t^M = \frac{1}{M^2} \log \int e^{\sum_{i=1}^k t_i M \mathrm{Tr}_V(X^M_{B_i})} \mu_0^{M,L}(dX^M)$$

To that end we observe that, by differentiating with respect to α, since $F_0^M = 0$,

$$\begin{aligned} F_t^M &= \sum_{i=1}^k t_i \int_0^1 \int \left(\frac{1}{M} \mathrm{Tr}_V(X^M_{B_i}) \right) d\mu^M_{\alpha t} d\alpha \\ &= \sum_{i=1}^k t_i \sum_v \mu(v)^2 \int_0^1 \int \left(\frac{1}{\mu(v)} \mathrm{tr}(d_v X^M_{B_i}) \right) d\mu^M_{\alpha t} d\alpha. \end{aligned}$$

Thus, by Proposition 11, for $|t_i| \leq t(c,L)$ $\int \left(\frac{1}{\mu(v)} \mathrm{tr}(d_v X^M_{B_i}) \right) d\mu^M_{\alpha t}$ converges to $\mathrm{tr}_{\alpha t}(B_i)$ and therefore by bounded convergence theorem, we get

$$\begin{aligned} \lim_{M\to\infty} F_t^M &= \sum_{i=1}^k t_i \sum_v \mu(v)^2 \sum_{n_i \geq 0} \prod_j \frac{1}{n_j!} \int_0^1 \prod (\alpha t_j)^{n_j + 1_{j=i}} \delta^{\sharp loops} d\alpha, \\ &= \left(\sum_v \mu(v)^2 \right) \sum_{\sum n_i \geq 1} \prod_j \frac{1}{n_j!} t_j^{n_j} \delta^{\sharp loops}. \end{aligned}$$

2.5.2. *The case of A_∞ and TL for $\delta \geq 2$.* The previous construction only allows a countable set of values of δ's (which however contains all the possible $\delta < 2$ and accumulates at 2). To get all possible values of δ, we need to consider infinite graphs. However, our construction below relies heavily on the fact that the entries of the eigenvector μ go to infinity exponentially fast with their distance to a distinguished vertex of the graph. We will therefore restrict ourselves to the graph A_∞ for which we shall prove that this property holds. Since not all subfactor planar algebras can be embedded into the graph planar algebra of A_∞, we shall restrict ourselves to Temperley–Lieb algebra TL in this section. Thus it is enough to consider the infinite graph A_∞ since this graph possesses an eigenvalue δ for any possible $\delta \geq 2$ with an eigenvector with positive entries. We let μ be the corresponding eigenvector and denote by $n \in \mathbb{N}$ the vertices of A_∞. Noting that by definition for $n \geq 3$

$$\mu(n-1) + \mu(n+1) = \delta \mu(n)$$

we see that $\mu(n) \approx \delta^n$ as n goes to infinity. We let, as in the finite graph case, μ_M be the law of the independent $M_{s(e)} \times M_{t(e)}$ matrices with covariance $1/M\sqrt{\mu(s(e))\mu(t(e))}$ for $e \in E_+$ (which is now infinite). Edges far from the origin will have variance decreasing exponentially fast with the distance to the origin (recall that $\delta > 1$). To construct the law on the infinite graph, we let Σ_n be the sigma-algebra generated by the X_e^M for $e \in E_n$, the set of edges with distance less than n from the origin. The idea is to consider the Gibbs measure indexed by infinite graphs as the limit of the conditional expectation with respect to Σ_n. And more precisely, if we let

$$\tilde{X}_B^n = \sum_{\substack{e_1\cdots e_{2k}\in L_B \\ \forall \ell, e_\ell \in E_n}} \left\{\prod \delta_{e_{i_p}=e_{j_p}^o}\sigma(e_{i_p})\right\}(e_1\cdots e_{2k})$$

and let $V^n = \sum t_i \tilde{X}_{B^i}^n$, W^n so that $M\mathrm{tr}W^n = \mathrm{Tr}_V V^n$ be the corresponding approximation of W

$$d\mu_t^{M,K,n}(dX^M) = \frac{\prod_{e\in V_n} 1_{\|X_e\|_\infty \le K}}{Z_V^{M,K,n}} e^{M\mathrm{Tr}_V(V^n)} d\mu_0^M|_{\Sigma_n}(dX^M).$$

We shall prove

Proposition 14. *Let $K > 2$. Then, there exists $t(K) > 0$ so that for $\max_{1\le i\le k}|t_i| \le t(K)$, for any vertex v for any loop $w \in L(v)$ there exists a limit*

$$\nu_t(X_w) = \frac{1}{\mu(v)} \lim_{n\to\infty}\lim_{M\to\infty} \mu_t^{M,K,n}(tr(X_w))$$

which does not depend on v. Moreover, $\nu_t(d_v X_B) = tr_t(B)$ for any Temperley–Lieb tangle B and any vertex v. Finally,

$$\lim_{n\to\infty}\lim_{M\to\infty} \frac{1}{M^2\sum_{v\in V_n}\mu(v)^2} \log Z_t^{M,K,n} \quad = \quad \sum_{\sum n_i \ge 1} \prod \frac{1}{n_j!} t_j^{n_j} \delta^{\sharp loops}$$

Proof. In fact, due again to strict convexity, the negative of the Hessian of the log-density of $\mu_t^{M,K,n}$ is bounded below independently of n on $\|X^M\|_\infty \le K$. In fact, because $(\mu(s(e))\mu(t(e)))^{\frac{1}{2}}$ is at least of order δ^n if $s(e) \in E_n^c$, we can choose t small enough so that this Hessian is bounded below by the diagonal matrix which takes the value $C\delta^n$ for $v \in E_{n+1}\backslash E_n$ with some positive constant C. This is enough to guarantee Brascamp-Lieb inequalities and concentration of measure which do not depend on the dimension and in particular on n. In particular

$$\mu_t^{M,K,n}(\max_{e\in E_{k+1}\backslash E_k} \|X_e^M\|_\infty \ge \ell\delta^{-k/2}) \le e^{-\eta M}$$

for some $\ell < \infty$ and $\eta > 0$. Moreover, we can apply concentration inequalities and use the fact that $\tilde{X}_B^n$ is a Lipschitz function of the entries with Lipschitz norm of order $M^{-\frac{1}{2}}$ with overwhelming probability because of the above control (since $\delta > 1$ we obtain absolutely converging sequences) to see that (6) still holds under $\mu_t^{M,K,n}$, with a constant $C(w,K)$ independent of n. Therefore, by the same arguments, we obtain the convergence of $\mu_t^{M,K,n}(\mathrm{tr}(\tilde{X}_w^n))$, for any loop w as M go to infinity. The limit satisfies a Schwinger–Dyson equation which has a unique solution, exactly as in Lemma 13, again because the controls are uniform in n, the range of t for which we have uniqueness does not depend on n. Note that this equation is the same for μ_t^n and μ_t^m provided we consider only words w and edges e in $E_n \cap E_m$. Define

$$\delta(k) := \max_{|w|\leq k} |\mu^n(w) - \mu^m(w)|$$

we obtain that $\delta(k)$ satisfies (12) for $k \leq n \wedge m$. For larger k we just bound the additional term uniformly by $A(t)K^{D-2+k}$. We then find that

$$\Delta(\gamma) \leq \frac{\gamma^2}{m(1-K\gamma)}\Delta(\gamma) + A(t)\gamma^{2-D}\Delta(\gamma) + \sum_{k\geq m\wedge n} A(t)K^{D-2+k}\gamma^{k-1}$$

yielding, if $(C')^{-1} = 1 - \frac{\gamma^2}{m(1-K\gamma)} - A(t)\gamma^{2-D}$ is positive (which is always possible if t is small enough for some $\gamma \in (0, K^{-1})$),

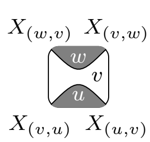
$$\Delta(\gamma) \leq C' \sum_{p\geq m\wedge n} A(t)K^{D-2+k}\gamma^{k-1} \approx C''(\gamma K)^{n\wedge m}$$

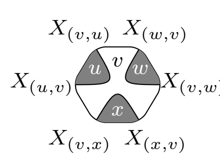
which goes to zero as $n \wedge m$ goes to infinity. This shows that μ_t^n converges as n goes to infinity. The limit satisfies Schwinger–Dyson equation with appropriate δ and therefore corresponds, when restricted to P, with tr_t. The statement on the free energy is derived as in the previous subsection. $\square$

3. Combinatorics of a few models

In this section, we actually compute the generating functions of a few loop models we have just constructed.

3.1. **The cup matrix model.** We consider the case of the matrix model with tangles $B_n = \mathrm{cup}^n$ obtained by drawing n non nested strings in a white tangle with black inside. With k finite and fixed, we thus consider the law tr_t corresponding to the tangles $(B_1, \ldots, B_k)$ and the coefficients $(t_1, \ldots, t_k)$. By Propositions 11 and 14, B_n is represented in P^Γ by $(\sum_{e\in E_+} \sigma(e)X_eX_e^*)^n$, e.g.

$B_2 =$ 

, $B_3 =$ 

, etc

and the associated Gibbs measure is, by Example 10,

$$d\mu_t^{M,K}(X^M) = \frac{\mathbf{1}_{\|X^M\|\leq K}}{Z_t^{M,K}} e^{M\sum_{i=1}^n t_i \mathrm{Tr}_V((\sum_{e\in E_+}\sigma(e)X_eX_e^*)^i)} d\mu_M(X^M).$$

Note that

$$\mathrm{Tr}_V((\sum_{e\in E_+} \sigma(e)X_eX_e^*)^i) = M\sum_{v\in V_+} \mu(v)\mathrm{tr}((\sum_{s(e)=v} \sigma(e)X_eX_e^*)^i)$$

so that the families $(X_e, s(e) = v)$, $v \in V_+$, are independent. Since we evaluate the above expression at words in $Z_v = (\sum_{e:s(e)=v} \sigma(e) X_e X_e^*)$ for a given v we may as well consider the law of Z_v for a fixed v. Recalling that X_e has variance $\sqrt{M_{s(e)} M_{t(e)}}^{-1}$, we find that $Z_v = \sum_{e:s(e)=v} \sigma(e) X_e X_e^* = Y_v Y_v^*$ with a $M_v \times (\sum_{e:s(e)=v} M_{t(e)})$ matrix Y_v with i.i.d. centered Gaussian entries with covariance M_v^{-1}. The law of the eigenvalues $(\lambda_1, \cdots, \lambda_{\mu(v)M})$ of such a matrix is asymptotically equivalent (since we can again by Brascamp–Lieb inequality remove the cutoff on X^M and transform it as a cutoff on Z_v for some K' large enough) to

$$d\mu_t^{M,K'}(\lambda) = \frac{1}{Z_t^{M,K'}} \prod_{i \neq j} |\lambda_i - \lambda_j| \prod_i \lambda_i^{\sum_{e:s(e)=v} M_{t(e)}} e^{M_v \sum_{i=1}^{M_v} (\sum_{n=1}^k t_n \lambda_i^n - \lambda_i)} \prod 1_{\lambda_i \in [0,K']} d\lambda_i .$$

Since $(\sum_{e:s(e)=v} M_{t(e)})/M_v$ converges as M goes to infinity to δ, it is classical [1, Theorem 2.6.1] to prove a large deviation principle for the law of the empirical measure of the λ_i under $\mu_t^{M,K'}$ from which one easily derives the convergence of the empirical measure. Therefore we deduce that

Lemma 15. *For all $K > 2$, and $t = (t_1, \ldots, t_k)$ small enough, all $n \geq 0$, all $v \in V_+$,*

$$tr_t(B_n) \quad = \quad \lim_{M \to \infty} \frac{1}{\mu(v)} \mu_t^{M,K}(tr(d_v X_{B_k})) = \nu_t(x^n) \tag{15}$$

with ν_t the only probability measure on $[-K, K]$ which maximizes, with $P(x) = \sum_{n=1}^k t_n x^n$,

$$I_t(\nu) := \Sigma(\nu) + \delta \int \log |x| d\nu(x) + \int P(x) d\nu(x) - \int x d\nu(x)$$

where Σ is the free entropy

$$\Sigma(\mu) = \iint \log |x - y| d\nu(x) d\nu(y).$$

We can obtain an equation for ν_t by writing that $I_t(\nu_t) \geq I_t(\nu_t^\epsilon)$ with ν_t^ϵ the law of $x + \epsilon h(x)$ under ν_t and h a smooth real-valued function. Expressing that the linear term in ϵ must vanish and ultimately taking $h(x) = (z - x)^{-1}$ we deduce that ν_t is solution of the Schwinger–Dyson type equation

$$\left(\int \frac{1}{z - x} d\nu_t(x) \right)^2 + \delta \int \frac{1}{x(z - x)} d\nu_t(x) + \int \frac{P'(x) - 1}{z - x} d\nu_t(x) = 0$$

which yields with $G_t(z) = \int \frac{1}{z-x} d\nu_t(x), Q(z) = \int \frac{P'(x)-P'(z)}{z-x} d\nu_t(x)$, and $c = \delta \int x^{-1} d\nu_t(x)$,

$$G_t(z)^2 + \left(\frac{\delta}{z} + P'(z) - 1\right) G_t(z) + Q(z) + \frac{c}{z} = 0$$

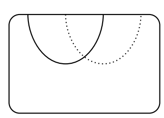
so that

$$G_t(z) = \frac{1}{2z}\left(z(1-P'(z)) - \delta - \sqrt{(z(P'(z)-1)+\delta)^2 - 4z(zQ(z)-c)}\right).$$

It can be shown that for t small enough, ν_t has connected support and deduce a formula for G_t, see e.g. [5]. Indeed, according to [1, Lemma 2.6.2], for small t_i's, ν_t is characterized by the fact that a certain strictly concave function is smaller than some constant outside of its support and equal to it at the boundary of the support; this is only possible if the support is connected. Since G_t must be analytic outside the support $[a,b]$ of ν_t, the formula for G_t can be uniquely determined by the fact that there exists a polynomial R with degree $k-1$, and $a < b$ so that

$$(z(P'(z)-1)+\delta)^2 - 4z(zQ(z)-c) = (z-a)(b-z)R(z)^2$$

3.2. **The $O(n,m)$ model.** Consider the potential

$V =$ 

considered in the planar algebra of two stitched Temperley–Lieb algebras (see §2.1.2) and and recall the notation of §2.2.2.

In the finite-depth case, the random matrix model is given by considering block matrices indexed by pairs (e,v) where e is an edge in Γ_r and v is a vertex in Γ_b or e is an edge in Γ_b and v is a vertex in Γ_r with the potential of the form

$$\sum_{\substack{e\in\Gamma_r \\ v \text{ black vertex}}} \sigma(e) X_{(e,v)} X^*_{(e,v)} + \sum_{\substack{e\in\Gamma_b \\ v \text{ red vertex}}} \sigma(e) X_{(e,v)} X^*_{(e,v)}$$
$$+\beta \sum_{e\in\Gamma_r} \sum_{f\in\Gamma_b} \sigma(e)\sigma(f) X_{(e,s(f))} X_{(f,t(e))} X^*_{(e,t(f))} X^*_{(f,s(e))}.$$

If $\delta_r = n$ and $\delta_b = m$, we can choose Γ_r (resp., Γ_b) to be the graph with one odd vertex and n (resp., m) even vertices, with exactly one edge between the odd vertex and every even vertex. the resulting combinatorics is exactly the same as of the $O(n,m)$ model with the potential

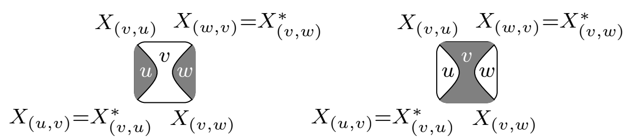
$$\beta \sum_{i=1}^{n} \sum_{j=1}^{m} X_i Y_j X_i^\dagger Y_j^\dagger$$

which was studied for instance in [7], in connection with the problem of enumerating meanders.

3.3. **The double cup matrix model.** The potential is a linear combination of two tangles; the tangle with two cups with black inside (with coefficient A) and the tangle with two cups with white inside (with coefficient B respectively). We denote by V the element of the planar algebra associated to it, namely (see Example 10):

$$V = A\left(\sum_{e\in E_-}\sigma(e)X_eX_e^*\right)^2 + B\left(\sum_{e\in E_+}\sigma(e)X_eX_e^*\right)^2 .$$

Diagramatically, this corresponds to tangles of the form



where the only difference between the two pictures is that on the left, $v \in V_+$ and $u, w \in V_-$, and vice versa on the right.

The associated Gibbs measure $\mu_t^{M,K}(dX^M)$ is given by

$$\frac{\mathbf{1}_{\|X^M\|\le K}}{Z_t^{M,K}} e^{tM^2\sum_{v\in V}(A1_{v\in V_-}+B1_{v\in V_+})\mu(v)\mathrm{tr}\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2}\mu_M(dX^M)$$

To analyze the asymptotics of this measure, we shall first perform a Hubbard–Stratonovitch transformation and then study the resulting Gibbs measure. We first relate this measure with our problem. For the sake of simplicity, we restrict ourselves to δ's corresponding to finite graphs, and even further to the graphs A_n with n vertices and $n-1$ edges. This is enough to characterize the generating functions by analyticity (since the index δ corresponding to these graphs has an accumulation point at 2). In fact the restriction to finite graphs allows us to avoid dealing with a Gibbs measure on infinitely many matrices whereas the restriction to A_n ensures the uniqueness of the minimizers of the entropy described in Proposition 17, and therefore their characterization.

3.3.1. *The partition function and an auxiliary matrix model.* Let us first consider the partition function

$$Z_t^{M,K} = \int \mathbf{1}_{\|X^M\|\le K} e^{tM^2\sum_{v\in V}(A1_{v\in V_-}+B1_{v\in V_+})\mu(v)\mathrm{tr}\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2}\mu^M(dX^M)$$

and introduce independent $M_v \times M_v$ matrices G_v from the GUE with covariance $1/M_v$. Then, assuming tA, tB positive and putting $\alpha(v) = \sqrt{8tA}$ if $v \in V_-$,

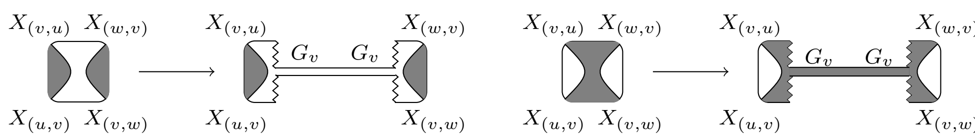
$\alpha(v) = \sqrt{8tB}$ if $v \in V_+$, $\mu(v)' = \mu(v)\sqrt{M_v(\mu(v)M)^{-1}}$ (which approximately equals $\mu(v)$), we get

$$Z_t^{M,K} = \int \mathbf{1}_{\|X^M\|\le K} e^{-\frac{M^2}{2}\sum_{v\in V}\alpha(v)\mu(v)'\mathrm{tr}\left(G_v\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)}\mu^M(dX^M, dG^M).$$

Note that again for $\alpha(v)$ small enough, the integral has a strictly log-concave density and therefore Brascamp-Lieb inequalities show that the matrices G_v are also bounded with large probability and so there exists K' large enough so that, we have

$$Z_t^{M,K} \sim \int \mathbf{1}_{\|G^M\|\le K'} e^{-\frac{M^2}{2}\sum_{v\in V}\alpha(v)\mu(v)'\mathrm{tr}\left(G_v\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)}\mu^M(dX^M, dG^M)$$

where we used the standard notation $A_M \sim B_M$, for two sequences A_M, B_M, for a shorthand for $A_M B_M^{-1}$ converges to one as M goes to infinity. Diagramatically this corresponds to the following "breaking" of the tangles:



We next integrate over the matrices X^M, recalling that the entries of X_e have covariance $(M_{s(e)}M_{t(e)})^{-1/2}$. Up to a constant, this provides the term

$$\prod_{e\in E_+} e^{-M^2\mathrm{tr}\otimes\mathrm{tr}(\log(I+\alpha(s(e))'I\otimes G_{s(e)}+\alpha(t(e))'G_{t(e)}\otimes I))}$$

where we noticed that $\int e^{-\gamma x^2}dx = \sqrt{2\pi}\gamma^{-\frac{1}{2}}$ and the matrices X_e have complex entries (so each term appears twice). Here $\alpha(s(e))'\alpha(s(e))^{-1} = \alpha(t(e))'\alpha(t(e))^{-1} = (\mu(t(e))'/\mu(t(e)))^{1/2}$ is approximately equal to one. We can finally diagonalize the matrices G_v to get

$$Z_t^{M,K} \sim \int \prod_{v\in V} 1_{|\lambda^v|\le K'}\Delta(\lambda^v)d\lambda^v \exp\left[-\sum_{v\in V}\frac{M_v}{2}\sum_{i=1}^{M_v}(\lambda_i^v)^2\right]$$
$$\cdot \prod_{e\in E_+}\exp\left[-\sum_{\substack{1\le i\le M_{s(e)}\\ 1\le j\le M_{t(e)}}}\log(1+\alpha(s(e))\lambda_i^{s(e)}+\alpha(t(e))\lambda_j^{t(e)})\right]$$

with $\lambda^v = (\lambda_1^v, \ldots, \lambda_{M_v}^v)$ the eigenvalues of G_v and $\Delta(\lambda^v) = \prod_{1\le i\ne j\le M_v}|\lambda_i^v - \lambda_j^v|$. The asymptotics of $\frac{1}{M^2}\log Z_t^{M,K}$ can be obtained via the global asymptotics of the eigenvalues $(\lambda_v, v\in V)$, that is the convergence

$$\lim_{M\to\infty} E[\frac{1}{M_v}\sum_{i=1}^{M_v}(\lambda_i^v)^p] = \int x^p d\nu_v(x) \qquad p\in\mathbb{N}, \quad v\in V_\pm.$$

under the associated Gibbs measure $P_t^{M,K}(d\lambda)$ given by

$$\frac{1_{|\lambda^v|\le K'}}{Z_t^{M,K}}\prod_{v\in V}\Delta(\lambda^v)d\lambda^v\prod_{e\in E_+}\exp\left[-\sum_{i=1}^{M_{s(e)}}\sum_{j=1}^{M_{t(e)}}\log(1+\alpha(s(e))\lambda_i^{s(e)}+\alpha(t(e))\lambda_j^{t(e)})\right]$$
$$\cdot\exp\left[-\sum_{v\in V}\frac{M_v}{2}\sum_{i=1}^{M_v}(\lambda_i^v)^2\right].$$

In the sequel, we shall prove not only this convergence but the existence of two probaility measures ν_- and ν_+ so that

$$\int x^p d\nu_v(x)=\int x^p d\nu_\pm(x)\qquad p\in\mathbb{N},\quad v\in V_\pm\,. \tag{16}$$

Before attacking this question, let us summarize what information the auxiliary probability measure $P_t^{M,K}$ and (16) tells us about our original question. Recall that tr_t is the tracial states constructed with the two tangles with two strings and opposite shading. We claim that $\nu_\pm$ gives the law of a cup under tr_t in the following sense.

Proposition 16. *Assume (16) and recall that $\alpha=\sqrt{8tA}$. Let B_n be the tangle with n non nested strings and black shading inside and put for small z*

$$C(z,A,B)=\sum_{n\ge0}z^n tr_t(B_n)$$

and $M(z)=\int\sum_{n\ge0}z^nx^nd\nu_+(x)$. Then, $\gamma(z)=\frac{\sqrt{8tA}z}{1-z^2M(z)}$ is invertible from a neighborhood of the origin into a neighborhood of the origin, with inverse $z(\gamma)$ and

$$C(\gamma,A,B)=\frac{\alpha z(\gamma)}{\gamma}M(z(\gamma))=\frac{\alpha}{\gamma z(\gamma)}\left(1-\frac{\alpha z(\gamma)}{\gamma}\right).$$

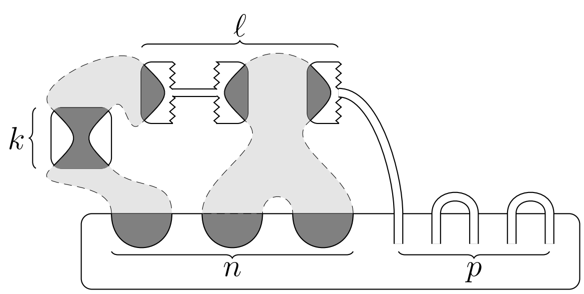
Proof. Observe first that $P_t^{M,K}$ is the law of the eigenvalues of $(G_v^M, v\in V)$ under

$$\nu_t^{M,K}(dX^M,dG^M)=\frac{\mathbf{1}_{\|G^M\|\le K'}}{Z_t^{M,K}}e^{-\frac{M^2}{2}\sum_{v\in V}\alpha(v)\mathrm{tr}\left(G_v\sum_{(vt)\in E}\sigma(e)X_{vt}X_{vt}^*\right)}\mu^M(dX^M,dG^M).$$

Adapting the previous considerations we see that $G_v\sum_{e:s(e)=v}\sigma(e)X_eX_e^*$ correspond to an element of a planar algebra with one string and one strip, the strip being in the white shading iff $v\in V_-$, both being independent in the sense that they can not be glued together, and the strips requiring to be coupled with another strip corresponding to the same vertex, the weight being one. We can also show the convergence (in a small parameters regime) of the law $\nu_t^{M,K}$ restricted to the planar algebra generated by elements with non-crossing strings and strips.

We denote the limit by τ_t. tr_t corresponds to the case where we restrict ourselves to elements with only strings (since then expectation over the strips, that is the Gaussian variables can be taken) whereas $\nu_\pm$ corresponds to restricting ourselves to element with strips only (inside a white or a black shading). To relate both states let us consider the expectation of an element $B_{n,p}$ with n non nested cups with black shading inside, followed by p strips in the white region. Let $C(p,n,\ell,k)$ be the number of possible configurations build above this tangle with ℓ (resp. k) tangles with one string and one strip in the white (resp. black) shaded region. We get an induction relation by gluing the first strip which yields for $p \geq 1$,

$$\begin{aligned} C(p,n,\ell,k) &= \sum_{p_1=0}^{p-2} \sum_{\ell_1 \leq \ell} \sum_{k_1 \leq k} C_k^{k_1} C_\ell^{\ell_1} C(p_1,0,\ell_1,k_1) C(p-p_1-2,n,\ell-\ell_1,k-k_1) \\ &\quad + \ell C(p-1,n+1,\ell-1,k). \end{aligned}$$



The first term appears when the strip is glued with another strip of the tangle whereas the second one shows up when it is glued with a strip of an element with a strip and a string with the right shading.

We let

$$C(z,\gamma,\alpha,\beta) = \sum \frac{z^p \gamma^n \alpha^\ell \beta^k}{\ell! k!} C(p,n,\ell,k)$$

and conclude from the induction relation that

$$C(z,\gamma,\alpha,\beta) - C(0,\gamma,\alpha,\beta) = z^2 C(z,0,\alpha,\beta) C(z,\gamma,\alpha,\beta) + \frac{\alpha z}{\gamma}(C(z,\gamma,\alpha,\beta) - C(z,0,\alpha,\beta))$$

which gives:

$$C(z,\gamma,\alpha,\beta) = \frac{\gamma C(0,\gamma,\alpha,\beta) - \alpha z C(z,0,\alpha,\beta)}{\gamma - \alpha z - \gamma z^2 C(z,0,\alpha,\beta)}$$

Since $C(z,\gamma,\alpha,\beta)$ is analytic in z, γ small enough, we deduce that if the denominator vanishes so that

$$\gamma = (1 - z^2 C(z,0,\alpha,\beta))^{-1} \alpha z =: \gamma(z)$$

then the numerator must vanish too. Therefore, with $z(\gamma)$ the inverse of $\gamma(z)$ (which exists by the implicit function theorem in a neighborhood of the origin) we deduce

$$C(0,\gamma,\alpha,\beta) = \frac{\alpha z(\gamma)}{\gamma} C(z(\gamma),0,\alpha,\beta).$$

Since if we choose $\alpha = \sqrt{2tA}, \beta = \sqrt{2tB}$, we have $C(0,\gamma,\alpha,\beta) = C(\gamma, A, B)$ and $C(z,0,\alpha,\beta) = M(z)$, we have proved the proposition. □

3.4. **Solving the auxiliary matrix model.** We study in this section the law $P_t^{M,K}$ and prove (16).

3.4.1. *Large deviation estimates and limit points.* Using standard large deviation theory [1, Section 2.6], and putting

$$\Sigma(\mu,\nu) := \iint \log|x-y| d\mu(x) d\nu(y), \quad \Sigma(\mu) := \Sigma(\mu,\mu)\,,$$

we deduce that

Proposition 17. *Set $\alpha = \sqrt{8tA}$ and $\beta = \sqrt{8tB}$. The law of the spectral measures $(\frac{1}{M_v}\sum_{i=1}^{M_v} \delta_{\lambda_i^v}, v \in V)$ of the matrices $G_v, v \in V$ under $P_t^{M,K}$ satisfies a large deviation principle in the scale M^2 and with good rate function*

$$I_t(\nu_v, v \in V) := \frac{1}{2}\sum_{v\in V} \mu(v)^2 \int x^2 d\nu_v(x) - \sum_{v\in V} \mu(v)^2 \Sigma(\nu_v)$$
$$+ \sum_{e\in E_+} \mu(t(e))\mu(s(e)) \int \log(1+\alpha(s(e))x + \alpha(t(e))y) d\nu_{s(e)}(x) d\nu_{t(e)}(y).$$

Take $\Gamma = A_n, n \geq 1$. Then, I_t achieves its minimal value at a unique point so that $\nu_v = \nu_\pm$ if $v \in V_\pm$ with (ν_+,ν_-) the unique minimizer of

$$S(\nu_+,\nu_-) = \sum_{\epsilon=\pm}\left(\frac{1}{2}\int x^2 d\nu_\epsilon(x) - \Sigma(\nu_\epsilon)\right) + \delta \iint \log|1+\alpha x + \beta y| d\nu_+(x) d\nu_-(y).$$

In particular, for any $p \geq 0$, we have

$$\lim_{M\to\infty} E[\frac{1}{M_v}\sum_{i=1}^{M_v} (\lambda_i^v)^p] = \int x^p d\nu_\pm(x) \quad \text{if } v \in V_\pm$$

Remark 18. By large deviation techniques as in [2], it is easy to check that (ν_+,ν_-) is also the limit of the spectral measures of the two Hermitian $M\times M$ random matrices G_+, G_- with joint law given by

$$\frac{1_{\|G_\pm\|\leq K'}}{Z^{M,\delta}} e^{-\delta M^2 \mathrm{tr}\otimes\mathrm{tr}\,\log(I+\alpha I\otimes G_+ + \beta G_-\otimes I)} e^{-\frac{M^2}{2}\mathrm{tr}(G_+^2+G_-^2)} dG_+ dG_-$$

when K' is large enough. This last formula is easily obtained for integer values of δ starting from the graphical rules of the double cup matrix model, and usually in the physics literature such expressions are then analytically continued to δ non integer.

Proof of Proposition 17. The first point is to show the uniqueness of the minimizers of I_t. We let $\tilde{\nu}_v$ be the law of $-\alpha x$ (resp. $1+\beta y$) under ν_v for $v \in V_+$ (resp. $v \in V_-$). Then, we have to minimize

$$I_t(\nu_v, v \in V) := H(\tilde{\nu}_v, v \in V) + L(\tilde{\nu}_v, v \in V)$$

with, for probability measures $p_v, v \in V$ on the real line,

$$H(p_v, v \in V) := \sum_{e \in E_+} \mu(t(e))\mu(s(e))\Sigma(p_{s(e)}, p_{t(e)}) - \sum_{v \in V} \mu(v)^2 \Sigma(p_v)$$

and

$$L(p_v, v \in V) := \frac{1}{2} \sum_{v \in E_-} (\frac{\mu(v)}{\alpha})^2 \int x^2 dp_v(x) + \sum_{v \in E_+} (\frac{\mu(v)}{\beta})^2 \int (1-x)^2 \, dp_v(x).$$

Since I_t is a good rate function, it has compact level sets (see the case $V = \{0\}$ in [2]) and therefore I_t achieves its maximal value. We next prove that its maximizer is unique. Note that L is linear in the measures. We shall prove that H is strictly convex. Indeed, put

$$d(v) = \sharp\{e \in E_+ : s(e) = v\} + \sharp\{e \in E_+ : t(e) = v\}$$

and observe that when $\Gamma \subset A_\infty$ the degree $d(v)$ of each vertex is bounded by one (for the boundary points) or by two. Therefore, the quadratic form

$$Q(x) = \sum_{v \in V} x_v^2 - \sum_{e \in E_+} x_{s(e)} x_{t(e)} = \frac{1}{2} \sum_{e \in E_+} (x_{s(e)} - x_{t(e)})^2 + \sum_{v \in V} x_v^2 (1 - \frac{d(v)}{2})$$

is positive definite. We let (γ_i, v_i) be the eigenvalues and eigenvectors of the corresponding matrix in $\mathbb{R}^{+*} \times \mathbb{R}^{|V|}$ and write

$$H(p_v, v \in V) = -\sum_i \gamma_i \Sigma(\sum_{u \in V} v_i(u)\mu(u)p_u).$$

Finally Σ is strictly log-concave on the set of real valued measures with fixed mass, as it was proved in [2] on the set of probability measures. Applying it to the measure $p_i = \sum_{u \in V} \mu(u)v_i(u)p_u$ with given mass $\sum_{u \in V} v_i(u)\mu(u)$ and using that the γ_i's are positive, we deduce that H is strictly convex. Therefore, I_t (and by the same argument S) achieves its minimal value at a unique point $(\nu_v, v \in V)$ (resp. (ν_+, ν_-)). We next show that it has to be $\nu_v = \nu_\pm$ if $v \in V_\pm$. In fact, the infimum of I_t is characterized by the fact that for all $v \in V^+$, the function $f_v(x)$ given by

$$2\mu(v) \int \log|x-y| d\nu_v(y) - \sum_{e:s(e)=v} \mu(t(e)) \int \log(1+\alpha x + \beta y) d\nu_{t(e)}(y) - \frac{\mu(v)}{2} x^2$$

is constant on the support of ν_v and non positive outside. We have the same equation for $v \in E_-$ with α and β exchanged. The same characterization holds for ν_+ which is such that the function f_+ given by

$$-\delta \int \log(1+\alpha x+\beta y) d\nu_-(y) + 2\int \log|x-y| d\nu_+(y) - \frac{x^2}{2}$$

is constant on the support of ν_+ and non positive outside (and again a similar equation for ν_- with α and β exchanged). Putting $\nu_v = \nu_\pm$ for $v \in V_\pm$, we find that $f_v = \mu(v) f_\pm$ as $\sum_{e:s(e)=v} \mu(t) = \delta\mu(v)$, and therefore is indeed constant on the support of ν_v and non positive outside. Thus $\nu_v = \nu_\pm$ for $v \in V_\pm$ is a solution, and by the first part the unique solution. $\square$

3.4.2. *Properties of the minimizers of the rate function.* We can give the following characterization of the minimizer (ν_+, ν_-) of S.

Lemma 19. *Let $\alpha = \sqrt{8tA}$ and $\beta = \sqrt{8tB}$. There exists $t_0 > 0$ so that for $|t| \le t_0$,*

- *$\nu_\pm$ has a connected support $S_\pm$ included in $[-2-A(t), 2+A(t)]$ with $A(t)$ going to zero as t goes to zero.*
- *There exist functions $(f_\pm, g_\pm)$ which are analytic in a neighborhood of $S_\pm$ and so that for $z \in \mathbb{C}$*

$$G_\pm(z) = \int \frac{1}{z-x} d\nu_\pm(x) = f_\pm(z) - \sqrt{g_\pm(z)} \tag{17}$$

  with the branch of the square root chosen on $\mathbb{R}^-$. Moreover $g_\pm$ is real on the real line and $S_\pm = \{x : g_\pm(x) < 0\}$. We define $G_\pm(z+i0)$ and $G_\pm(z-i0)$ as the limit of $G_\pm$ when z goes to an element of $S_\pm$ from above or from below.
- *For all $x \in S_\pm$ we have with $\alpha_+ = \alpha, \alpha_- = \beta$,*

$$\delta \frac{\alpha_\pm}{\alpha_\mp} G_\mp\left(\frac{1-\alpha_\pm x}{\alpha_\mp}\right) + x = G_\pm(x+i0) + G_\pm(x-i0). \tag{18}$$

- *There exists at most one solution to (18) so that $G_\pm$ are the Cauchy–Stieltjes transform of probability measures $\nu_\pm$ supported by $[-3,3]$.*

Proof. The fact that the support of ν_- and ν_+ is connected is a consequence of Remark 18 and [15, Theorem 4.4 and Theorem 4.2] which asserts that the limiting measures $\nu_\pm$ have connected supports when the potential is strictly locally convex, which is the case when t is small enough (note here that the potential is not a polynomial, however it expands in absolutely converging power series when t is small enough and x, y are bounded by K' so that we can apply the techniques from [15] to represent the measures as the law of an element of the C^* algebra generated by the free Brownian motion). Another way to see this is to notice that the function f_+ is strictly concave outside of the support and continuous on the support where it takes only one constant value; hence the support can not be disconnected. The fact that the support is bounded is a direct consequence of Brascamp-Lieb inequalities.

To deduce the second point, we obtain an equation on (ν_+, ν_-) by writing

$$S(\nu_+^\zeta, \nu_-^\zeta) \geq S(\nu_+, \nu_-)$$

with $\nu_\pm^\zeta$ the law of $x+\zeta h_\pm(x)$ for bounded continuous functions $h_\pm$ on $\mathbb{R}$. Writing that the linear term in ζ must vanish results with the equation

$$\sum_{\epsilon=\pm}\left(\int x h_\epsilon(x) d\nu_\epsilon(x) - \iint \frac{h_\epsilon(x)-h_\epsilon(y)}{x-y} d\nu_\epsilon(x) d\nu_\epsilon(y)\right) = -\delta \int \frac{\alpha h_+(x) + \beta h_-(y)}{1+\alpha x+\beta y} d\nu_+(x) d\nu_-(y). \tag{19}$$

Taking $h_+(x) = -\frac{\beta}{\alpha}(z + \frac{1+\alpha x}{\beta})^{-1}$ and $h_-(x) = (z-x)^{-1}$ we get that, with $m(z) = G_+(-(1+\beta z)/\alpha)$,

$$\frac{\beta^2}{\alpha^2} m(z)^2 + G_-(z)^2 + \delta\frac{\beta}{\alpha} m(z) G_-(z) - zG_-(z) + 1 + \frac{\beta^2}{\alpha^2}(1-(1+\frac{\beta z}{\alpha})m(z)) = 0$$

which gives

$$G_-(z) = \frac{1}{2}(b(z) - \sqrt{b(z)^2 - 4a(z)})$$

with the cut of the square root on $\mathbb{R}^-$ and

$$b(z) = z - \delta\frac{\beta}{\alpha} m(z), \qquad a(z) = \frac{\beta^2}{\alpha^2} m(z)^2 + 1 + \frac{\beta^2}{\alpha^2}(1-(1+\frac{\beta z}{\alpha})m(z)).$$

But, for small t, when z is in the neighborhood of S_-, $-(1+\beta z)/\alpha$ is in the neighborhood of $-1/\alpha$ which is far from the support S_+. Hence, $a(z)$ and $b(z)$ are analytic in the neighborhood of S_- and also take real values. This completes the proof of the second point.

For the third point, it is enough to take $h_+ = 0$ or $h_- = 0$ in (19) with the remark that

$$\int\int \frac{h(x)-h(y)}{x-y} d\nu_\pm(x) d\nu_\pm(y) = \int h(x)[G_\pm(x+i0) + G_\pm(x-i0)] d\nu_\pm(x)$$

and use the continuity of $G_\pm$ above and below the cut due to the previous point to obtain the desired equations almost surely and then everywhere.

For the last point, since (18) is equivalent to (19), we show the uniqueness of the solution by taking $h_+(x) = (z-x)^{-1}$ to deduce that

$$-1 + zG_\pm(z) - G_\pm(z)^2 = \delta\alpha_\pm \int \frac{1}{z-x} \alpha_\mp G_\mp(-\frac{1+\alpha_\pm x}{\alpha_\mp}) d\nu_\pm(x) =: \epsilon_\pm(z)$$

so that for z sufficiently large

$$G_{\pm}(z) = \frac{1}{2}(z - \sqrt{z^2 - \epsilon_{\pm}(z)})$$

where we have taken the usual determination of the square root as $\epsilon_{\pm}(z)$ is small since $\alpha_{\mp} G_{\mp}(-\frac{1+\alpha_{\pm} x}{\alpha_{\mp}})$ is uniformly close to one for $x \in S_{\pm}$ and t small. Therefore, if we have two solutions G and $\tilde{G}$, we find that there exists $M(t)$ finite so that for t small,

$$\sup_{\substack{z \in \mathbb{R} \\ |z| \geq M(t)}} |G_{\pm}(z) - \tilde{G}_{\pm}(z)| \leq \frac{1}{2} \sup_{\substack{z \in \mathbb{R} \\ |z| \geq M(t)}} |G_{\mp}(z) - \tilde{G}_{\mp}(z)|$$

which results with $G_{\mp}(z) = \tilde{G}_{\mp}(z)$ for z large enough and real, and then for all z in the complement of $S_{\pm}$ by analyticity. □

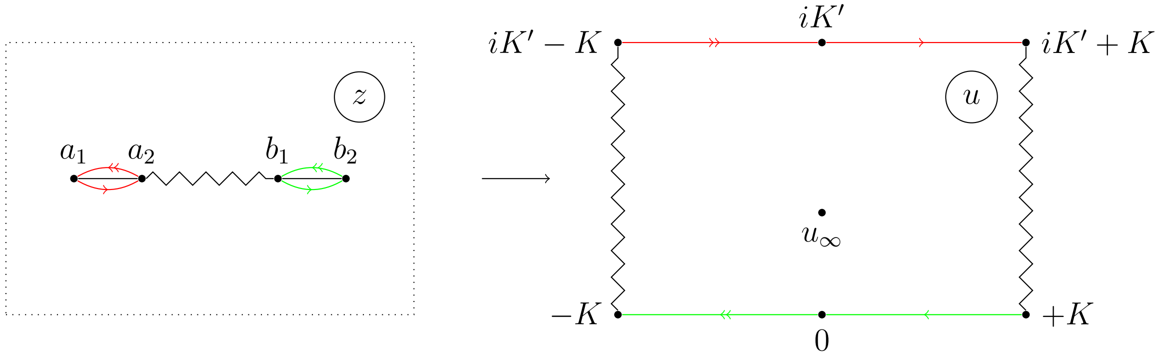
3.4.3. *Characterization of the minimizers of the rate function.* In this section we completely characterize the measures (ν_+, ν_-) by their Cauchy–Stieltjes transform. To simplify the notations, we let $\tilde{\nu}_+$ and $\tilde{\nu}_-$ be the law of $-\alpha x$ and $1 + \beta x$ under ν_+ and ν_- respectively. By Lemma 19, for t small enough, $\tilde{\nu}_+$ and $\tilde{\nu}_-$ have disjoint connected supports $[a_1, a_2]$ and $[b_1, b_2]$ around the origin and the unity respectively. Our study will be restricted to this case, which therefore include small t's but eventually a larger class of parameters. We will first proceed by a reparametrization of the Cauchy–Stieltjes transforms of $\tilde{\nu}_+$ and $\tilde{\nu}_-$, which will allow to obtain very simple equations, and then solve these equations.

• **Parametrization**. Consider the standard parameterization of the elliptic curve $y^2 = (z - a_1)(z - a_2)(z - b_1)(z - b_2)$:

$$u(z) = \frac{i}{2}\sqrt{(b_1 - a_1)(b_2 - a_2)} \int_{b_2}^{z} \frac{dz'}{\sqrt{(z' - a_1)(z' - a_2)(z' - b_1)(z' - b_2)}} \tag{20}$$

where the path of integration avoids the segment $[a_1, b_2]$ (for $z \in [a_2, b_1]$, we choose to come from the upper half plane, though this choice is irrelevant, see the comment below on $2K$-periodicity), and the square root in the denominator is defined as having cuts $[a_1, a_2]$ and $[b_1, b_2]$ and such that at infinity it behaves like z^2.

The image of $z \in \mathbb{C} \cup \{\infty\} - ([a_1, a_2] \cup [b_1, b_2]) \mapsto u(z) = (Re\ u(z), \operatorname{Im} u(z))$ is a rectangle of the form $[-K, K[\times]0, iK'[$, where K and K' are usually called quarter-periods (though they will be half-periods in what follows). Now, note that crossing the line $[a_2, b_1]$ corresponds to $u \to u + 2K$. Since all the functions of z we shall consider are smooth when one crosses this line, they can be made into periodic functions on the strip $0 < \operatorname{Im} u < K'$. The map $z \mapsto u(z)$ is then an analytic isomorphism from the Riemann sphere $\mathbb{C} \cup \{\infty\}$ minus the two segments $[a_1, a_2]$ and $[b_1, b_2]$ to $\mathbb{R}/2K\mathbb{Z}\times]0, iK'[$. If one extends the map to the cuts $[a_1, a_2]$ and $[b_1, b_2]$, then the result depends on whether one approaches them from the upper or lower half-plane, and they get sent to $\operatorname{Im} u = K'$ et $\operatorname{Im} u = 0$ as described on the figure. More precisely, if $a \in]a_1, a_2[$, then $u(a+i0)+u(a-i0) = 2iK'$ (the two images of a are symmetric w.r.t. iK' on the line $\operatorname{Im} u = K'$), and similarly if $b \in]b_1, b_2[$, then $u(b+i0)+u(b-i0) = 0$.



The inverse map $z(u)$ can be expressed in terms of Jacobi's elliptic function sn, and can be deduced from the following identity:

$$\operatorname{sn}^2(u, k^2) = \frac{a_1 - b_1}{b_2 - b_1} \frac{b_2 - z}{a_1 - z}$$

with $k^2 = \frac{(b_2-b_1)(a_2-a_1)}{(b_2-a_2)(b_1-a_1)}$. Note finally that

$$\lim_{z\to\infty} u(z) = \frac{i}{2} \int_{b_2}^{+\infty} \frac{\sqrt{(b_1-a_1)(b_2-a_2)}dx}{\sqrt{(x-a_1)(x-a_2)(x-b_1)(x-b_2)}} =: u_\infty \in i\mathbb{R}$$

is a pole of $z(u)$. It is a simple pole as one easily sees that for z large $u(z) \sim u_\infty + \frac{z_{-1}}{z} + o(1/z)$ with a non vanishing constant z_{-1}. Moreover, $z(u)$ is analytic everywhere else in $[-K; K] \times [0, iK']$ by the implicit function theorem, and, once analytically continued to the whole complex plane, is even and *elliptic* (doubly periodic), with poles at $\pm u_\infty \pmod{2K, 2iK'}$.

• **Resolvents**. We let $\omega_\pm$ be the reparametrization of the Cauchy–Stieltjes transform of $\tilde\nu_+$ and $\tilde\nu_-$ respectively;

$$\omega_\pm(u) = \int \frac{1}{z(u) - x} d\tilde\nu_\pm(x)$$

The functions $\omega_\pm(u)$ are analytic in the strip $0 < \operatorname{Im} u < iK'$, and according to the second point (17) of Lemma 19, they can in fact be analytically continued to some neighborhood of the closed strip $0 \le \operatorname{Im} u \le iK'$. Indeed, if $0 < |\operatorname{Re} u(z)| < K$ the mapping $z \mapsto u(z)$ is invertible around z and the extension in the z variable translates directly into the u variable. If $z = a_1, a_2$ (resp. b_1, b_2 for G_-), one has according to (17) (with a more detailed study of $g_\pm$ which shows that for t small enough $g'_\pm$ does not vanish in a neighborhood of $S_\pm$), $G_\pm(z) \propto (z - z')^{1/2}$ as

$z' \to z$; but this matches the behavior of $u(z)$ in (20), so once again $\omega_\pm(u)$ is a well-defined analytic function in the neighborhood of $0, iK', \pm K, iK' \pm K$.

By (18), we have

$$\omega_+(u) + \omega_+(2iK' - u) - \delta\omega_-(u) = P_+(u) \qquad \operatorname{Im} u = K' \tag{21}$$
$$\omega_-(u) + \omega_-(-u) - \delta\omega_+(u) = P_-(u) \qquad \operatorname{Im} u = 0 \tag{22}$$

where $P_+(u) = z(u)/\alpha$ and $P_-(u) = (z(u) - 1)/\beta$. $P_\pm$ are even elliptic functions (with periods $2K$, $2iK'$).

We also have the following additional conditions:

$$\omega_+(u) = \omega_+(-u) \qquad \operatorname{Im} u = 0 \tag{23}$$
$$\omega_-(u) = \omega_-(2iK' - u) \qquad \operatorname{Im} u = K' \tag{24}$$

expressing the fact that the Cauchy–Stieltjes transform of $\tilde{\nu}_+$ is analytic in a neighborhood of $[b_1, b_2]$, so its values at $z \pm i0$, $z \in [b_1, b_2]$ should be equal; and similarly for $\tilde{\nu}_-$.

Now these equations can be repeatedly used to extend $\omega_\pm$ to the whole complex plane: for example $u \to 2iK' - u$ maps the strip $0 \le \operatorname{Im} u \le K'$ to the strip $K' \le \operatorname{Im} u \le 2K'$, so we can use Eq. (21) as a definition of ω_+ in this new strip and equation (21) precisely ensure that the two definitions coincide at their common boundary $\operatorname{Im} u = K'$; and so on. This way we obtain meromorphic functions $\omega_\pm(u)$ defined on the whole complex plane, and by uniqueness of analytic functions that coincide on a set with accumulation points, we deduce that the above equations are true for all u:

$$\omega_+(u + 2K) = \omega_+(u) \tag{25}$$
$$\omega_-(u + 2K) = \omega_-(u) \tag{26}$$
$$\omega_+(u) = \omega_+(-u) \tag{27}$$
$$\omega_-(u) = \omega_-(2iK' - u) \tag{28}$$
$$\omega_+(u) + \omega_+(2iK' - u) - \delta\omega_-(u) = P_+(u) \tag{29}$$
$$\omega_-(u) + \omega_-(-u) - \delta\omega_+(u) = P_-(u) \tag{30}$$

Furthermore, these functions must possess a zero at $u_\infty := u(z = \infty)$ and a prescribed derivative at u_∞.

• **General solution of the saddle point equations**. Now we have a well-posed analytic problem, which can be solved explicitly. Set $\delta = q + q^{-1}$. If $|\delta| < 2$, q is not real and has modulus one, e.g. if $\delta = 2\cos(\pi/n)$, $q = e^{i\pi/n}$. If $\delta > 2$, q is real and can be chosen in $]0, 1[$. Set

$$\varphi_\pm(u) = q^{\pm 1}\omega_+(u) - \omega_-(u) - R_\pm(u)$$

where $R_\pm(u) = \frac{1}{1-q^{\pm 2}}(q^{\pm 1}P_+(u) + q^{\pm 2}P_-(u))$. Then the equations can be recombined into:

$$\varphi_\pm(u+2K) = \varphi_\pm(u) \tag{31}$$

$$\varphi_\pm(u+2iK') = q^{\pm 2}\varphi_\pm(u) \tag{32}$$

where the first point is a direct consequence of (25) and (26), whereas the second is obtained by multiplying (29) by $q^\pm$ and (30) by $q^{\pm 2}$ and adding the two corresponding equations. Moreover, (27) and (30) implies that

$$\varphi_\pm(-u) = -\varphi_\mp(u). \tag{33}$$

Thus we may consider φ_+ only. Furthermore we know that the only poles of $\varphi_\pm$ in the fundamental domain $[-K;K]\times[-iK';iK']$ are at $\pm u_\infty$ i.e. $z\to\infty$; they appear because of the inhomogeneous terms $R_\pm$, which have such poles. We can therefore express $\varphi_+(u)$ in terms of θ functions. Define Θ to be the rescaled θ_1 function, or explicitly

$$\Theta(u) = 2\sum_{n=0}^{\infty} e^{-(n+1/2)^2\pi K'/K}\sin(2n+1)\frac{\pi u}{2K} \tag{34}$$

which satisfies

$$\Theta(u+2K) = -\Theta(u)$$
$$\Theta(u+2iK') = -e^{\pi(K'-iu)/K}\Theta(u)$$

and with a unique simple zero at $u=0 \pmod{2K, 2iK'}$.

Then we have

Proposition 20. *Write $q=e^{i\pi\nu}$ with ν real if $\delta<2$ and purely imaginary if $\delta>2$. Then,*

$$\varphi_+(u) = c_+\frac{\Theta(u-u_\infty-2\nu K)}{\Theta(u-u_\infty)} + c_-\frac{\Theta(u+u_\infty-2\nu K)}{\Theta(u+u_\infty)} \tag{35}$$

where

$$c_\pm = \mp z_{-1}\frac{\Theta'(0)}{\Theta(2\nu K)}\frac{1}{q-1/q}(\alpha^{-1}+q^{\pm 1}\beta^{-1})$$

if $z(u) = z_{-1}/(u-u_\infty) + O(1)$ as u goes to u_∞.

Proof. This can be viewed as a consequence of the Riemann–Roch theorem, but we give here an elementary proof.

If $\delta > 2$, let us first rule out the possibility that $q^2 = e^{-2\pi nK'/K}$ for some integer number n. Indeed then $e^{-ni\pi u/K}\varphi_+(u)$ is also elliptic and therefore the sum of its residues vanishes. But the latter is given by $e^{-ni\pi u_\infty/K}(\alpha^{-1}+q\beta^{-1}) + e^{ni\pi u_\infty/K}(\alpha^{-1}+q^{-1}\beta^{-1})$ which can not vanish. Thus, $q^2\neq e^{-2\pi nK'/K}$ and therefore $\Theta(2\nu K)\neq 0$ (which allows in particular to define $c_\pm$ above).

Next, let $\tilde{\varphi}_+$ denote the right hand side of (35) and observe that by the properties of the functions Θ, $\tilde{\varphi}_+$ satisfies (32) and (31).

The formula for $c_\pm$ is obtained by requiring that $\tilde{\varphi}_+$ have the same residues at $u \sim \pm u_\infty$ as our solution φ_+. Indeed, $\Theta(u) \sim \Theta'(0)u$ as u goes to zero so that (35) shows that

$$\tilde{\varphi}_+(u) \sim_{u \to u_\infty} c_+ \frac{\Theta(2\nu K)}{\Theta'(0)(u - u_\infty)} + O(1)$$

whereas both ω_+ and ω_- go to zero and

$$R_+(u) \sim \frac{1}{1-q^2}(\alpha^{-1}q + \beta^{-1}q^2)z(u) \approx \frac{1}{z_{-1}(1-q^2)(u-u_\infty)}.$$

The formula $\varphi_+ = q\omega_+ - \omega_- - R_+$ allows to conclude. The same reasoning works as $u \to -u_\infty$ by using $\varphi_+(-u) = -\varphi_-(u)$.

Let us finally show that $f := \varphi_+ - \tilde{\varphi}_+$ must vanish. Indeed f is holomorphic and therefore $g := f'/f$ is holomorphic except where f vanishes, where it has only simple poles, with non-negative residues. But since f satisfies (32) and (31), g is elliptic and therefore the sum of its residues vanishes. Hence, the residues of g vanish, and therefore by Liouville's Theorem, g is constant, resulting with $f(u) = e^{\gamma u}$ for some constant γ. But then (32) implies that $\gamma = i\pi n/K$ and $q^2 = e^{-2\pi n K'/K}$, which we excluded earlier.

□

Finally, we need to fix the parameters a_1, a_2, b_1, b_2.

The first way is to notice that $G_\pm(z)$ is an analytic function in α, β, z in a neighborhood of the origin as, by Remark 18, it is the Stieltjes function of the limiting spectral measure of a matrix model with strictly log-concave density which, even though not polynomial, expands as a power series, see [15]. The coefficients of these series can be computed recursively by the Schwinger–Dyson equation. Finally, by (17), the boundary of the support $[a_1, a_2]$ are determined by $g_+(a_i) = 0, i = 1, 2$ which shows that there is a polynomial P so that $P(a_i, G_+(a_i), \alpha, \beta) = 0$. The implicit function theorem then implies that a_i is an analytic function of α, β for $i = 1, 2$ whose expansion can be deduced from the expansion of G_+. The same applies for b_1, b_2.

The second way to determine these boundary points uses the explicit formula in terms of θ functions and the reparametrization of the problem in terms of $p := \exp(-\pi K'/K)$ and of

$$\kappa := p e^{-2i\pi u_\infty/K}$$

Note that because (a_1, a_2, b_1, b_2) expand analytically in α, β, so do (K, K', u_∞) with $u_\infty = \sum_{n+m \geq -1} u_{n,m}\beta^n\alpha^m$, $K = u(b_1) = \sum_{n+m\geq -1} K_{n,m}\beta^n\alpha^m$ and $K' = -iu(a_1) = \sum_{n+m\geq 0} K'_{n,m}\beta^n\alpha^m$. As a consequence, (p, κ) also expand in terms of (α, β), with $\kappa \sim \sqrt{\alpha/\beta}$ and $p \sim \sqrt{\alpha\beta}$ when α, β are small but α/β of order one. Again by the implicit function theorem, we can invert this expansion and obtain

α, β as a power series in (p, κ), and therefore also (a_1, a_2, b_1, b_2), z_{-1} and $z(u)$. We can then identify the expansion of $(q - q^{-1})\omega_+$ and $\varphi_+ + R_+ - \varphi_- R_-$ around u_∞ to compute ω_+ recursively. In appendix A, we provide the first few orders of the power series expansion of some quantities as a function of p and κ, and their diagrammatic meaning.

Proposition 21. *If $\delta = 2\cos(\pi/n)$, $n \geq 3$, $G_\pm$ satisfy an algebraic equation.*

Proof. Observe that since $q = e^{i\pi/n}$, by equations (31,32), $\varphi_\pm$ are elliptic with periods $(2K, 2niK')$, and therefore so are $\omega_\pm$. The function $z(u)$ is also elliptic with these periods. But a fundamental theorem of elliptic functions [29, section 20.54] states that two elliptic functions with the same periods are related by an algebraic equation: there exist two polynomials $P_\pm$ so that

$$P_\pm(\omega_\pm(u), z(u)) = 0\,.$$

Composing with $u(z)$ shows the existence of an algebraic relation.

We can in fact determine the degree of $P_\pm$ by a slightly more explicit construction of these polynomials; we find it is at most $2n - 2$ in z and $2n$ in ω. □

This is to be compared with Theorem 15 of [3]. Indeed, their generating series $M(q, \nu, t, w, z; x, y)$ is closely related to our $G_\pm(z; \alpha, \beta; \delta)$. The correspondence of parameters goes as follows: $q = \delta^2$, $\nu = 1 + \delta\,\alpha/\beta$; among the three parameters t, w, z, one is redundant and if z is set to 1 one has $t = \beta$, but w is fixed to be $w = 1/\delta$ (w and z are parameters weighing in our language white and black regions between tangles; note that introducing an extra paramater in our matrix model to let w vary is possible and would make no difference in the exact solution presented so far, so that Prop. 21 would still hold). x, y are "boundary" parameters similar to our parameter z, in the sense that they give a weight to a particular edge (or vertex, or face) of the planar map. However they are not exactly the same, and therefore direct identification of our generating series is not possible; only specific identities can be written, such as

$$\frac{1}{\alpha}\int x\, d\tilde{\nu}_+(x) = M(\delta^2, 1 + \delta\,\alpha/\beta, \beta, 1/\delta, 1; 1, 1)$$

where the left hand side is the $1/z^2$ term in the $z \to \infty$ expansion of $G_+(z)$, and the right hand side is the generating series of [3] with certain specializations of its parameters.

Appendix A. Analyticity of tr

Let P be a planar algebra. Let $S_1, \ldots, S_k$ and S be elements of this planar algebra and set for complex parameters $t_1, \ldots, t_k$, and a fugacity $\delta \in \mathbb{C}$,

$$\mathrm{tr}_{t,\delta}(S) = \sum_{n_1,\ldots,n_k=0}^{\infty} \prod_{i=1}^{n_k} \frac{t_i^{n_i}}{n_i!} \sum_{P \in P(n_1,\ldots,n_k,S)} \delta^{\#\text{ loops in } P} \tag{36}$$

where we sum over all admissible planar maps built on $S_1, \ldots, S_k, S$. Then we state that

Lemma 22. *There exists a positive constant B so that for all $t_1, \ldots, t_k, \delta \in \mathbb{C}^{k+1}$ so that $\max_{1\le i\le k} |\delta|^{\frac{1}{2}}|t_i| < B$, $tr_{t,\delta}(S)$ is a well defined absolutely converging series, and therefore $t, \delta \to tr_{t,\delta}(S)$ is analytic on this set.*

Indeed, the number of loops is bounded by the number of elements $S_1, \ldots, S_k, S$ given by $n_1 + \cdots + n_k + 1$ times their maximal number of boundary points divided by two. Therefore, the coefficients of the series are simply bounded by

$$C(n_1, \ldots, n_k) := |\delta|^{\frac{1}{2}} \prod_{i=1}^{k} \frac{(|t_i||\delta|^{\frac{1}{2}})^{n_i}}{n_i!}$$

and the sum can be enlarged to all planar maps that can be built over n_i (resp. one) vertices with degree given by the number of boundary points of S_i, $1 \le i \le k$ (resp. S). It is well known, see e.g. [14, p. 255], that the number of such maps grows as $\prod_{i=1}^{k} n_i! A^{n_1+\cdots+n_k}$ for some finite constant A. Hence, $\mathrm{tr}_{t,\delta}(S)$ is an absolutely converging series on $\max_{1\le i\le k} |t_i||\delta|^{\frac{1}{2}} A < 1$, domain on which it is analytic.

Note that it is expected that as one increases the t_i, one should eventually reach a hypersurface of singularities which signals the boundary of the analyticity region in the variables $t = (t_1, \ldots, t_k)$. This singularity is usually present in matrix models and is explained by the proliferation of planar maps: typically the number of planar maps grows exponentially with its number of vertices and this produces a finite radius of convergence of the corresponding generating series. This is certainly what happens in the cases studied in section 3. The model of 3.1 is closely related to the so-called one-matrix-model, whose possible "critical behaviors" (i.e., types of singularities) are well-known [6]. No exact solution is known for the model of 3.2, but a conjecture on its critical exponent is proposed in [7]. Finally, it is expected that the model of 3.3 has a critical behavior of the type of the $O(n)$ model on the line $A = B$, and that of pure gravity on other lines of constant ratio $A/B \neq 1$. It would be interesting to find an interpretation of these critical behaviors in the present context, i.e. in terms of properties of tr_t.

Appendix B. First few diagrams of the double cup matrix model

The parameters a_1, a_2, b_1, b_2 of the auxiliarly model of sect. 3.4 can, as already mentioned, be fixed by appropriate expansion of $\omega_+(u)$ around $\pm u_\infty$. In practice, the resulting equations reduce to first degree equations on the condition that one solves them parametrically in terms of the elliptic nome $p := \exp(-\pi K'/K)$ and of u_∞. All other quantities can then be obtained in the same parameterization.

As a check, we shall here write the first few orders of the expansion for small α, β. The correct scaling as $\alpha, \beta \to 0$ at fixed ratio is to keep the quantity

$$\kappa := pe^{-2i\pi u_\infty/K}$$

fixed while sending p to zero, so that $\alpha/\beta \sim \kappa^2$ and $\alpha\beta \sim p^2$. We find the following expansions:

$$\begin{aligned}
\alpha =& \kappa\Big[p - (\kappa(3\delta+2) + \kappa^{-1}(2\delta+6))p^2 \\
&+ \big((8\delta^2+5\delta+3)\kappa^2 + (8\delta^2+45\delta+24) + (5\delta^2+12\delta+17)\kappa^{-2}\big)\, p^3 + O(p^4)\Big] \\
\beta =& \kappa^{-1}\Big[p - (\kappa^{-1}(3\delta+2) + \kappa(2\delta+6))p^2 \\
&+ \big((8\delta^2+5\delta+3)\kappa^{-2} + (8\delta^2+45\delta+24) + (5\delta^2+12\delta+17)\kappa^{2}\big)\, p^3 + O(p^4)\Big] \\
a_1 =& \kappa\left[-2p^{1/2} + \delta p + 2((3+\delta)\kappa^{-2} + (1+\delta)\kappa^2)p^{3/2} + O(p^2)\right] \\
a_2 =& \kappa\left[2p^{1/2} + \delta p - 2((3+\delta)\kappa^{-2} + (1+\delta)\kappa^2)p^{3/2} + O(p^2)\right] \\
b_1 =& 1 - \kappa^{-1}\left[2p^{1/2} + \delta p - 2((1+\delta)\kappa^{-2} + (3+\delta)\kappa^2)p^{3/2} + O(p^2)\right] \\
b_2 =& 1 - \kappa^{-1}\left[-2p^{1/2} + \delta p + 2((1+\delta)\kappa^{-2} + (3+\delta)\kappa^2)p^{3/2} + O(p^2)\right] \\
\int x\, d\tilde{\nu}_+(x) =& \kappa\Big[\delta p - \delta(\kappa(2\delta+1) + \kappa^{-1}(\delta+5))p^2 \\
&+ \delta\left(\kappa^2(4\delta^2+1) + 3\delta^2 + 24\delta + 1 + \kappa^{-2}(2\delta^2+4\delta+11)\right) p^3 + O(p^4)\Big] \\
\int x^2\, d\tilde{\nu}_+(x) =& \kappa\Big[p + (\kappa(\delta^2-2\delta-2) - 2\kappa^{-1}(3+\delta))p^2 \\
&+ (\kappa^2(-4\delta^3+3\delta^2+3\delta+3) + (-2\delta^3-4\delta^2+36\delta+24) + \kappa^{-2}(5\delta^2+12\delta+17))p^3 + O(p^4
\end{aligned}$$

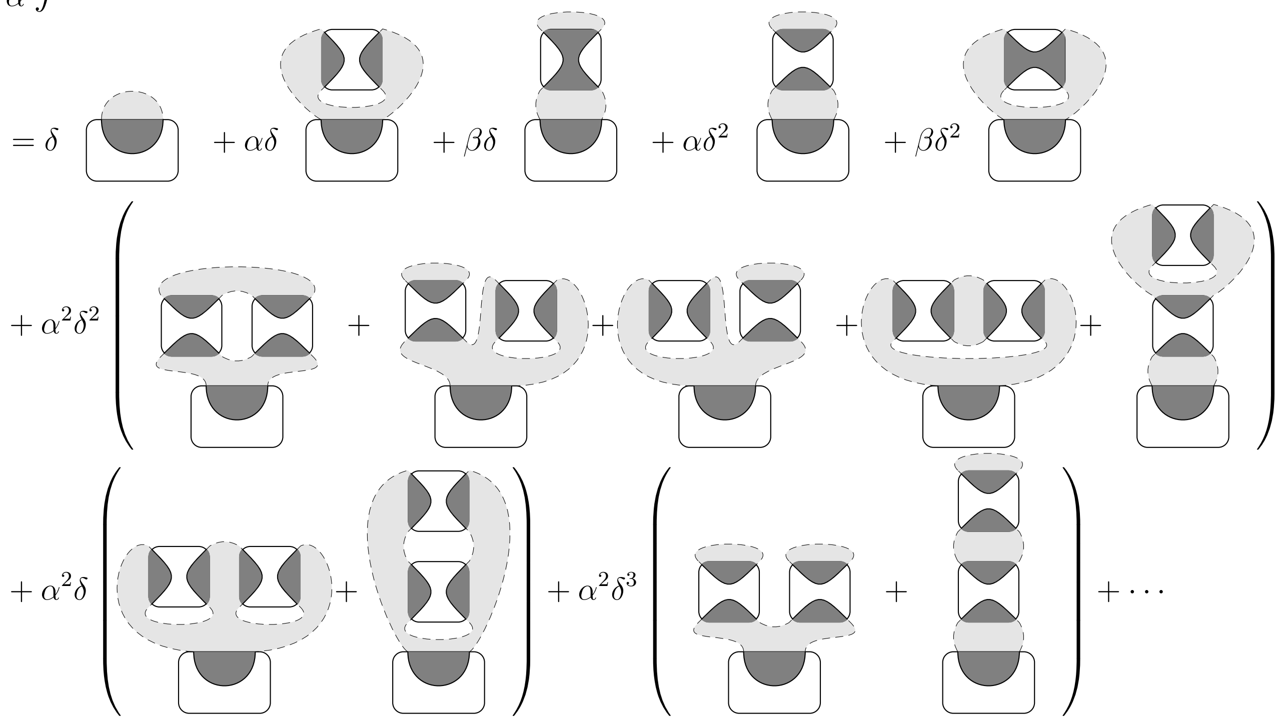
(the expansions of a_1, a_2, b_1, b_2 are only given up to order $p^{3/2}$ because of issues of space).

Inverting the first two expansions and inserting the result in the last two yields

$$\begin{aligned}
\frac{1}{\alpha}\int x\, d\tilde{\nu}_+(x) &= \delta + \delta(1+\delta)(\alpha+\beta) + \delta((2+5\delta+2\delta^2)\alpha^2 + (6+8\delta+4\delta^2)\alpha\beta + (2+5\delta+2\delta^2)\beta^2) + \\
\frac{1}{\alpha^2}\Big(\int x^2\, d\tilde{\nu}_+(x) - \alpha\Big) &= \delta(1+\delta) + \delta(2+5\delta+2\delta^2)\alpha + \delta(3+4\delta+2\delta^2)\beta + \cdots
\end{aligned}$$

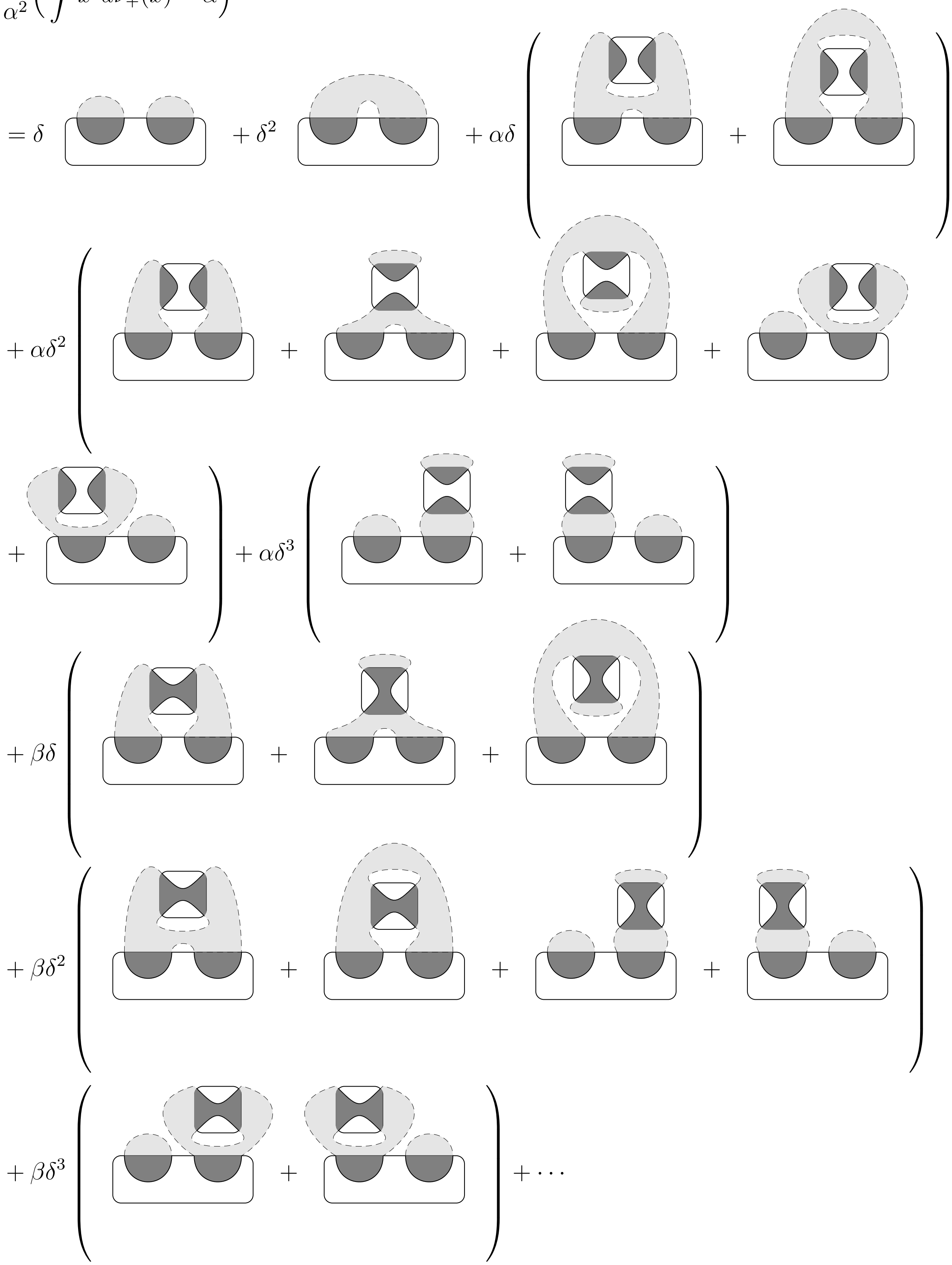
According to Prop. 16, $\int x\, d\tilde{\nu}_+(x)/\alpha$ corresponds diagramatically to:

$$\frac{1}{\alpha}\int x\, d\tilde{\nu}_+(x)$$



Similarly, we have:

$$\frac{1}{\alpha^2}\Big(\int x^2 d\tilde{\nu}_+(x) - \alpha\Big)$$



Appendix C. The stitching of two planar algebras.

Let $\mathcal{P} = P_n^{\pm}$ and $\mathcal{Q} = Q_n^{\pm}$ be two subfactor planar algebras (assumed spherical for simplicity). We define a new subfactor planar algebra

$$\mathcal{P} \circledcirc \mathcal{Q} = (P \circledcirc Q)_n^{\pm}$$

which will not be irreducible even if both $\mathcal{P}$ and $\mathcal{Q}$ are.

The vector spaces of $\mathcal{P} \circledcirc \mathcal{Q}$ are defined as follows. Fix the even number $2n$ and consider the possible partititions π of $\{1,2,3,...,2n\}$ into two subsets of even sizes $2(p_\pi)$ and $2(q_\pi)$, whose elements we call of type $\mathcal{P}$ and $\mathcal{Q}$ respectively. Then

$$(P \circledcirc Q)_n^{\pm} = \bigoplus_\pi (P \circledcirc Q)_\pi^{\pm} \qquad \text{with } (P \circledcirc Q)_\pi^{\pm} = P_{p_\pi}^{\pm} \otimes Q_{q_\pi}^{\pm}.$$

To define the action of planar tangles on $\mathcal{P} \circledcirc \mathcal{Q}$ we use the notion of a $s$tring-coloring. Given a planar tangle T with discs D_i as in section 2.1, a string-colouring σ of T is an assignment of $\mathcal{P}$ or $\mathcal{Q}$ to every string of T so that if one removes all the strings of either color, one gets two planar tangles $T_{\mathcal{P}}^{\sigma}$ and $T_{\mathcal{Q}}^{\sigma}$ when one takes as initial segments, with shading, the intervals containing the initial segments of T. In particular there must be an even number of strings of each color incident to every disc of T, but this is not a sufficient condition for the shadings to be coherent.

A string-coloring σ of T defines:

(a) a partition π_σ of the boundary points for each disc (numbered from 1 to $2b_j$ starting at the first boundary point after the initial segment in clockwise order)into $\mathcal{P}$ and $\mathcal{Q}$ points, and
(b) two planar tangles $T_{\mathcal{P}}^{\sigma}$ and $T_{\mathcal{Q}}^{\sigma}$ by removing all the strings of the other color and taking as initial segments for discs the ones containing the initial segments of T. The shadings of $T_{\mathcal{P}}^{\sigma}$ and $T_{\mathcal{Q}}^{\sigma}$ are determined by that of the initial segment of the outside boundary of T.

By multilinearity it suffices to define the action Z_T of a planar tangle T, with k internal discs, on a $k-$tuple $(x_1, x_2, ..., x_k)$ of elements of $\mathcal{P} \circledcirc \mathcal{Q}$ where $x_i = \sum_\pi v_i^\pi \otimes w_i^\pi$ with $v_i \in P_{p_\pi}^{\pm}$ and $w_i Q_{q_\pi}^{\pm}$.

Suppose such an element x_i of $(P \circledcirc Q)_{b_i}^{\pm}$ is assigned to each D_i, then we define

$$M_T(x_1, x_2, ..., x_k) = \sum_\sigma M_{T_{\mathcal{P}}^{\sigma}}(v_1^{\pi_\sigma}, v_2^{\pi_\sigma}, ..., v_k^{\pi_\sigma}) \otimes M_{T_{\mathcal{Q}}^{\sigma}}(w_i^{\pi_\sigma}, w_2^{\pi_\sigma}, ..., w_k^{\pi_\sigma})$$

where σ runs over all the string-colorings of T.

It is clear that this action is compatible with the gluings.

The $*$-structure on $\mathcal{P} \circledcirc \mathcal{Q}$ is derived in the obvious way from those of $\mathcal{P}$ and $\mathcal{Q}$.

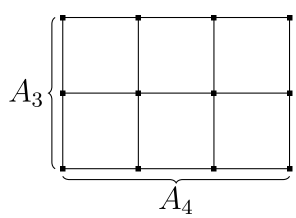
Notes. (i) Another way to describe the action is as follows: suppose elements $v_i \otimes w_i \in (P \circledcirc Q)^{\pm}_{\pi_i}$ are assigned to the internal discs D_i of T. Then the value of M_T is zero unless the colouring of the boundary points implied by the π_i extends to a string colouring of T and then this value is the sum over all such extensions σ of $M_{T^{\sigma}_{\mathcal{P}}}(v_1, v_2, ...v_k) \otimes M_{T^{\sigma}_{\mathcal{Q}}}(w_1, w_2, ..., w_k)$. If there are strings connecting the outside boundary of T to itself this element will lie in more than one direct summand of $(P \circledcirc Q)^{\pm}_n$.

(ii) It is clear from (i) that the loop parameter of $\mathcal{P} \circledcirc \mathcal{Q}$ is the *sum* of the loop parameters of $\mathcal{P}$ and $\mathcal{Q}$ respectively.

(iii) Positive definiteness of the inner product is also clear from (i)-the various $(P \circledcirc Q)_\pi$ are orthogonal and in each one the inner product is just the tensor product inner product for $P^{\pm}_{p_\pi} \otimes Q^{\pm}_{q_\pi}$.

(iv) It is the exponential generating functions for $\mathcal{P}$ and $\mathcal{Q}$ that behaves well under this operation.

(v) For the inductive limit algebra structure of $\mathcal{P} \circledcirc \mathcal{Q}$ a complete set of centrally orthogonal minimal projections is given by the tensor products of such sets of projections for $\mathcal{P}$ and $\mathcal{Q}$. Thus the vertices of the principal graph of $\mathcal{P} \circledcirc \mathcal{Q}$ is the Cartesian product of the vertices of the principal graphs of $\mathcal{P}$ and $\mathcal{Q}$ and (p, q) is adjacent to (p', q') iff p is adjacent to p' or q is adjacent to q' but not both. For instance if $\mathcal{P}$ has principal graph A_3 and $\mathcal{Q}$ has principal graph A_4, then that of $\mathcal{P} \circledcirc \mathcal{Q}$ is:



(vi) We see that in the loop basis description off $\mathcal{P} \circledcirc \mathcal{Q}$, from each vertex of a loop one may choose to travel on the principal graph of $\mathcal{P}$ or that of $\mathcal{Q}$.

(vii) If we were dealing with unshaded planar algebras we would simply remove the restrictions on the parity of the numbers of boundary points and the partitions π.

(viii) A TL basis diagram in $\mathcal{P} \circledcirc \mathcal{Q}$ consists of a sum over all *planar* partitions of the boundary points into $\mathcal{P}$ points and $\mathcal{Q}$ points with that basis diagram regarded as a tensor product of its $\mathcal{P}$ part and its $\mathcal{Q}$ part.

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