First order asymptotics of matrix integrals ; a rigorous approach towards the understanding of matrix models

ALICE GUIONNET*

30/10/2002

Abstract : We investigate the limit behaviour of the spectral measures of matrices following the Gibbs measure for the Ising model on random graphs, Potts model on random graphs, matrices coupled in a chain model or induced QCD model. For most of these models, we prove that the spectral measures converge almost surely and describe their limit via solutions to an Euler equation for isentropic flow with negative pressure $p(\rho) = -3^{-1}\pi^2\rho^3$.

Keywords : Large deviations, random matrices, analysis of first-order non linear pde.

Mathematics Subject of Classification : 60F10, 15A52, 35F20.

1 Introduction

It appears since the work of 't Hooft that matrix integrals can be seen, via Feynman diagrams expansion, as generating functions for enumerating maps (or triangulated surfaces). We refer here to the very nice survey of A. Zvonkin's [30]. One matrix integrals are used to enumerate maps with a given genus and given vertices degrees distribution whereas several matrices integrals can be used to consider the case where the vertices can additionally be coloured (i.e can take different states).

Matrix integrals are usually of the following form

$$Z_N(P) = \int e^{-N\mathrm{tr}(P(A_1^N,\cdots,A_d^N))} dA_1^N \cdots dA_d^N$$

with some polynomial function P of d-non-commutative variables and the Lebesgue measure dA on some well chosen ensemble of $N \times N$ matrices such as the set $\mathcal{H}_N$ (resp. $\mathcal{S}_N$, resp. $\mathcal{S}ymp_N$) of $N \times N$ Hermitian (resp. symmetric, resp. symplectic) matrices. One would like to understand the full expansion of $Z_N(P)$ in powers of N. For instance, in the case where the matrices live on $\mathcal{H}_N$, the formal expansion linked with Feynamn diagrams is of the type

$$\frac{1}{N^2} \log Z_N(P) = \sum_{g \geq 0} \frac{1}{N^{2g}} C_P(g)$$

*Ecole Normale Superieure de Lyon, Unite de Mathematiques pures et appliquees, UMR 5669, 46 Allee d'Italie, 69364 Lyon Cedex 07, France

where $C_P(g)$ enumerates some maps with genus g. Such an expansion was proved to hold rigorously in the one matrix case by K. McLaughlin and N. Ercolani in 2002.

A related issue is to understand the asymptotic behaviour of the corresponding Gibbs measure

$$\mu_P^N(dA_1^N \cdots dA_d^N) = \frac{1}{Z_N(P)} e^{-N\mathrm{tr}(P(A_1^N,\cdots,A_d^N))} dA_1^N \cdots dA_d^N.$$

More precisely, if for a $N \times N$ matrix A, $(\lambda_1(A), \cdots, \lambda_N(A))$ denotes its eigenvalues and $\hat{\mu}_A^N := N^{-1} \sum_{i=1}^N \delta_{\lambda_i(A)}$ its spectral measure, one would like to understand the asymptotic behaviour of $(\hat{\mu}_{A_1^N}^N, \cdots, \hat{\mu}_{A_d^N}^N)$ under the Gibbs measure μ_P^N when N goes to infinity. Of course, this understanding is intimately related with the first order asymptotic of the free energy $F_N(P) = N^{-2} \log Z_N(P)$. In fact, the rigorous approach of the full expansion of matrix integrals when $d = 1$ given by K. McLaughlin and N. Ercolani is based on Riemann Hilbert problems techniques which themselves require a precise understanding of such asymptotics of the spectral measures.

However, only very few matrix integrals could be evaluated in the physics litterature, even on a non rigorous ground. These cases corresponds in general to the case where integration holds over Hermitian matrices. Using orthogonal polynomial methods, Mehta [21] obtained the limiting free energy for the Ising model on random graphs, corresponding to $d = 1$ and $P(A, B) = P(A) + Q(B) - AB$ when $P(x) = Q(x) = gx^4 + x^2$. He extended this work [9, 17] with coauthors to matrices coupled in a chain, model corresponding to $P(A_1, \cdots, A_d) = \sum_{i=1}^d P_i(A_i) - \sum_{i=2}^d A_{i-1}A_i$. However, he did not discuss in these works the limiting spectral distribution of the matrices under the corresponding Gibbs measure. On a less rigorous ground, P. Zinn Justin [28, 29] discussed the limiting spectral measures of the matrices following the Gibbs measure of the so-called Potts model on random graphs, described by $P(A_1, \cdots, A_d) = \sum_{i=1}^d P_i(A_i) - \sum_{i=2}^d A_1 A_i$. Very interesting work was also achieved by V. Kazakov (in particular for the so-called $ABAB$ interaction case), A. Migdal and B. Eynard for instance. We refer to the review [13] of B. Eynard for a general survey. Matytsin [18] obtained the first order asymptotics for spherical integrals, from which he could study the phase transition of diverse matrix models (see [19] for instance). O. Zeitouni and myself [15] gave a complete proof of part of his derivation in [15] and the present paper is actually finishing to put his article [18] on a firm ground.

In this paper, we investigate the problem of the first order asymptotics of matrix integrals with AB interaction, including the above Ising model, Potts model, matrix model coupled in a chain and induced QCD models. The integration will hold over either Hermitian matrices or symmetric matrices. The case of symplectic matrices could be handle similarly. We obtain, as a consequence of [15], the convergence of the free energy and represent its limit as the solution of a variational problem. We here study this variational problem and characterize its critical points. One of the main outcome of this study is to show that under the Gibbs measure μ_P^N of the Ising model described by

$$P(A, B) = P(A) + Q(B) - AB$$

with $P(x) \geq ax^4 + b$ and $Q(x) \geq cx^4 + d$ with $a, b > 0$, the spectral measures of (A_1^N, A_2^N) converges almost surely and to characterize its limit. More precisely, we shall prove that

Theorem 1.1 *1) $(\hat{\mu}_A^N, \hat{\mu}_B^N)$ converges almost surely towards a unique couple (μ_A, μ_B) of probability measures on $\mathbb{R}$.*

2) (μ_A, μ_B) are compactly supported with finite non-commutative entropy

$$\Sigma(\mu) = \int\int \log|x-y| d\mu(x) d\mu(y).$$

3) There exists a couple $(\rho^{A\to B}, u^{A\to B})$ of measurable functions on $\mathbb{R}\times(0,1)$ such that $\rho_t^{A\to B}(x)dx$ is a probability measure on $\mathbb{R}$ for all $t\in(0,1)$ and $(\mu_A, \mu_B, \rho^{A\to B}, u^{A\to B})$ are characterized uniquely as the minimizer of a strictly convex function under a linear constraint (see Theorem 3.3).

In particular, $(\rho^{A\to B}, u^{A\to B})$ are solution of the Euler equation for isentropic flow with negative pressure $p(\rho) = -\frac{\pi^2}{3}\rho^3$ such that, for all (x,t) in the interior of $\Omega = \{(x,t)\in\mathbb{R}\times[0,1]; \rho_t^{A\to B}(x)\neq 0\}$,

$$\begin{cases} \partial_t \rho_t^{A\to B} + \partial_x(\rho_t^{A\to B} u_t^{A\to B}) = 0 \\ \partial_t(\rho_t^{A\to B} u_t^{A\to B}) + \partial_x(\rho_t^{A\to B}(u_t^{A\to B})^2 - \frac{\pi^2}{3}(\rho_t^{A\to B})^3) = 0 \end{cases} \tag{1.1}$$

with the probability measure $\rho_t^{A\to B}(x)dx$ weakly converging towards $\mu_A(dx)$ (resp. $\mu_B(dx)$) as t goes to zero (resp. one).
Moreover, we have

$$P'(x) - x - \frac{\beta}{2}u_0^{A\to B}(x) - \frac{\beta}{2}H\mu_A(x) = 0 \quad \mu_A - a.s \quad \text{and } Q'(x) - x + \frac{\beta}{2}u_1^{A\to B}(x) - \frac{\beta}{2}H\mu_B(x) = 0 \quad \mu_B - a.s.$$

A more detailed characterization of $(\mu_A, \mu_B, \rho^{A\to B}, u^{A\to B})$ is given in Theorem 3.3.

Here, $H\mu$ stands for the Hilbert transform of the probability measure μ given by

$$H\mu(x) = PV\int \frac{1}{x-y} d\mu(y) = \lim_{\epsilon\downarrow 0}\int \frac{(x-y)}{(x-y)^2+\epsilon^2} d\mu(y)$$

To obtain such a result, we shall first study the limit obtained in [15] for spherical integrals. This limit was indeed given by the infimum of a rate function over measure-valued processes with given initial and terminal data. We show in section 2 that this infimum is in fact taken at a unique probability measure-valued path, solution of the Euler equation for isentropic flow described in (1.1). Using a saddle point method, we derive from [15] in Theorem 3.1 formulae for the limiting free energy of some matrix models with AB interaction. In the Ising model case, this free energy is indeed written as the infimum of a strictly convex function, from which uniqueness of the minimizers is obtained. As a consequence, we obtain the convergence of the spectral measures under the Gibbs measure for Ising model. A variational study then shows that the limiting spectral measures satisfies the above set of equations (see Theorem 3.3) . For the other considered models (q-Potts model, matrix coupled in a chain, induced QCD), obvious convexity arguments and therefore uniqueness is lost in general, but still holds in certain cases. However, we can still specify some properties of the limit points (see Theorem 3.4).

In this paper, we shall denote $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ the set of continuous processes with values in the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$, endowed with its usual weak topology. For a measurable set Ω of $\mathbb{R}\times[0,1]$, $\mathcal{C}_b^{2,1}(\Omega)$ denotes the set of real-valued functions on Ω which are p times continuously differentiable with respect to the (first) space variable and q times continuously differentiable with respect to the (second) time variable with bounded derivatives. $\mathcal{C}_c^{p,q}(\Omega)$ will denote the functions of $\mathcal{C}_b^{p,q}(\Omega)$ with compact support in the interior of the measurable set Ω. $L_p(d\mu)$ will denote the space of measurable functions with finite p^{th} moment under a given measure μ. We shall say that an equality holds in the sense of distribution on a measurable set Ω if it holds, once integrated with respect to any $\mathcal{C}_c^{\infty,\infty}(\Omega)$ functions.

2 Study of the rate function governing the asymptotic behaviour of spherical integrals

In [15], Ofer Zeitouni and I studied the so-called spherical integral

$$I_N^{(\beta)}(D_N, E_N) := \int \exp\{N\mathrm{tr}(UD_NU^*E_N)\}dm_N^\beta(U),$$

where m_N^β denotes the Haar measure on the orthogonal group $\mathcal{O}_N$ when $\beta = 1$ and on the unitary group $\mathcal{U}_N$ when $\beta = 2$, and D_N, E_N are diagonal real matrices whose spectral measures converge to μ_D, μ_E. We proved (see Theorem 1.1 in [15]) the existence and represent as solution to a variational problem the limit

$$I^{(\beta)}(\mu_D, \mu_E) := \lim_{N\to\infty} N^{-2} \log I_N^{(\beta)}(D_N, E_N).$$

This result in fact was obtained under the additionnal technical assumptions that there exists a compact subset $\mathcal{K}$ of $\mathbb{R}$ such that supp $\hat{\mu}_{D_N}^N \subset \mathcal{K}$ for all $N \in \mathbb{N}$ and that $\hat{\mu}_{E_N}^N(x^2)$ is uniformly bounded (in N). These hypotheses will be made throughout this section.

In this section, we investigate the variational problem which defines $I^{(\beta)}$ and study its minimizer. We indeed prove Matytsin's heuristics [18] outlined in section 6 of [15]. Let us recall the formula obtained in [15] for $I^{(\beta)}$:

$$I^{(\beta)}(\mu_D, \mu_E) := -J_\beta(\mu_D, \mu_E) + I_\beta(\mu_E) - \inf_{\mu\in M_1(\mathbb{R})} I_\beta(\mu) + \frac{1}{2}\int x^2 d\mu_D(x)$$

where, for any $\mu \in M_1(\mathbb{R})$,

$$I_\beta(\mu) = \frac{1}{2}\int x^2 d\mu(x) - \frac{\beta}{2}\Sigma(\mu).$$

$J_\beta(\mu_D, .)$ is the rate function governing the deviations of the law of the spectral measure of $X_N = D_N + W_N$ with a Hermitian (resp. symmetric) Gaussian Wigner matrix W_N and a deterministic diagonal matrix $D_N = \mathrm{diag}(d_1, \cdots, d_N)$, $(d_i)_{1\le i\le N} \in \mathbb{R}^N$, with spectral measure $\hat{\mu}_{D_N}^N = N^{-1}\sum_{i=1}^N \delta_{d_i}$ weakly converging towards $\mu_D \in \mathcal{P}(\mathbb{R})$. It is given (see [15]) by

$$J_\beta(\mu_D, \mu) = \frac{\beta}{2}\inf\{S_{\mu_D}(\nu_.); \nu \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R})) : \nu_1 = \mu\}. \tag{2.1}$$

if

$$S_{\mu_D}(\nu) := \begin{cases} +\infty, & \text{if } \nu_0 \neq \mu_D, \\ S^{0,1}(\nu) := \sup_{f\in\mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])} \sup_{0\le s\le t\le 1} \bar{S}^{s,t}(\nu, f), & \text{otherwise.} \end{cases}$$

Here, we have set, for any $f, g \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$, any $s \le t \in [0,1]$, and any $\nu_. \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$,

$$\begin{aligned} S^{s,t}(\nu, f) &= \int f(x,t) d\nu_t(x) - \int f(x,s) d\nu_s(x) \\ &\quad - \int_s^t \int \partial_u f(x,u) d\nu_u(x) du - \frac{1}{2} \int_s^t \int \int \frac{\partial_x f(x,u) - \partial_x f(y,u)}{x-y} d\nu_u(x) d\nu_u(y) du, \end{aligned} \quad (2.2)$$

$$< f, g >_{s,t}^{\nu} = \int_s^t \int \partial_x f(x,u) \partial_x g(x,u) d\nu_u(x) du \,, \quad (2.3)$$

and

$$\bar{S}^{s,t}(\nu, f) = S^{s,t}(\nu, f) - \frac{1}{2} < f, f >_{s,t}^{\nu} \,. \quad (2.4)$$

It can be shown by Riesz's theorem (see such a derivation in [7] for instance) that any measure-valued path $\nu_. \in \mathcal{C}([0,1], M_1(\mathbb{R}))$ in $\{S_{\mu_D} < \infty\}$ is such that there exists a process $k_.$ so that

1. $$\inf_{f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])} < f - k, f - k >_{0,1}^{\nu} = 0$$

2. $\nu_0 = \mu_D$ and for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$, any $0 \le s \le t \le 1$,

$$S^{s,t}(\nu, f) =< f, k >_{s,t}^{\nu} \,. \quad (2.5)$$

Then, it is not hard to show that

$$S_{\mu_D}(\nu_.) = \frac{1}{2} < k, k >_{0,1}^{\nu} \,.$$

Therefore, $J_\beta(\mu_D, \mu)$ is given also by

$$J_\beta(\mu_D, \mu) = \frac{\beta}{4} \inf\{< k, k >_{0,1}^{\nu}; \quad (\nu, k) \text{ satisfies } (C)\}. \quad (2.6)$$

with (C) the condition

(C) : $\nu_0 = \mu_D$, $\nu_1 = \mu$, $\partial_x k \in \overline{\mathcal{C}^{1,1}(\mathbb{R} \times [0,1])}^{L^2(d\nu_t dt)}$, *and, for any* $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$, *any* $s, t \in [0,1]$,

$$S^{s,t}(\nu, f) =< f, k >_{s,t}^{\nu}$$

The main Theorem of this section states as follows

Theorem 2.1 *Let $\mu_E \in \{J_\beta(\mu_D, .) < \infty\}$ with finite entropy Σ. Then, the infimum in $J_\beta(\mu_D, \mu_E)$ is reached at a unique probability measures-valued path $\mu^* \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ such that*

- $\mu_0^* = \mu_D$, $\mu_1^* = \mu_E$.
- *For any $t \in (0,1)$, μ_t^* is absolutely continuous with respect to Lebesgue measure ; $\mu_t^*(dx) = \rho_t^*(x) dx$. $t \in [0,1] \to \mu_t^* \in \mathcal{P}(\mathbb{R})$ is continuous and therefore $\lim_{t \downarrow 0} \mu_t^* = \mu_D$, $\lim_{t \uparrow 1} \mu_t^* = \mu_E$.*

- *Let k^* be such that the couple (μ^*, k^*) satisfies (C). Then, if we set*

$$u_t^* = \partial_x k_t^* + H\mu_t^*(y),$$

(ρ^*, u^*) satisfies the Euler equation for isentropic flow described by the equations, for $t \in (0,1)$,

$$\begin{aligned} \partial_t \rho_t^*(x) &= -\partial_x(\rho_t^*(x) u_t^*(x)) & (2.7) \\ \partial_t(\rho_t^*(x) u_t^*(x)) &= -\partial_x(\rho_t^*(x) u_t^*(x)^2 - \frac{\pi^2}{3}\rho_t^*(x)^3) & (2.8) \end{aligned}$$

in the sense of distributions that for all $f \in \mathcal{C}_c^{\infty,\infty}(I\!R \times [0,1])$,

$$\int_0^1 \int \partial_t f(t,x) d\mu_t^*(x) dt + \int_0^1 \int \partial_x f(t,x) u_t^*(x) d\mu_t^*(x) dt = 0$$

and, for any $\epsilon > 0$, any $f \in \mathcal{C}_c^{\infty,\infty}(\Omega_\epsilon)$ with $\Omega_\epsilon := \{(x,t) \in I\!R \times [0,1] : \rho_t^*(x) > \epsilon\}$,

$$\int \left(2u_t^*(x)\partial_t f(x,t) + \left(u_t^*(x)^2 - \pi^2 \rho_t^*(x)^2\right) \partial_x f(x,t)\right) dxdt = 0. \qquad (2.9)$$

If we assume that (μ_D, μ_E) are compactly supported probability measures, we additionnally know that (ρ^*, u^*) are smooth in the interior of Ω_0, which guarantees that (2.7) and (2.8) hold everywhere in the interior of Ω_0. Moreover, Ω_0 is bounded in $I\!R \times [0,1]$. Furthermore, there exists a sequence $(\phi^\epsilon)_{\epsilon > 0}$ of functions such that if we set

$$\rho_t^\epsilon(x) := \pi^{-1}(\max\{\partial_t \phi^\epsilon + 4^{-1}(\partial_x \phi^\epsilon)^2, 0\})^{\frac{1}{2}}$$

then

$$\int (u_t^*(x) - \partial_x \frac{\phi_t^\epsilon(x)}{2})^2 d\mu_t^*(x) dt + \frac{\pi^2}{3} \int \left(\rho_t^*(x) - \rho_t^\epsilon(x)\right)^2 \left(\rho_t^*(x) + \rho_t^\epsilon(x)\right) dxdt$$
$$+\pi^2 \int |\partial_t \phi^\epsilon + 4^{-1}(\partial_x \phi^\epsilon)^2 - \pi^2 \rho_t^\epsilon(x)^2| d\mu_t^*(x) dt \leq \epsilon.$$

As a consequence, if we let $\Pi_t^*(x) = \int^x u_t^*(y)dy$, which should be thought as the limit in $H_1(\rho_t^*(x)dxdt)$ of the sequence $(2^{-1}\phi^\epsilon)_{\epsilon>0}$, we find that it satisfies, in the sense of distributions in Ω_0,

$$\partial_t \Pi_t^* = -\frac{1}{2}(\partial_x \Pi_t^*)^2 + \frac{\pi^2}{2}(\rho_t^*)^2,$$

which is Matytsin 's equation [18].

The (non trivial) existence of solutions to the Euler equation for isentropic flow (2.7), (2.8), is a consequence of our variational study. The uniqueness of the solutions to these equations could be derived, under some additional regularity properties, from a convexity property of our rate function S_{μ_D}. Even when such solutions are not unique, we know that our minimizer is unique due to a convex property of S_{μ_D} which is a consequence of its representation of Property 2.2.1) below (see Property 2.5).

Property 2.2 *Let $\mu_E \in \{J_\beta(\mu_D,.) < \infty\}$ having finite entropy Σ. Then,*

1) For any $\nu \in \mathcal{C}([0,1],\mathcal{P}(I\!R))$, if (ν,k) verifies (C) and $u_t(x) = \partial_x k_t(x) + H\nu_t(x)$,

$$S_{\mu_0}(\nu) = \frac{1}{2}\int_0^1 \int (u_t(x))^2 d\nu_t(x)dt + \frac{1}{2}\int_0^1 \int (H\nu_t(x))^2 d\nu_t(x)dt - \frac{1}{2}(\Sigma(\mu_1) - \Sigma(\mu_0)).$$

2) Consequently, we can write J_β under the following form

$$J_\beta(\mu_D,\mu_E) = \frac{\beta}{4}\left\{\int_0^1 \int (u_t^*(x))^2 d\mu_t^*(x)dt + \int_0^1 \int (H\mu_t^*(x))^2 d\mu_t^*(x)dt - (\Sigma(\mu_E) - \Sigma(\mu_D))\right\} \quad (2.10)$$

with (μ^*,u^*) as in Theorem 2.1. Note here that $\mu_t^*(dx) = \rho_t^*(x)dx$ for $t \in (0,1)$ and $\rho_.^* \in L_3(dxdt)$, so that

$$\int_0^1 \int (H\mu_t^*(x))^2 d\mu_t^* dt = \frac{\pi^2}{3}\int_0^1 \int (\rho_t^*(x))^3 dxdt.$$

3) As a consequence,

$$\begin{aligned} I^{(\beta)}(\mu_D,\mu_E) &= -\frac{\beta}{4}\left\{\int_0^1 \int (u_t^*(x))^2 d\mu_t^*(x)dt + \int_0^1 \int (H\mu_t^*(x))^2 d\mu_t^*(x)dt\right\} \\ &\quad -\frac{\beta}{4}(\Sigma(\mu_E) + \Sigma(\mu_D)) + \frac{1}{2}\int x^2 d\mu_D(x) + \frac{1}{2}\int x^2 d\mu_E(x) - \inf I_\beta. \end{aligned} \quad (2.11)$$

In [18], a similar result was announced (see formulae (1.4) and (2.8) of [18]). However, it seems (as far as I could understand) that in formulae (2.10,2.11) of [18], the first term as the opposite sign. But, in [19], formula (2.18), the very same result is stated.

Let us also notice that the minimizer μ_t^* has the following representation in the free probability context. Let $(\mathcal{A},\tau)$ be a non-commutative probability space on which an operator D with distribution μ_D, an operator E with distribution μ_E and a semi-circular variable S, free with (D,E), live. Then, there exists a joint distribution of (D,E) such that $(\mu_t^*)_{t\in[0,1]}$ is the law of a free Brownian bridge

$$X_t = tE + (1-t)D + \sqrt{t(1-t)}S.$$

The isentropic Euler equation which governs μ^* hence partially specify the joint law of (D,E). More specifically, for any $t \in (0,1)$, our result implies that for any $p \in I\!N$,

$$\lim_{N\to\infty} \int (tUE_NU^* + (1-t)D_N)^p \frac{e^{N\mathrm{tr}(UE_NU^*D_N)}}{I_N^{(\beta)}(D_N,E_N)} dm_N^\beta(U) = \tau((tE + (1-t)D)^p) = \int x^p d\nu_t^*(x)$$

if ν_t^* is the unique compactly supported probability measure such that, if S is a semicircular variable free with (D,E), for any $p \in I\!N$,

$$\int x^p d\mu_t^*(x) = \tau\left((tE + (1-t)D + \sqrt{t(1-t)}S)^p\right) = \int x^p d\nu_t^* \boxed{+} \sigma_{t(1-t)}(x)$$

where $\boxed{+}$ denotes the free convolution and σ_δ the semicircular variable with covariance δ.

2.1 Study of S_{μ_0}

Hereafter and to simplify the notations, $\mu_D = \mu_0$ and $\mu_E = \mu_1$ with some probability measures (μ_0, μ_1) on $\mathbb{R}$. We shall in this section study the rate function S_{μ_0} and show that it achieves its minimal value on $\{\nu \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R})) : \nu_1 = \mu_1\}$ at a unique continuous measure-valued path μ^*.

2.1.1 S_{μ_0} achieves its minimal value

Recall that for any probability measure $\mu_0 \in \mathcal{P}(\mathbb{R})$, S_{μ_0} is a good rate function on $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ (see Theorem 2.4(1) of [15]). Therefore, the infimum defining $J_\beta(\mu_0, \mu_1)$ is, when it is finite, achieved in $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$. We shall in the sequel restrict ourselves to (μ_0, μ_1) such that $J_\beta(\mu_0, \mu_1)$ is finite.

2.1.2 A new formula for S_{μ_0}

In this section, we shall give a simple formula of $S_{\mu_0}(\nu)$ in terms of $u_. = H\nu_. + \partial_x k_.$ and ν when (k, ν) satisfies (C). We begin with the following preliminary Lemma

Lemma 2.3 *Let $(\mu_0, \mu_1) \in \{\mu \in \mathcal{P}(\mathbb{R}) : \Sigma(\mu) > -\infty\}$ and $\nu_. \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ such that $\nu_0 = \mu_0,\ \nu_1 = \mu_1$ and $\nu_. \in \{S_{\mu_0} < \infty\}$. Then, for almost all $t \in (0,1)$, $\nu_t(dx) \ll dx$ and*

$$\int_0^1 \int (H\nu_t(x))^2 \, d\nu_t(x) dt = \frac{\pi^2}{3} \int_0^1 \int \left(\frac{d\nu_t(x)}{dx} \right)^3 dx dt < \infty.$$

The idea of the proof of the lemma is quite simple ; we make, in the definition of S_{μ_0}, the change of variable $f(x,t) \to f(x,t) - \int \log|x-y| d\nu_t(x)$. However, because $(x,t) \to \int \log|x-y| d\nu_t(x)$ is not in $\mathcal{C}^{2,1}(\mathbb{R} \times [0,1])$ in general, the full proof requires approximations of the path $\nu_.$ and becomes rather technical. This is the reason why I defer it to the appendix, section 4.2. We shall now prove the following

Property 2.4 *Let $(\mu_0, \mu_1) \in \{\mu \in \mathcal{P}(\mathbb{R}) : \Sigma(\mu) > -\infty\}$ and $\nu_. \in \mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ such that $\nu_0 = \mu_0,\ \nu_1 = \mu_1$ and $\nu_. \in \{S_{\mu_0} < \infty\}$. Then, if (ν, k) satisfies (C) and if we set $u_t := \partial_x k_t(x) + H\nu_t(x)$, we have*

$$S_{\mu_0}(\nu) = \frac{1}{2} \int_0^1 \int (u_t(x))^2 d\nu_t(x) dt + \frac{1}{2} \int_0^1 \int (H\nu_t(x))^2 d\nu_t(x) dt - \frac{1}{2} (\Sigma(\mu_1) - \Sigma(\mu_0)).$$

Proof.

Let us recall that (ν, k) satisfying condition (C) implies that for any $f \in \mathcal{C}_b^{2,1}(\mathbb{R} \times [0,1])$,

$$\begin{aligned}
\int f(x,t) d\nu_t(x) - \int f(x,s) d\nu_s(x) &= \int_s^t \int \partial_v f(x,v) d\nu_v(x) ds \\
&\quad + \frac{1}{2} \int_s^t \int \int \frac{\partial_x f(x,v) - \partial_x f(y,v)}{x-y} d\nu_v(x) d\nu_v(y) dv \\
&\quad + \int_s^t \int \partial_x f(x,s) \partial_x k(x,s) d\nu_s(x) \qquad (2.12)
\end{aligned}$$

with $\partial_x k \in L^2(d\nu_t(x) \times dt)$. Observe that by [24], p. 170, for any $s \in [0,1]$ such that ν_s is absolutely continuous with respect to Lebesgue measure with density $\rho_s \in L_3(dx)$, for any compactly supported measurable function $\partial_x f(.,s)$,

$$\int\int \frac{\partial_x f(x,s)-\partial_x f(y,s)}{x-y} d\nu_s(x) d\nu_s(y) = 2\int \partial_x f(x,s) H\nu_s(x) dx ds.$$

Since by Lemma 2.3, for almost all $s \in [0,1]$, $\nu_s(dx) \ll dx$ with a density $\rho_s \in L_3(dx)$ we conclude that, in the sense of distributions on $\mathbb{R}\times[0,1]$, (2.12) implies

$$\partial_s \rho_s + \partial_x(u_s\rho_s) = 0, \tag{2.13}$$

i.e for any compactly supported $f \in \mathcal{C}_c^{\infty,\infty}(\mathbb{R}\times[0,1])$ vanishing at the boundary of $\mathbb{R}\times[0,1]$,

$$\int_0^1 \int_{\mathbb{R}} (\partial_s f(x,s) + u_s \partial_x f(x,s))\rho_s(x) dx = 0.$$

Note here that, by dominated convergence theorem, we can equivalently take $f \in \mathcal{C}_b^{2,1}(\mathbb{R}\times[0,1])$.

Moreover, since $H\nu_.$ belongs to $L^2(d\nu_s \times ds)$ by Lemma 2.3, we can write

$$\begin{aligned} 2S_{\mu_0}(\nu_.) = < k, k >_{0,1}^{\nu} \quad &= \int_0^1 \int_{\mathbb{R}} (u_s(x))^2 d\nu_s(x) ds + \int_0^1 \int_{\mathbb{R}} (H\nu_s(x))^2 d\nu_s(x) ds \\ &-2 \int_0^1 \int_{\mathbb{R}} H\nu_s(x) u_s(x) d\nu_s(x) ds \end{aligned} \tag{2.14}$$

We shall now see that the last term in the above right hand side only depends on (μ_0,μ_1). The only difficulty in the proof of this point lies in the fact that $x,s \in \mathbb{R}\times[0,1] \to \int \log|x-y| d\nu_s(y)$ is not in $\mathcal{C}_c^{2,1}(\mathbb{R}\times[0,1])$.

However, following Lemma 5.16 in [8], if $\boxed{+}$ denotes the free convolution (see [26] for a definition), if for any $\delta>0$, σ_δ denotes the semicircular law with covariance δ, and if u_t^δ denotes the field corresponding to $\nu_t \boxed{+} \sigma_\delta$,

$$\Sigma(\nu_1 \boxed{+} \sigma_\delta) - \Sigma(\nu_0 \boxed{+} \sigma_\delta) = 2\int_0^1 \int_{\mathbb{R}} H(\nu_s \boxed{+} \sigma_\delta)(x) u_s^\delta(x) d\nu_s \boxed{+} \sigma_\delta(x) ds. \tag{2.15}$$

It is well known that $\delta \to \Sigma(\nu \boxed{+} \sigma_\delta)$ is continuous (see [23], Theorem 2.1 for the lower semicontinuity and use the well known upper semi-continuity). Moreover, if X_s is a random variable with distribution ν_s and S a semicircular variable, free with X_s, living in a non commutative probability space $(\mathcal{A},\tau)$, by Theorem 4.2 in [8], the field u^δ is given, $\nu_s \boxed{+} \sigma_\delta$ almost surely, by

$$u_s^\delta = \tau(\partial_x k(X_s,s) | X_s + \sqrt{\delta} S) + H\nu_s \boxed{+} \sigma_\delta.$$

Consequently,

$$\int_{\mathbb{R}} H(\nu_s \boxed{+} \sigma_\delta)(x) u_s^\delta(x) d\nu_s \boxed{+} \sigma_\delta(x) = \tau\left(\partial_x k(X_s,s) H(\nu_s \boxed{+} \sigma_\delta)(X_s + \sqrt{\delta}S)\right) + \tau\left(H(\nu_s \boxed{+} \sigma_\delta)(X_s + \sqrt{\delta}S)^2\right).$$

Moreover, by Voiculescu [25], Proposition 3.5 and Corollary 6.13, if $\nu_s(dx) = \rho_s(x)dx \in L_3(dx)$,

$$\lim_{\delta\to 0} \tau\left((H(\nu_s \boxed{+} \sigma_\delta)(X_s + \sqrt{\delta}S) - H\nu_s(X_s))^2 \right) = 0.$$

Therefore, for any such $s \in [0,1]$,

$$\lim_{\delta\to 0}\int_{I\!R} H(\nu_s \boxplus \sigma_\delta)(x)u_s^\delta(x)d\nu_s \boxplus \sigma_\delta(x) = \int_{I\!R} H\nu_s(x)u_s(x)d\nu_s(x). \tag{2.16}$$

Note that by Lemma 2.3, this convergence holds for almost all $s \in [0,1]$ since $\rho_. \in L_3(dxdt)$. Finally, by Propositions 3.5 and 3.7 of [25], for any s such that $H\nu_s$ is well defined,

$$H\nu_s \boxplus \sigma_\delta(X_s + \sqrt{\delta}S) = \tau(H\nu_s(X_s)|X_s + \sqrt{\delta}S)$$

so that for any $\delta > 0$

$$\tau\left((H\nu_s \boxplus \sigma_\delta(X_s + \sqrt{\delta}S))^2\right) \le \tau\left((H\nu_s(X_s))^2\right).$$

Therefore, dominated convergence theorem and (2.16) imply that

$$\lim_{\delta\to 0}\int_0^1\int_{I\!R} H\nu_s \boxplus \sigma_\delta(x)u_s^\delta(x)d\nu_s \boxplus \sigma_\delta(x)ds = \int_0^1\int_{I\!R} H\nu_s(x)u_s(x)d\nu_s(x)ds.$$

Thus, (2.15) extends to $\delta = 0$ which proves, with (2.14), Property 2.4.

∎

2.1.3 Uniqueness of the minimizers of S_{μ_0}

We shall use the formula for S_{μ_0} obtained in the last section to prove that

Property 2.5 *For any $(\mu_0, \mu_1) \in \mathcal{P}(I\!R)$ with finite entropy Σ, there exists a unique measures-valued path μ^* such that*

$$J_\beta(\mu_0,\mu_1) = \frac{\beta}{2}\inf\{S_{\mu_0}(\nu_.) \quad : \quad \nu_1 = \mu_1\} = \frac{\beta}{2}S_{\mu_0}(\mu^*).$$

In the following, μ^* shall always denote the minimizer of Property 2.5 and $\partial_x k^*, u^*$ its associated fields.
Proof. According to the previous section, the minimizers of S_{μ_0} also minimize

$$S(u,\rho) = \frac{\pi^2}{3}\int_0^1\int_{I\!R}(\rho_t(x))^3 dxdt + \int_0^1\int_{I\!R}(u_t(x))^2\rho_t(x)dxdt$$

under the constraint $\partial_t\rho_t + \partial_x(\rho_t u_t) = 0$ in the sense of distributions, $\rho_t \ge 0$ almost surely w.r.t Lebesgue measure and $\int \rho_t(x)dx = 1$, and with given initial and terminal data for ρ given by

$$\lim_{t\downarrow 0}\rho_t(x)dx = \mu_0(dx), \quad \lim_{t\uparrow 1}\rho_t(x)dx = \mu_1(dx)$$

where convergence holds in the weak sense (with respect to bounded continuous functions) and is simply due to the fact that S_{μ_0} is finite only on continuous measure-valued paths.

Let $m = u\rho$ be the corresponding momentum. In the variables (m,ρ), $S(\rho,m)$ reads

$$\mathcal{S}(m,\rho) = \frac{\pi^2}{3}\int_0^1\int_{I\!R}(\rho_t(x))^3 dxdt + \int_0^1\int_{I\!R}\frac{(m_t(x))^2}{\rho_t(x)}dxdt$$

with the convention $\frac{0}{0} = 0$, whereas the constraint becomes **linear**

$$\partial_t(\rho_t(x)) + \partial_x(m_t(x)) = 0, \quad \rho_t(x)dx \in \mathcal{P}(I\!R) \quad \forall t \in [0,1], \quad \lim_{t\downarrow 0}\rho_t(x)dx = \mu_0(dx), \quad \lim_{t\uparrow 1}\rho_t(x)dx = \mu_1(dx).$$

We now observe that $\mathcal{S}$ is a **strictly convex** function. Indeed, if (m^1, ρ^1) and (m^2, ρ^2) are any two couples of measurable functions in $\{\mathcal{S} < \infty\}$, it is easy to see that for any $\alpha \in (0,1)$

$$\begin{aligned}\partial_\alpha^2 \mathcal{S}(\alpha m^1 + (1-\alpha)m^2, \alpha\rho^1 + (1-\alpha)\rho^2) &= 2\pi^2 \int_0^1 \int_{I\!R} (\rho_t^1(x) - \rho_t^2(x))^2(\alpha\rho_t^1(x) + (1-\alpha)\rho_t^2(x))dxdt \\ &\quad + 2\int_0^1 \int_{I\!R} \frac{(\rho_t^1(x)m_t^2(x) - \rho_t^2(x)m_t^1(x))^2}{(\alpha\rho_t^1(x) + (1-\alpha)\rho_t^2(x))^3} dxdt.\end{aligned}$$

Hence, $\partial_\alpha^2 \mathcal{S}(\alpha m^1 + (1-\alpha)m^2, \alpha\rho^1 + (1-\alpha)\rho^2) > 0$ for some $\alpha \in (0,1)$ unless for almost all $t \in [0,1]$

$$\rho_t^1(x) = \rho_t^2(x) = \rho_t(x), \text{ and } u_t^1(x) = \frac{m_t^1(x)}{\rho_t^1(x)} = \frac{m_t^2(x)}{\rho_t^2(x)} = u_t^2(x) \quad \rho_t(x)dxdt \text{ a.s.}$$

In other words, $\mathcal{S}$ is strictly convex. By standard convex analysis, the strict convexity of $\mathcal{S}$ results with the uniqueness of its minimizers given a linear constraint, and in particular in J_β. More precisely, from the above, the minimizer μ^* in J_β is defined uniquely for almost all $t \in [0,1]$ (and then everywhere by continuity of μ^*) and its field u^*, or equivalently $\partial_x k^*$, is then defined uniquely $\mu_t^*(dx)dt$ almost surely.

∎

2.2 A priori properties of the minimizer μ^*

In this section, we shall see that the minimizer μ^* has to be the distribution of a free Brownian bridge when at least one of the probability measure μ_0 or μ_1 are compactly supported, the other having finite variance (since we rely on [15]'s results). To simplify the statements, we shall assume throughout this section that both probability measures are compactly supported. This property will unable us to obtain a priori properties on the laws of the minimizers, such as existence, boundedness, and smoothness of their densities.

2.2.1 Free Brownian bridge characterization of the minimizer

Let us state more precisely the theorem obtained in this section. A free Brownian bridge between μ_0 and μ_1 is the law of

$$X_t = (1-t)X_0 + tX_1 + \sqrt{t(1-t)}S \tag{2.17}$$

with a semicircular variable S, free with X_0 and X_1, with law μ_0 and μ_1 respectively. We let $\text{FBB}(\mu_0, \mu_1) \subset \mathcal{C}([0,1], \mathcal{P}(I\!R))$ denote the set of such laws (which depend of course not only on μ_0, μ_1 but on the joint distribution of (X_0, X_1) too). Then, we shall prove that

Theorem 2.6 *Assume μ_0, μ_1 compactly supported. Then,*

$$\begin{aligned}J_\beta(\mu_0, \mu_1) &= \frac{\beta}{2}\inf\{S(\nu), \nu_0 = \mu_0, \nu_1 = \mu_1\} \\ &= \frac{\beta}{2}\inf\{S(\nu) \quad ; \quad \nu \in FBB(\mu_0, \mu_1)\}.\end{aligned}$$

Therefore, since $FBB(\mu_0,\mu_1)$ is a closed subset of $\mathcal{C}([0,1],\mathcal{P}(I\!R))$, the unique minimizer μ^* in the above infimum belongs to $FBB(\mu_0,\mu_1)$.

The proof of Theorem 2.6 is rather technical and goes back through the large random matrices origin of J_β. We therefore defer it to the appendix.

2.2.2 Properties of the free Brownian motion paths

As a consequence of Theorem 2.6, we shall prove that

Corollary 2.7 *Assume μ_0 and μ_1 compactly supported. Then,*

a) There exists a compact set $K \subset I\!R$ so that for all $t\in[0,1]$, $\mu_t^*(K^c)=0$. For all $t\in(0,1)$, the support of μ_t^* is the closure of its interior.

b) $\mu_t^*(dx) \ll dx$ for all $t\in[0,1]$. Let $\rho_t^*(x)=\frac{d\mu_t^*(x)}{dx}$.

c) There exists a finite constant C (independent of t) so that, μ_t^* almost surely,

$$\rho_t^*(x)^2 + (H\mu_t^*(x))^2 \le (t(1-t))^{-1}$$

and

$$|u_t^*(x)| \le C(t(1-t))^{-\frac{1}{2}}.$$

d) (ρ^*,u^*) are analytic in the interior of $\Omega=\{x,t\in I\!R\times[0,1]:\rho_t^*(x)>0\}$.

e) At the boundary of $\Omega_t=\{x\in I\!R:\rho_t^*(x)>0\}$, for $x\in\Omega_t$,

$$|\rho_t^*(x)^2\partial_x\rho_t^*(x)| \le \frac{1}{4\pi^3t^2(1-t)^2} \quad\Rightarrow\quad \rho_t^*(x) \le \left(\frac{3}{4\pi^3t^2(1-t)^2}\right)^{\frac{1}{3}}(x-x_0)^{\frac{1}{3}}$$

if x_0 is the nearest point of x in Ω_t^c.

Consequently, the minimizer μ^* may only have shocks at the boundary of its support.

Proof. This corollary is a direct consequence of Theorem 2.6 and we shall collect these properties for any free brownian bridge law. Indeed, let $(\mathcal{A},\tau)$ be a non-commutative probability space in which two operators X_0,X_1 with laws μ_0 and μ_1 and a semicircular variable S, free with (X_0,X_1), live. We assume throughout that X_0 and X_1 are bounded by C for the operator norm (i.e $\mu_0([-C,C]^c)=\mu_1([-C,C]^c)=0$).

Let μ_t be the distribution of

$$X_t = tX_1 + (1-t)X_0 + \sqrt{t(1-t)}S.$$

Clearly, since S is bounded by 2 for the operator norm, X_t is bounded by $C+2$ for all $t\in[0,1]$. Thus, proposition 4 in [3] finishes the proof of a). Following Voiculescu (see Proposition 3.5 and Corollary 3.9 in [25]), the Hilbert transform of μ_t is given, μ_t-almost surely, by

$$H\mu_t(x) = \tau((2\sqrt{t(1-t)})^{-1}S|X_t)$$

with $\tau(\quad|X_t)$ the conditionnal expectation with respect to X_t, i.e the orthogonal projection on the sigma algebra generated by X_t. We deduce that since S is bounded for the operator norm by 2, μ_t-almost surely,

$$|H\mu_t(x)| \le \frac{1}{\sqrt{t(1-t)}}.$$

Further, following [4], the stochastic differential equation satisfied by X_t shows that, for any twice continuously differentiable function f on $\mathbb{R}$,

$$\mu_t(f) = \mu_0(f) + \frac{1}{2}\int_0^t \int\int \frac{\partial_x f(x) - \partial_x f(y)}{x-y} d\mu_s(x) d\mu_s(y) ds + \int_0^t \int \partial_x f(x) \partial_x k_s(x) d\mu_s(x) ds \tag{2.18}$$

with k the element of $L^2(d\mu_s ds)$ given by

$$\partial_x k_s(x) = \tau(\frac{X_s - X_1}{s-1}|X_s).$$

Hence,

$$u_t := \partial_x k_t + H\mu_t = \tau(\frac{X_t - X_1}{t-1}|X_t) + \tau(\sqrt{4t(1-t)}^{-1} S|X_t) = \tau(X_1 - X_0 + \frac{(1-2t)}{2\sqrt{t(1-t)}} S|X_t). \tag{2.19}$$

Therefore, μ_t-almost surely,

$$|u_t| \le 2C + \frac{1}{\sqrt{t(1-t)}}. \tag{2.20}$$

Moreover, by Biane's results [3], we know that, for $t \in (0,1)$, μ_t is absolutely continuous with respect to Lebesgue measure. We denote by ρ_t its density. Then, we also know that for all $t \in (0,1)$, μ_t-almost surely,

$$\rho_t(x)^2 + (H\mu_t)^2(x) \le \frac{1}{t(1-t)}. \tag{2.21}$$

Let us mention the regularity properties that $(\mu_t)_{t\in(0,1)}$ will inherite from its free Brownian bridge formula. If ν_t denotes the law of $tX_1 + (1-t)X_0$, we have, following Biane [3], corollary 3, that if we set

$$\begin{aligned} v(u,t) &= \inf\{v \ge 0 | \int \frac{d\nu_t(x)}{(u-x)^2 + v^2} \le (t(1-t))^{-1}\}, \\ &= \inf\{v \ge 0 | \tau\left(((tX_1 + (1-t)X_0 - u)^2 + v^2)^{-1}\right) \le (t(1-t))^{-1}\}, \end{aligned}$$

$$\psi(u,t) = u + t(1-t)\int \frac{(u-x)d\nu_t(x)}{(u-x)^2 + v(u,t)^2},$$

then

$$H\mu_t(\psi(u,t)) = \int \frac{(u-x)d\nu_t(x)}{(u-x)^2 + v(u,t)^2},$$

while

$$\rho_t(\psi(u,t)) = \frac{v(u,t)}{\pi t(1-t)}.$$

From these formulae, we observe that ψ^{-1} is analytic in the interior of Ω since ψ' is bounded below by a positive constant there (see Biane [3], p 713 and the obvious analyticity in the time parameter $t \in (0,1)$), it is clear that ρ is $\mathcal{C}^\infty$ in Ω. Hence, the weak equation (2.18) is verified in the strong sense in Ω and we find that in Ω, $u_t(x) = \rho_t(x)^{-1}\int_x^\infty \partial_t \rho_t(y) dy$ is $\mathcal{C}^\infty$.

At the boundary of $\Omega_t = \{x : (x,t) \in \Omega\}$, Biane ([3], corollary 5) also noticed that

$$|\rho_t(x)^2 \partial_x \partial_t(x)| \le \frac{1}{4\pi^3 t^2 (1-t)^2} \Rightarrow \rho_t(x) \le \left(\frac{3}{4\pi^3 t^2(1-t)^2}\right)^{\frac{1}{3}} (x - x_0)^{\frac{1}{3}}$$

with x_0 the nearest point of the boundary of Ω_t from x.

2.3 The variational problem

We now turn to the analysis of the variational problem defining J_β ; we shall prove that

Property 2.8 *Assume that μ_0 and μ_1 are probability measures on $I\!R$ such that $\Sigma(\mu_0)$ and $\Sigma(\mu_1)$ are finite. Then, the path $\mu^* \in \mathcal{C}([0,1],\mathcal{P}(I\!R))$ minimizing $J_\beta(\mu_0,\mu_1)$ satisfies ;*

1)$\mu^*_0 = \mu_0$ and $\mu^*_1 = \mu_1$.

2) For any $t \in (0,1)$, $\mu^*_t(dx) \ll dx$. Let $(\rho^*_t)_{t\in(0,1)}$ denote the corresponding density. By continuity of μ^*, $\mu^*_t(dx) = \rho^*_t(x)dx$ converges towards μ_0 (resp. μ_1) as t goes to zero (resp. one) in the usual weak sense on $\mathcal{P}(I\!R)$.

3) μ^* is characterized as the unique continuous measure-valued path such that $\mu^*_0 = \mu_0$ and $\mu^*_1 = \mu_1$ and, for any $\nu \in \{S_{\mu_0} < \infty\}$ so that (ν, k) satisfies (C) and $\nu_1 = \mu_1$, we have, with $u = \partial_x k + H\nu$,

$$\int[\int 2u^*_t(u_t d\nu_t - u^*_t d\mu^*_t) - \int (u^*_t)^2(d\nu_t - d\mu^*_t) + \pi^2 \int (\rho^*_t)^2(d\nu_t - d\mu^*_t)]dt \geq 0 \tag{2.22}$$

4)As a consequence, (ρ^*, u^*) satisfies the Euler equation for isentropic flow described by the equations, for $t \in (0,1)$,

$$\begin{aligned} \partial_t \rho^*_t(x) &= -\partial_x(\rho^*_t(x)u^*_t(x)) & (2.23)\\ \partial_t(\rho^*_t(x)u^*_t(x)) &= -\partial_x(\rho^*_t(x)u^*_t(x)^2 - \frac{\pi^2}{3}\rho^*_t(x)^3) & (2.24) \end{aligned}$$

in the sense of distributions that for all $f \in \mathcal{C}^{\infty,\infty}_c(I\!R \times [0,1])$,

$$\int_0^1 \int \partial_t f(t,x) d\mu^*_t(x)dt + \int_0^1 \int \partial_x f(t,x) u^*_t(x) d\mu^*_t(x)dt = 0$$

and, for any $\epsilon > 0$, any $f \in \mathcal{C}^{\infty,\infty}_c(\Omega_\epsilon)$ with $\Omega_\epsilon := \{(x,t) \in I\!R \times [0,1] : \rho^*_t(x) > \epsilon\}$,

$$\int \left(2u^*_t(x)\partial_t f(x,t) + \left(u^*_t(x)^2 - \pi^2\rho^*_t(x)^2\right)\partial_x f(x,t)\right) dxdt = 0. \tag{2.25}$$

Let us now assume that (μ_0,μ_1) are compactly supported. Then

5) (2.24) is true everywhere in the interior of Ω_0. Moreover, (2.25) can be improved by the statement that

$$\int_0^1 \int_{I\!R} u^*_t(x)(\partial_t f(t,x) + u^*_t(x)\partial_x f(t,x)) d\mu^*_t(x)dt = \frac{\pi^2}{3}\int_0^1 \int_{I\!R} (\rho^*_t(x))^3 \partial_x f(t,x) dxdt$$

for all $f \in \mathcal{C}^{1,1}_b(I\!R \times (0,1))$.

6) There exists a sequence $(\phi^\epsilon)_{\epsilon>0}$ of functions such that if we set

$$\rho^\epsilon_t(x) := \pi^{-1}(\max\{\partial_t \phi^\epsilon + 4^{-1}(\partial_x \phi^\epsilon)^2, 0\})^{\frac{1}{2}}$$

then

$$\int (u_t^*(x) - \partial_x \frac{\phi_t^\epsilon(x)}{2})^2 d\mu_t^*(x)dt + \frac{\pi^2}{3} \int \left(\rho_t^*(x) - \rho_t^\epsilon(x)\right)^2 \left(\rho_t^*(x) + \rho_t^\epsilon(x)\right) dxdt$$
$$+\pi^2 \int |\partial_t \phi^\epsilon + 4^{-1}(\partial_x \phi^\epsilon)^2 - \pi^2 \rho_t^\epsilon(x)^2| d\mu_t^*(x)dt \leq \epsilon.$$

Discussion 2.9 *Matytsin [18] noticed that if we set*

$$f(x,t) = u_t^*(x) + i\pi\rho_t^*(x),$$

then the Euler equation for isentropic flow implies that f the Burgers equation. Hence, if one assumes that f can be smoothly extended to the complex plan, we find by usual characteristic methods that for $z \in \mathbb{C}$

$$f(f(z,0)t + z, t) = f(z,0)$$

and therefore, setting $G_+(z) = z + f(z,0)$ and $G_-(z) = z - f(z,1)$, we see that our problem boils down to solve

$$G_+ \circ G_-(z) = G_- \circ G_+(z) = z$$

with $\Im(G_+)(x) = \pi\rho_0(x)$ and $\Im(G_-)(x) = -\pi\rho_1(x)$ if ρ_0 and ρ_1 are the densities of μ_0, μ_1 respectively. This kind of characterization is in fact reminiscent to the description of minimizers provided by P. Zinn Justin [29]. However, such a result would require more smoothness of (ρ^*, u^*) than what we proved here.

Proof of Property 2.8 : By property 2.4, we want to minimize

$$S(\rho, u) := \int_0^1 \int (u_t(x))^2 \rho_t(x) dxdt + \frac{\pi^2}{3} \int_0^1 \int (\rho_t(x))^3 dxdt$$

under the constraint (C') :

$$\partial_t \rho_t + \partial_x(u_t\rho_t) = 0, \quad \lim_{t\downarrow 0} \rho_t(x)dx = \mu_0, \quad \lim_{t\uparrow 1} \rho_t(x)dx = \mu_1$$

and when $\rho_t(x)dx \in \mathcal{P}(\mathbb{R})$ for all $t \in [0,1]$. To study the variational problem associated with this energy, I know essentially three ways. The first is to make a perturbation with respect to the source. This strategy was followed by D. Serre in [22] but applies only when we know a priori that $(\rho^*, u^*\rho^*)$ are uniformly bounded. Since this case corresponds to the case where μ_0, μ_1 are compactly supported, we shall consider it in the second part of the proof. One can also use a target type perturbation, which is a standard perturbation on the space of probability measure, viewed as a subspace of the vector space of measures. This method gives (3) in Property 2.8 as we shall see. The last way is to use convex analysis, following for instance Y. Brenier (see [5], section 3.2). We shall also detail these arguments, since it provides some approximation property of the field u^*, as described in Property 2.8.6).

We begin with the target type perturbation. In the following, we denote (ρ^*, u^*) the minimizer of S under the constraint (C'). Let $(\rho, u) \in \{S < \infty\}$ satisfying the constraint (C'). Then, for any $\alpha \in [0,1]$, we set, with $m = \rho u$ and $m^* = \rho^* u^*$,

$$\rho^\alpha = (1-\alpha)\rho^* + \alpha\rho, \quad m^\alpha := (1-\alpha)(\rho^* u^*) + \alpha(\rho u) := \rho^\alpha u^\alpha, \quad u^\alpha = (m^\alpha/\rho^\alpha).$$

It is then not hard to check that $S(\rho^\alpha, u^\alpha) < \infty$ for all $\alpha \in [0,1]$. Moreover, by the convexity of $\phi : \alpha \to (\rho_t^\alpha(x))^{-1}(m_t^\alpha(x))^2 + 3^{-1}\pi^2(\rho_t^\alpha(x))^3$ for all admissible $(\rho, m), (\rho', m')$, we see that $\alpha^{-1}(\phi(\alpha) - \phi(0))$ decreases as $\alpha \to 0$ showing, by monotone convergence theorem the existence of $\partial_\alpha S(\rho^\alpha, u^\alpha)(0^+)$ and

$$\begin{aligned}\partial_\alpha S(\rho^\alpha, u^\alpha)(0^+) &= \int [-(u^*)^2\rho^* - (u^*)^2\rho + 2mu^* + \pi^2(\rho^*)^2(\rho - \rho^*)]dxdt \\ &= \int [2u^*(m - m^*) - (u^*)^2(\rho - \rho^*) + \pi^2(\rho^*)^2(\rho - \rho^*)]dxdt.\end{aligned}$$

Hence, for any $(\rho, u) \in \{S < \infty\}$, we have

$$\partial_\alpha S(\rho^\alpha, u^\alpha)(0^+) = \int [2u^*(m - m^*) - (u^*)^2(\rho - \rho^*) + \pi^2(\rho^*)^2(\rho - \rho^*)]dxdt \geq 0 \tag{2.26}$$

Reciprocally, since S is convex in (ρ, m), we know that

$$S(\rho^\alpha, u^\alpha) \geq S(\rho^*, u^*) + \partial_\alpha S(\rho^\alpha, u^\alpha)(0^+)\alpha$$

so that (2.26) implies that $S(\rho^\alpha, u^\alpha) \geq S(\rho^*, u^*)$ for all $\alpha \in [0,1]$ and $(\rho, u) \in \{S < \infty\}$. Hence, (2.26) characterizes our unique minimizer, which proves Property 2.8.3). We can apply this result with

$$\rho = \rho^* + \epsilon\partial_x\phi, \quad m = m^* - \epsilon\partial_t\phi$$

for some $\phi \in \mathcal{C}_c^{1,1}(\Omega_\epsilon)$, $\epsilon > 0$, such that $\partial_x\phi(.,0) = \partial_x\phi(.,1) = 0$, insuring that $S(\rho, u)$ has finite entropy. This yields the second point of Property 2.8.3). Conditions at the boundary of the support can also be deduced from (2.26), but they are hardly understandable, since the conditions over the potentials ϕ become more stringent.

To prove the last points of our property which concerns the case where (μ_0, μ_1) are compactly supported, we follow D. Serre [22] and Y. Brenier [5].

The idea developped in [22] is basically to set $a_t(x) = a(t,x) = (\rho_t^*(x), \rho_t^*(x)u_t^*(x))$ so that $\mathrm{div}(a_t(x)) = 0$ and perturbe a by considering a family

$$a^g = J_g(a.\nabla_{x,t}h) \circ g = J_g(\rho^*(\partial_t h + u^*\partial_x h)) \circ g$$

with a $\mathcal{C}^\infty$ diffeomorphism g of $Q = [0,1] \times I\!R$ with inverse $h = g^{-1}$ and Jacobian J_g. Such an approach yields the Euler's equation (2.25) of Property 2.8 (use the boundedness of (ρ^*, u^*) obtained in Corollary 2.7 to apply theorem 2.2 of [22]). Moreover, since we saw in Corollary 2.7.d) that ρ^* and u^* are smooth in the interior of Ω_0, (2.24) results with Property 2.8.5).

We now developp convex analysis for our problem following [5]. By Corollary 2.7.a), we see that there exists a compact K such that $\mu_t^*(K^c) = 0$ for all $t \in [0,1]$. We set $Q = K \times [0,1]$ and $E = \mathcal{C}_b(Q) \times \mathcal{C}_b(Q) \times \mathcal{C}_b(Q)$.

For any continuous functions $(F, G, H) \in E$, we set

$$\alpha(F, G, H) = \frac{2}{3\pi}\int_Q |H(x,t)|^{\frac{3}{2}} dxdt$$

if $H \geq 0$ and $F + (\frac{G}{2})^2 \leq 0$, on Q, and $+\infty$ otherwise. For any $(\mu, M, \widetilde{\mu}) \in E'$, let us consider

$$\alpha^*(\mu, M, \widetilde{\mu}) = \sup\{\int_Q F(x,t)\mu(dx,dt) + \int_Q G(x,t)M(dx,dt) + \int_Q H(x,t)\widetilde{\mu}(dx,dt) - \alpha(F,G,H)\}.$$

It is not hard to see that $\alpha^*(\mu, M, \widetilde{\mu}) < \infty$ iff μ is non negative, M is absolutely continuous w.r.t μ and $\widetilde{\mu}$ is absolutely continuous w.r.t Lebesgue measure with density in $L_3(dxdt)$. Moreover, if we denote $\widetilde{\mu}(dx,dt) = \widetilde{\rho}_t(x)dxdt$, $M(dx,dt) = u_t(x)d\mu(x,t)$, it is not hard to see that $\alpha^*(\mu, M, \widetilde{\mu}) = \int u^2(x,t)d\mu(x,t) + \frac{\pi^2}{3}\int \widetilde{\rho}_t(x)^3 dxdt$. Now, let

$$\beta(F,G,H) = \int_Q F(x,t)\rho_t^*(x)dxdt + \int_Q G(x,t)u_t^*(x)\rho_t^*(x)dxdt + \int_Q H(x,t)\rho_t^*(x)dxdt$$

if there exists $\phi \in \mathcal{C}_b^{1,1}(Q)$ such that

$$F(x,t) + H(x,t) = \partial_t\phi(x,t), \quad G(x,t) = \partial_x\phi(x,t)$$

for all $(x,t) \in Q$, and is equal to $+\infty$ otherwise. We consider

$$\beta^*(\mu, M, \widetilde{\mu}) \quad = \quad \sup\{\int_Q F(x,t)\mu(dx,dt) + \int_Q G(x,t)M(dx,dt) + \int_Q H(x,t)\widetilde{\mu}(dx,dt) - \beta(F,G,H)\}$$

Then, β^* is infinite unless $\int_Q \partial_t\phi(x,t)(\mu(x,t) - \rho_t^*(x))dxdt + \int_Q \partial_x\phi(x,t)(x,t)(M(x,t) - m_t^*(x))dxdt = 0$ for all $\phi \in \mathcal{C}^{1,1}(Q)$ and $\int H(x,t)(\mu(dx,dt) - \widetilde{\mu}(dx,dt)) = 0$ for all $H \in \mathcal{C}_b(Q)$. Therefore, $\mu = \widetilde{\mu}$ and $\partial_t\mu + \partial_x M = 0$ in the sense of distributions, $\int \mu(x,t)dx = 1$ for almost all $t \in [0,1]$ and $\lim_{t\downarrow 0}\mu(dx,dt) = d\mu_0(x)$, $\lim_{t\uparrow 1}\mu(dx,dt) = d\mu_1(x)$. As a consequence,

$$\begin{aligned}\inf\{\alpha^*(\mu, M, \widetilde{\mu}) + \beta^*(\mu, M, \widetilde{\mu})\} &= \inf\{\mathcal{S}(\rho, m) \quad : \quad (\rho, m) \text{ satisfies (C') and } \rho_t|_{K^c} = 0 \quad \forall t \in [0,1]\} \\ &= 2\inf\{S_{\mu_0}(\nu) \quad : \quad \nu_1 = \mu_1\} + (\Sigma(\mu_0) - \Sigma(\mu_1)) := Z(\mu_0, \mu_1)\end{aligned}$$

where in the last line we have used Property 2.4 and Corollary 2.7.a).

Observe that α, β are convex functions with values in $]-\infty, \infty]$. Moreover, there is at least one point $(F,G,H) \in E$, namely $F = -1, \quad G = 0, \quad H = 1$ for which α is continuous for the uniform topology on E and β finite (this is the reason why we need to work on a compact set K instead of $\mathbb{R}$). Thus, following [5], by the Fenchel-Rockafellar duality theorem (see théorème 1.11 in [6]), we have

$$\inf\{\alpha^*(\mu, M, \widetilde{\mu}) + \beta^*(\mu, M, \widetilde{\mu}), \quad (\mu, M, \widetilde{\mu}) \in E'\} \quad = \quad \sup\{-\alpha(F,G,H) - \beta(-F,-G,-H) : (F,G,H) \in E\}$$

and the infimum is achieved. More precisely,

$$Z(\mu_0, \mu_1) = \sup\{\int_Q \partial_t\phi_t(x)\rho_t^*(x)dxdt + \int_Q \partial_x\phi_t(x)m_t^*(x)dxdt - \frac{2}{3\pi}\int_Q (H)^{\frac{3}{2}}\, dxdt\}$$

where the supremum is taken over $\phi \in \mathcal{C}_b^{1,1}(Q)$ and H in $\mathcal{C}_b(Q)$ such that $H \geq 0$, $\partial_t\phi + (\partial_x\phi/2)^2 \leq H$. Optimizing over H yields

$$Z(\mu_0,\mu_1) = \sup\{\int_Q \partial_t\phi_t(x)\rho_t^*(x)dxdt + \int_Q \partial_x\phi_t(x)m_t^*(x)dxdt - \frac{2}{3\pi}\int_Q \left(\max\{\partial_t\phi + (\partial_x\phi/2)^2, 0\}\right)^{\frac{3}{2}} dxdt\}$$

As a consequence, there exists a sequence of functions ϕ^ϵ in $\mathcal{C}_b^{1,1}(Q)$ such that if we set

$$\pi^2(\rho^\epsilon)^2 = \max\{\partial_t\phi^e + 4^{-1}(\partial_x\phi)^2, 0\},$$

$$\int u_t^*(x)^2 d\mu_t^*(x)dt + \frac{\pi^2}{3}\int \rho_t^*(x)^3 dxdt \quad \leq \quad \int \left(\partial_t\phi_t^\epsilon(x) + u_t^*(x)\partial_x\phi^\epsilon\right) d\mu_t^*(x)dt - \frac{2\pi^2}{3}\int \rho_t^\epsilon(x)^3 dxdt + \epsilon^2$$

for all $\epsilon > 0$, which implies

$$\begin{aligned}
\int (u_t^*(x) - \partial_x\frac{\phi_t^\epsilon(x)}{2})^2 d\mu_t^*(x)dt \quad &\leq \quad \pi^2\int \rho_t^\epsilon(x)^2\rho_t^*(x)dxdt - \frac{2\pi^2}{3}\int \rho_t^\epsilon(x)^3 dxdt - \frac{\pi^2}{3}\int \rho_t^*(x)^3 dxdt \\
&\quad -\pi^2\int |\partial_t\phi^\epsilon + (\frac{\partial_x\phi^\epsilon}{2})^2 - (\pi^2(\rho^\epsilon)^2|\rho_t^*(x)dxdt + \epsilon^2 \\
&= \quad -\frac{\pi^2}{3}\int (\rho_t^*(x) - \rho_t^\epsilon(x))^2(2\rho_t^\epsilon(x) + \rho_t^*(x))dxdt \\
&\quad -\pi^2\int |\partial_t\phi^\epsilon + (\frac{\partial_x\phi^\epsilon}{2})^2 - (\pi^2(\rho^\epsilon)^2|\rho_t^*(x)dxdt + \epsilon^2 \qquad (2.27)
\end{aligned}$$

which completes the proof of the Property.

∎

3 Applications to matrix integrals

In physics, several matrix integrals have been of interests in the 80's and 90's for their applications to quantum fields theory as well as string theory. We refer here to the works of M. Mehta, A. Matytsin, A. Migdal, V. Kazakov, P. Zinn Justin and B. Eynard for instance. Among these integrals, are often considered the following :

- The random Ising model on random graphs described by the Gibbs measure

  $$\mu_{Ising}^N(dA, dB) = \frac{1}{Z_{Ising}^N} e^{N\mathrm{tr}(AB) - N\mathrm{tr}(P_1(A)) - N\mathrm{tr}(P_2(B))} dAdB$$

  with Z_{Ising}^N the partition function

  $$Z_{Ising}^N = \int e^{N\mathrm{tr}(AB) - N\mathrm{tr}(P_1(A)) - N\mathrm{tr}(P_2(B))} dAdB$$

  and two polynomial functions P_1, P_2. The limiting free energy for this model was calculated by M. Mehta [21] in the case $P_1(x) = P_2(x) = x^2 + gx^4$ and integration holds over $\mathcal{H}_N$. However, the limiting spectral measures of A and B under μ_{Ising}^N were not considered in that paper. A discussion about this problem can be found in P. Zinn Justin [29].

- One can also define the Potts model on random graphs described by the Gibbs measure

$$\mu^N_{Potts}(dA_1, ..., dA_q) = \frac{1}{Z^N_{Potts}} \prod_{i=2}^{q} e^{N\mathrm{tr}(A_1 A_i) - N\mathrm{tr}(P_i(A_i))} dA_i e^{-N\mathrm{tr}(P_1(A_1))} dA_1.$$

  The limiting spectral measures of $(A_1, \cdots, A_q)$ are discussed in [29] when $P_i = gx^3 - x^2$ (!).

- As a straightforward generalization, one can consider matrices coupled by a chain following S. Chadha, G. Mahoux and M. Mehta [9] given by

$$\mu^N_{chain}(dA_1, ..., dA_q) = \frac{1}{Z^N_{chain}} \prod_{i=2}^{q} e^{N\mathrm{tr}(A_{i-1} A_i) - N\mathrm{tr}(P_i(A_i))} dA_i e^{-N\mathrm{tr}(P_1(A_1))} dA_1.$$

  q can eventually go to infinity as in [19].

- Finally, we can mention the so-called induced QCD studied in [18]. It is described, if $\Lambda = [-q, q]^D \subset \mathbb{Z}^D$, by

$$\mu^N_{QCD}(dA_i, i \in \Lambda) = \frac{1}{Z^N_{QCD}} \prod_{i\in\Lambda} \int e^{N\sum_{j=1}^{2D} \mathrm{tr}(U_j A_{i+e_j} U_j^* A_i)} \prod_{j=1}^{2D} dm^\beta_N(U_j) \prod_{i\in\Lambda} e^{-N\mathrm{tr}(P(A_i))} dA_i$$

  where $(e_j)_{1\le j\le 2D}$ is a basis of $\mathbb{Z}^D$. The description of the limit behaviour of the spectral measures of $A_1, \cdots, A_q$ is given in [18] in the case $q = \infty$. We impose periodic boundary conditions at the boundary of the lattice points Λ.

In this section, we shall study the asymptotic behaviour of the free energy of these models as well as describe the limit behaviour of the spectral measures of the matrices under the corresponding Gibbs measures.

The theorem states as follows

Theorem 3.1 *Assume that $P_i(x) \ge c_i x^4 + d_i$ with $c_i > 0$ and some finite constants d_i. Hereafter, $\beta = 1$ (resp. $\beta = 2$) when dA denotes the Lebesgue measure on $\mathcal{S}_N$ (resp. $\mathcal{H}_N$). Then,*

$$
\begin{aligned}
F_{Ising} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z_{Ising}^N \\
&= -\inf\{\mu(P) + \nu(Q) - I^{(\beta)}(\mu,\nu) - \frac{\beta}{2}\Sigma(\mu) - \frac{\beta}{2}\Sigma(\nu)\} - 2 \inf_{\nu\in\mathcal{P}(I\!R)} I_\beta(\nu) \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
F_{Potts} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z_{Potts}^N \\
&= -\inf\{\sum_{i=1}^q \mu_i(P_i) - \sum_{i=2}^q I^{(\beta)}(\mu_1,\mu_i) - \frac{\beta}{2}\sum_{i=1}^q \Sigma(\mu_i)\} - q \inf_{\nu\in\mathcal{P}(I\!R)} I_\beta(\nu) \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
F_{chain} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z_{chain}^N \\
&= -\inf\{\sum_{i=1}^q \mu_i(P_i) - \sum_{i=2}^q I^{(\beta)}(\mu_{i-1},\mu_i) - \frac{\beta}{2}\sum_{i=1}^q \Sigma(\mu_i)\} - q \inf_{\nu\in\mathcal{P}(I\!R)} I_\beta(\nu) \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
F_{QCD} &= \lim_{N\to\infty} \frac{1}{N^2} \log Z_{QCD}^N \\
&= -\inf\{\sum_{i\in\Lambda} \mu_i(P) - \sum_{i\in\Lambda}\sum_{j=1}^{2D} I^{(\beta)}(\mu_{i+e_j},\mu_i) - \frac{\beta}{2}\sum_{i\in\Lambda} \Sigma(\mu_i)\} - 2D|\Lambda| \inf_{\nu\in\mathcal{P}(I\!R)} I_\beta(\nu) \qquad (3.4)
\end{aligned}
$$

Remark 3.2: The above theorem actually extends to polynomial functions going to infinity like x^2. However, the case of quadratic polynomials is trivial since it boils down to the Gaussian case and therefore the next interesting case is quartic polynomial as above. Moreover, Theorem 3.3 fails in the case where P, Q go to infinity only like x^2. However, all our proofs would extends easily for functions $P_i's$ such that $P_i(x) \geq a|x|^{2+\epsilon} + b$ with some $a > 0$ and $\epsilon > 0$.

Theorem 3.1 will be proved in the next section, but merely boils down to a Laplace's (or saddle point) method.

We shall then study the variational problems for the above energies. We prove the following for the Ising model.

Theorem 3.3 *Assume $P_1(x) \geq ax^4 + b, P_2(x) \geq ax^4 + b$ for some positive constant a. Then*

- *The infimum in F_{Ising} is achieved at a unique couple (μ_A, μ_B) of probability measures.*
- *(μ_A, μ_B) are compactly supported measures with finite entropy Σ.*
- *Let $(\rho^{A\to B}, u^{A\to B})$ be the minimizer of S_{μ_A} on $\{\nu_1 = \mu_B\}$ as described in Theorem 2.8. Then, $(\mu_A, \mu_B, \rho^{A\to B}, m^{A\to B} = \rho^{A\to B} u^{A\to B})$ is the unique minimizer of the strictly convex energy*

$$
\begin{aligned}
\mathcal{L}(\mu, nu, \rho^*, m^*) &:= \mu(P_1 - \frac{1}{2}x^2) + \nu(P_2 - \frac{1}{2}x^2) - \frac{\beta}{4}(\Sigma(\mu) + \Sigma(\nu)) \\
&\quad + \frac{\beta}{4}\left(\int_0^1 \int \frac{(m_t^*(x))^2}{\rho_t^*(x)} dx dt + \frac{\pi^2}{3}\int_0^1 \int \rho_t^*(x)^3 dx dt\right)
\end{aligned}
$$

Thus, we find that $(\mu_A, \mu_B, \rho^{A\to B}, m^{A\to B})$ *are characterized by the property that for any* $(\mu, \nu, \rho^*, m^*) \in \{\mathcal{L} < \infty\}$,

$$
\begin{aligned}
&\int (P_1 - \frac{1}{2}x^2) d(\mu - \mu_A) + \int (P_2 - \frac{1}{2}x^2) d(\nu - \mu_B) \\
&-\frac{\beta}{2} \int\int \log|x-y| d\mu_A(y)(d\mu - d\mu_A)(x) - \frac{\beta}{2}\int\int \log|x-y| d\mu_B(y)(d\mu - d\mu_B)(x) \\
&+\frac{\beta}{4}\int [2u^{A\to B}(m^* - m^{A\to B}) - (u^{A\to B})^2(\rho^* - \rho^{A\to B}) + \pi^2(\rho^{A\to B})^2(\rho^* - \rho^{A\to B})] dx dt \geq 0
\end{aligned}
$$

- $(\rho^{A\to B}, m^{A\to B})$ *satisfies the Euler equation for isentropic flow with pressure* $p(\rho) = -\frac{\pi^2}{3}\rho^3$ *in the strong sense in the interior of* $\Omega = \{(x,t) \in I\!R \times [0,1] : \rho_t^{A\to B}(x) \neq 0\}$ *and satisfy the conclusions of Property 2.8.*

- *Moreover,*

$$\beta H\mu_A(x) = P_1'(x) - x - \frac{\beta}{2}u_0^{A\to B}(x), \quad \mu_A \, a.s,$$

  and

$$\beta H\mu_B(x) = P_2'(x) - x + \frac{\beta}{2}u_1^{A\to B}(x), \quad \mu_B \, a.s.$$

For the other models, we unfortunately loose obvious convexity, and therefore uniqueness of the minimizers in general. We can still prove the following

Theorem 3.4 • *For any given* μ_1, *there exists at most one minimizer* $(\mu_2, \cdots, \mu_q)$ *in* F_{Potts} *but uniqueness of* μ_1 *is unclear in general, except in the case* $q = 3$ *The critical points in* F_{Potts} *are compactly supported, with finite entropy* Σ.

Let $(\mu_1, \cdots, \mu_q)$ *be a critical point and for* $i \in \{2, \cdots, q\}$, *denote* (ρ^i, u^i) *the unique minimizer described in Theorem 2.8 with* $\mu_0^i(dx) = \mu_1(dx)$ *and* $\mu_1^i(dx) = \mu_i(dx)$. *Then*

$$P_1'(x) = qx + \frac{\beta}{2}\sum_{i=2}^{q} u_0^i(x) - \frac{\beta}{2}(q-3)H\mu_1(x)$$

μ_1*-almost surely and*

$$P_i'(x) = x - \frac{\beta}{2}u_1^i(x) - \frac{\beta}{2}H\mu_i(x), \quad 2 \leq i \leq q$$

μ_i*-almost surely.*

- *There exists at most one minimizer in* $F_{Chain,}$. *The minimizer* $(\mu_1, \cdots, \mu_q)$ *is compactly supported with finite entropy* Σ. *The critical points* $(\mu_1, \cdots, \mu_q)$ *in* F_{chain} *are such that for* $i \in \{2, \cdots, q\}$, *W It is such that if we denote* (ρ^i, u^i) *the minimizer described in Theorem 2.8 with* $\mu_0^i(dx) = \mu_{i-1}(dx)$ *and* $\mu_1^i(dx) = \mu_i(dx)$, *we have*

$$P_1'(x) = x + \frac{\beta}{2}u_0^2 - \frac{\beta}{2}H\mu_1(x) \text{ and } P_i'(x) = 2x - \frac{\beta}{2}(u_1^i - u_0^{i+1}), \quad 2 \leq i \leq q.$$

μ_1*-almost surely and* μ_i*-almost surely respectively.*

- *Again, uniqueness of the critical points in F_{QCD} is unclear in general, except in the case $D=1$ where uniqueness holds. In this case, the minimizer μ_i is symmetric, yielding $\mu_i = \mu$ for all $i \in \Lambda$ and the unique path (ρ, u) described in Theorem 2.8 with boundary data (μ, μ), satisfies $u_0^*(x) = -u_1^*(x)$ and*

$$P'(x) - 2x - \beta u_0^*(x) = 0 \quad \mu \ a.s.$$

3.1 Proof of Theorem 3.1

The proof of Theorem 3.1 follows a standard Laplace's method. We shall only detail it in the Ising model case, the generalization to the other models being straightforward.

Let P, Q be two polynomial functions and define, for $N \in \mathbb{N}$, $\Lambda_N(P,Q) \in \mathbb{R} \cup \{+\infty\}$ by

$$\Lambda_N^\beta(P,Q) = \int \exp\{-N\mathrm{tr}(P(A)) - N\mathrm{tr}(Q(B)) + N\mathrm{tr}(AB)\} dA dB$$

where the integration holds over orthogonal (resp. Hermitian) matrices if $\beta = 1$ (resp. $\beta = 2$).

We claim that

Lemma 3.5 *Assume that there exists $a, c \in \mathbb{R}^{+*}$, and $b, d \in \mathbb{R}$ such that*

$$P(x) \geq ax^4 + b \text{ and } Q(x) \geq cx^4 + d, \qquad \text{for all } x \in \mathbb{R}.$$

Then, we have

$$\lim_{N\to\infty} \frac{1}{N^2} \log \Lambda_N(P,Q) = \sup_{\mu,\nu \in \mathcal{P}(\mathbb{R})} \{-\mu(P) - \nu(Q) + I^{(\beta)}(\mu,\nu) + \frac{\beta}{2}(\Sigma(\mu) + \Sigma(\nu))\} - 2 \inf_{\nu \in \mathcal{P}(\mathbb{R})} I_\beta(\nu)$$

Remark here that the result could be extend to $P(x) \geq ax^2 + b$ and $Q(x) \geq cx^2 + d$ with $ac > 1$ but that the Gaussian case being uninteresting, we shall use the above and simpler hypothesis.

Proof.

Observe that for any $\epsilon > 0$,

$$\begin{aligned}
|\mathrm{tr}_N(AB) - \mathrm{tr}_N\left(\frac{A}{1+\epsilon A^2}\frac{B}{1+\epsilon B^2}\right)| &\leq \epsilon \left|\mathrm{tr}_N\left(\frac{A^3}{1+\epsilon A^2}B\right)\right| + \epsilon\left|\mathrm{tr}_N\left(\frac{A}{1+\epsilon A^2}\frac{B^3}{1+\epsilon B^2}\right)\right| \\
&\leq \epsilon\left(\left(\mathrm{tr}_N\left(\frac{A^6}{(1+\epsilon A^2)^2}\right)\right)^{\frac{1}{2}} \left(\mathrm{tr}_N B^2\right)^{\frac{1}{2}}\right. \\
&\quad \left. + \left(\mathrm{tr}_N\left(\frac{B^6}{(1+\epsilon B^2)^2}\right)\right)^{\frac{1}{2}} \left(\mathrm{tr}_N A^2\right)^{\frac{1}{2}}\right) \\
&\leq \sqrt{\epsilon}\left(\left(\mathrm{tr}_N(A^4)\right)^{\frac{1}{2}}\left(\mathrm{tr}_N(B^2)\right)^{\frac{1}{2}} + \left(\mathrm{tr}_N(B^4)\right)^{\frac{1}{2}}\left(\mathrm{tr}_N A^2\right)^{\frac{1}{2}}\right) \\
&\leq \sqrt{\epsilon}\left(\mathrm{tr}_N(A^4) + \mathrm{tr}_N(B^4) + \mathrm{tr}_N(A^2) + \mathrm{tr}_N(B^2)\right)
\end{aligned}$$

Therefore, if we set

$$\mu_{Ising}^N(dA, dB) = \frac{1}{\Lambda_N^\beta(P,Q)} \exp\{-N\mathrm{tr}(P(A)) - N\mathrm{tr}(Q(B)) + N\mathrm{tr}(AB)\} dA dB$$

and

$$\begin{aligned}\Delta_N(\epsilon) &:= \left|\frac{1}{N^2}\log\frac{\int\exp\{-N\mathrm{tr}(P(A))-N\mathrm{tr}(Q(B))+N\mathrm{tr}(\frac{A}{1+\epsilon A^2}\frac{B}{1+\epsilon B^2})\}dAdB}{\Lambda_N^\beta(P,Q)}\right|,\\ &= \left|\frac{1}{N^2}\log\mu_{Ising}^N\left(\exp\{N\mathrm{tr}(\frac{A}{1+\epsilon A^2}\frac{B}{1+\epsilon B^2})-N\mathrm{tr}(AB)\}\right)\right|,\end{aligned}$$

we get

$$\begin{aligned}\Delta_N(\epsilon) &\leq \frac{1}{N^2}\log\mu_{Ising}^N\left(\exp\{\sqrt{\epsilon}N\mathrm{tr}(A^4+A^2)+\sqrt{\epsilon}N\mathrm{tr}(B^4+B^2)\}\right)\\ &\leq \frac{1}{qN^2}\log\mu_{Ising}^N\left(\exp\{q\sqrt{\epsilon}N\mathrm{tr}(A^4+A^2)+q\sqrt{\epsilon}N\mathrm{tr}(B^4+B^2)\}\right)\end{aligned}$$

where we used Jensen's inequality with $q>1$. Now, under our hypothesis, and since $2|AB|\leq A^2+B^2$, it is clear that if $q\sqrt{\epsilon}$ is chosen small enough (e.g smaller than $a\wedge c$), the above right hand side is bounded uniformly. Hence, we take $q=\frac{1}{2a\wedge c\sqrt{\epsilon}}$ and obtain

$$\limsup_{N\to\infty}\Delta_N(\epsilon) \quad\leq\quad C\sqrt{\epsilon} \tag{3.5}$$

with a finite constant C. Moreover, for any $\epsilon>0$, we can use saddle point method (see [1] for a full rigorous derivation) and Theorem 1.1 of [15] to obtain

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^2}\log\int\exp\{-N\mathrm{tr}(P(A))-N\mathrm{tr}(Q(B))-N\mathrm{tr}(\frac{A}{1+\epsilon A^2}\frac{B}{1+\epsilon B^2})\}dAdB\\ &= \sup_{\mu,\nu\in\mathcal{P}(I\!R)}\{-\mu(P)-\nu(Q)+I^{(\beta)}(\mu\circ\phi_\epsilon^{-1},\nu\circ\phi_\epsilon^{-1})+\frac{\beta}{2}(\Sigma(\mu)+\Sigma(\nu))\}-2\inf I_\beta\end{aligned}$$

with $\phi_\epsilon(x)=(1+\epsilon x^2)^{-1}x$ and $\mu\circ\phi_\epsilon^{-1}(f)=\mu(f\circ\phi_\epsilon)$. Thus, (3.5) results with

$$\lim_{N\to\infty}\frac{1}{N^2}\log\Lambda_N(P,Q)=\lim_{\epsilon\to 0}\sup_{\mu,\nu\in\mathcal{P}(I\!R)}\{-\mu(P)-\nu(Q)+I^{(\beta)}(\mu\circ\phi_\epsilon^{-1},\nu\circ\phi_\epsilon^{-1})+\frac{\beta}{2}(\Sigma(\mu)+\Sigma(\nu))\}-2\inf I_\beta.$$

Moreover, we can prove as for (3.5) that for any μ,ν such that $\mu(x^4)\leq M$ and $\nu(x^4)\leq M$,

$$|I^{(\beta)}(\mu\circ\phi_\epsilon^{-1},\nu\circ\phi_\epsilon^{-1})-I^{(\beta)}(\mu,\nu)|\leq C(M)\sqrt{\epsilon}.$$

Using the fact that

$$|I^{(\beta)}(\mu,\nu)|\leq\frac{1}{2}(\mu(x^2)+\nu(x^2)),$$

as well as

$$\Sigma(\mu)+\Sigma(\nu)\leq C(\mu(x^2)+\nu(x^2)+1)$$

for some finite constant C, we see that the supremum above is taken at μ,ν such that $\mu(x^4)$ and $\nu(x^4)$ are bounded by some finite constant depending only on P,Q. Hence, we can take the limit ϵ going to zero above and conclude. ∎

3.2 Proof of Theorem 3.3 and 3.4

3.2.1 The Ising model

Let us recall that

$$F_{Ising} + 2 \inf_{\nu \in \mathcal{P}(I\!R)} I_\beta(\nu) \quad = \quad -\inf\{\mu(P_1) + \nu(P_2) - I^{(\beta)}(\mu,\nu) - \frac{\beta}{2}\Sigma(\mu) - \frac{\beta}{2}\Sigma(\nu)\}$$

Observe that since $I^{(\beta)}(\mu,\nu) \leq 2^{-1}\mu(x^2) + 2^{-1}\nu(x^2)$, the minimizer (μ_A, μ_B) in the above right hand side is such that

$$\mu_A(P_1 - \frac{1}{2}x^2) + \mu_B(P_2 - \frac{1}{2}x^2) - \frac{\beta}{2}\Sigma(\mu_A) - \frac{\beta}{2}\Sigma(\mu_B) \leq -F_{Ising} - 2 \inf_{\nu \in \mathcal{P}(I\!R)} I_\beta(\nu) < \infty.$$

Hence, since $P_1 - 2^{-1}x^2$ and $P_2 - 2^{-1}x^2$ are bounded below under our hypotheses (for well chosen a), we conclude that $\Sigma(\mu_A)$ and $\Sigma(\mu_B)$ are bounded below and hence finite. Further, if $2n_1$ (resp. $2n_2$) is the degree of P_1 (resp. P_2) for $n_1, n_2 \geq 2$, we also see that

$$\mu_A(x^{2n_1}) < \infty, \quad \mu_B(x^{2n_1}) < \infty. \tag{3.6}$$

Thus, we can use Property 2.4 to get

$$\begin{aligned}
F_{Ising} + 2 \inf_{\nu \in \mathcal{P}(I\!R)} I_\beta(\nu) \quad &= \quad -\inf \Big\{ \mu(P_1 - \frac{1}{2}x^2) + \nu(P_2 - \frac{1}{2}x^2) - \frac{\beta}{4}(\Sigma(\mu) + \Sigma(\nu)) \\
&\quad + \frac{\beta}{4} \inf_{(u^*,\mu^*) \in (C)_{\mu,\nu}} \{ \int_0^1 \int u_t^*(x)^2 d\mu_t^*(x) dt + \int_0^1 \int H\mu_t^*(x)^2 d\mu_t^*(x) dt \} \Big\} \\
&= \quad - \inf_{\substack{\mu,\nu \in \mathcal{P}(I\!R) \\ (u^*,\mu^*) \in (C)_{\mu,\nu}}} \Big\{ \mu(P_1 - \frac{1}{2}x^2) + \nu(P_2 - \frac{1}{2}x^2) - \frac{\beta}{4}(\Sigma(\mu) + \Sigma(\nu)) \\
&\quad + \frac{\beta}{4} \Big(\int_0^1 \int u_t^*(x)^2 d\mu_t^*(x) dt + \frac{\pi^2}{3} \int_0^1 \int \rho_t^*(x)^3 dx dt \Big) \Big) \\
&:= \quad - \inf_{\substack{\mu,\nu \in \mathcal{P}(I\!R) \\ (u^*,\mu^*) \in (C)_{\mu,\nu}}} L(\mu,\nu,\mu^*,u^*)
\end{aligned}$$

where $(u^*, \mu^*) \in (C)_{\mu,\nu}$ means that in the sense of distributions

$$\partial_t \rho_t^* + \partial_x(\rho_t^* u_t^*) = 0, \quad \lim_{t \downarrow 0} \mu_t^*(dx) = \mu, \quad \lim_{t \uparrow 1} \mu_t^*(dx) = \nu$$

and we have used in the last line that when the above infimum is finite, μ_t^* is absolutely continuous with respect to Lebesgue measure for almost all $t \in [0,1]$ and with density $\rho^* \in L_3(dxdt)$ (see Lemma 2.3).

Observe that if $L(\mu,\nu,\mu^*,u^*) = \mathcal{L}(\mu,\nu,\rho^*,m^*)$ with $m^* = \rho^* u^*$, $\mathcal{L}$ is a strictly convexe function of (μ,ν,ρ^*,m^*) (recall that $-\Sigma$ is convex, see [1] for instance) and that the constraint $(C)_{\mu,\nu}$ is linear in the variables (μ,ν,ρ^*,m^*). Therefore, the above minimum is achieved at a unique point $(\mu_A, \mu_B, \mu_.^{A\to B}, m_.^{A\to B})$.

We now perform a measure type perturbation to characterize the infimum. Take $(\mu,\nu,\rho^*,m^*) \in \{\mathcal{L} < \infty\}$ and set, for $\alpha \in [0,1]$,

$$(\mu^\alpha, \nu^\alpha, \rho^\alpha, m^\alpha) = \alpha(\mu,\nu,\rho^*,m^*) + (1-\alpha)(\mu_A, \mu_B, \rho_.^{A\to B}, u_.^{A\to B}).$$

Then, we find that we must have

$$
\begin{aligned}
&\int (P_1 - \frac{1}{2}x^2) d(\mu - \mu_A) + \int (P_2 - \frac{1}{2}x^2) d(\nu - \mu_B) \\
&- \frac{\beta}{2} \int\int \log|x-y| d\mu_A(y)(d\mu - d\mu_A)(x) - \frac{\beta}{2} \int\int \log|x-y| d\mu_B(y)(d\mu - d\mu_B)(x) \\
&+ \frac{\beta}{4} \int [2u^{A\to B}(m^* - m^{A\to B}) - (u^{A\to B})^2(\rho^* - \rho^{A\to B}) + \pi^2(\rho^{A\to B})^2(\rho^* - \rho^{A\to B})] dx dt \geq 0 \quad (3.7)
\end{aligned}
$$

Taking $\mu = \mu_A$ and $\nu = \mu_B$, we see that $(\rho^{A\to B}, u^{A\to B})$ must satisfy Property 2.8. Now, if $\mu(dx) = \mu_A(dx) + \partial_x \phi_0(x) dx$, $\nu(dx) = \mu_B(dx) + \partial_x \phi_1(x) dx$ and $m^* = m^{A\to B} - \partial_t \phi$, $\rho^*_t = \rho^{A\to B} + \partial_x \phi$ with $\phi \in \mathcal{C}_b^{\infty,\infty}(\mathbb{R} \times [0,1])$ such that

$$
\int_{\rho^{A\to B}\neq 0} \frac{(m_t^{A\to B} + \epsilon\partial_t \phi)^2}{\rho^{A\to B} + \epsilon \partial_x \phi} dx dt < \infty, \quad \int_{\rho^{A\to B}=0} \frac{(\partial_t \phi)^2}{\partial_x \phi} dx dt < \infty, \tag{3.8}
$$

we obtain by (3.7)

$$
\begin{aligned}
&\int (P_1 - \frac{1}{2}x^2) \partial_x \phi_0(x) dx + \int (P_2 - \frac{1}{2}x^2) \partial_x \phi_1(x) dx \\
&- \frac{\beta}{2} \int\int \log|x-y| d\mu_A(y) \partial_x \phi_0(x) dx - \frac{\beta}{2} \int\int \log|x-y| d\mu_B(y) \partial_x \phi_1(x) dx \\
&+ \frac{\beta}{4} \int [2u^{A\to B}\partial_t \phi - (u^{A\to B})^2 \partial_x \phi + \pi^2 (\rho^{A\to B})^2 \partial_x \phi] dx dt \geq 0
\end{aligned} \tag{3.9}
$$

which becomes an equality if ϕ is supported in $\Omega = \{(x,t) \in \mathbb{R} \times [0,1] : \rho^{A\to B} \neq 0\}$ by symmetry. If we assume that $u^{A\to B}$ is sufficiently smooth, in particular continuously differentiable with respect to the time variable around $t=0$ and $t=1$, we can use integration by parts to see that

$$
\int [2u^{A\to B}\partial_t \phi - (u^{A\to B})^2 \partial_x \phi + \pi^2 (\rho^{A\to B})^2 \partial_x \phi] dx dt \geq 2[\int \Pi_t^{A\to B} \partial_x \phi_t dx]_0^1
$$

yielding that there exists two constants l_1, l_2 such that

$$
\begin{aligned}
P_1(x) - \frac{1}{2}x^2 - \frac{\beta}{2} \int \log|x-y| d\mu_A(y) - 2\Pi_0^{A\to B}(x) &= l_1 \quad \mu_A \quad a.s \quad (3.10) \\
P_2(x) - \frac{1}{2}x^2 - \frac{\beta}{2} \int \log|x-y| d\mu_B(y) - 2\Pi_1^{A\to B}(x) &= l_2 \quad \mu_B \quad a.s \quad (3.11) \\
P_1(x) - \frac{1}{2}x^2 - \frac{\beta}{2} \int \log|x-y| d\mu_A(y) - 2\Pi_0^{A\to B}(x) &\geq l_1 \quad \text{if } x \in \mathrm{supp}(\mu_A)^c \\
P_2(x) - \frac{1}{2}x^2 - \frac{\beta}{2} \int \log|x-y| d\mu_B(y) - 2\Pi_1^{A\to B}(x) &\geq l_2 \quad \text{if } x \in \mathrm{supp}(\mu_B)^c
\end{aligned}
$$

Such a result would generalize the usual equations obtained in the one matrix case. However, since we could not prove such a regularity property of $(\rho^{A\to B}, u^{A\to B})$, we shall now obtain a Schwinger-Dyson type formula following [8], theorem 2.15 and proposition 2.17, to obtain a weak form of (3.10),(3.11). Let us briefly recall the ideas in the case $\beta = 2$ (the case $\beta = 1$ being similar), which is based on an infinitesimal change of variables.

If, in Z^N_{Ising}, we change $A \to A + N^{-1}h(A,B)$ with some smooth bounded functions h of two non-commutative variables (take for instance h belonging to the set $\mathcal{CC}_{st}(\mathbb{C})$ of Stieljes functionals defined in [7, 8](see also its definition in appendice 4.1), it turns out that, due to Kadison-Fuglede determinant formula (see [8], the proof of theorem 2.15 and proposition 2.17)

$$Z^N_{Ising} = \int e^{\operatorname{tr}(h(A,B)(-P_1'(A)+B))+N^{-1}\operatorname{tr}\otimes\operatorname{tr}(D_A h(A,B))+O(1)-N\operatorname{tr}(P_1(A)+P_2(B)-AB)} dAdB$$

with D_A the non commutative derivation with respect to A given by

$$D_A(hg) = D_A h \times 1 \otimes g + h \otimes 1 \times D_A g, \quad \forall h, g \in \mathcal{CC}_{st}(\mathbb{C}), \quad D_A B = 0, \quad D_A A = 1 \otimes 1.$$

Therefore, we can find a finite constant $C(h)$ such that for any $\epsilon > 0$

$$\mu^N_{Ising}\left(|\hat{\mu}^{(N)} \otimes \hat{\mu}^{(N)}(D_A h(A,B)) + \hat{\mu}^{(N)}((-P_1'(A) + B)h(A,B))| \le \epsilon\right) \ge 1 - 2e^{-\epsilon N + C(h)} \tag{3.12}$$

with $\hat{\mu}^{(N)}$ the empirical distribution of A, B defined by

$$\hat{\mu}^{(N)}(h) = \operatorname{tr}_N(h(A,B)), \quad \forall h \in \mathcal{CC}_{st}(\mathbb{C}).$$

Of course, the same type of formula holds when A is replaced by B. It is not hard to see that $\hat{\mu}^{(N)}$ is tight under μ^N_{Ising} for the topology described in [8], corresponding to the $\mathcal{CC}_{st}(\mathbb{C})$-weak topology (see [8] for proof of similar statements). Let τ be a limit point. Taking, for $\epsilon > 0$ and $\delta > 0$, $h(A,B) = (1+\delta A^2)^{-p} j(A)(1 + \epsilon B^2)^{-1}$ with $j(x) = \prod_{1\le i\le n}(z_i - x)^{-1}$ for some $z_i \in \mathbb{C}\backslash\mathbb{R}$ and $n \in \mathbb{N}$, and p large enough (p larger than half the degree of P_1') so that $D_A h(A,B) \in \mathcal{CC}_{st}(\mathbb{C}) \otimes \mathcal{CC}_{st}(\mathbb{C})$ and $(1+\delta A^2)^{-p}(P_1'(A) - B)(1+\epsilon B^2)^{-1} j(A) \in \mathcal{CC}_{st}(\mathbb{C})$, we deduce from (3.12) that τ must satisfies for any $\epsilon, \delta > 0$ and p large enough,

$$\tau \otimes \tau(D_A(1+\delta A^2)^{-p} j(A) \times 1 \otimes (1+\epsilon B^2)^{-1}) = \tau((P_1'(A) - B)(1+\delta A^2)^{-p} j(A)(1+\epsilon B^2)^{-1}). \tag{3.13}$$

Similarly for any $\epsilon, \delta > 0$, and p large enough,

$$\tau \otimes \tau(D_B(1+\delta B^2)^{-p} j(B) \times 1 \otimes (1+\epsilon B^2)^{-1}) = \tau((P_2'(B) - A)(1+\delta A^2)^{-p} j(B)(1+\epsilon A^2)^{-1}). \tag{3.14}$$

Now, by (3.6), $P_1'(A) - B$ and $P_2'(B) - A$ belongs to $L^1(\tau)$ so that we can let δ, ϵ going to zero to conclude by dominated convergence theorem that

$$\tau \otimes \tau(D_A j(A)) = \tau((P_1'(A) - B)j(A)), \quad \tau \otimes \tau(D_B j(B)) = \tau((P_2'(B) - A)j(B)). \tag{3.15}$$

We next show that (3.15) implies that μ_A and μ_B are compactly supported when $n_1 \ge 2$ and $n_2 \ge 2$, and first that all their moments are finite. To this end, take $j(x) = \left((1+\epsilon x^2)^{-1}x\right)^n = \epsilon^{-n}\left(1 + i\epsilon^{-1}(x - i\epsilon^{-1})^{-1}\right)^n (x + i\epsilon^{-1})^{-n}$ for $n \in \mathbb{N}$, yielding

$$\mu_A\left(P_1'(x)\left((1+\epsilon x^2)^{-1}x\right)^n\right) = \tau\left(\tau(B|A)\left((1+\epsilon A^2)^{-1}A\right)^n\right) + \tau \otimes \tau(D_A j(A)) \tag{3.16}$$

with, since Df can be represented in the tensor product space as $Df(x,y) = (x-y)^{-1}(f(x)-f(y))$,

$$\tau\otimes\tau(D_A j(A)) = \sum_{p=0}^{n-1}\mu_A\left(((1+\epsilon x^2)^{-1}x)^p\right)\mu_A\left(((1+\epsilon x^2)^{-1}x)^{n-1-p}\right)$$
$$-\epsilon\sum_{p=0}^{n-1}\mu_A\left(((1+\epsilon x^2)^{-1}x)^{p+1}\right)\mu_A\left(((1+\epsilon x^2)^{-1}x)^{n-p}\right).$$

When n is odd, it is not hard to see that we can find $c>0, d_n\in\mathbb{R}$ such that $P'(x)x^n \geq cx^{2n_1-1+n} - d_n$, so that we deduce from (3.16) that

$$c\mu_A\left(|\frac{x}{1+\epsilon x^2}|^{2n_1-1+n}\right) \leq d_n + 2n\sup_{p\leq n}\mu_A\left((\frac{x}{1+\epsilon x^2})^p\right)^2 + \mu_A\left(|\frac{x}{1+\epsilon x^2}|^{nq}\right)^{\frac{1}{q}}\mu_B(|x|^p)^{\frac{1}{p}} \tag{3.17}$$

where we have used in the last line Hölder's inequality with conjugate exponents p,q. We take $q = n^{-1}(2n_1 - 1+n)$, $p = (2n_1-1)^{-1}(2n_1-1+n)$. Similarly, we obtain for μ_B, and $q = n^{-1}(2n_2-1+n)$, $p = (2n_2-1)^{-1}(2n_2-1+n)$,

$$c\mu_B\left(|\frac{x}{1+\epsilon x^2}|^{2n_2-1+n}\right) \leq d_n + 2n\sup_{p\leq n}\mu_B\left((\frac{x}{1+\epsilon x^2})^p\right)^2 + \mu_B\left(|\frac{x}{1+\epsilon x^2}|^{nq}\right)^{\frac{1}{q}}\mu_A(|x|^p)^{\frac{1}{p}}. \tag{3.18}$$

Now, we have seen that

$$\mu_A(x^{2n_1})<\infty, \quad \mu_B(x^{2n_2})<\infty$$

so that (3.17),(3.18) yields

$$\begin{aligned}\mu_A(x^{2n_1-1+n}) &= \sup_{\epsilon\geq 0}\mu_A\left(((1+\epsilon x^2)^{-1}x)^{2n_1-1+n}\right) < \infty \text{ for } 2n_1-1+n\leq m_1^A := 2n_2(2n_1-1)\\ \mu_B(x^{2n_2-1+k}) &= \sup_{\epsilon\geq 0}\mu_B\left(((1+\epsilon x^2)^{-1}x)^{2n_2-1+k}\right) < \infty \text{ for } 2n_2-1+k\leq m_1^B := 2n_2(2n_2-1)\end{aligned}$$

and then by induction for $2n_1-1+n\leq m_p^A := m_{p-1}^B(2n_1-1)$, $2n_2-1+k\leq m_p^B := m_{p-1}^A(2n_2-1)$ for all $p\geq 2$. Since $2n_1-1>1$ and $2n_1-1>1$, m_p^A and m_p^B go to infinity with p, which proves that μ_A and μ_B have finite moments of all orders.

As a consequence, we can extend by dominated convergence theorem (3.16) to polynomial functions (i.e. take $\epsilon=0$) resulting with

$$\mu_A\left(P_1'(x)x^n\right) = \tau\left(\tau(B|A)A^n\right) + \sum_{p=0}^{n-1}\mu_A\left(x^p\right)\mu_A\left(x^{n-1-p}\right). \tag{3.19}$$

and a similar equation for the moments of μ_B. Let us write $P_1'(x) = \alpha_1 x^{2n_1-1} + \sum_{p=2}^{2n_1}\alpha_p x^{2n_1-p}$, $P_2'(x) = \beta_1 x^{2n_2-1} + \sum_{p=2}^{2n_2}\beta_p x^{2n_2-p}$ with $\alpha_1>0, \beta_1>0$. Setting $a_n = |\mu_A(x^n)|$ and $b_n = |\mu_B(x^n)|$, we deduce that

$$\alpha_1 a_{2n_1-1+n} \leq \sum_{p=2}^{2n_1}|\alpha_p|a_{2n_1-p+n} + \sum_{p=0}^{n-1}a_p a_{n-1-p} + a_{qn}^{\frac{1}{q}}b_p^{\frac{1}{p}} \tag{3.20}$$

$$\beta_1 b_{2n_1-1+n} \leq \sum_{p=2}^{2n_2}|\beta_p|b_{2n_1-p+n} + \sum_{p=0}^{n-1}b_p b_{n-1-p} + b_{qn}^{\frac{1}{q}}a_p^{\frac{1}{p}} \tag{3.21}$$

with conjuguate exponents (p,q) to be chosen later.

Now, we make the induction hypothesis that for some $R \in \mathbb{R}^+$, for some $m \in \mathbb{N}$,

$$a_p \leq R^p C_p, \quad b_p \leq R^p C_p, \quad \text{for } p \leq m$$

with C_p the Catalan numbers given by

$$C_p = \sum_{n=0}^{p-1} C_n C_{p-1-n}, \quad C_0 = 1.$$

Of course, up to take R big enough, we can always assume that $m \geq 2n_1 \vee n_2$. Now, plugging this hypothesis into (3.20),(3.21) with $m+1 = 2n_1 - 1 + n$ and $q = mn^{-1}$, we obtain

$$\begin{aligned}
\alpha_1 a_{2n_1-1+n} &\leq \sum_{p=2}^{2n_1} |\alpha_p| R^{2n_1-p+n} C_{2n_1-p+n} + R^n C_n + R^{n+1} (C_m)^{\frac{n}{m}} (C_{[\frac{m}{m-n}]+1})^{\frac{m-n}{m}} \\
&\leq C_{m+1} R^{m+1} (\sum_{p=2}^{2n_1} |\alpha_p| R^{-p} + R^{n-2-m} + R^{n-m})
\end{aligned}$$

where we have used that C_m increases with m. Thus, our induction hypothesis is verified as soon as

$$\begin{aligned}
\sum_{p=2}^{2n_1} |\alpha_p| R^{-p} + R^{-2n_1} + R^{2(1-n_1)} &\leq \alpha_1 \\
\sum_{p=2}^{2n_2} |\beta_p| R^{-p} + R^{-2n_1} + R^{2(1-n_2)} &\leq \beta_1
\end{aligned}$$

which is clearly the case for R large enough since we asumed $n_1 \wedge n_2 \geq 2$. Since $m^{-1} \log C_m$ goes to 4 as m goes to infinity, we deduce that

$$\limsup_{m\to\infty} \frac{1}{2m} \log \mu_A(x^{2m}) \leq R+4, \quad \limsup_{m\to\infty} \frac{1}{2m} \log \mu_B(x^{2m}) \leq R+4,$$

implying that μ_A and μ_B are supported into $[-R-4, R+4]$ for R finite satisfying the above induction hypothesis (plus the condition imposed by the first $2n_1 \vee n_2$ moments).

Let us now go back to (3.15) and notice that since the Stieljes functions are dense in $\mathcal{C}_c(\mathbb{R})$ and $P_1' - \tau(A|B)$ belongs to $L^1(\tau)$, it can be extended to $j \in \mathcal{C}_b^1(\mathbb{R})$;

$$\int\int \frac{j(x) - j(y)}{x - y} d\mu_A(x) d\mu_A(y) = \tau((P'(x) - \tau(B|A)) j(x)) \tag{3.22}$$

Since (μ_A, μ_B) are compactly supported, we can use the conclusions of section 2.2.2. We see that $\mu_t^{A\to B}$-almost surely,

$$u_t^{A\to B} = \tau(B - A|X_t) + (1-2t) H\mu_t^{A\to B}(x)$$

so that

$$u_0^{A\to B} = \tau(B|A) - x + H\mu_A$$

at least in the sense of distribution as in (3.22). Thus, by uniqueness of the solutions to the Euler equation given the initial and final data (μ_A, μ_B) proved in Property 2.5, we conclude that

$$H\mu_A(x) = P_1'(x) - x - u_0^{A\to B}(x)$$

in the sense of distribution that

$$\frac{1}{2}\int\int \frac{h(x)-h(y)}{x-y} d\mu_A(x)d\mu_A(y) = \int (P_1'(x) - x - u_0^{A\to B}(x))h(x)d\mu_A(x) \tag{3.23}$$

for all $h \in \mathcal{C}_b^1(I\!R)$. We now show that this weak equality in fact yields the almost equality. Indeed, taking $h = P_\epsilon * g$ with P_ϵ the Cauchy law with parameter ϵ, one obtains from the weak equality

$$\int H(P_\epsilon * \mu_A)(x)g(x)dP_\epsilon * \mu_A(x) = \tau((P_1'(A) - A - u_0^{A\to B}(A))g(A + C_\epsilon)).$$

Therefore, for any bounded measurable function g, if we set $M_A := \sup_{x\in \mathrm{supp}(\mu_A)} |P_1'(x) - x - u_0^{A\to B}(x)|$,

$$|\int H(P_\epsilon * \mu_A)(x)g(x)dP_\epsilon * \mu_A(x)| \le M_A \int |g(x)|dP_\epsilon * \mu_A(x)$$

from which we deduce that, since $P_\epsilon * \mu_A \ll dx$ with a non zero density everywhere on $I\!R$, for all $\epsilon > 0$

$$|H(P_\epsilon * \mu_A)(x)| \le M_A \qquad \text{a.s.}$$

Consequently, for any $\epsilon > 0$,

$$\int \left(\frac{dP_\epsilon * \mu_A}{dx}\right)^3 (x)dx = \frac{3}{\pi^2}\int H(P_\epsilon * \mu_A)(x)^2 dP_\epsilon * \mu_A(x) \le \frac{3}{\pi^2} M_A^2.$$

As a consequence, we claim that $\mu_A \ll dx$ and

$$\int \left(\frac{d\mu_A}{dx}\right)^3 (x)dx \le \frac{3}{\pi^2} M_A^2.$$

Indeed, if f is a Lipschitz function with Lipschitz constant $|f|_{\mathcal{L}}$, we know that

$$|\int f(x)d\mu_A(x)| \le \epsilon|f|_{\mathcal{L}} + |\int f(x)dP_\epsilon * \mu_A(x)| \le \epsilon|f|_{\mathcal{L}} + \left(\frac{3}{\pi^2}M_A^2\right)^{\frac{1}{3}} \left(\int |f(x)|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}.$$

We can now let ϵ going to zero to conclude that

$$|\int f(x)d\mu_A(x)| \le \left(\frac{3}{\pi^2}M_A^2\right)^{\frac{1}{3}} \left(\int |f(x)|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}.$$

which proves the claim.

As a consequence, by Tricomi ??, (3.23) gives for all $h \in L^\infty(d\mu_A)$

$$\int (P_1'(x) - x - u_0^{A\to B}(x) - H(\mu_A)(x))h(x)d\mu_A(x) = 0,$$

and hence the μ_A almost sure equality.

The second equation is derived similarly and one finds that

$$2H\mu_B(x) = P_2'(x) - \tau(A|B)(x) = P_2'(x) - (x - u_1^{A\to B}(x) - H\mu_B(x))$$

resulting with

$$H\mu_B(x) = P_2'(x) - x + u_1^{A\to B}(x)$$

μ_B almost surely. Note also that by Property 2.8, the fact that (μ_A, μ_B) are compactly supported implies that $(\rho^{A\to B}, u^{A\to B})$ satisfies the isentropic Euler equation in the strong sense in Ω.

3.3 q-Potts model

In this case, we find that

$$\begin{aligned}
F_{Potts} &= -\inf\{\sum_{i=1}^{q}\mu_i(P_i) - \sum_{i=2}^{q} I^{(\beta)}(\mu_1,\mu_i) - \frac{\beta}{2}\sum_{i=1}^{q}\Sigma(\mu_i)\} - q\inf_{\nu\in\mathcal{P}(I\!R)} I_\beta(\nu) \\
&= -\inf\{\mu_1(P_1 - \frac{q}{2}x^2) + \sum_{i=2}^{q}\mu_i(P_i - \frac{x^2}{2}) \\
&\quad +\frac{\beta}{4}\sum_{i=2}^{q}\inf_{(u^*,\mu^*)\in(C)_{\mu_1,\mu_i}}\left\{\int_0^1\int(u_t^*(x))^2 d\mu_t^*(x)dt + \int_0^1\int(H\mu_t^*(x))^2 d\mu_t^*(x)dt\right\} \\
&\quad -\frac{\beta}{4}\sum_{i=2}^{q}\Sigma(\mu_i) + \frac{\beta}{4}(q-3)\Sigma(\mu_1)\} - q\inf_{\nu\in\mathcal{P}(I\!R)} I_\beta(\nu)
\end{aligned} \tag{3.24}$$

When $q > 3$, the above functionnal is not anymore clearly convex in μ_1 since $\Sigma(\mu_1)$ is concave. Hence, the uniqueness of the minimizers is now unclear. Note however that the Euler-Lagrange term may still contain sufficient convexity in μ_1 to insure uniqueness but simply that the above formula does not show it. In the case $q = 3$, the functional is still convex, and strictly convex in the arguments (μ^*, m^*). Therefore, uniqueness of the minimizers still holds since if (μ, ν, μ^*, u^*) and $(\widetilde{\mu}, \widetilde{\nu}, \widetilde{\mu}^*, \widetilde{u}^*)$, we would still find that by convexity $\widetilde{\mu}^* = \mu^*$ and therefore $\mu = \mu_0^* = \widetilde{\mu}_0^* = \widetilde{\mu}$, $\nu = \mu_1^* = \widetilde{\mu}_1^* = \widetilde{\nu}$. The above formula already shows that the critical points satisfy $\mu_i(P_i) < \infty$ and have finite entropy Σ. We can also obtain the Schwinger-Dyson equations in this case and deduce as for the Ising model that the critical points are compactly supported and satisfy the equations of Theorem 3.4.

3.4 Chain model

In this case,

$$
\begin{aligned}
F_{chain} &= -\inf\{\sum_{i=1}^{q}\mu_i(P_i) - \sum_{i=2}^{q} I^{(\beta)}(\mu_{i-1},\mu_i) - \frac{\beta}{2}\sum_{i=1}^{q}\Sigma(\mu_i)\} - q\inf_{\nu\in\mathcal{P}(\mathbb{R})} I_\beta(\nu) \qquad (3.25)\\
&= -\inf\{\mu_1(P_1 - \frac{x^2}{2}) + \sum_{i=2}^{q}\mu_i(P_i - x^2)\\
&\quad + \frac{\beta}{4}\sum_{i=2}^{q}\inf_{(u^*,\mu^*)\in(C)_{\mu_i,\mu_{i+1}}}\left\{\int_0^1\int(u_t^*(x))^2 d\mu_t^*(x)dt + \int_0^1\int(H\mu_t^*(x))^2 d\mu_t^*(x)dt\right\}\\
&\quad - \frac{\beta}{4}\Sigma(\mu_1)\} - q\inf_{\nu\in\mathcal{P}(\mathbb{R})} I_\beta(\nu) \qquad (3.26)
\end{aligned}
$$

Here, we still have convexity and strict convexity on the term coming from $I^{(\beta)}$. Hence, uniqueness of the minimizers hold. Again, we can prove the conclusions of Theorem 3.4 as for the Ising model.

3.5 Induced QCD model

$$
\begin{aligned}
F_{QCD} &= -\inf\{\sum_{i=1}^{q}\mu_i(P) - \sum_{i\in\Lambda}\sum_{j=1}^{2D} I^{(\beta)}(\mu_{i+e_j},\mu_i) - \frac{\beta}{2}\sum_{i\in\Lambda}\Sigma(\mu_i)\} - 2D|\Lambda|\inf_{\nu\in\mathcal{P}(\mathbb{R})} I_\beta(\nu)\\
&= -\inf\{\sum_{i=1}^{q}\mu_i(P - Dx^2) - \frac{\beta}{2}(1-D)\sum_{i\in\Lambda}\Sigma(\mu_i)\\
&\quad + \sum_{i\in\Lambda}\sum_{j=1}^{2D}\inf_{(u^{i,\mu},\mu^{i,\mu})\in(C)_{\mu_i,\mu_{i+\mu}}}\left\{\int_0^1\int(u_t^{i,\mu}(x))^2 d\mu_t^{i,\mu}(x)dt + \int_0^1\int(H\mu_t^{i,\mu}(x))^2 d\mu_t^{i,\mu}(x)dt\right\}\}\\
&\quad -2D|\Lambda|\inf_{\nu\in\mathcal{P}(\mathbb{R})} I_\beta(\nu)
\end{aligned}
$$

Again, obvious convexity disappears and uniqueness of the minimizers becomes unclear when $D > 1$. Uniqueness of the minimizers still holds when $D = 1$. Then, clearly $\mu_i = \mu$ for all $i \in \Lambda$ and $u_0^* = -u_1^*$ at the minimizing path with (ρ^*, u^*) the solution of the Euler equation with with boundary data (μ, μ). μ then satisfies

$$
P'(x) - 2x - \beta u_0^*(x) \quad = \quad 0
$$

in the sense of distributions in supp(μ), which corresponds to the result obtained by Matytsin [[18], (4.3)] when $\beta = 2$. Actually, since we can prove as for the Ising model that μ is compactly supported, it turns out that $P'(x) - 2x - \beta u_0^*(x)$ is in every $L_p(d\mu)$ and therefore that $P'(x) - 2x - \beta u_0^*(x) = 0$ almost everywhere in the support of μ.

4 Appendice

4.1 Free Brownian motion description of the minimizers

Let us return to the probability aspect of the story. In fact, by definition, if

$$X_t^N = X_0^N + H_t^N$$

with a Hermitian (if $\beta = 2$, otherwise symmetric if $\beta = 1$) matrix X_0^N with spectral measure $\hat{\mu}_0^N$ and a Hermitian (resp. symmetric) Brownian motion H^N, if we denote $\hat{\mu}_t^N$ the spectral measure of X_t^N, then, if $\hat{\mu}_0^N$ converges towards a compactly supported probability measure μ_0, for any $\mu_1 \in \mathcal{P}(I\!R)$,

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat{\mu}_1^N,\mu_1)<\delta) = \liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat{\mu}_1^N,\mu_1)<\delta) = -J_\beta(\mu_0,\mu_1).$$

Let us now reconsider the above limit and show that the limit must be taken at a free Brownian bridge. More precisely, we shall see that, if τ denotes the joint law of (X_0, X_1) (the precise sense of which being given below) and μ^τ the law of the free Brownian bridge (2.17) associated with (X_0, X_1),

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat{\mu}_1^N,\mu_1)<\delta) \le \sup_{\substack{\tau\circ X_0^{-1}=\mu_0\\ \tau\circ X_1^{-1}=\mu_1}}\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(\max_{1\le k\le n} d(\hat{\mu}_{t_k}^N,\mu_{t_k}^\tau)\le\delta)$$

for any family $\{t_1,\cdots,t_n\}$ of times in $[0,1]$. Therefore, the large deviation estimate obtained in [15] yields

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat{\mu}_1^N,\mu_1)<\delta) \le -\frac{\beta}{2}\inf\{S(\mu^\tau), \tau\circ X_0^{-1}=\mu_0, \tau\circ X_1^{-1}=\mu_1\}.$$

The lower bound estimate obtained in [15] therefore guarantees that

$$\inf\{S(\nu), \nu_0=\mu_0, \nu_1=\mu_1\} = \inf\{S(\mu^\tau), \tau\circ X_0^{-1}=\mu_0, \tau\circ X_1^{-1}=\mu_1\}.$$

Such kind of result were already obtained in [8] and [4].

Let us now be more precise. We recall that we can define the joint law of the two matrices X_0^N, X_1^N by the family

$$\hat{\mu}_{0,1}^N(F) = \mathrm{tr}_N(F(X_0^N, X_1^N))$$

when F is taken into a natural set $\mathcal{F}$ of test functions of two non-commutatives variables and $\mathrm{tr}_N(A) = N^{-1}\sum_{i=1}^N A_{ii}$. It is common in free probability to consider polynomial test functions. In [7], bounded analytic test functions were introduced for self-adjoint non-commutatives variables. $\mathcal{F} = \mathcal{CC}_{st}(\mathbb{C})$ is there the complex vector space generated by

$$F(X_1,X_2) = \prod_{1\le i\le n}^{\rightarrow}\frac{1}{z_i-\alpha_i^1X_1-\alpha_i^2X_2}$$

where $(z_i)_{1\le i\le n}$ belongs to $(\mathbb{C}\backslash I\!R)^n$, $(\alpha_i^k, 1\le k\le 2)_{i=1}^n$ to $(I\!R^2)^n$, and $\overrightarrow{\prod}$ is the non-commutative product.

We shall here use the very same set of functions and recall then that the space

$$\mathcal{M}_{0,1} = \{\tau\in\mathcal{F}^* : \tau(I)=1, \tau(FF^*)\ge 0, \tau(FG)=\tau(GF)\}$$

is a compact metric space. We denote by D a metric on $\mathcal{M}_{0,1}$. Let us recall [7] that if one considers the restriction $\mu_k = \tau \circ X_k^{-1}$ of τ to functions which only depends on one of the variables X_k, $k = 1, 2$, then μ_k is a probability measure on $\mathbb{R}$ (in fact the spectral measure of X_k) and that the topology inherited by duality from $\mathcal{F}$ is the *vague* topology, i.e. the topology generated by continuous compactly supported functions.

Since $\mathcal{M}_{0,1}$ is compact, for any $\epsilon > 0$, we can find $M \in \mathbb{N}$, $(\tau_k)_{1\le k\le M}$ so that $\mathcal{M}_{0,1} \subset \cup_{1\le k\le M}\{\tau : D(\tau, \tau_k) < \epsilon\}$ and therefore

$$\limsup_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N^2}\log \mathbb{P}(d(\hat{\mu}_1^N, \mu_1) < \delta) \quad \le \quad \max_{1\le k\le M} \limsup_{N\to\infty} \frac{1}{N^2}\log \mathbb{P}(d(\hat{\mu}_1^N, \mu_1) < \delta; D(\hat{\mu}_{0,1}^N, \tau_k) < \epsilon)$$

Now, conditionnally to X_1^N,

$$dX_t^N = dH_t^N - \frac{X_t^N - X_1^N}{1-t}dt$$

or equivalently

$$X_t^N = tX_1^N + (1-t)X_0^N + (1-t)\int_0^t (1-s)^{-1}dH_s^N.$$

Let us assume that $\hat{\mu}_{0,1}^N$ converges towards $\tau \in \mathcal{M}_{0,1}$ when N goes to infinity and that X_1^N, X_0^N remains uniformly bounded for the operator norm. In particular, $\hat{\mu}^N_{tX_1^N+(1-t)X_0^N}$ converges for any $t \in [0, 1]$ towards $\nu_t^\tau = \tau \circ (tX_1 + (1-t)X_0)^{-1}$;

$$\nu_t^\tau(f) = \tau(f(tX_1 + (1-t)X_0))$$

for any test function f. Therefore, Voiculescu's result implies that $\hat{\mu}^N_{X_t^N}$ converges towards the distribution μ_t^τ of $tX_1 + (1-t)X_0 + (1-t)\int_0^t (1-s)^{-1}dS_s$ with a free Brownian motion S, free with $tX_1 + (1-t)X_0$. We shall now extend this result in our topology and also control the dependence of this convergence with respect to the speed of convergence of the distribution of (X_0^N, X_1^N) towards τ.

We shall work below with given $(X_0^N, X_1^N) \in \{d(\hat{\mu}_0^N, \mu_0) < \delta; d(\hat{\mu}_1^N, \mu_1) < \delta; D(\hat{\mu}_{0,1}^N, \tau) < \epsilon\}$.

Let, for $u \le t$, $X_u^{N,t}$ denote the process

$$X_u^{N,t} = tX_1^N + (1-t)X_0^N + (1-t)\int_0^u (1-s)^{-1}dH_s^N.$$

Then, one deduces from Ito's calculus that for any test function f

$$\hat{\mu}^N_{X_u^{N,t}}(f) = \hat{\mu}^N_{tX_1^N+(1-t)X_1^N}(f) + \frac{(1-t)^2}{2}\int_0^u \hat{\mu}^N_{X_s^{N,t}} \otimes \hat{\mu}^N_{X_s^{N,t}}(\frac{f'(x)-f'(y)}{x-y})\frac{ds}{(1-s)^2} + M_f^N(u)$$

with a martingale $M_f^N(u)$ such that

$$\mathbb{E}[\sup_{u\in[0,t]}(M_f^N(u))^2] \le \frac{||f'||_\infty^2}{N^2}.$$

Moreover, it is not hard to check that $(\hat{\mu}^N_{X_u^{N,t}}, u \le t)$ is tight in $\mathcal{C}([0,1], \mathcal{P}(\mathbb{R}))$ (see the proof of exponential tightness of the spectral process of $X_0^N + H_t^N$ given in [15]). The limit points $(\mu_{X_u^t}, u \le t)$ (when $D(\hat{\mu}_{0,1}^N, \tau)$ goes to zero) satisfy the equation

$$\mu_{X_u^t}(f) = \nu_t^\tau(f) + \frac{(1-t)^2}{2}\int_0^u \mu_{X_s^t} \otimes \mu_{X_s^t}(\frac{f'(x)-f'(y)}{x-y})\frac{ds}{(1-s)^2}.$$

This equation admits a unique solution, as can be proved following the arguments of [8] or [15], p. 494. Taking $f(x) = e^{i\xi x}$, and substracting both equations, we find, with

$$\Delta_u^N(R) = \sup_{|\xi|\le R} \mathbb{E}[|\hat\mu^N_{X_u^{N,t}}(e^{i\xi x}) - \mu_{X_u^t}(e^{i\xi x})|],$$

that for $u \le t$

$$\Delta_u^N(R) \le \Delta_0^N(R) + 4R^2\int_0^u \Delta_s^N(R)ds + \frac{R}{N}$$

which yields thanks to Gronwall lemma and taken at $u = t$, since $\mu_t^\tau = \mu_{X_t^t}$,

$$\sup_{|\xi|\le R} \mathbb{E}[|\hat\mu^N_{X_t^N}(e^{i\xi x}) - \mu_t^\tau(e^{i\xi x})|] \le (\frac{R}{N} + \sup_{|\xi|\le R} \mathbb{E}[|\hat\mu^N_{tX_1^N+(1-t)X_0^N}(e^{i\xi x}) - \nu_t^\tau(e^{i\xi x})|])e^{4R^2 t}.$$

Therefore, if we define the distance d_F on $\mathcal{P}(\mathbb{R})$ by

$$d_F(\mu,\mu') = \int |\mu(e^{i\xi x}) - \mu'(e^{i\xi x})|e^{-4\xi^2}d\xi$$

we have proved that there exists a finite constant C such that for all $t \in [0,1]$,

$$\mathbb{E}[d_F(\hat\mu^N_{X_t^N}, \mu_t^\tau)] \le C d_F(\hat\mu^N_{tX_1^N+(1-t)X_0^N}, \nu_t^\tau) + \frac{C}{N}.$$

It is not hard to convince ourselves that d_F is a distance compatible with the weak topology on $\mathcal{P}(\mathbb{R})$.

Observe now that on $\{d(\hat\mu_1^N, \mu_1) < \delta, d(\hat\mu_0^N, \mu_0) < \delta\}$, $(\hat\mu^N_{tX_1+(1-t)X_0}, t \in [0,1])$ is tight for the usual weak topology so that for any $\epsilon > 0$ we can find $\kappa > 0$ so that for any τ and $t \in [0,1]$, $D(\tau, \hat\mu^N_{0,1}) < \epsilon$ implies

$$d_F(\hat\mu^N_{tX_1^N+(1-t)X_0^N}, \nu_t^\tau) < \kappa.$$

Therefore, for any $t_1, \cdots, t_n \in [0,1]$, for any $(X_0^N, X_1^N) \in \{d(\hat\mu_0^N, \mu_0) < \delta; d(\hat\mu_1^N, \mu_1) < \delta; D(\hat\mu^N_{0,1}, \tau) < \epsilon\}$, Chebyshev inequality yields

$$\mathbb{P}(\max_{1\le k\le n} d_F(\hat\mu^N_{X^N_{t_k}}, \mu^\tau_{t_k}) > \eta | X_1^N) \le nC(\kappa + \frac{1}{N})$$

with $\mu_t^\tau = \mu_{X_t}$ the distribution of $X_t = tX_1 + (1-t)X_0 + \sqrt{t(1-t)}S$ when the law of (X_0, X_1) is τ. Hence for any η, when κ (i.e ϵ) is small enough and N large enough,

$$\mathbb{P}(\max_{1\le k\le n} d_F(\hat\mu^N_{X^N_{t_k}}, \mu^\tau_{t_k}) < \eta | X_1^N) > \frac{1}{2}.$$

Hence

$$\mathbb{P}(d(\hat\mu_1^N, \mu_1) < \delta; D(\hat\mu^N_{0,1}, \tau) < \epsilon) \quad \le \quad 2\mathbb{P}(d(\hat\mu_1^N, \mu_1) < \delta; D(\hat\mu^N_{0,1}, \tau) < \epsilon, \max_{1\le k\le n} d_F(\hat\mu^N_{X^N_{t_k}}, \mu^\tau_{t_k}) < \eta).$$

We arrive at, for ϵ small enough and any $\tau \in \mathcal{M}_{0,1}$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(d(\hat\mu_1^N, \mu_1) < \delta; D(\hat\mu^N_{0,1}, \tau) < \epsilon) \le \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\max_{1\le k\le n} d_F(\hat\mu^N_{t_k}, \mu^\tau_{t_k}) < \delta).$$

Using the large deviation upper bound for the law of $(\hat\mu_t^N, t \in [0,1])$ from [15], we deduce

$$\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat\mu_1^N,\mu_1)<\delta) \quad\le\quad -\frac{\beta}{2}\min_{1\le p\le M}\inf_{\max_{1\le k\le n} d_F(\nu_{t_k},\mu_{t_k}^{\tau_p})\le\delta} S(\nu)$$

We can now let ϵ going to zero, and then δ going to zero, and then n going to infinity, to obtain, since S is a good rate function, that

$$\limsup_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat\mu_1^N,\mu_1)<\delta) \quad\le\quad -\frac{\beta}{2}\inf_{\substack{\tau:\tau\circ X_0^{-1}=\mu_0\\ \tau\circ X_1^{-1}=\mu_1}} S(\mu^\tau).$$

Since it was also proved in [15] that

$$\liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}(d(\hat\mu_1^N,\mu_1)<\delta)\ge -\frac{\beta}{2}\inf_{\substack{\nu_0=\mu_0\\ \nu_1=\mu_1}} S(\nu)$$

we obtain

$$\inf_{\substack{\nu_0=\mu_0\\ \nu_1=\mu_1}} S(\nu) = \inf_{\substack{\tau:\tau\circ X_0^{-1}=\mu_0\\ \tau\circ X_1^{-1}=\mu_1}} S(\mu^\tau).$$

Hence, if $\mathrm{FBB}(\mu_0,\mu_1)$ is the set of laws of free Brownian bridges between μ_0 and μ_1, i.e

$$\mathrm{FBB}(\mu_0,\mu_1)=\{\mu^\tau,\tau\circ X_0^{-1}=\mu_0,\tau\circ X_1^{-1}=\mu_1\},$$

we have seen that

$$\inf\{S(\nu),\nu_0=\mu_0,\nu_1=\mu_1\}=\inf\{S(\nu),\nu\in\mathrm{FBB}(\mu_0,\mu_1)\}.$$

To finish the proof of Theorem 2.6, we need to show that $\mathrm{FBB}(\mu_0,\mu_1)$ is a closed subset of $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$ so that indeed the infimum is reached in $\mathrm{FBB}(\mu_0,\mu_1)$.

Observe here that μ^τ does depend only partially on τ since it only depends on $\{\nu_t^\tau,t\in[0,1]\}$. Noting that

$$\nu_t^\tau(x^p)=\sum_{r=0}^p t^r\,\tau(P_{r,p}(X_1-X_0,X_0))$$

with $P_{r,p}(X,Y)$ the sum over all the monomial functions with total degree p and degree r in X, we see that μ^τ only depends on the restriction of τ to polynomial functions $P\in\mathcal{S}=\{P_{r,p},0\le r\le p<\infty\}$. Of course,

$$\mathcal{M}_{0,1}^{\mathcal{S},C}=\{\tau|_{\mathcal{S}},\tau\in\mathcal{M}_{0,1},\tau(X^{2p}+Y^{2p})\le 2C^{2p},\forall p\in\mathbb{N}\}$$

is closed for the dual topology generated by the polynomial functions of $\mathcal{S}$. Here C denotes a common uniform bound on X_0 and X_1, and we have

$$\mathrm{FBB}(\mu_0,\mu_1)=\{\mu^{\tau|_{\mathcal{S}}},\tau\in\mathcal{M}_{0,1}\}=\{\mu^\kappa,\kappa\in\mathcal{M}_{0,1}^{\mathcal{S},C}\}.$$

We denote, for $\kappa\in\mathcal{M}_{0,1}^{\mathcal{S},C}$ and $t\in[0,1]$, $\nu_t^\kappa\in\mathcal{P}(\mathbb{R})$ the distribution of $tX_1+(1-t)X_0$ when the joint distribution of (X_0,X_1) restricted to $\mathcal{S}$ is κ. Then, $\mu_t^\kappa=\nu_t^\kappa\boxed{+}\sigma_{t(1-t)}$. We now show that $\mathrm{FBB}(\mu_0,\mu_1)$ is a closed set of $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, which insures, since S is a good rate function on $\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, that the infimum is achieved on $\mathrm{FBB}(\mu_0,\mu_1)$. Indeed, if μ^n is a sequence of $\mathrm{FBB}(\mu_0,\mu_1)$ given by $\{\nu_t^{\kappa_n} \boxed{+} \sigma_{t(1-t)}, t\in[0,1]\}$, the weak convergence of μ^n implies the weak convergence of κ^n. Indeed, for any $p\in\mathbb{N}$, any $t\in[0,1]$,

$$\mu_t^n(x^p) = \nu_t^{\kappa_n}(x^p) + P_t(\mu_t^n(x^l), l\le p-1)$$

with a polynomial function P_t. Hence, by induction, the convergence of $(\mu_t^n(x^p))_{p\in\mathbb{N}}$ (recall that μ^n is supported by $[-C-2,C+2]$ for any n so that weak convergence is equivalent to moments convergence) results with the convergence of $(\nu_t^{\kappa_n}(x^p)))_{p\in\mathbb{N}}$, and again, since $(\nu_t^{\kappa_n})_{n\in\mathbb{N}}$ is supported by $[-C,C]$, with the weak convergence of $\nu_t^{\kappa_n}$ towards some probability measure ν_t. Since this convergence holds for any $t\in[0,1]$, we can expend the moments in powers of the time variable to conclude that κ_n converges towards $\kappa\in\mathcal{M}_{0,1}^{\mathcal{S},C}$. Again by free convolution calculus, this convergence results with the convergence of μ^n towards $\mu^\kappa\in\mathrm{FBB}(\mu_0,\mu_1)$. Hence, $\mathrm{FBB}(\mu_0,\mu_1)$ is closed.

4.2 Proof of Lemma 2.3:

In [15] (see (2.13) and Lemma 2.10) O. Zeitouni and I proved that for any path $\nu\in\mathcal{C}^1([0,1],\mathcal{P}(\mathbb{R}))$, there exists a path $\nu^{\epsilon,\Delta}$ such that

$$\limsup_{\epsilon,\Delta\downarrow 0} S^{0,1}(\nu^{\epsilon,\Delta}) = S_{\mu_0}(\nu).$$

This path was constructed as follows. Let P_ϵ be the Cauchy law with parameter ϵ and set $\mu^\epsilon = P_\epsilon * \mu$ be the convoluted path with the Cauchy law. Moreover, if $0=t_1<t_2<\ldots<t_n=1$ with $t_i=(i-1)\Delta$, we set, for $t\in[t_k,t_{k+1}[$,

$$\nu_t^{\epsilon,\Delta} = \nu_{t_k}^\epsilon + \frac{(t-t_k)}{\Delta}[\nu_{t_{k+1}}^\epsilon - \nu_{t_k}^\epsilon].$$

Let us therefore consider $S^{0,1}(\nu^{\epsilon,\Delta})$. Because we took the convolution with respect to the Cauchy law, the Hilbert transform $H\nu_t^{\epsilon,\Delta}$ is well defined, and actually a continuously differentiable function with respect to the time variable and an analytic function with respect to the space variable. Henceforth, in the supremum defining $S^{0,1}(\nu^{\epsilon,\Delta})$, we can actually make the change of function $f(t,x)\to f(t,x)-\int\log|x-y|d\nu_t^{\epsilon,\Delta}(y)$. Observing that, with $\nu_i^\epsilon=\nu_i * P_\epsilon$ for $i\in\{0,1\}$,

$$\int_0^1\int\partial_t\left(\int\log|x-y|^{-1}d\nu_t^{\epsilon,\Delta}(y)\right)d\nu_t^{\epsilon,\Delta}(x)dt = \frac{1}{2}\left(\Sigma(\nu_1^\epsilon)-\Sigma(\nu_0^\epsilon)\right),$$

we find that

$$\begin{aligned}
S^{0,1}(\nu^{\epsilon,\Delta}) &= \frac{1}{2}\left(\Sigma(\nu_1^\epsilon)-\Sigma(\nu_0^\epsilon)\right)+\frac{1}{2}\int_0^1\int(H\nu_t^{\epsilon,\Delta})^2 d\nu_t^{\epsilon,\Delta}dt \\
&\quad + \sup_{f\in\mathcal{C}_b^{2,1}([0,1]\times\mathbb{R})}\{\int f_1 d\mu_1^\epsilon - \int f_0 d\nu_0^\epsilon - \int_0^1\int\partial_t f_t d\nu_t^{\epsilon,\Delta}dt - \frac{1}{2}< f,f >_{\nu^{\epsilon,\Delta}}^{0,1}\} \\
&\ge \frac{1}{2}\left(\Sigma(\nu_1^\epsilon)-\Sigma(\nu_0^\epsilon)\right)+\frac{1}{2}\int_0^1\int(H\nu_t^{\epsilon,\Delta})^2 d\mu_t^{\epsilon,\Delta}dt
\end{aligned}$$

where in the last line we observed that

$$
\begin{aligned}
&\sup_{f\in\mathcal{C}_b^{2,1}([0,1]\times\mathbb{R})}\{\int f_1 d\nu_1^\epsilon - \int f_0 d\nu_0^\epsilon - \int_0^1\int \partial_t f_t d\nu_t^{\epsilon,\Delta}dt - \frac{1}{2} < f,f >_{\nu^{\epsilon,\Delta}}^{0,1}\} \\
= &\sup_{f\in\mathcal{C}_b^{2,1}([0,1]\times\mathbb{R})}\sup_{\lambda\in\mathbb{R}}\{\lambda\int f_1 d\nu_1^\epsilon - \lambda\int f_0 d\nu_0^\epsilon - \lambda\int_0^1\int \partial_t f_t d\nu_t^{\epsilon,\Delta}dt - \frac{\lambda^2}{2} < f,f >_{\nu^{\epsilon,\Delta}}^{0,1}\} \\
= &\frac{1}{2}\sup_{f\in\mathcal{C}_b^{2,1}([0,1]\times\mathbb{R})}\left(\frac{(\int f_1 d\nu_1^\epsilon - \int f_0 d\nu_0^\epsilon - \int_0^1\int \partial_t f_t d\nu_t^{\epsilon,\Delta})^2}{< f,f >_{\nu^{\epsilon,\Delta}}^{0,1}}\right) \\
\geq &\ 0
\end{aligned}
$$

Observing that

$$\int_0^1\int (H\nu_t^{\epsilon,\Delta})^2 d\mu_t^{\epsilon,\Delta}dt = \Delta\sum_{k=0}^{[\frac{1}{\Delta}]}\int (H\nu_{t_k}^\epsilon)^2 d\nu_{t_k}^\epsilon$$

converges since $t\to H\nu_t^\epsilon$ and $t\to\nu_t^\epsilon$ are continuous for any $\nu\in\mathcal{C}([0,1],\mathcal{P}(\mathbb{R}))$, we arrive at

$$\liminf_{\Delta\downarrow 0} S^{0,1}(\nu^{\epsilon,\Delta}) \quad\geq\quad \frac{1}{2}\left(\Sigma(\nu_1^\epsilon)-\Sigma(\nu_0^\epsilon)\right)+\frac{1}{2}\int_0^1\int (H\nu_t^\epsilon)^2 d\nu_t^\epsilon dt \tag{4.1}$$

Notice that for $t\in\{0,1\}$,

$$\Sigma(\nu_t^\epsilon) = \int \log|x-y|^{-1} dP_\epsilon * \nu_t(x) dP_\epsilon * \nu_t(y) = \frac{1}{2}\int \log((x-y)^2+\epsilon^2)^{-1} d\nu_t(x)d\nu_t(y).$$

Hence, monotone convergence theorem asserts that

$$\lim_{\epsilon\downarrow 0}\Sigma(\nu_t^\epsilon) = \Sigma(\nu_t).$$

Now, recall that for any $\rho\in L_3$, Tricomi [24] p. 169 asserts that

$$\frac{\pi^2}{2}\rho(x)^2 = \frac{1}{2}(H\rho)^2(x) - H(\rho(H\rho))(x)\,,$$

so that

$$\int (H\rho)^2(x)\rho(x)dx = \frac{\pi^2}{3}\int (\rho(x))^3 dx.$$

Since, for any $\epsilon>0$, ν_t^ϵ is absolutely continuous with respect to Lebesgue measure with density $\rho_t^\epsilon\in L_3(dx)$ for almost all $t\in[0,1]$, we conclude by Fatou's lemma that that

$$+\infty > \liminf_{\epsilon\downarrow 0}\liminf_{\Delta\downarrow 0} S^{0,1}(\nu^{\epsilon,\Delta}) \quad\geq\quad \frac{1}{2}\left(\Sigma(\nu_1)-\Sigma(\nu_0)\right)+\frac{\pi^2}{6}\int_0^1\int \liminf_{\epsilon\downarrow 0}(\rho_t^\epsilon(x))^3 dx dt.$$

Finally, it is easy to see that (4.1) implies that $\mu_t(dx)=\rho_t(x)dx$ for almost all $t\in(0,1)$ and then that $\rho_t^\epsilon(x)$ converges towards ρ_t almost surely. Hence, we have proved that

$$S_{\mu_0}(\nu_.) \geq \frac{1}{2}(\Sigma(\nu_1) - \Sigma(\nu_0)) + \frac{\pi^2}{6}\int_0^1 \int (\rho_t(x))^3 dx dt.$$

∎

Acknowledgments : I am very much indebted towards C. Villani and O. Zeitouni whose careful reading of preliminary versions of the manuscript, wise remarks and encouragments were crucial in this research. I am also very grateful to D. Serre and Y. Brenier for stimulating discussions.

BIBLIOGRAPHY

[1] G. BEN AROUS, A. GUIONNET, Large deviations for Wigner's law and Voiculescu's non-commutative entropy, *Prob. Th. Rel. Fields 108*, 517–542 (1997).

[2] H. BERCOVICI and D. VOICULESCU ; Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, **42** (1993), 733–773.

[3] P.BIANE On the Free convolution with a Semi-circular distribution *Indiana University Mathematics Journal* **46** (1997) 705-718

[4] P. BIANE, M. CAPITAINE, A. GUIONNET ; Large deviation bounds for matrix Brownian motion. Preprint (2002)

[5] Y. BRENIER, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations *Comm. Pure. Appl. Math.* **52** 411-452(1999)

[6] H. BRÉZIS, Functional analysis, Masson, Paris(1983)

[7] T. CABANAL-DUVILLARD, A. GUIONNET ; Large deviations upper bounds and non commutative entropies for some matrices ensembles, *Annals Probab.* **29** 1205-1261 (2001).

[8] T. CABANAL-DUVILLARD, A. GUIONNET ; Discussions around non-commutative entropies, To appear in *Adv. Math.* (2002).

[9] S. CHADHA, G. MADHOUX, M. L. MEHTA; A method of integration over matrix variables II, *J. Phys. A.* **14** (1981), 579–586.

[10] P. DEIFT, T. KRIECHERBAUER, K.T.-R. McLAUGHLIN ; New results on the equilibrium measure for logarithmic potentials in the presence of an external field *J. Approx. Theory* **95** 388-475(1998)

[11] A. DEMBO, O. ZEITOUNI ; *Large deviations techniques and applications*, second edition, Springer (1998).

[12] B. EYNARD ; Eigenvalue distribution of large random matrices, from one matrix to several coupled matrices, *Nuclear Phys. B.* **506**, 633–664 (1997).

[13] B. EYNARD; Random matrices, http://www-spht.cea.fr/lectures-notes.shtml

[14] A. GUIONNET ; Large deviation upper bounds and central limit theorems for band matrices, *Ann. Inst. H. Poincaré Probab. Statist* **38** 341-384 (2002)

[15] A. GUIONNET ; O. ZEITOUNI ; Large deviations asymptotics for spherical integrals Vol. **188**, 461-515 (2001)

[16] J. HARER, D. ZAGIER, The Euler caracteristic of the moduli space of curves *Invent. Math.* **85** 457-485(1986)

[17] G. MAHOUX, M. MEHTA ; A method of integration over matrix variables III, *Indian J. Pure Appl. Math.***22** 531-546 (1991)

[18] A. MATYTSIN ; On the large N-limit of the Itzykson-Zuber integral, *Nuclear Physics* **B411** (1994), 805–820.

[19] A. MATYTSIN, P. ZAUGG; Kosterlitz-Thouless phase transitions on discretized random surfaces, *Nuclear Physics* **B497** , 699–724(1997).

[20] M. L. MEHTA; *Random matrices*, 2nd ed. Academic Press (1991).

[21] M. L. MEHTA; A method of integration over matrix variables, *Comm. Math. Phys.* **79**, 327–340(1981).

[22] D. SERRE ; Sur le principe variationnel des équations de la mécanique des fluides parfaits. *Math. Model. Num. Analysis* **27** 739-758(1993)

[23] SZAREK, S and D. VOICULESCU; Volumes of restricted Minkowsky Sums and the Free analogue of the Entropy Power Inequality *Com. Math. Phys.* **178** 563-570 (1996)

[24] F. G. TRICOMI ; *Integral equations*, Interscience, New York (1957).

[25] D. VOICULESCU ; The analogues of Entropy and Fisher's Information Measure in Free Probability Theory, V : Noncommutative Hilbert Transforms *Invent. Math.* **132** 189-227(1998)

[26] D. VOICULESCU ; Lectures on free probability theory, *Lecture Notes in Mathematics*, **1738**, 283–349 (2000).

[27] E. WIGNER ; On the distribution of the roots of certain symmetric matrices, *Ann. Math.* **67**, 325-327 (1958).

[28] P. ZINN-JUSTIN ; Universality of correlation functions of hermitian random matrices in an external field, *Comm. Math. Phys.* **194**, 631–650 (1998).

[29] P. ZINN-JUSTIN ; The dilute Potts model on random surfaces, *J. Stat. Phys.* **98**, 245–264 (2000).

[30] A. ZVONKIN, Matrix integrals and Map enumeration ; an accessible introduction*Math. Comput. Modelling* **26** 281-304 (1997)