CUGLIANDOLO-KURCHAN EQUATIONS FOR DYNAMICS OF SPIN-GLASSES.

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ABSTRACT. We study the Langevin dynamics for the family of spherical p-spin disordered mean-field models of statistical physics. We prove that in the limit of system size N approaching infinity, the empirical state correlation and integrated response functions for these N-dimensional coupled diffusions converge almost surely and uniformly in time, to the non-random unique strong solution of a pair of explicit non-linear integro-differential equations, first introduced by Cugliandolo and Kurchan.

1. INTRODUCTION AND MAIN RESULTS

Markovian dynamics with random interactions can produce very complex phase transitions, and fascinating long time behaviors, for strong disorder (or low temperature). The physics literature has shown that dynamics of mean-field spin glasses is a very good field to get an accurate sample of possible and generic long time phenomena (as aging, memory, rejuvenation, failure of the Fluctuation-Dissipation theorem, see [8] for a good survey).

This class of problems can be roughly described as follows. Let Γ (a compact metric space) be the state space for spins and ν be a probability measure on Γ. Typically, in the discrete or Ising spins context $\Gamma = \{-1, 1\}$ and $\nu = 1/2(\delta_1 + \delta_{-1})$. In the continuous or soft spin context, $\Gamma = I$ a compact interval of the real line and $\nu(dx) = Z^{-1}e^{-U(x)}dx$, where $U(x)$ is the "one-body potential". For each configuration of the spin system, i.e. for each $\mathbf{x} = (x_1, ..., x_N) \in \Gamma^N$ one defines a random Hamiltonian, $H^N_{\mathbf{J}}(\mathbf{x})$, as a function of the configuration $\mathbf{x}$ and of an exterior source of randomness $\mathbf{J}$, i.e. a random variable defined on another probability space. The Gibbs measure at inverse temperature β is then defined on the configuration space Γ^N by

$$\mu^N_{\beta,\mathbf{J}}(d\mathbf{x}) = \exp(-\beta H^N_{\mathbf{J}}(\mathbf{x}))\nu(d\mathbf{x})/Z^N_{\mathbf{J}}$$

The statics problem amounts to understanding the large N behavior of these measures for various classes of random Hamiltonians ([21] is a recent and beautiful book on the mathematical results pertaining to these equilibrium problems). The dynamics question consists of understanding the behavior of Markovian processes on the configuration space Γ^N, for which the Gibbs measure is invariant and even reversible, in the limit of large systems (large N) and long times, either when the randomness $\mathbf{J}$ is fixed (the quenched case) or when it is averaged (often called the annealed case, in the mathematics literature, but not in the physics papers).

These dynamics are typically Glauber dynamics for the discrete spin setting, or Langevin dynamics for continuous spins. Defining precisely what we mean here by large system size and long time is a very important question, and very different results can be expected (and sometimes proved, see [3] and references therein), for various time-scales as functions of the size of the system. We will restrict ourselves to the case where the system size is first taken to infinity, i.e for the shortest possible long time scales, much too short typically to allow any escape from meta-stable states (as opposed to the situation in [4] for instance). The first step is then to derive limiting equations for various quantities, when N tends to ∞. The second step is to understand the large time behavior of these limiting macroscopic equations. Let us be more specific by describing one of the main initial objectives of the theory, which is the long time behavior of the Langevin dynamics of the

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Sherrington-Kirkpatrick model and its generalization, the p-spin model. In the SK model, either for discrete or for continuous spins, the Hamiltonian is given by:

$$H_{\mathbf{J}}^N(\mathbf{x}) = \sum_{1 \le i,j \le N} J_{\{ij\}} x^i x^j ,$$

where the randomness is due to the couplings $J_{\{ij\}}$ which are assumed to be i.i.d Gaussian centered, of variance N^{-1} in case $i \neq j$ and $2N^{-1}$ in case $i = j$. In the p-spins model the Hamiltonian contains interaction between subsets of spins of size p, that is,

$$H_{\mathbf{J}}^N(\mathbf{x}) = \sum_{1 \le i_1 < \ldots < i_p \le N} J_{i_1 \ldots i_p} x^{i_1} \ldots x^{i_p} , \tag{1.1}$$

where the couplings $J_{i_1 \ldots i_p}$ are assumed to be i.i.d. Gaussian centered and of variance of $O(N^{-(p-1)})$. The initial SK model corresponds of course to the case $p = 2$.

Propagation of chaos for dynamics of the SK model, or equivalently the large N limit for the behavior of the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{x^i(t)}$ has been obtained some time ago both for continuous spins (see [19, 20], in the physics literature and [6, 7, 15] for a mathematical treatment) or discrete ones (see [14]). The limiting equations (the so called self-consistent single-spin dynamics) are very complex and have resisted so far all attempts to understand their long time behavior. This is due in part to the fact that the empirical measure is a much too rich object. Finding an autonomous system of tractable equations for a proper well chosen set of lower dimensional quantities is still a very open question, even in the physics literature.

But a large range of interesting and related models have been recently analyzed more successfully in the physics literature, i.e the spherical p-spin models (see [8, 10]). The spherical version of the spin glass models consists of a classical simplification, that is, replacing the product structure of the configuration space Γ^N by the sphere $S^{N-1}(\sqrt{N})$ of radius $\sqrt{N}$ in $\mathbb{R}^N$, in effect, imposing on the configuration $\mathbf{x}$ a hard constraint $\frac{1}{N}\sum_{i=1}^N x_i^2 = 1$. The spherical p-spin Gibbs measure is thus the probability measure on the sphere $S^{N-1}(\sqrt{N})$ given by

$$\mu_{\beta,\mathbf{J}}^N(d\mathbf{x}) = \exp(-\beta H_{\mathbf{J}}^N(\mathbf{x}))\nu_N(d\mathbf{x})/Z_{\mathbf{J}}^N ,$$

where the Hamiltonian is given by (1.1) and the measure ν_N is the uniform measure on the sphere $S^{N-1}(\sqrt{N})$. One can also study a very similar problem by replacing the hard spherical constraint by a soft one, i.e by replacing the uniform measure ν_N on the sphere $S^{N-1}(\sqrt{N})$ by a measure on $\mathbb{R}^N$

$$\nu_N(d\mathbf{x}) = \exp\big(- N f(\frac{1}{N}\sum_{i=1}^N x_i^2)\big) d\mathbf{x}/Z^N ,$$

where f is a smooth function growing fast enough at infinity. This study has been done successfully (see [5], and [11]) in the case where $p = 2$, the spherical SK model. There one could obtain a very complete description of the limiting dynamics using only one quantity, the empirical state correlation function

$$C_N(s,t) = \frac{1}{N}\sum_{i=1}^N x^i(s) x^i(t) .$$

Indeed, this quantity was shown to have a non-random limit $C(s,t)$, when N tends to ∞, which satisfies an autonomous integro-differential equation. This is a rather simple setting, when $p = 2$, since this quadratic case can use the well know tools pertaining to the study of spectra of GOE random matrices, after diagonalization of the random matrix (J_{ij}). The results, though showing an aging phenomenon, are very far from what is expected for the true (i.e non spherical) SK model. The case of $p > 2$ is a completely different story. The physics literature, mainly Cugliandolo and Kurchan (see [8, 10, 12]), has given a very rich picture of the behavior of the Langevin dynamics for these spherical p-spin models, which are believed to mimic in certain cases the behavior of the dynamics of the true SK model. The main innovation is the fact that the description of the limiting dynamics relies now on two objects, the empirical correlation function and the response function (and not only one as in the spherical $p = 2$ case or the whole infinite dimensional object of the true, non-spherical, SK model). These two functions are believed to satisfy a set of coupled non-linear integro-differential equations (which of course decouple in the $p = 2$ case). Our work establishes rigorously the asymptotic validity of these Cugliandolo-Kurchan equations (see [12] for the physical derivation of these). We shall devote a future work to the fascinating question of understanding the large time behavior of the solution to these equations (see [8] or [10] to see what is expected). Some of the needed mathematical tools for attacking this problem can be found in [16], as well as an intriguing link with equations arising naturally in non-commutative probability.

We now describe more precisely our result and then the structure of this paper.

Fixing a positive integer N (denoting the system size), we consider the mean field random Hamiltonian

$$H^N_{\mathbf{J}}(\mathbf{x}) = -2\sum_{p=1}^{m} \frac{a_p}{p!} \sum_{1\le i_1,\dots,i_p\le N} J_{i_1\dots i_p} x^{i_1}\dots x^{i_p}\,, \tag{1.2}$$

where $m \ge 2$, the state variable is $\mathbf{x} = (x^1,\dots,x^N) \in \mathbb{R}^N$, and the disorder parameters $J_{i_1\dots i_p} = J_{\{i_1,\dots,i_p\}}$ are independent (modulo the permutation of the indices) centered Gaussian variables. The variance of $J_{i_1\dots i_p}$ is $c(\{i_1,\dots,i_p\})N^{-p+1}$, where

$$c(\{i_1,\dots,i_p\}) = \prod_k l_k!\,, \tag{1.3}$$

and $(l_1, l_2, \dots)$ are the multiplicities of the different elements of the set $\{i_1,\dots,i_p\}$ (for example, $c = 1$ when $i_j \ne i_{j'}$ for any $j \ne j'$, while $c = p!$ when all i_j values are the same).

Let f be a differentiable function on $\mathbb{R}_+$ with f' locally Lipschitz, such that

$$\sup_{\rho\ge 0} |f'(\rho)|(1+\rho)^{-r} < \infty \tag{1.4}$$

for some $r < \infty$, and for some $A, \delta > 0$,

$$\inf_{\rho\ge 0}\{f'(\rho) - A\rho^{m/2+\delta-1}\} > -\infty \tag{1.5}$$

(typically, $f(\rho) = \kappa(\rho-1)^r$ for some $r > m/2$ and $\kappa \gg 1$). At temperature $\beta^{-1} > 0$, the soft spherical version of the system corresponds to the equilibrium probability measure $\mu^N_{\beta,\mathbf{J}}$ on $\mathbb{R}^N$ whose density (with respect to Lebesgue measure) is

$$\frac{d\mu^N_{\beta,\mathbf{J}}}{d\mathbf{x}} = Z^{-1}_{\beta,\mathbf{J}} e^{-\beta H^N_{\mathbf{J}}(\mathbf{x}) - Nf(N^{-1}\|\mathbf{x}\|^2)}\,,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^N$ and the normalization factor $Z_{\beta,\mathbf{J}} = \int e^{-\beta(H^N_{\mathbf{J}}(\mathbf{x}) - Nf(N^{-1}\|\mathbf{x}\|^2)}d\mathbf{x}$ is a.s. finite (by (1.5)). Recall that $\mu^N_{\beta,\mathbf{J}}$ is the invariant measure of the randomly interacting particles described by the (Langevin) stochastic differential system:

$$dx^j_t = dB^j_t - f'(N^{-1}\|\mathbf{x}_t\|^2)x^j_t dt + \beta G^j(\mathbf{x}_t)dt\,, \tag{1.6}$$

where $\mathbf{B} = (B^1,\dots,B^N)$ is an N-dimensional standard Brownian motion, independent of both the initial condition $\mathbf{x}_0$ and the disorder $\mathbf{J}$, while

$$G^i(\mathbf{x}) := -\frac{1}{2}\partial_{x^i}\left(H^N_{\mathbf{J}}(\mathbf{x})\right) = \sum_{p=1}^{m} \frac{a_p}{(p-1)!} \sum_{1\le i_1,\dots,i_{p-1}\le N} J_{ii_1\dots i_{p-1}} x^{i_1}\dots x^{i_{p-1}}\,, \tag{1.7}$$

for $i = 1,\dots,N$.

In Proposition 2.1 we prove that for a.e. disorder $\mathbf{J}$, initial condition $\mathbf{x}_0$ and Brownian path $\mathbf{B}$, there exists a unique strong solution of (1.6) for all $t \ge 0$, whose law we denote by $\mathbb{P}^N_{\beta,\mathbf{x}_0,\mathbf{J}}$.

We are interested in the time evolution of the empirical state correlation function

$$C_N(s,t) := \frac{1}{N}\sum_{i=1}^{N} x_s^i x_t^i \,, \tag{1.8}$$

and that of the empirical integrated response function

$$\chi_N(s,t) := \frac{1}{N}\sum_{i=1}^{N} x_s^i B_t^i \,, \tag{1.9}$$

under the quenched law $\mathbb{P}^N_{\beta,\mathbf{x}_0,\mathbf{J}}$, as the system size $N \to \infty$. Note that since $\{\chi_N(s,t), 0 \le t \le s \le T\}$ is not determined by $N^{-1}\sum \delta_{x^i_{[0,T]}}$, the strategy by which the limiting equations in [6, 7] are derived, does not apply directly in our case.

We assume hereafter that the initial condition $\mathbf{x}_0$ is independent of the disorder $\mathbf{J}$, and the limit

$$\lim_{N\to\infty} \mathbb{E} C_N(0,0) = C(0,0) \,, \tag{1.10}$$

exists, and is a finite. Further, we assume that the tail probabilities $\mathbb{P}(|C_N(0,0) - C(0,0)| > x)$ decay exponentially fast in N (so the convergence $C_N(0,0) \to C(0,0)$ holds almost surely), and that for each $k < \infty$, the sequence $N \mapsto \mathbb{E}[C_N(0,0)^k]$ is uniformly bounded.

To be more specific, we consider hereafter the product probability space $\mathcal{E}_N = \mathbb{R}^N \times \mathbb{R}^{d(N,m)} \times \mathbb{C}([0,T],\mathbb{R}^N)$ (here T is a fixed time and $d(N,m)$ is the dimension of the space of the interactions $\mathbf{J}$), equipped with the natural Euclidean norms for the finite dimensional parts, i.e $(\mathbf{x}_0, \mathbf{J})$, and the sup-norm for the Brownian motion $\mathbf{B}$. The space $\mathcal{E}_N$ is endowed with the product probability measure $\mathbb{P} = \mu_N \otimes \gamma_N \otimes P_N$, where μ_N denotes the distribution of $\mathbf{x}_0$, γ_N is the (Gaussian) distribution of the coupling constants $\mathbf{J}$, and P_N is the distribution of the N-dimensional Brownian motion.

Hypothesis 1.1. *For $(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) \in \mathcal{E}_N$ we introduce the norms*

$$\|(\mathbf{x}_0,\mathbf{J},\mathbf{B})\|^2 = \sum_{i=1}^{N} (x_0^i)^2 + \sum_{p=1}^{m} \sum_{1\le i_1 \dots i_p \le N} (N^{\frac{p-1}{2}} J_{i_1\cdots i_p})^2 + \sup_{0\le t\le T} \sum_{i=1}^{N} (B_t^i)^2 \,.$$

We shall assume that μ_N is such that the following concentration of measure property holds on $\mathcal{E}_N$; there exists two finite positive constants C and α, independent on N, such that, if V is a Lipschitz function on $\mathcal{E}_N$, with Lipschitz constant K, then for all $\rho > 0$,

$$\mu_N \otimes \gamma_N \otimes P_N[|V - \mathbb{E}[V]| \ge \rho] \le C^{-1} \exp(-C(\frac{\rho}{K})^\alpha) \,.$$

The above concentration inequality holds for any Lipschitz function V that does not depend on $\mathbf{x}_0$, since the concentration of measure property holds for the Gaussian measures $\gamma_N \otimes P_N$, with $\alpha = 2$ (c.f. [2]). Unfortunately, assuming that the concentration of measure property holds for the measures μ_N on $\mathbb{R}^N$ does not assure the concentration of measure for the product measure $\mu_N \otimes \gamma_N \otimes P_N$. Hence, we shall have to assume a property which implies the concentration inequality of Hypothesis 1.1 and further tensorizes (see [2] for a more thorough discussion). For example, if μ_N satisfy the Poincaré inequality uniformly in N, then the product measures $\mu_N \otimes \gamma_N \otimes P_N$ also satisfy the Poincaré inequality uniformly in N, since the Gaussian measure $\gamma_N \otimes P_N$ does. The required uniform in N concentration property is then satisfied, with $\alpha = 1$ (c.f. [1, 2]).

Our main result is the proof that as $N \to \infty$ the functions $C_N(s,t)$ and $\chi_N(s,t)$ converge to non-random functions $C(s,t)$ and $\chi(s,t)$, that are characterized by the following theorem.

Theorem 1.2. *Let $\psi(r) = \nu'(r) + r\nu''(r)$ and*

$$\nu(r) := \sum_{p=1}^{m} \frac{a_p^2}{p!} r^p . \tag{1.11}$$

Suppose μ_N satisfies hypothesis 1.1. Fixing any $T < \infty$, as $N \to \infty$ the random functions C_N and χ_N converge uniformly on $[0,T]^2$, almost surely and in L_p with respect to $\mathbf{x}_0$, $\mathbf{J}$ and $\mathbf{B}$ to non-random functions $\chi(s,t) = \int_0^t R(s,u)du$ and $C(s,t) = C(t,s)$. Further, $R(s,t) = 0$ for $t > s$, $R(s,s) = 1$, and for $s > t$ the absolutely continuous functions C, R and $K(s) = C(s,s)$ are the unique solution in the space of bounded, continuous functions, of the integro-differential equations

$$\partial_s R(s,t) = -f'(K(s))R(s,t) + \beta^2 \int_t^s R(u,t)R(s,u)\nu''(C(s,u))du, \tag{1.12}$$

$$\partial_s C(s,t) = -f'(K(s))C(s,t) + \beta^2 \int_0^s C(u,t)R(s,u)\nu''(C(s,u))du + \beta^2 \int_0^t \nu'(C(s,u))R(t,u)du, \tag{1.13}$$

$$\partial_s K(s) = -2f'(K(s))K(s) + 1 + 2\beta^2 \int_0^s \psi(C(s,u))R(s,u)du, \tag{1.14}$$

where the initial condition $K(0) = C(0,0)$ is determined by (1.10).

We note in passing that for $p = 2$, i.e. $\nu(r) = r^2/2$, we get from (1.12) the autonomous equation

$$\partial_s H(s,t) = \beta^2 \int_t^s H(u,t)H(s,u)du\,, \quad H(t,t) = 1\,,$$

for $H(s,t) = R(s,t)\exp(\int_t^s f'(K(u))du)$, whose unique solution is the Laplace transform of the semi-circle probability measure, evaluated at $\beta(s-t)$. Plugging this expression in (1.13) and (1.14), upon scaling both the function f and time by a factor β, we recover the limiting equation of [5, (2.16)] after some integrations by parts. Further, setting $K(s) = 1$ and $\partial_s K(s) = 0$ in (1.14), while replacing $f'(K(s))$ in (1.12)–(1.14) by a time varying constant $z(s)$, corresponds to the hard spherical constraint of [12]. Indeed, the limiting equations of [12] are thus recovered.

Note that the constant β can be embedded into $\{a_p\}$ resulting with $\beta G^j(\cdot) \mapsto G^j(\cdot)$ and then having $\beta = 1$ in the stochastic differential system (1.6). Adopting this convention, we thus take hereafter $\beta = 1$. It is trivial to check that the explicit dependence of (1.12)–(1.14) on β is indeed as stated, i.e., with each appearance of $\nu'(\cdot)$, $\nu''(\cdot)$ (and $\psi(\cdot)$) multiplied by β^2.

The empirical quantities $K_N(s) := C_N(s,s)$ and

$$A_N(s,t) := \frac{1}{N}\sum_{i=1}^{N} G^i(\mathbf{x}_s)x_t^i\,, \qquad F_N(s,t) := \frac{1}{N}\sum_{i=1}^{N} G^i(\mathbf{x}_s)B_t^i\,, \tag{1.15}$$

play a key role in the derivation of Theorem 1.2. Indeed, with

$$D_N(s,t) := -f'(\mathbb{E}(K_N(t)))C_N(s,t) + A_N(t,s)\,, \qquad E_N(s,t) := -f'(\mathbb{E}(K_N(s)))\chi_N(s,t) + F_N(s,t)\,, \tag{1.16}$$

the key step of the proof of Theorem 1.2 is summarized by

Proposition 1.3. *Fixing any $T<\infty$, in case $\beta=1$, any limit point of the sequence $(\mathbb{E}C_N, \mathbb{E}\chi_N, \mathbb{E}D_N, \mathbb{E}E_N)$ with respect to uniform convergence on $[0,T]^2$, satisfies the integral equations*

$$\begin{aligned}
(1.17)\quad C(s,t) &= C(s,0)+\chi(s,t)+\int_0^t D(s,u)du,\\
(1.18)\quad \chi(s,t) &= s\wedge t+\int_0^s E(u,t)du,\\
D(s,t) &= -f'(C(t,t))C(t,s)-\int_0^{t\vee s}\nu'(C(t,u))D(s,u)du-\int_0^{t\vee s}C(s,u)\nu''(C(t,u))D(t,u)du\\
(1.19)\quad &\quad +C(s,t\vee s)\nu'(C(t\vee s,t))-C(s,0)\nu'(C(0,t)),\\
E(s,t) &= -f'(C(s,s))\chi(s,t)-\int_0^s \nu'(C(s,u))E(u,t)du-\int_0^s \chi(u,t)\nu''(C(s,u))D(s,u)du\\
(1.20)\quad &\quad +\chi(s,t)\nu'(C(s,s))-\int_0^{t\wedge s}\nu'(C(s,u))du,
\end{aligned}$$

in the space of bounded continuous functions on $[0,T]^2$ subject to the symmetry condition $C(s,t)=C(t,s)$ and the boundary conditions $E(s,0)=0$ for all s, and $E(s,t)=E(s,s)$ for all $t\geq s$.

We next detail the organization of the paper, and hence, that of the proof of Theorem 1.2.

In Section 2 we prove the existence of strong solutions $\mathbf{x}_t$ for the Stochastic Differential System (in short **SDS**) given in (1.6), for any $N<\infty$ (see Proposition 2.1). We then prove that the functions A_N, F_N, χ_N and C_N associated with these solutions have uniformly bounded (in N) finite moments of all order, and form pre-compact sequences with respect to uniform convergence on compacts both almost surely and in the mean (see Proposition 2.3). Applying the "localized concentration of measure" of Lemma 2.5, we complete the preliminary analysis of the finite size **SDS** by proving in Proposition 2.4 that as $N\to\infty$ each of the four functions A_N, F_N, χ_N and C_N "self-averages", namely, concentrates around its mean. These results rely on the bounding in Appendix B norms of the disorder $\mathbf{J}$ that are expressed as the supremum of certain Gaussian fields.

The proof of Proposition 1.3, namely, that each limit point of $(\mathbb{E}C_N, \mathbb{E}\chi_N, \mathbb{E}D_N, \mathbb{E}E_N)$ must satisfy (1.17)–(1.20), is the subject of Section 3. Of these equations, upon multiplying the integrated form (3.1) of our **SDS** by x_t^i or by B_t^i, then averaging over i and the probability space, both (1.17) and (1.18) are immediate consequences of self-averaging. The crux of the proof is thus Proposition 3.1, where we show that as $N\to\infty$, both $\mathbb{E}A_N$ and $\mathbb{E}F_N$ are well approximated by certain combination of our four functions, thereby leading to (1.19) and (1.20). The emergence of $\nu'(C(s,u))$ and $\nu''(C(s,u))$ in the latter pair of equations, and hence in Theorem 1.2, is a consequence of the structure of the covariance kernel $k_{ts}(\mathbf{x})=\mathbb{E}_{\mathbf{J}}[G^i(\mathbf{x}_t)G^j(\mathbf{x}_s)]$, obtained by integrating over the disorder parameters $\mathbf{J}$ assuming their independence of the frozen path $\mathbf{x}_t$ (see Lemma 3.2). However, the main difficulty of the proof is the intricate dependence between $\mathbf{J}$ and $\{\mathbf{x}_t, 0\leq t\leq T\}$. Taking full advantage of the Gaussian law of $\mathbf{J}$ and the Brownian law of $\mathbf{B}$, this difficulty is dealt with by combining the Itô's calculus identities of Appendix A with Girsanov's theorem and the resulting Gaussian change of measure identities that are derived in Appendix C. This approach succeeds in deriving a "closed" system of finitely many limiting equations thanks to the fact that apart from our self-averaging global quantities, the kernel k_{ts} is a quadratic form of $\mathbf{x}$. Note that certain Hamiltonians other than (1.2) also have such a property, hence are amenable to a similar treatment (one such example is $H_{\mathbf{J}}^N(\mathbf{x}/\|\mathbf{x}\|)$ for $H_{\mathbf{J}}^N(\cdot)$ of (1.2)).

Section 4 mostly deals with analytic considerations. Its starting point is Lemma 4.1, showing that any solution of (1.17)–(1.20) is sufficiently differentiable to give rise to a solution of (1.12)–(1.14). This is followed by Proposition 4.2, establishing the uniqueness of the latter system of equations by a Gronwall type argument. Using these two ingredients, as well as the pre-compactness and self-averaging of our four functions, we conclude by deducing Theorem 1.2 out of Proposition 1.3.

We start with the almost sure existence of the strong solution $\mathbf{x}_t$ of (1.6).

Proposition 2.1. *Assume that f' is locally Lipschitz, satisfying (1.5). Then, for any $N \in \mathbb{N}$, almost any $\mathbf{J}$, initial condition $\mathbf{x}_0$ and Brownian path $\mathbf{B}$, there exists a unique strong solution to (1.6). This solution is also unique in law for almost any $\mathbf{J}$, and $\mathbf{x}_0$, it is a probability measure on $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^N)$ which we denote $\mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}$. Further, with*

$$||\mathbf{J}||^N_\infty = \max_{1\le p\le m} \sup_{||\mathbf{u^i}||\le 1, 1\le i\le p} \left|\sqrt{N}^{-1} \sum_{1\le i_k\le N, 1\le k\le p} N^{\frac{p-1}{2}} J_{i_1\cdots i_p} u^1_{i_1}\cdots u^p_{i_p}\right| \tag{2.1}$$

we have for $\delta > 0$ of (1.5), $q := m/(2\delta) + 1$, some $\kappa < \infty$, all N, $z > 0$, $\mathbf{J}$, and $\mathbf{x}_0$, that

$$\mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}\Big(\sup_{t\in\mathbb{R}^+} K_N(t) \ge K_N(0) + \kappa(1+||\mathbf{J}||^N_\infty)^q + z\Big) \le e^{-zN}\,. \tag{2.2}$$

Consequently, for any $L > 0$, there exists $z = z(L) < \infty$ such that

$$\mathbb{P}\Big(\sup_{t\in\mathbb{R}^+} K_N(t) \ge z\Big) \le e^{-LN}\,. \tag{2.3}$$

Proof: For every $M > 0$ we introduce a bounded globally Lipschitz function ϕ_M on $\mathbb{R}^N$ which we choose such that $\phi_M(\mathbf{x}) = \mathbf{x}$ when $||\mathbf{x}|| \le \sqrt{NM}$, and then consider the truncated drift $b^M(\mathbf{u}) = (b^M_1(\mathbf{u}),\ldots, b^M_N(\mathbf{u}))$ given by $b^M_i(\mathbf{u}) = G^i(\phi_\mathbf{M}(\mathbf{u})) - f'(N^{-1}|\mathbf{u}|^2 \wedge M)u^i$.

Since f' is locally Lipschitz, and since $||\mathbf{J}||^N_\infty$ is finite almost surely for all p and N , it is thus clear that the drift $b^M(\mathbf{u})$ is globally Lipschitz. The existence and uniqueness of a square-integrable strong solution $\mathbf{u}^{(M)}$ for the **SDS**

$$du^i_t = b^M_i(\mathbf{u}_t)dt + dB^i_t$$

is thus standard (for example, see [17, Theorems 5.2.5, 5.2.9]). With $\mathbf{u}^{(M)}$ defined for all M on the same probability space and filtration, consider the stopping times $\tau_M = \inf\{t : ||\mathbf{u}^{(M)}_t|| \ge \sqrt{NM}\}$. Note that $\mathbf{u}^{(M)}$ is the unique strong solution of (1.6) for $t \in [0, \tau_M]$, with τ_M a non-decreasing sequence. By the Borel-Cantelli lemma, it suffices for the existence of a unique strong solution $\mathbf{u} = \lim_{M\to\infty} \mathbf{u}^{(M)}$ of the **SDS** (1.6) in $[0, T]$, to show that

$$\sum_{M=1}^{\infty} \mathbb{P}\left(\tau_M \le T\right) < \infty. \tag{2.4}$$

To this end, fix M and let $\mathbf{x}_t = \mathbf{u}^{(M)}_{t\wedge\tau_M}$ and $Z_s = 2N^{-1}\sum_{i=1}^N \int_0^{s\wedge\tau_M} x^i_t dB^i_t$. Applying Ito's formula for $C_N(t) = N^{-1}||\mathbf{x}_t||^2$ we see that

$$C_N(s) \le C_N(0) + 2\sum_{p=1}^m \frac{a_p||\mathbf{J}||^N_\infty}{(p-1)!}\int_0^{s\wedge\tau_M} C_N(t)^{\frac{p}{2}}dt - 2\int_0^{s\wedge\tau_M} f'(C_N(t))C_N(t)dt + Z_s + s\wedge\tau_M\,. \tag{2.5}$$

Since $x^{1-\frac{m}{2}}f'(x) \to \infty$, it follows from (2.5) that there is an almost surely finite constant $c(||\mathbf{J}||^N_\infty)$, independent of M, such that

$$C_N(s) \le C_N(0) + c(||\mathbf{J}||^N_\infty)s + Z_s \tag{2.6}$$

As the quadratic variation of the martingale Z_s is $(4/N)\int_0^{s\wedge\tau_M} C_N(t)dt \le 4sN^{-1}M$, applying Doob's inequality (c.f. [17, Theorem 3.8, p. 13]) for the exponential martingale $L^\lambda_s = \exp(\lambda Z_s - 2(\lambda^2/N)\int_0^{s\wedge\tau_M} C_N(t)dt)$ (with respect to the filtration $\{\mathcal{H}_t\}$ of $\mathbf{B}_t$), yields that

$$\mathbb{P}\left(\sup_{s\le T}\{Z_s - 2\int_0^s C_N(t)dt\} \ge z\right) \le \mathbb{P}\left(\sup_{s\le T} L^N_s \ge e^{zN}\right) \le e^{-zN}\,, \tag{2.7}$$

for any $z > 0$. Therefore, (2.6) shows that with probability greater than $1 - e^{-zN}$,

$$C_N(s \wedge \tau_M) \leq C_N(0) + c(||\mathbf{J}||_\infty^N)T + z + 2\int_0^{s\wedge\tau_M} C_N(t)dt \ ,$$

for all $s \leq T$, and by Gronwall's lemma then also

$$\sup_{t\leq T} N^{-1}|\mathbf{u}_{t\wedge\tau_M}^{(M)}|^2 \leq [C_N(0) + c(||\mathbf{J}||_\infty^N)T + z]e^{2T} . \tag{2.8}$$

Setting $z = M/3$, for large enough M (depending of N, $\mathbf{J}$, $\mathbf{x}_0$ and T which are fixed here), the right-side of (2.8) is at most $M/2$, resulting with

$$\mathbb{P}(\tau_M \leq T) \leq e^{-MN/3},$$

and hence with (2.4). We also have weak uniqueness of our solutions for almost all $\mathbf{J}$ since the restriction of any weak solution to the stopped σ-field $\mathcal{H}_{\tau_M}$ for the filtration $\mathcal{H}_t$ of $\mathbf{B}_t$ is unique. We denote this unique weak solution of (1.6) by $\mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}$.

Turning to the proof of (2.2), by (1.5), for any $c > 0$ there exists $\kappa < \infty$ such that for all $r, x \geq 0$,

$$2[f'(x)x - r\sum_{p=1}^m \frac{a_p x^{\frac{p}{2}}}{(p-1)!}] - 1 \geq cx - \kappa(1+r)^q.$$

Taking $r = ||\mathbf{J}||_\infty^N$, we see that by (2.5), for all N and $s \geq 0$,

$$C_N(s\wedge\tau_M) \leq C_N(0) - \int_0^{s\wedge\tau_M} [cC_N(t) - \kappa(1+||\mathbf{J}||_\infty^N)^q]dt + Z_s \ ,$$

where $(Z_s)_{s\geq 0}$ is a martingale with bracket $(4N^{-1}\int_0^{s\wedge\tau_M} C_N(t)dt, s \geq 0)$.

By Doob's inequality (2.7), with probability at least $1 - e^{-zN}$,

$$\sup_{u\leq s\wedge\tau_M} Z_u \leq 2\int_0^{s\wedge\tau_M} C_N(t)dt + z,$$

for all $s \geq 0$. Setting $c = 3$ we then have that

$$C_N(s\wedge\tau_M) \quad \leq \quad C_N(0) + z - \int_0^{s\wedge\tau_M} C_N(t)dt + \kappa(1+||\mathbf{J}||_\infty^N)^q(s\wedge\tau_M) \ , \tag{2.9}$$

so that by Gronwall's lemma,

$$C_N(s\wedge\tau_M) \leq e^{-s\wedge\tau_M}(C_N(0)+z) + \kappa(1+||\mathbf{J}||_\infty^N)^q \int_0^{s\wedge\tau_M} e^{-t}dt$$

from which the conclusion (2.2) is obtained by considering $M \to \infty$.

In view of the assumed exponential in N decay of the tail probabilities for $K_N(0)$ and the bound (B.7) on the corresponding probabilities for $||\mathbf{J}||_\infty^N$ we thus get also the bounds of (2.3). $\square$

Our next lemma provides bounds on $G^i(\mathbf{x})$ of (1.7) which we often use en-route to the uniform boundedness of moments, pre-compactness, and concentration around the mean of the functions A_N, F_N, χ_N, and C_N.

Lemma 2.2. *There exists a constant $c < \infty$ such that, for all $N < \infty$ and every $\mathbf{x}$, $\widetilde{\mathbf{x}} \in \mathbb{R}^N$,*

$$\Big[\sum_{i=1}^N (G^i(\mathbf{x}) - G^i(\widetilde{\mathbf{x}}))^2\Big]^{\frac{1}{2}} \leq c||\mathbf{J}||_\infty^N[1 + (\frac{1}{N}||\mathbf{x}||^2)^{\frac{m-2}{2}}][1 + (\frac{1}{N}||\widetilde{\mathbf{x}}||^2)^{\frac{m-2}{2}}]||\mathbf{x} - \widetilde{\mathbf{x}}|| \ , \tag{2.10}$$

and in particular,

$$\Big[\frac{1}{N}\sum_{i=1}^N G^i(\mathbf{x})^2\Big]^{\frac{1}{2}} \leq c||\mathbf{J}||_\infty^N[1 + (\frac{1}{N}||\mathbf{x}||^2)^{\frac{m-1}{2}}] \, . \tag{2.11}$$

Proof: Fixing N, $\mathbf{x}$, $\widetilde{\mathbf{x}}$ and $\mathbf{y} \in \mathbb{R}^N$, note the telescoping sum

$$\sum_{i=1}^{N}(G^i(\mathbf{x}) - G^i(\widetilde{\mathbf{x}}))y^i = \sum_{p=1}^{m} \frac{a_p}{(p-1)!} \sum_{l=1}^{p-1} \sum_{i_k} J_{i_1 \cdots i_p} (\prod_{h=1}^{l-1} x^{i_h})(x^{i_l} - \widetilde{x}^{i_l})(\prod_{h=l+1}^{p-1} \widetilde{x}^{i_h}) y^{i_p}$$

Recall the definition (2.1) leading to the bounds,

$$|\sum_{i_k} J_{i_1 \cdots i_p}(\prod_{h=1}^{l-1} x^{i_h})(x_{i_l} - \widetilde{x}_{i_l})(\prod_{h=l+1}^{p-1} \widetilde{x}^{i_h}) y^{i_p}| \leq ||\mathbf{J}||_\infty^N (\frac{1}{N}||\mathbf{x}||^2)^{\frac{l-1}{2}} (\frac{1}{N}||\widetilde{\mathbf{x}}||^2)^{\frac{p-l-1}{2}} ||\mathbf{x} - \widetilde{\mathbf{x}}|| ||\mathbf{y}||,$$

for $1 \leq l \leq p-1 \leq m-1$. Consequently, we get for some $c = c(a_1, \ldots, a_m) < \infty$ which is independent of N, that

$$\sum_{i=1}^{N}(G^i(\mathbf{x}) - G^i(\widetilde{\mathbf{x}}))y^i \leq c||\mathbf{J}||_\infty^N [1 + (\frac{1}{N}||\mathbf{x}||^2)^{\frac{m-2}{2}}][1 + (\frac{1}{N}||\widetilde{\mathbf{x}}||^2)^{\frac{m-2}{2}}] ||\mathbf{x} - \widetilde{\mathbf{x}}|| ||\mathbf{y}||,$$

for all $\mathbf{x}$, $\widetilde{\mathbf{x}}$ and $\mathbf{y}$. Taking $y^i = G^i(\mathbf{x}) - G^i(\widetilde{\mathbf{x}})$ results with the bound of (2.10), out of which we get the bound (2.11) by considering $\widetilde{\mathbf{x}} = \mathbf{0}$. □

By the estimate (B.6) of Appendix B, we know that for any $k < \infty$,

$$\sup_N \mathbb{E}[(||\mathbf{J}||_\infty^N)^k] < \infty, \tag{2.12}$$

for the norm $||\mathbf{J}||_\infty^N$ of (2.1). By the boundedness of $N \mapsto \mathbb{E}[K_N(0)^k]$ and (2.2) the bound (2.12) immediately implies that for each $k < \infty$, also

$$\sup_N \mathbb{E}\Big[\sup_{t \in \mathbb{R}+} K_N(t)^k\Big] < \infty. \tag{2.13}$$

Building upon (2.12) and (2.13) we next prove uniform moment bounds and pre-compactness of the functions of interest to us.

Proposition 2.3. *Let U_N denote any one of the functions A_N, F_N, χ_N and C_N. Then, $\mathbb{E}\big[\sup_{s,t \leq T} |U_N(s,t)|^k\big]$ is uniformly bounded in N, for any fixed $T < \infty$ and $k < \infty$. Further, for any fixed $T < \infty$, the sequence of continuous functions $U_N(s,t)$ is pre-compact almost surely and in expectation with respect to the uniform topology on $[0,T]^2$.*

Proof: We start with the uniform bound on the moments of A_N, F_N, χ_N and C_N. To this end, let $B_N(t) := \frac{1}{N}\sum_{i=1}^N |B_t^i|^2$ and $G_N(t) := \frac{1}{N}\sum_{i=1}^N |G^i(\mathbf{x}_t)|^2$. Fixing $T, k < \infty$ let $||K_N||_\infty := \sup\{K_N(t) : 0 \leq t \leq T\}$, and similarly define $||B_N||_\infty$, $||G_N||_\infty$ and $||U_N||_\infty := \sup\{U_N(s,t) : 0 \leq s,t \leq T\}$.

A key estimate is then the following,

$$\sup_N \mathbb{E}[(||\mathbf{J}||_\infty^N)^k] + \sup_N \mathbb{E}[||K_N||_\infty^k] + \sup_N \mathbb{E}[||B_N||_\infty^k] + \sup_N \mathbb{E}[||G_N||_\infty^k] < \infty, \tag{2.14}$$

holding for each fixed k. Indeed, the bounds on $||\mathbf{J}||_\infty^N$ and $||K_N||_\infty$ are already obtained in (2.12) and (2.13), whereas by (2.11) we have that

$$(G_N(t))^{\frac{1}{2}} \leq c||\mathbf{J}||_\infty^N [1 + K_N(t)^{\frac{m-1}{2}}], \tag{2.15}$$

yielding by (2.12) and (2.13) the uniform moment bound on $||G_N||_\infty$. Finally, it is easy to show that

$$\mathbb{P}(||B_N||_\infty \geq 4T(3+x)) \leq e^{-Nx}, \tag{2.16}$$

for all $T, N < \infty$ and $x > 0$ (c.f. [5, (3.44)]), thereby providing a uniform bound for each moment of $||B_N||_\infty$ and concluding the derivation of (2.14).

Similarly, by (2.3), (2.15), (2.16) and the exponential in N bound of (B.7) on the tail of $||\mathbf{J}||_\infty^N$, we have for each $L > 0$, that there exists $M = M(L) < \infty$ such that for all N,

$$\mathbb{P}(||\mathbf{J}||_\infty^N + ||K_N||_\infty + ||B_N||_\infty + ||G_N||_\infty \geq M) \leq e^{-LN}. \tag{2.17}$$

In view of (2.14) and (2.17), we get the uniform in N bounds on moments of $\|U_N\|_\infty$ and the exponential in N bounds on the tail probabilities for $\|U_N\|_\infty$ upon observing that

$$\begin{aligned} |C_N(s,t)| &\le K_N(s)+K_N(t), \quad |\chi_N(s,t)| \le K_N(s)+B_N(t), \\ |A_N(s,t)| &\le G_N(s)+K_N(t), \quad |F_N(s,t)| \le G_N(s)+B_N(t). \end{aligned}$$

With the previous controls on $\|U_N\|_\infty$ already established, by the Arzela-Ascoli theorem, the pre-compactness of U_N follows by showing that it is an equi-continuous sequence. To this end, observe that such $U_N(s,t)$ are all of the form $\frac{1}{N}\sum_{i=1}^N a_s^i b_t^i$ hence,

$$\begin{aligned} |U_N(s,t)-U_N(s',t')| &\le \frac{1}{N}\sum_{i=1}^N |a_s^i - a_{s'}^i||b_t^i| + \frac{1}{N}\sum_{i=1}^N |a_{s'}^i||b_t^i - b_{t'}^i| \\ &\le \big[\frac{1}{N}\sum_{i=1}^N |a_s^i - a_{s'}^i|^2\big]^{1/2}\big[\frac{1}{N}\sum_{i=1}^N |b_t^i|^2\big]^{1/2} + \big[\frac{1}{N}\sum_{i=1}^N |b_t^i - b_{t'}^i|^2\big]^{1/2}\big[\frac{1}{N}\sum_{i=1}^N |a_{s'}^i|^2\big]^{1/2}. \end{aligned}$$

Here the functions $\mathbf{a}_t$ and $\mathbf{b}_t$ are either $\mathbf{x}_t$, $\mathbf{B}_t$ or $G(\mathbf{x}_t)$. So, in view of (2.14) and (2.17), it suffices to show that for any $\epsilon > 0$, some function $L(\delta,\epsilon)$ going to infinity as δ goes to zero and all N,

$$\mathbb{P}\big(\sup_{|t-t'|<\delta} \Big[\frac{1}{N}\sum_{i=1}^N |b_t^i - b_{t'}^i|^2\Big] > \epsilon\big) \le e^{-L(\delta,\epsilon)N}, \quad \sup_{|t-t'|<\delta} \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N |b_t^i - b_{t'}^i|^2\Big] \le L(\delta,\epsilon)^{-1},$$

where $\mathbf{b} = \mathbf{x}$, $\mathbf{B}$ or $G(\mathbf{x})$. Obviously, this holds for $\mathbf{b} = \mathbf{B}$, in which case the expectation equals δ regardless of N and the tail probability bound follows upon considering the union of such probabilities for $t,t' \in [i\delta,(i+2)\delta]$, $i = 0,1,\ldots,T/\delta$ and applying (2.16) with $T = 2\delta$. In case $\mathbf{b} = G(\mathbf{x})$, note that by (2.10) we have that

$$\frac{1}{N}\sum_{i=1}^N (G^i(\mathbf{x}_t) - G^i(\mathbf{x}_{t'}))^2 \le 4c^2(\|\mathbf{J}\|_\infty^N)^2(\frac{1}{N}\|\mathbf{x}_t - \mathbf{x}_{t'}\|^2)(1+K_N(t)^{(m-2)})(1+K_N(t')^{(m-2)})$$

for all t,t'. Thus, in view of the bounds (2.14) and (2.17), everything reduces to showing that

$$\mathbb{P}\big(\sup_{|t-t'|<\delta} \frac{1}{N}\sum_{i=1}^N |x_t^i - x_{t'}^i|^2 > \epsilon\big) \le e^{-L'(\delta,\epsilon)N}, \ \sup_{|t-t'|<\delta} \mathbb{E}\Big[\big(\frac{1}{N}\sum_{i=1}^N |x_t^i - x_{t'}^i|^2\big)^2\Big] \le L'(\delta,\epsilon)^{-1} \tag{2.18}$$

for some L' with the same properties as L. To this end, note that by (1.6)

$$|x_t^i - x_{t'}^i| \le |B_t^i - B_{t'}^i| + \|f'(K_N)\|_\infty \int_t^{t'} |x_u^i| du + \int_t^{t'} |G^i(\mathbf{x}_u)| du.$$

So by (1.4), for some universal constant $\rho_1 < \infty$, all t,t' and N,

$$\frac{1}{N}\sum_{i=1}^N |x_t^i - x_{t'}^i|^2 \le \frac{3}{N}\sum_{i=1}^N |B_t^i - B_{t'}^i|^2 + 3|t-t'|^2\Big[\rho_1(1+\|K_N\|_\infty)^{2r}\|K_N\|_\infty + \|G_N\|_\infty\Big]$$

Thus, by (2.14), (2.16) and (2.17), we readily obtain (2.18). □

A key ingredient in the derivation of the limiting dynamics is the "self-averaging" of the functions A_N, F_N, C_N, χ_N (and hence of D_N and E_N), as we next state and prove.

Proposition 2.4. *Assume that the concentration of measure of Hypothesis 1.1 holds, and as before let U_N denote any one of the functions A_N, F_N, χ_N and C_N. Then, for any $T < \infty$ and $\rho > 0$,*

$$\sum_N \mathbb{P}\big[\sup_{s,t\le T} |U_N(s,t) - \mathbb{E}(U_N(s,t))| \ge \rho\big] < \infty, \tag{2.19}$$

implying by the uniform moment bounds on $||U_N||_\infty$ *that*

$$\lim_{N\to\infty} \sup_{s,t\le T} \mathbb{E}\Big[|U_N(s,t) - \mathbb{E}U_N(s,t)|^2\Big] = 0\,. \tag{2.20}$$

Our strategy relies on the following general "localized" version of the concentration of measure property, which might be of some independent interest.

Lemma 2.5. *Suppose* V_N *are functions on normed spaces* $\mathcal{E}_N$ *in which the concentration of measure of Hypothesis 1.1 holds. Assume that* $K = \sup_N \mathbb{E}[V_N^2]^{1/2} < \infty$ *and that for every* $M < \infty$ *there exists a subset* $\mathcal{L}_{N,M}$ *of* $\mathcal{E}_N$ *on which* $|V_N| \le 2M$ *and* V_N *is Lipschitz with Lipschitz constant* $A_{N,M} \le \frac{D(M)}{\sqrt{N}}$. *Further assume that, for every* $L > 0$ *there exists an* $M = M(L)$ *such that* $\mathbb{P}[\mathcal{L}^c_{N,M}] \le \exp(-LN)$ *for all* N. *Then,*

$$\mathbb{P}[|V_N - \mathbb{E}[V_N]| \ge \rho] \le C^{-1} \exp\big(-C(\frac{\rho}{2D(M(L))})^\alpha) N^{\frac{\alpha}{2}}\big) + 4(K + M(L))\rho^{-1} e^{-LN/2} + e^{-NL}\ . \tag{2.21}$$

Proof: Recall that for every $M < \infty$ there exists a globally Lipschitz function

$$U_N^M(z) = \sup_{y\in\mathcal{L}_{N,M}} \{V_N(y) - A_{N,M}||z-y||\}$$

on $\mathcal{E}_N$, of Lipschitz constant $A_{N,M}$, which coincides with V_N on the set $\mathcal{L}_{N,M}$. These properties are inherited by $V_N^M = \max(U_N^M, -2M)$ for which also $|V_N^M| \le 2M$. We thus get (2.21) by combining the triangle inequality

$$|V_N - \mathbb{E}[V_N]| \le |V_N - V_N^M| + |V_N^M - \mathbb{E}[V_N^M]| + |\mathbb{E}[V_N^M] - \mathbb{E}[V_N]|\,,$$

with the Cauchy-Schwartz inequality

$$|\mathbb{E}[V_N - V_N^M]| \le \mathbb{E}|V_N - V_N^M| \le 2M\mathbb{P}(\mathcal{L}^c_{N,M}) + \mathbb{E}[|V_N|\mathbf{1}_{\mathcal{L}^c_{N,M}}] \le (K+2M)\mathbb{P}[\mathcal{L}^c_{N,M}]^{1/2}\,,$$

and applying the concentration of measure inequality of Hypothesis 1.1 for V_N^M. □

Proof of Proposition 2.4: We wish to apply the estimate (2.21) to $V_N = U_N(s,t)$ for any of our four functions, and any fixed pair of times s,t. To this end, for each $M < \infty$ and any N define the subset

$$\mathcal{L}_{N,M} = \{(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) \in \mathcal{E}_N\ :\ ||\mathbf{J}||^N_\infty + ||B_N||_\infty + ||K_N||_\infty + ||G_N||_\infty \le M\ \}$$

of $\mathcal{E}_N$. For M sufficiently large, the probability of the complement set $\mathcal{L}^c_{N,M}$ decays exponentially in N by (2.17), whereas by Proposition 2.3 we have the uniform moment bounds for the functions $U_N(s,t)$, as well as the stated pointwise bound in $\mathcal{L}_{N,M}$. It thus suffices to prove the stated Lipschitz constant of V_N on $\mathcal{L}_{N,M}$. We start by showing next that restricted to the set $\mathcal{L}_{N,M}$, the solution $\mathbf{x}$ of (1.6) is a Lipschitz function of the triple $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$.

Lemma 2.6. *Let* $\mathbf{x}, \tilde{\mathbf{x}}$ *be the two strong solutions of (1.6) constructed from* $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ *and* $(\tilde{\mathbf{x}}_0, \tilde{\mathbf{J}}, \tilde{\mathbf{B}})$, *respectively. If* $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ *and* $(\tilde{\mathbf{x}}_0, \tilde{\mathbf{J}}, \tilde{\mathbf{B}})$ *are both in* $\mathcal{L}_{N,M}$, *then we have the Lipschitz estimate:*

$$\sup_{t\le T} \frac{1}{N} \sum_{1\le i\le N} |x_t^i - \tilde{x}_t^i|^2 \le \frac{D_o(M,T)}{N} ||(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\tilde{\mathbf{x}}_0, \tilde{\mathbf{J}}, \tilde{\mathbf{B}})||^2\,, \tag{2.22}$$

where the finite constant $D_o(M,T)$ *is independent of* N.

Proof: We denote by $G^i(\cdot)$ (resp. $\widetilde{G}^i(\cdot)$) the Gaussian fields constructed from the $\mathbf{J}$ (resp. $\widetilde{\mathbf{J}}$). We write the following natural decomposition:

$$
\begin{aligned}
e_N(t) := \frac{1}{N}\sum_{i=1}^N |x_t^i - \widetilde{x}_t^i|^2 &= \frac{1}{N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)\Big((x_0^i - \widetilde{x}_0^i) + (B_t^i - \widetilde{B}_t^i) \\
&\quad - \int_0^t (f'(K_N(u)) - f'(\widetilde{K}_N(u)))x_u^i du - \int_0^t f'(\widetilde{K}_N(u))(x_u^i - \widetilde{x}_u^i) du \\
&\quad + \int_0^t (G^i(\mathbf{x}_u) - G^i(\widetilde{\mathbf{x}}_u))du + \int_0^t (G^i(\widetilde{\mathbf{x}}_u) - \widetilde{G}^i(\widetilde{\mathbf{x}}_u))du\Big) \\
&= I_1 + I_2 + \cdots + I_6
\end{aligned} \tag{2.23}
$$

Then, since $\|K_N\|_\infty \leq M$ and $\|\widetilde{K}_N\|_\infty \leq M$, we have that for some finite $C(M,T)$ that is independent of N and for all $c > 0$,

$$
\begin{aligned}
I_1 &\leq \frac{c}{2N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)^2 + \frac{1}{2cN}\sum_{i=1}^N (x_0^i - \widetilde{x}_0^i)^2 \\
I_2 &\leq \frac{c}{2N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)^2 + \frac{1}{2cN}\sum_{i=1}^N (B_t^i - \widetilde{B}_t^i)^2 \\
I_3 + I_4 &\leq C(M,T)\Big(\frac{c}{2N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)^2 + \int_0^t \frac{1}{2cN}\sum_{i=1}^N (x_u^i - \widetilde{x}_u^i)^2 du\Big) \\
I_6 &\leq C(M,T)\Big(\frac{c}{2N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)^2 + \frac{1}{2cN}\sum_{p=1}^m \sum_{1\leq i_1 \ldots i_p \leq N} (N^{\frac{p-1}{2}}(J_{i_1\cdots i_p} - \widetilde{J}_{i_1\cdots i_p}))^2\Big)
\end{aligned}
$$

We can bound the term I_5 on $\mathcal{L}_{N,M}$, using (2.10) of Lemma 2.2 by

$$
I_5 \leq C(M)||\mathbf{J}||_\infty^N \int_0^t \left(\frac{1}{N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)^2\right)^{\frac{1}{2}} \left(\frac{1}{N}\sum_{i=1}^N (x_u^i - \widetilde{x}_u^i)^2\right)^{\frac{1}{2}} du
$$

$$
\leq C(M,T)\Big(\frac{c}{2N}\sum_{i=1}^N (x_t^i - \widetilde{x}_t^i)^2 + \int_0^t \frac{1}{2cN}\sum_{i=1}^N (x_u^i - \widetilde{x}_u^i)^2 du\Big),
$$

for some $C(M,T)$ independent of N and all $c > 0$. Adding these estimates we get the bound

$$
e_N(t) \leq \widetilde{C}(M,T)\Big[c e_N(t) + \frac{1}{cN}||(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})||^2 + \frac{1}{c}\int_0^t e_N(u)du\Big],
$$

on $e_N(t)$ of (2.23), so for $c = c(M,T) > 0$ small enough, Gronwall's lemma yields the stated bound (2.22). □

Equipped with Lemma 2.6 it is now easy to prove that

Lemma 2.7. *Let $\mathbf{x}, \widetilde{\mathbf{x}}$ be the two strong solutions of (1.6) constructed from $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ and $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})$, respectively. If $(\mathbf{x}_0, \mathbf{J}, \mathbf{B})$ and $(\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})$ are both in $\mathcal{L}_{N,M}$, then we have the Lipschitz estimate for each of the four functions $U_N(s,t)$ of interest,*

$$
\sup_{s,t\leq T} |U_N(s,t) - \widetilde{U_N}(s,t)| \leq \frac{D(M,T)}{\sqrt{N}}||(\mathbf{x}_0, \mathbf{J}, \mathbf{B}) - (\widetilde{\mathbf{x}}_0, \widetilde{\mathbf{J}}, \widetilde{\mathbf{B}})||, \tag{2.24}
$$

where the constant $D(M,T)$ depends only on M and T and not on N.

Proof: Since each of the four functions $U_N(s,t)$ is of the form $\frac{1}{N}\sum_{i=1}^N a_s^i b_t^i$, we have that

$$|U_N(s,t) - \widetilde{U_N}(s,t)| \leq \frac{1}{N}\sum_{i=1}^N |a_s^i - \widetilde{a}_s^i||b_t^i| + \frac{1}{N}\sum_{i=1}^N |\widetilde{a}_s^i||b_t^i - \widetilde{b}_t^i|$$
$$\leq \quad \big[\frac{1}{N}\sum_{i=1}^N |a_s^i - \widetilde{a}_s^i|^2\big]^{1/2}\big[\frac{1}{N}\sum_{i=1}^N |b_t^i|^2\big]^{1/2} + \big[\frac{1}{N}\sum_{i=1}^N |b_t^i - \widetilde{b}_t^i|^2\big]^{1/2}\big[\frac{1}{N}\sum_{i=1}^N |a_s^i|^2\big]^{1/2}.$$

Here the functions $\mathbf{a}_t$ and $\mathbf{b}_t$ are either $\mathbf{x}_t$, $\mathbf{B}_t$ or $G(\mathbf{x}_t)$. This bound and Lemma 2.6 are sufficient to prove the conclusion of the lemma for the two functions $C_N(s,t)$ and $\chi_N(s,t)$. To prove it for the other two functions $A_N(s,t)$ and $F_N(s,t)$, note that by (2.10) we have the bound

$$\big[\frac{1}{N}\sum_{i=1}^N |G^i(\mathbf{x}_s) - G^i(\widetilde{\mathbf{x}}_s)|^2\big]^{1/2} \leq C(M)||\mathbf{J}||_\infty^N(1 + M^{\frac{p-1}{2}})\Big(\frac{1}{N}\sum_{1\leq i\leq N}|x_s^i - \widetilde{x}_s^i|^2\Big)^{1/2},$$

holding on $\mathcal{L}_{N,M}$, so combining the obvious consequence of Cauchy-Schwartz

$$\big[\frac{1}{N}\sum_{i=1}^N |G^i(\widetilde{\mathbf{x}}_s) - \widetilde{G}^i(\widetilde{\mathbf{x}}_s)|^2\big]^{1/2} \leq C(M)\big[\frac{1}{N}\sum_{p=1}^m \sum_{1\leq i_1\ldots i_p\leq N}(N^{\frac{p-1}{2}}(J_{i_1\cdots i_p} - \widetilde{J}_{i_1\cdots i_p}))^2\big]^{1/2},$$

on $\mathcal{L}_{N,M}$, with Lemma 2.6, one gets

$$\big[\frac{1}{N}\sum_{i=1}^N |G^i(\mathbf{x}_s) - \widetilde{G}^i(\widetilde{\mathbf{x}}_s)|^2\big]^{1/2} \leq \frac{C(M,T)}{\sqrt{N}}||(\mathbf{x}_0,\mathbf{J},\mathbf{B}) - (\widetilde{\mathbf{x}}_0,\widetilde{\mathbf{J}},\widetilde{\mathbf{B}})||.$$

Moreover, the estimate (2.11) gives the bound

$$\big[\frac{1}{N}\sum_{i=1}^N |G^i(\mathbf{x}_t)|^2\big]^{1/2} \leq c||\mathbf{J}||_\infty^N(1 + M^{m-1}) \leq C(M).$$

The last two estimates and Lemma 2.6 thus yield the conclusion (2.24) for both $A_N(s,t)$ and $F_N(s,t)$. □

In view of Lemma 2.7, the inequality (2.21) applies for $V_N = U_N(s,t)$, for any fixed $s,t\leq T$ with constants K and $D = D(M(L),T)$ that are independent of s,t, ρ and N. Consequently, by the union bound, for any finite subset $\mathcal{A}$ of $[0,T]^2$, and any $\rho > 0$, the sequence $N \mapsto \mathbb{P}[\sup_{(s,t)\in\mathcal{A}}|U_N(s,t) - \mathbb{E}U_N(s,t)| \geq \rho/3]$ is summable.

Recall that in the course of proving Proposition 2.3 we showed that for any $\epsilon > 0$ there exists $\widetilde{L}(\delta,\epsilon)\to\infty$ for $\delta\to 0$, such that for all N,

$$\mathbb{P}(\sup_{|s-s'|+|t-t'|<\delta}|U_N(s,t) - U_N(s',t')| > \epsilon) \leq e^{-\widetilde{L}(\delta,\epsilon)N}, \qquad \sup_{|s-s'|+|t-t'|<\delta}|\mathbb{E}U_N(s,t) - \mathbb{E}U_N(s',t')| \leq \widetilde{L}(\delta,\epsilon)^{-1}.$$

Choosing $\delta > 0$ small enough so that $\widetilde{L}(2\delta,\rho/3) > 3/\rho > 0$, we thus get (2.19) by considering the finite subset $\mathcal{A} = \{(i\delta, j\delta) : i,j = 0,1,\ldots,T/\delta\}$ of all points of $[0,T]^2$ on a δ-mesh. □

We shall often apply the following direct consequence of Propositions 2.3 and 2.4.

Corollary 2.8. *Suppose that $\Psi : \mathbb{R}^\ell \to \mathbb{R}$ is locally Lipschitz with $|\Psi(z)| \leq M||z||_k^k$ for some $M,\ell,k < \infty$, and $Z_N \in \mathbb{R}^\ell$ is a random vector, where for $j = 1,\ldots,\ell$, the j-th coordinate of Z_N is one of the functions A_N, F_N, χ_N or C_N evaluated at some $(s_j,t_j)\in[0,T]^2$. Then,*

$$\lim_{N\to\infty}\sup_{s_j,t_j}|\mathbb{E}\Psi(Z_N) - \Psi(\mathbb{E}Z_N)| = 0.$$

Proof: It follows from Proposition 2.3 that $R = \sup_{s_j,t_j,N} \|\mathbb{E}(Z_N)\|_k < \infty$. For each $r \geq R$ let c_r denote the finite Lipschitz constant of $\Psi(\cdot)$ (with respect to $\|\cdot\|_2$), on the compact set $\Gamma_r := \{z : \|z\|_k \leq r\}$. Then,

$$\begin{aligned} |\mathbb{E}\Psi(Z_N) - \Psi(\mathbb{E}Z_N)| &\leq \mathbb{E}|\Psi(Z_N) - \Psi(\mathbb{E}Z_N)|\mathbf{1}_{Z_N\in\Gamma_r} + \mathbb{E}|\Psi(Z_N)|\mathbf{1}_{Z_N\notin\Gamma_r} + |\Psi(\mathbb{E}Z_N)|\mathbb{P}(Z_N \notin \Gamma_r) \\ &\leq c_r\mathbb{E}[\|Z_N - \mathbb{E}Z_N\|_2] + 2\ell M r^{-k}\mathbb{E}\|Z_N\|_k^{2k} . \end{aligned}$$

We have by (2.20) and the uniform moment bounds of Proposition 2.3 that $\sup_{s_j,t_j} \mathbb{E}[\|Z_N - \mathbb{E}Z_N\|_2] \to 0$ as $N \to \infty$, while $c' = \sup_{s_j,t_j,N} \mathbb{E}\|Z_N\|_k^{2k} < \infty$, implying that

$$\lim_{N\to\infty} \sup_{s_j,t_j} |\mathbb{E}\Psi(Z_N) - \Psi(\mathbb{E}Z_N)| \leq 2c'\ell M r^{-k} ,$$

which we make arbitrarily small by taking $r \to \infty$. □

3. Limiting equations: proof of Proposition 1.3

We shall denote in short

$$C_N^a(s,t) = \mathbb{E}[C_N(s,t)], \qquad \chi_N^a(s,t) = \mathbb{E}[\chi_N(s,t)]$$

where expectation is over the Brownian path $\mathbf{B}$, the disorder $\mathbf{J}$ and the initial condition $\mathbf{x}_0$, and adopt a similar notation for other functions of interest.

Integrating the **SDS** (1.6) we have that

$$x_s^i = x_0^i + B_s^i - \int_0^s f'(K_N(u))x_u^i du + \int_0^s G^i(\mathbf{x}_u)du . \tag{3.1}$$

Hence, upon multiplying by x_t^i and B_t^i followed by averaging over i and taking the expected value, we get that for any $s,t \in \mathbb{R}^+$,

$$C_N^a(s,t) = C_N^a(0,t) + \chi_N^a(t,s) - \int_0^s \mathbb{E}[f'(K_N(u))C_N(u,t)]du + \int_0^s A_N^a(u,t)du \tag{3.2}$$

$$\chi_N^a(s,t) = \chi_N^a(0,t) + t \wedge s - \int_0^s \mathbb{E}[f'(K_N(u))\chi_N(u,t)]du + \int_0^s F_N^a(u,t)du . \tag{3.3}$$

In the following, we use $a_N \simeq b_N$ when $a_N(\cdot,\cdot) - b_N(\cdot,\cdot) \to 0$ as $N \to \infty$, uniformly on $[0,T]^2$. Applying Corollary 2.8 (for $\Psi(z) = z_1 f'(z_2)$ whose polynomial growth is guaranteed by our assumption (1.4)), we deduce that

$$\mathbb{E}[f'(K_N(u))C_N(u,t)] \simeq f'(K_N^a(u))C_N^a(u,t), \quad \mathbb{E}[f'(K_N(u))\chi_N(u,t)] \simeq f'(K_N^a(u))\chi_N^a(u,t).$$

Our next proposition, approximates the terms A_N^a and F_N^a which we need in order to compute the limits of (3.2) and (3.3) as $N \to \infty$.

Proposition 3.1. *We have that*

$$\begin{aligned} A_N^a(t,s) &\simeq \nu'(C_N^a(t,t\vee s))C_N^a(s,t\vee s) - \nu'(C_N^a(t,0))C_N^a(s,0) - \int_0^{s\vee t} \nu'(C_N^a(t,u))D_N^a(s,u)du \\ &- \int_0^{s\vee t} \nu''(C_N^a(t,u))C_N^a(s,u)D_N^a(t,u)du . \end{aligned} \tag{3.4}$$

Further,

$$\begin{aligned} F_N^a(s,t) &\simeq \chi_N^a(s,t\wedge s)\nu'(C_N^a(s,s)) - \int_0^{t\wedge s} \nu'(C_N^a(s,u))du - \int_0^s \nu'(C_N^a(s,u))E_N^a(u,t\wedge u)du \\ &- \int_0^s \chi_N^a(u,t\wedge u)\nu''(C_N^a(s,u))D_N^a(s,u)du . \end{aligned} \tag{3.5}$$

Deferring the proof of Proposition 3.1, we first use its conclusion to complete the proof of Proposition 1.3. To this end, note that $\mathbf{B}_0 = 0$ so $F_N^a(s,0) = 0$, for all N and s. Further, with $\mathcal{G}_s = \sigma(\mathbf{J}, \mathbf{x}_0, \mathbf{B}_u, u \le s)$, we have by (1.15) that for all N and $t > s$,

$$F_N^a(s,t) = F_N^a(s,s) + \frac{1}{N}\sum_{i=1}^N \mathbb{E}[G^i(\mathbf{x}_s)\mathbb{E}(B_t^i - B_s^i|\mathcal{G}_s)] = F_N^a(s,s)$$

(recall the independence of $\mathbf{B}_t - \mathbf{B}_s$ and $\mathcal{G}_s$). By the same reasoning also $\chi_N^a(s,0) = 0$, $E_N^a(s,0) = 0$, $E_N^a(s,t) = E_N^a(s,s)$ and $\chi_N^a(s,t) = \chi_N^a(s,s)$ for all N and any $t \ge s \ge 0$ (cf. (1.9) and (1.16)). Likewise, by definition $C_N^a(t,s) = C_N^a(s,t)$.

Let $\Phi(C,\chi,D,E) : \mathcal{C}([0,T]^2)^4 \to \mathcal{C}([0,T]^2)^4$ denote the difference between the two sides of equations (1.17)–(1.20). In view of the above boundary and symmetry conditions, comparing (3.2)–(3.5) with (1.17)–(1.20) we see that $\Phi(C_N^a, \chi_N^a, D_N^a, E_N^a) \to 0$ as $N \to \infty$, uniformly on $[0,T]^2$. It is not hard to check that $\Phi(\cdot)$ is continuous with respect to the topology of uniform convergence, hence $\Phi(\cdot) = 0$ at any limit point of $(C_N^a, \chi_N^a, D_N^a, E_N^a)$, which also necessarily satisfies the same boundary and symmetry conditions, thus completing the proof of Proposition 1.3. $\square$

As already noted, the first step in proving Proposition 3.1 is,

Lemma 3.2. *Let $\mathbb{E}_{\mathbf{J}}$ denotes the expectation with respect to the Gaussian law $\mathbb{P}_{\mathbf{J}}$ of the disorder $\mathbf{J}$. Then, for each continuous path $\mathbf{x} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)$ and all $s,t \in [0,T]$ and $i,j \in \{1,\dots,N\}$,*

$$k_{ts}^{ij}(\mathbf{x}) := \mathbb{E}_{\mathbf{J}}[G^i(\mathbf{x}_t)G^j(\mathbf{x}_s)] = \frac{x_t^j x_s^i}{N}\nu''(C_N(s,t)) + \mathbf{1}_{i=j}\nu'(C_N(s,t)) \,. \tag{3.6}$$

Proof: Observe that $G^i(\mathbf{x})$ of (1.7) are, for any given $\mathbf{x}$, centered, jointly Gaussian random variables. Further, by our choice of $c(\{i_1,\dots,i_p\})$ it is not hard to verify that for any $i, i_1, \dots, i_{p-1} = 1,\dots,N$

$$c(\{i,i_1,\dots,i_{p-1}\}) \sum_{1\le j_1,\dots,j_{p-1}\le N} \mathbf{1}_{\{i,i_1,\dots,i_{p-1}\}=\{j,j_1,\dots,j_{p-1}\}} = (p-1)!(\mathbf{1}_{j=i} + \sum_{r=1}^{p-1}\mathbf{1}_{j=i_r})\,. \tag{3.7}$$

Hence, by (1.7) and (3.7), we have that for any given vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\begin{aligned}
\mathbb{E}[G^i(\mathbf{x})G^i(\mathbf{y})] &= \sum_{p=1}^m \left(\frac{a_p}{(p-1)!}\right)^2 \sum_{i_1,\cdots,i_{p-1},j_1,\cdots,j_{p-1}} \mathbb{E}(J_{ii_1\cdots i_{p-1}}J_{ij_1\cdots j_{p-1}})x^{i_1}\dots x^{i_{p-1}}y^{j_1}\dots y^{j_{p-1}} \\
&= \sum_{p=1}^m \frac{a_p^2}{(p-1)!}N^{-(p-1)} \sum_{i_1,\cdots,i_{p-1}} (1 + \sum_{r=1}^{p-1}\mathbf{1}_{i=i_r})x^{i_1}\dots x^{i_{p-1}}y^{i_1}\dots y^{i_{p-1}} \\
&= \sum_{p=1}^m \frac{a_p^2}{(p-1)!}\Big(\frac{1}{N}\sum_{\ell=1}^N x^\ell y^\ell\Big)^{p-1} + \frac{x^i y^i}{N}\sum_{p=2}^m \frac{a_p^2}{(p-2)!}\Big(\frac{1}{N}\sum_{\ell=1}^N x^\ell y^\ell\Big)^{p-2},
\end{aligned}$$

so that with $\nu(\rho) = \sum_{p=1}^m \frac{a_p^2}{p!}\rho^p$ we have,

$$\mathbb{E}[G^i(\mathbf{x})G^i(\mathbf{y})] = \nu'\Big(\frac{1}{N}\sum_{\ell=1}^N x^\ell y^\ell\Big) + \frac{x^i y^i}{N}\nu''\Big(\frac{1}{N}\sum_{\ell=1}^N x^\ell y^\ell\Big)\,. \tag{3.8}$$

Further, if $i \neq j$, then by (3.7) we also have that

$$\begin{aligned}
x^i y^j \mathbb{E}[G^i(\mathbf{x})G^j(\mathbf{y})] &= \sum_{p=2}^{m} \left(\frac{a_p}{(p-1)!}\right)^2 \sum_{i_1,\cdots,i_{p-1},j_1,\cdots,j_{p-1}} \mathbb{E}(J_{ii_1\cdots i_{p-1}} J_{jj_1\cdots j_{p-1}}) x^i x^{i_1} \dots x^{i_{p-1}} y^j y^{j_1} \dots y^{j_{p-1}} \\
&= \sum_{p=2}^{m} \frac{a_p^2}{(p-1)!} N^{-(p-1)} \sum_{i_1,\cdots,i_{p-1}} \sum_{r=1}^{p-1} \mathbf{1}_{j=i_r} x^i x^{i_1} \dots x^{i_{p-1}} y^i y^{i_1} \dots y^{i_{p-1}} \\
&= \frac{x^i y^i x^j y^j}{N} \sum_{p=2}^{m} \frac{a_p^2}{(p-2)!} \Big(\frac{1}{N}\sum_{\ell=1}^{N} x^\ell y^\ell\Big)^{p-2},
\end{aligned}$$

implying that when $i \neq j$,

$$\mathbb{E}[G^i(\mathbf{x})G^j(\mathbf{y})] = \frac{x^j y^i}{N} \nu'' \Big(\frac{1}{N}\sum_{\ell=1}^{N} x^\ell y^\ell\Big), \tag{3.9}$$

so replacing $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x}_t$ and $\mathbf{x}_s$ respectively, we immediately get (3.6) out of (3.8), (3.9) and the definition of $C_N(\cdot,\cdot)$. ☐

Proof of Proposition 3.1: Fixing a continuous path $\mathbf{x}$, let k_t denote the operator on $L_2(\{1,\cdots N\} \times [0,t])$ with the kernel $k = k(\mathbf{x})$ of (3.6). That is, for $f \in L_2(\{1,\cdots N\} \times [0,t])$, $u \le t$, $i \in \{1,\cdots,N\}$

$$[k_t f]^i_u = \sum_{j=1}^{N} \int_0^t k^{ij}_{uv} f^j_v \, dv, \tag{3.10}$$

which is clearly also in $L_2(\{1,\cdots N\} \times [0,t])$. We next extend the definition (3.10) to the stochastic integrals of the form

$$[k_t \circ dZ]^i_u = \sum_{j=1}^{N} \int_0^t k^{ij}_{uv} dZ^j_v,$$

where Z^j_v is a continuous semi-martingale with respect to the filtration $\mathcal{F}_t = \sigma(\mathbf{x}_u : 0 \le u \le t)$ and is composed for each j, of a squared-integrable continuous martingale and a continuous, adapted, squared-integrable finite variation part. In doing so, recall that by (3.6) each $k^{ij}_{uv}(\mathbf{x})$ is the finite sum of terms such as $x^{i_1}_u \cdots x^{i_a}_u x^{j_1}_v \cdots x^{j_b}_v$, where in each term a, b and $i_1,\dots,i_a, j_1,\dots,j_b$ are some non-random integers. Keeping for simplicity the implicit notation $\int_0^t k^{ij}_{uv} dZ^j_v$ we thus adopt hereafter the convention of accordingly decomposing such integral to a finite sum, taking for each of its terms the variable $x^{i_1}_u \cdots x^{i_a}_u$ outside the integral, resulting with the usual Itô adapted stochastic integrals. The latter are well defined, with $[k_t \circ dZ]^i_u$ being in $L_2(\{1,\cdots N\} \times [0,t])$ (recall Proposition 2.1 that $\mathbf{x}_t$ has uniformly bounded finite moments of all orders under the joint law $\mathbb{P}_{\mathbf{J}} \otimes \mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}$, hence so does the kernel $k^{ij}_{ts}(\mathbf{x})$).

Equipped with these definitions, we next claim that

Lemma 3.3. *Fixing $\tau \in \mathbb{R}_+$, let $V^i_s(\mathbf{x}) = \mathbb{E}[G^i(\mathbf{x}_s)|\mathcal{F}_\tau]$ and $Z^i_s(\mathbf{x}) = \mathbb{E}[B^i_s|\mathcal{F}_\tau]$ for $s \in [0,\tau]$. Then, under $\mathbb{P}_{\mathbf{J}} \otimes \mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}$ we can choose a version of these conditional expectations such that the stochastic processes*

$$U^i_s(\mathbf{x}) = x^i_s - x^i_0 + \int_0^s f'(K_N(u)) x^i_u du \tag{3.11}$$

$$Z^i_s(\mathbf{x}) = U^i_s(\mathbf{x}) - \int_0^s V^i_u(\mathbf{x}) du, \tag{3.12}$$

are both continuous semi-martingales with respect to the filtration $\mathcal{F}_s$, composed of squared-integrable continuous martingales and finite variation parts. Moreover, such choice satisfies for any i and $s \in [0,\tau]$,

$$V^i_s + [k_\tau V]^i_s = [k_\tau \circ dU]^i_s, \tag{3.13}$$

and $V^i_s = [k_\tau \circ dZ]^i_s$ *for any* i *and all* $s \le \tau$. *Further, for any* $u, v \in [0,\tau]$ *and* $i, j \le N$, *let*

$$\Gamma^{ij}_{uv}(\mathbf{x}) := \mathbb{E}\Big[(G^i(\mathbf{x}_u) - V^i_u(\mathbf{x}))(G^j(\mathbf{x}_v) - V^j_v(\mathbf{x}))|\mathcal{F}_\tau\Big] \tag{3.14}$$

Then, we can choose a version of Γ^{il}_{uv} *such that for any* $s, v \le \tau$ *and all* $i, l \le N$,

$$\sum_{j=1}^N \int_0^\tau k^{ij}_{su} \Gamma^{jl}_{uv} du + \Gamma^{il}_{sv} = k^{il}_{sv} \ . \tag{3.15}$$

Proof: Since $U^i_s(\mathbf{x}) = \int_0^s G^i(\mathbf{x}_u)du + B^i_s$ (see (3.1)), the relation (3.12) between $\mathbb{E}[B^i_s|\mathcal{F}_\tau]$, $U^i_s(\mathbf{x})$ and $\mathbb{E}[G^i(\mathbf{x}_u)|\mathcal{F}_\tau]$ follows, as well as the continuity and integrability properties of the semi-martingales U_s and Z_s. Using hereafter G^i_s to denote $G^i(\mathbf{x}_s)$, let

$$\Lambda^N_\tau = \exp\Big\{\sum_{i=1}^N \int_0^\tau G^i_s dU^i_s(\mathbf{x}) - \frac{1}{2}\sum_{i=1}^N \int_0^\tau (G^i_s)^2 ds\Big\} . \tag{3.16}$$

By Girsanov formula we have that the restriction to $\mathcal{F}_\tau$ satisfies,

$$\mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}|_{\mathcal{F}_\tau} = \Lambda^N_\tau \mathbb{P}^N_{\mathbf{x}_0,0}|_{\mathcal{F}_\tau}$$

Hence, with $\tau \ge s$, for any bounded $\mathcal{F}_\tau$-measurable random variable Φ,

$$\mathbb{E}[G^i_s\Phi] = \mathbb{E}_{\mathbf{J}}\,\mathbb{E}_{\mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}}[G^i_s\Phi] = \mathbb{E}_{\mathbb{P}^N_{\mathbf{x}_0,0}}[\mathbb{E}_{\mathbf{J}}\,[G^i_s\Lambda^N_\tau]\Phi] = \mathbb{E}\Big[\frac{\mathbb{E}_{\mathbf{J}}\,[G^i_s\Lambda^N_\tau]}{\mathbb{E}_{\mathbf{J}}\,[\Lambda^N_\tau]}\Phi\Big] , \tag{3.17}$$

where the right-most identity is due to the change of measure formula $\mathbb{Q}^N_{\mathbf{x}_0} = \mathbb{E}_{\mathbf{J}}\,(\Lambda^N_\tau)\mathbb{P}^N_{\mathbf{x}_0,0}$ for the annealed law $\mathbb{Q}^N_{\mathbf{x}_0} = \mathbb{E}_{\mathbf{J}}\,\mathbb{P}^N_{\mathbf{x}_0,\mathbf{J}}$, restricted to $\mathcal{F}_\tau$. With (3.17) holding for all bounded $\mathcal{F}_\tau$-measurable Φ, it follows that,

$$V^i_s = \mathbb{E}[G^i_s|\mathcal{F}_\tau] = \frac{\mathbb{E}_{\mathbf{J}}\,[G^i_s\Lambda^N_\tau]}{\mathbb{E}_{\mathbf{J}}\,[\Lambda^N_\tau]} ,$$

and the identity (3.13) follows by the Gaussian change of measure identity (C.2) of Proposition C.1. Exactly the same line of reasoning shows that,

$$\Gamma^{ij}_{uv} = \frac{\mathbb{E}_{\mathbf{J}}\,[(G^i_u - V^i_u)(G^j_v - V^j_v)\Lambda^N_\tau]}{\mathbb{E}_{\mathbf{J}}\,[\Lambda^N_\tau]} ,$$

and the identity (3.15) follows by the identity (C.3) of Proposition C.1. Noting that $dZ = dU - V$ (see (3.12)), we have by (3.13) that for all i and $s \in [0,\tau]$,

$$[k_\tau \circ dZ]^i_s = [k_\tau \circ dU]^i_s - [k_\tau V]^i_s = V^i_s$$

as claimed. □

We now apply (3.13) to derive (3.4), the easy part of Proposition 3.1. To this end, fixing $s, t \in [0,T]^2$ let

$$\widehat{A}_N(t,s) = \frac{1}{N}\sum_{i=1}^N V^i_t(\mathbf{x})x^i_s ,$$

for $\tau = t \vee s$, noting that since x^i_s is measurable on $\mathcal{F}_\tau$, by (1.15),

$$A^a_N(t,s) = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N \mathbb{E}[G^i_t x^i_s|\mathcal{F}_\tau]\Big] = \mathbb{E}[\widehat{A}_N(t,s)] = \widehat{A}^a_N(t,s) .$$

With $t \le \tau$, combining (3.13) and (3.11) we get that

$$\widehat{A}_N(t,s) + \frac{1}{N}\sum_{i,j=1}^N \int_0^\tau x^i_s k^{ij}_{tu} V^j_u du = \frac{1}{N}\sum_{i,j=1}^N \int_0^\tau f'(K_N(u))x^i_s k^{ij}_{tu} x^j_u du + \frac{1}{N}\sum_{i,j=1}^N \int_0^\tau x^i_s k^{ij}_{tu} dx^j_u$$

(suppressing the dependence of k^{ij}_{tu} and V^j_u on $\mathbf{x}$, and following our convention regarding stochastic integrals such as $\int_0^\tau x^i_s k^{ij}_{tu} dx^j_u$). Using the explicit expression of $k^{ij}_{tu}(\mathbf{x})$, and collecting terms while changing the order of summation and integration, we arrive at the identity,

$$
\begin{aligned}
&\widehat{A}_N(t,s) + \int_0^\tau C_N(s,u)\nu''(C_N(t,u))\widehat{A}_N(u,t)du + \int_0^\tau \nu'(C_N(t,u))\widehat{A}_N(u,s)du \\
&= \int_0^\tau f'(K_N(u))C_N(s,u)\nu''(C_N(t,u))C_N(u,t)du + \int_0^\tau f'(K_N(u))\nu'(C_N(t,u))C_N(u,s)du \\
&+ \int_0^\tau C_N(s,u)\nu''(C_N(t,u))d_uC_N(u,t) + \int_0^\tau \nu'(C_N(t,u))d_uC_N(u,s)\,. \qquad (3.18)
\end{aligned}
$$

Applying Lemma A.1 for the semi-martingales $x = y = z = w = \mathbf{x}$, and the polynomials $Q(\rho) = \nu'(\rho)$ and $P(\rho) = \rho$ evaluated at $\sigma = \tau$, $\theta = s$ and $v = t$, we replace the stochastic integrals of (3.18) by

$$
\begin{aligned}
&\nu'(C_N(\tau,t))C_N(\tau,s) - \nu'(C_N(0,t))C_N(0,s) \\
&- \frac{1}{2N}C_N(t,t)\int_0^\tau C_N(u,s)\nu''(C_N(u,t))du - \frac{1}{N}C_N(s,t)\int_0^\tau \nu'(C_N(u,t))du\,. \qquad (3.19)
\end{aligned}
$$

Clearly, $\widehat{A}_N(t,s) = \mathbb{E}[A_N(t,s)|\mathcal{F}_\tau]$ has the same uniform moment bounds of Proposition 2.3 as A_N and further inherits the self-averaging property (2.20) from A_N. Hence, we may and shall apply Corollary 2.8 with possibly $\widehat{A}_N$ as one of the arguments of the locally Lipschitz function $\Psi(z)$ of at most polynomial growth at infinity. Doing so for the functions $z_1z_2\nu''(z_3)$, and $z_1\nu'(z_2)$ and applying Corollary 2.8 also for $f'(z_1)z_2\nu''(z_3)z_3$ and $f'(z_1)\nu'(z_2)z_3$, upon utilizing the uniform convergence with respect to the points $(s_j,t_j) \in [0,T]^2$, we deduce from (3.18) and (3.19) that

$$
\begin{aligned}
\widehat{A}^a_N(t,s) &+ \int_0^\tau C^a_N(s,u)\nu''(C^a_N(t,u))\widehat{A}^a_N(u,t)du + \int_0^\tau \nu'(C^a_N(t,u))\widehat{A}^a_N(u,s)du \\
&\simeq \int_0^\tau f'(K^a_N(u))C^a_N(s,u)\nu''(C^a_N(t,u))C^a_N(u,t)du \\
&+ \int_0^\tau f'(K^a_N(u))\nu'(C^a_N(t,u))C^a_N(u,s)du + \nu'(C^a_N(\tau,t))C^a_N(\tau,s) - \nu'(C^a_N(0,t))C^a_N(0,s)\,.
\end{aligned}
$$

Finally, recall that

$$
\widehat{A}^a_N(t,s) = A^a_N(t,s) = D^a_N(s,t) + f'(K^a_N(t))C^a_N(s,t)\,, \qquad (3.20)
$$

with the corresponding replacement for $\widehat{A}^a_N(u,t)$ and $\widehat{A}^a_N(u,s)$. With $\tau = t \vee s$, we indeed arrive at (3.4).

We now turn to the more involved part of Proposition 3.1, namely, the derivation of (3.5). To this end, as we have seen already, it suffices to consider $s \geq t$, as we do hereafter. To this end, taking $\tau = s$ we use the notation $V_u = \mathbb{E}[G_u|\mathcal{F}_s]$ of Lemma 3.3 while suppressing the dependence on $\mathbf{x}$. Since $\mathbf{B}_t = U_t - \int_0^t G_v dv$ with U_t being $\mathcal{F}_s$-measurable, we deduce that

$$
\mathbb{E}[G^i_s B^i_t] = \mathbb{E}[G^i_s U^i_t] - \int_0^t \mathbb{E}[G^i_v G^i_s]dv = \mathbb{E}\Big[\mathbb{E}[G^i_s|\mathcal{F}_s](U^i_t - \int_0^t V^i_v dv) - \int_0^t \Gamma^{ii}_{sv}dv\Big]\,,
$$

(recall that $\mathbb{E}(G_v - V_v|\mathcal{F}_s) = 0$ for all $v \leq s$ and $\Gamma^{ij}_{uv} := \mathbb{E}[(G^i_u - V^i_u)(G^j_v - V^j_v)|\mathcal{F}_s]$ is per (3.14)). Further, by (3.12) and the identity $\mathbb{E}[G^i_s|\mathcal{F}_s] = [k_s \circ dZ]^i_s$ of Lemma 3.3, we get

$$
\mathbb{E}[G^i_s B^i_t] + \mathbb{E}[\int_0^t \Gamma^{ii}_{sv}dv] = \mathbb{E}([k_s \circ dZ]^i_s Z^i_t)\,. \qquad (3.21)
$$

Since $Z^i_t = \mathbb{E}[B^i_t|\mathcal{F}_s]$ and $[k_s \circ dZ]^i_s$ is $\mathcal{F}_s$-measurable, we have that

$$
\mathbb{E}([k_s \circ dZ]^i_s Z^i_t) = \mathbb{E}([k_s \circ dZ]^i_s B^i_t) = \mathbb{E}(\mathbb{E}([k_s \circ d\mathbf{B}]^i_s|\mathcal{F}_s)B^i_t)\,,
$$

where the right-most identity holds since the kernel of the linear operator k_s is $\mathcal{F}_s$-measurable and $\mathbb{E}[\mathbf{B}_u|\mathcal{F}_s] = Z_u$ for all $u \leq s$ (recall that $[k_s \circ dZ]^i_s$ is the L_2-limit of discrete sums with mash size going to zero). In view of (3.1) and (3.6) we have that

$$
\begin{aligned}
[k_s \circ d\mathbf{B}]^i_s &= [k_s \circ d\mathbf{x}]^i_s + [k_s f'(K_N)\mathbf{x}]^i_s - [k_s G]^i_s \\
&= \int_0^s \nu'(C_N(s,u))dx^i_u + \int_0^s \nu''(C_N(s,u))x^i_u d_u C_N(s,u) \\
&+ \int_0^s \nu'(C_N(s,u))f'(K_N(u))x^i_u du + \int_0^s \nu''(C_N(s,u))f'(K_N(u))C_N(s,u)x^i_u du \\
&- \int_0^s \nu''(C_N(s,u))x^i_u A_N(u,s)du - \int_0^s \nu'(C_N(s,u))G^i_u du
\end{aligned}
\tag{3.22}
$$

Using Itô's formula for $x^i_u \nu'(C_N(s,u))$ we replace the two stochastic integrals in (3.22) by

$$
x^i_s \nu'(C_N(s,s)) - x^i_0 \nu'(C_N(s,u)) - \frac{1}{2N} C_N(s,s) \int_0^s \nu'''(C_N(s,u))x^i_u du - \frac{1}{N} x^i_s \int_0^s \nu''(C_N(s,u))du \,. \tag{3.23}
$$

Recall that by (1.15) and (3.21)

$$
F^a_N(s,t) + \mathbb{E}[\frac{1}{N}\sum_{i=1}^N \int_0^t \Gamma^{ii}_{sv} dv] = \mathbb{E}(\frac{1}{N}\sum_{i=1}^N \mathbb{E}([k_s \circ d\mathbf{B}]^i_s|\mathcal{F}_s)B^i_t)\,,
$$

which by the preceding is the expectation of

$$
\begin{aligned}
& \chi_N(s,t)\nu'(C_N(s,s)) - \chi_N(0,t)\nu'(C_N(s,0)) \\
-\ & \frac{1}{2N} C_N(s,s) \int_0^s \nu'''(C_N(s,u))\chi_N(u,t)du - \frac{1}{N}\int_0^s \nu''(C_N(s,u))\chi_N(s,t)du \\
+\ & \int_0^s f'(K_N(u))\nu'(C_N(s,u))\chi_N(u,t)du + \int_0^s f'(K_N(u))\nu''(C_N(s,u))C_N(s,u)\chi_N(u,t)du \\
-\ & \int_0^s \nu''(C_N(s,u))\chi_N(u,t)\widehat{A}_N(u,s)du - \int_0^s \nu'(C_N(s,u))F_N(u,t)du + \kappa_N(s,t)\,,
\end{aligned}
\tag{3.24}
$$

where in view of (1.15)

$$
\kappa_N(s,t) := \frac{1}{N}\sum_{i=1}^N \int_0^s \nu'(C_N(s,u))(G^i_u - V^i_u)B^i_t du\,.
$$

Recall (3.12) that $B^i_t = Z^i_t - \int_0^t (G^i_v - V^i_v)dv$, hence

$$
\kappa_N(s,t) = \frac{1}{N}\sum_{i=1}^N \int_0^s \nu'(C_N(s,u))(G^i_u - V^i_u)Z^i_t du - \frac{1}{N}\sum_{i=1}^N \int_0^s \nu'(C_N(s,u)) \int_0^t (G^i_u - V^i_u)(G^i_v - V^i_v)dvdu\,.
$$

As both $\nu'(C_N(u,s))$ and Z^t_i are $\mathcal{F}_s$-measurable while $\mathbb{E}(G_u - V_u|\mathcal{F}_s) = 0$ for all $u \leq s$, the expectation of the first term on the right-side vanishes. Further, conditioning on $\mathcal{F}_s$, we have in view of (3.14) that

$$
\mathbb{E}[\kappa_N(s,t)] = -\mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N \int_0^s \nu'(C_N(s,u)) \int_0^t \Gamma^{ii}_{uv} dvdu\Big]\,. \tag{3.25}
$$

All terms of (3.24) apart from κ_N are of the form covered by Corollary 2.8 for functions $\Psi(z)$ similar to those encountered in the derivation of (3.4), (namely, $z_1\nu'(z_2)$, $z_1\nu'''(z_2)z_3$, $z_1\nu''(z_2)$, $f'(z_1)\nu'(z_2)z_3$, $f'(z_1)z_2\nu''(z_3)z_3$ and $z_1z_2\nu''(z_3)$). Utilizing their uniform convergence and (3.25), while recalling that $\chi^a_N(0,t) = 0$, (3.20) and the analogous

$$
F^a_N(u,t) = E^a_N(u,t) + f'(K^a_N(u))C^a_N(s,u)\,,
$$

is not hard to check that we get (3.5), once we prove the following lemma.

Lemma 3.4. *For $v \in [0,s]$, let*

$$\phi_N(s,v) := \nu'(C_N(s,v)) - \frac{1}{N}\sum_{i=1}^N \Gamma_{sv}^{ii} - \frac{1}{N}\sum_{i=1}^N \int_0^s \nu'(C_N(s,u))\Gamma_{uv}^{ii}\,du$$

(where Γ is defined with $\tau = s$). Then, for any $t \le s$,

$$\int_0^t \mathbb{E}[\phi_N(s,v)]dv \simeq 0$$

Proof: Fixing $v \le s$, recall that by (3.6) we have that

$$\phi_N(s,v)+\frac{1}{N}\nu''(C_N(s,v))C_N(s,v)-\frac{1}{N^2}\sum_{i,j=1}^N \int_0^s \nu''(C_N(s,u))x_s^j x_u^i \Gamma_{uv}^{ji}\,du = \frac{1}{N}\sum_{i=1}^N \Big[k_{sv}^{ii}-\Gamma_{sv}^{ii}-\sum_{j=1}^N \int_0^s k_{su}^{ij}\Gamma_{uv}^{ji}\,du\Big]$$

Further, since we had $\tau = s$, the right-side is identically zero by (3.15), implying by the $\mathcal{F}_s$-measurability of $\mathbf{x}_s$, $\mathbf{x}_u$ and some algebraic manipulations that

$$\phi_N(s,v) + \frac{1}{N}\nu''(C_N(s,v))C_N(s,v) = \int_0^s \nu''(C_N(s,u))\frac{1}{N^2}\sum_{i,j=1}^N x_s^j x_u^i \mathbb{E}[(G_u^j - V_u^j)(G_v^i - V_v^i)|\mathcal{F}_s]du$$

$$= \int_0^s \nu''(C_N(s,u))\mathbb{E}\Big[(A_N(u,s) - \widehat{A}_N(u,s))(A_N(v,u) - \widehat{A}_N(v,u))|\mathcal{F}_s\Big]\,du\,.$$

Consequently, with $C_N(s,u)$ measurable on $\mathcal{F}_s$ we have that

$$\mathbb{E}[\phi_N(s,v)] = \int_0^s \mathbb{E}[\nu''(C_N(s,u))(A_N(u,s)-\widehat{A}_N(u,s))(A_N(v,u)-\widehat{A}_N(v,u))]\,du - \frac{1}{N}\mathbb{E}[\nu''(C_N(s,v))C_N(s,v)]\,.$$

In view of Proposition 2.3 and (2.20), the first term converges to zero uniformly in s,v by the uniform L_2 convergence of $A_N - A_N^a$ (hence also of $A_N - \widehat{A}_N$), and the uniform (in $[0,T]^2$ and N) bound on each moment of C_N and A_N (hence on those of $\widehat{A}_N$ as well). By same reasoning, the second term converges to zero at rate $1/N$, uniformly in s,v. Utilizing the uniformity of the convergence, we see that $\int_0^t \mathbb{E}[\phi_N(s,v)]dv \simeq 0$ as claimed. □

4. Differentiability and uniqueness for the limiting dynamics

We start the proof of Theorem 1.2 by the next lemma relating the solutions of (1.17)–(1.20) with those of (1.12)–(1.14).

Lemma 4.1. *Fixing $T < \infty$, suppose (C,χ,D,E) is a solution of the integral equations (1.17)–(1.20) in the space of bounded continuous functions on $[0,T]^2$ subject to the symmetry condition $C(s,t) = C(t,s)$ and the boundary conditions $E(s,0) = 0$ for all s, and $E(s,t) = E(s,s)$ for all $t \ge s$. Then, $\chi(s,t) = \int_0^t R(s,u)du$ where $R(s,t) = 0$ for $t > s$, $R(s,s) = 1$ and for $T \ge s > t$, the bounded and absolutely continuous functions C, R and $K(s) = C(s,s)$ necessarily satisfy the integro-differential equations (1.12)–(1.14).*

Proof: Consider the integral operator k_C on $\mathcal{C}([0,T])$ given by,

$$[k_C h](s) := -\int_0^s \nu'(C(s,u))h(u)du\,,$$

and let

$$h(s,t) := -f'(C(s,s))\chi(s,t) - \int_0^s \chi(u,t)\nu''(C(s,u))D(s,u)du + \chi(s,t)\nu'(C(s,s)) - \int_0^{t\wedge s} \nu'(C(s,u))du$$

Then, per fixed t, the equation (1.20) states that $E(s,t) = [k_C E(\cdot,t)](s) + h(s,t)$. Since the (continuous) kernel $\nu'(C(s,u))$ of k_C is uniformly bounded on $[0,T]^2$, it follows by Picard iterations (splitting $[0,T]$ to sufficiently small time intervals to guarantee convergence of the series $\sum_n k_C^n$), that

$$E(s,t) = \sum_{n\geq 0}[k_C^n h(.,t)](s) = h(s,t) + \int_0^s \kappa_C(s,v)h(v,t)dv\,, \tag{4.1}$$

with a uniformly bounded kernel κ_C. Plugging (4.1) into (1.18), we find by Fubini's theorem that

$$\chi(s,t) = s\wedge t + \int_0^s [\int_0^{t\wedge v} \nu'(C(v,u))du]\kappa_1(s,v)dv + \int_0^s \chi(v,t)\kappa_2(s,v)dv\,,$$

for some uniformly bounded functions κ_1 and κ_2 which depend only on C and D. Applying Picard's iterations once more, now with respect to the integral operator $[\kappa_2 g](s) = \int_0^s \kappa_2(s,v)g(v)dv$, we deduce that for some uniformly bounded κ_3 and κ_4,

$$\chi(s,t) = s\wedge t + \int_0^s \Big[(u\wedge t)\kappa_3(s,u) + [\int_0^{t\wedge u} \nu'(C(u,v))dv]\kappa_4(s,u)\Big]du\,.$$

On $s \geq t$, the function $s\wedge t = t$ is continuously differentiable, hence we easily conclude by Fubini's theorem that $t \to \chi(s,t)$ is continuously differentiable on $s \geq t$, with $\chi(s,t) = \int_0^t R(s,u)du$ for the bounded continuous function

$$R(s,t) = 1 + \int_t^s [\kappa_3(s,u) + \nu'(C(u,t))\kappa_4(s,u)]du\,.$$

In particular, $R(s,s) = 1$ for all s. The condition $E(s,t) = E(s,s)$ for $s \geq t$ implies by (1.18) that $\chi(s,t) = \chi(s,s)$ for $s \geq t$. Similarly, with $E(s,0) = 0$, it follows that $\chi(s,0) = 0$ for all s. From (1.17) we have that $C(s,t) - \chi(s,t)$ is differentiable with respect to its second argument t, with a bounded, continuous derivative $D = \partial_2(C-\chi)$. Consequently, $\partial_2 C = D + R$ where $R(s,t) = (\partial_2\chi)(s,t) = 0$ for all $t > s$ due to the boundary condition $\chi(s,t) = \chi(s,s)$. Further, $C(s,t) = C(t,s)$ implying that $\partial_1 C(s,t) = \partial_2 C(t,s) = D(t,s) + R(t,s)$ on $[0,T]^2$. Thus, combining the identity

$$\begin{aligned} C(s,t\vee s)\nu'(C(t\vee s,t)) - C(s,0)\nu'(C(0,t)) &= \int_0^{t\vee s} \nu'(C(t,u))(\partial_2 C)(s,u)du \\ &+ \int_0^{t\vee s} C(s,u)\nu''(C(t,u))(\partial_2 C)(t,u)du\,, \end{aligned}$$

with (1.19) we have that for all $t,s \in [0,T]^2$,

$$D(s,t) = -f'(K(t))C(t,s) + \int_0^{t\vee s} \nu'(C(t,u))R(s,u)du + \int_0^{t\vee s} C(s,u)\nu''(C(t,u))R(t,u)du\,. \tag{4.2}$$

Interchanging t and s in (4.2) and adding $R(t,s) = 0$ when $s > t$, results for $s > t$ with

$$(\partial_1 C)(s,t) = -f'(K(s))C(s,t) + \int_0^s \nu'(C(s,u))R(t,u)du + \int_0^s C(t,u)\nu''(C(s,u))R(s,u)du\,,$$

which is (1.13) for $\beta = 1$.

Since $K(s) = C(s,s)$, with $C(s,t) = C(t,s)$ and $\partial_2 C = D + R$, it follows that for all $h > 0$,

$$K(s) - K(s-h) = \int_{s-h}^s (D(s,u) + R(s,u))du + \int_{s-h}^s (D(s-h,u) + R(s-h,u))du\,.$$

Recall that $R(s,u) = 0$ for $u > s$, hence, dividing by h and taking $h \downarrow 0$, we thus get by the continuity of D and that of R for $s \geq t$ that $K(\cdot)$ is differentiable, with $\partial_s K(s) = 2D(s,s) + R(s,s) = 2D(s,s) + 1$, resulting by (4.2) with (1.14) for $\beta = 1$.

Further, it follows from (1.18) that $(\partial_1\chi)(u,t) = E(u,t) + 1_{u<t}$. Hence, combining the identity

$$\chi(s,t)\nu'(C(s,s)) - \chi(0,t)\nu'(C(s,0)) = \int_0^s \nu'(C(s,u))(\partial_1\chi)(u,t)du + \int_0^s \chi(u,t)\nu''(C(s,u))(\partial_2 C)(s,u)du\,,$$

with (1.20) we have that for all $T \geq s \geq t$,

$$E(s,t) = -f'(K(s))\chi(s,t) + \int_0^s \chi(u,t)\nu''(C(s,u))R(s,u)du \tag{4.3}$$

(recall that $\chi(0,t) = \chi(0,0) = 0$). Let

$$g(s,t) := -f'(K(s))R(s,t) + \int_0^s R(u,t)\nu''(C(s,u))R(s,u)du\,, \tag{4.4}$$

for $s,t \in [0,T]^2$. Recall that $\chi(s,t) = \int_0^t R(s,v)dv$, so by Fubini's theorem, (4.3) amounts to $E(s,t) = \int_0^t g(s,v)dv$ for all $s \geq t$. Further, with $E(s,t) = E(s,s)$ when $t > s$, it follows that

$$E(s,t) = \int_0^{t\wedge s} g(s,v)dv$$

for all $s,t \leq T$. Putting this into (1.18) we have by yet another application of Fubini's theorem that

$$\int_0^t R(s,u)du = \chi(s,t) = t + \int_0^s \int_0^{t\wedge u} g(u,v)dvdu = t + \int_0^t \int_v^s g(u,v)dudv\,,$$

for any $s \geq t$. Consequently, for every $t \leq s$,

$$R(s,t) = 1 + \int_t^s g(u,t)du\,,$$

implying that $\partial_1 R = g$ for a.e. $s > t$, which in view of (4.4) gives (1.12) for $\beta = 1$, thus completing the proof of the lemma. $\square$

We proceed by showing that the system of equations of interest to us admits at most one solution.

Proposition 4.2. *Let $T \geq 0$ and $\Delta_T = \{s,t \in (\mathbb{R}^+)^2 : 0 \leq t \leq s \leq T\}$. There exists at most one solution $(R,C,K) \in \mathcal{C}_b^1(\Delta_T)\times\mathcal{C}_b^1(\Delta_T)\times\mathcal{C}_b^1([0,T])$ to (S):=(1.12,1.13,1.14) with $C(s,t) = C(t,s)$ and boundary conditions*

$$\begin{aligned} R(s,s) &\equiv 1 \quad \forall s \geq 0 &(4.5)\\ C(s,s) &= K(s) \quad \forall s \geq 0 &(4.6)\\ C(0,0) &= K(0) \quad \textit{known.} \end{aligned}$$

Proof: As mentioned already, we may and shall take $\beta = 1$ (by scaling $\beta^2\nu \mapsto \nu$, with $\beta^2\psi \mapsto \psi$ accordingly), just as we have done throughout this paper.

Assume that there are two solutions (R,C,K) and $(\widetilde{R},\widetilde{C},\widetilde{K})$ of (S) with boundary condition (BC):=(4.5,4.6). We shall prove by Gronwall's type argument that these two solutions have to coincide. To do so we first show that the response function R is a Lipschitz function of the covariance functions (C,K) and then that the covariances obey integro-differential Gronwall type inequalities. We then use Gronwall arguments repeatedly to conclude. In what follows, T is fixed and all the constants (which eventually depend on T) will be denoted by M, even though they may change from line to line.

- *R is a Lipschitz function of the covariance*

  Let (R,C,K) be a solution to (S) and denote

$$H_C(s,t) := e^{\int_t^s f'(K(u))du} R(s,t).$$

Then, from (1.12) and (4.5), we deduce that H satisfies

$$\partial_s H_C(s,t) = \int_t^s H_C(u,t)H_C(s,u)\nu''(C(s,u))du, \quad \text{for} \quad s \geq t, \qquad H(t,t) = 1.$$

Note that H only depends on C. This equation was studied in [16] where it was shown that if NC_n denotes the set of involutions of $\{1,\cdots,2n\}$ without fixed points and without crossings and if $\mathrm{cr}(\sigma)$ is defined to be the set of indices $1 \leq i \leq 2n$ such that $i < \sigma(i)$, H is given by

$$H_C(s,t) = 1 + \sum_{n\geq 1}\int_{t\leq t_1\cdots\leq t_{2n}\leq s}\sum_{\sigma\in\mathrm{NC}_n}\prod_{i\in\mathrm{cr}(\sigma)}\nu''(C(t_{\sigma(i)},t_i))dt_1\cdots dt_{2n} \tag{4.7}$$

The number of non-crossing partitions $|\mathrm{NC}_n|$ of $2n$ elements is given by the Catalan number C_n which is at most 2^n. As $\int_{t\leq t_1\cdots\leq t_{2n}\leq s} dt_1\cdots dt_{2n} \leq M/(2n)!$ and $\sup_{(t,s)\in\Delta_T}\nu''(C(s,t))$ is uniformly bounded by hypothesis, the above sum converges absolutely. Further, telescoping each $\prod_{i\in\mathrm{cr}(\sigma)}\nu''(C(t_{\sigma(i)},t_i)) - \prod_{i\in\mathrm{cr}(\sigma)}\nu''(\widetilde{C}(t_{\sigma(i)},t_i))$, by the uniform Lipschitz continuity of $\nu''(\cdot)$ on compacts, we thus find a finite constant M such that for any pair $C,\widetilde{C}\in\mathcal{C}_b(\Delta_T)$ and any $t,s\in\Delta_T$,

$$|H_C(s,t) - H_{\widetilde{C}}(s,t)| \leq M\int_{t\leq t_2\leq t_1\leq s}|C(t_1,t_2)-\widetilde{C}(t_1,t_2)|dt_1dt_2. \tag{4.8}$$

Thus, if (R,C,K) and $(\widetilde{R},\widetilde{C},\widetilde{K})$ are two solutions of (S) in $\mathcal{C}_b^1(\Delta_T)\times\mathcal{C}_b^1(\Delta_T)\times\mathcal{C}_b^1([0,T])$, since K is uniformly bounded and $f'(\cdot)$ is locally Lipschitz, we obtain

$$|R(s,t)-\widetilde{R}(s,t)| \leq M\int_{t\leq t_2\leq t_1\leq s}|C(t_1,t_2)-\widetilde{C}(t_1,t_2)|dt_1dt_2 + M\int_t^s|K(u)-\widetilde{K}(u)|du. \tag{4.9}$$

- *Bounds on the difference of the covariances on $s \geq t$*

  Integrating (1.13) yields for $s \geq t$

$$\begin{aligned}C(s,t) &= K(t) - \int_t^s f'(K(u))C(u,t)du + \int_t^s du\int_0^t dv\nu'(C(u,v))R(t,v)\\&\quad + \int_t^s du\int_0^t dv\nu''(C(u,v))C(t,v)R(u,v) + \int_t^s du\int_t^u dv\nu''(C(u,v))C(v,t)R(u,v).\end{aligned}$$

Hence, if (R,C,K) and $(\widetilde{R},\widetilde{C},\widetilde{K})$ are two solutions of (S),

$$\begin{aligned}|C-\widetilde{C}|(s,t) &\leq M\Big[|K-\widetilde{K}|(t) + \int_t^s|K-\widetilde{K}|(u)du + \int_t^s|C-\widetilde{C}|(u,t)du + \int_t^s du\int_0^t dv|C-\widetilde{C}|(u,v)\\&\quad + \int_t^s du\int_0^t dv|C-\widetilde{C}|(t,v) + \int_t^s du\int_0^t dv|R-\widetilde{R}|(t,v) + \int_t^s du\int_0^t dv|R-\widetilde{R}|(u,v)\\&\quad + \int_t^s du\int_t^u dv|C-\widetilde{C}|(u,v) + \int_t^s du\int_t^u dv|C-\widetilde{C}|(v,t) + \int_t^s du\int_t^u dv|R-\widetilde{R}|(u,v)\Big]\\&:= I_1(s,t)+I_2(s,t)+\cdots+I_{10}(s,t)\end{aligned} \tag{4.10}$$

- *Bounds on the differences of the covariances on the diagonal*

  Similarly, integrating (1.14) gives

$$K(t) = K(0) - 2\int_0^t f'(K(u))K(u)du + t + 2\int_0^t du\int_0^u dv\psi(C(u,v))R(u,v)\,,$$

yielding in case $K(0)=\widetilde{K}(0)$ that

$$|K-\widetilde{K}|(t) \leq M\Big[\int_0^t|K-\widetilde{K}|(u)du + \int_0^t du\int_0^u|C-\widetilde{C}|(u,v)dv + \int_0^t du\int_0^u|R-\widetilde{R}|(u,v)dv\Big] \tag{4.11}$$

Plugging (4.9) into (4.11) yields

$$|K-\widetilde{K}|(t)\le M[\int_{0\le t_1\le t_2\le t}|C-\widetilde{C}|(t_2,t_1)dt_1dt_2+\int_0^t|K-\widetilde{K}|(u)du] \tag{4.12}$$

Recall that by Gronwall's lemma, if h,g are two non-negative functions such that

$$h(t)\le g(t)+A\int_0^t h(s)ds$$

for some $A\ge 0$, then

$$h(t)\le g(t)+A\int_0^t g(v)e^{A(t-v)}dv\le e^{At}g(t)$$

where the last inequality holds when g is non-decreasing. Applying this inequality with

$$g(t)=\int_{0\le t_1\le t_2\le t}|C-\widetilde{C}|(t_2,t_1)dt_1dt_2$$

which is non-negative and non-decreasing yields

$$|K-\widetilde{K}|(t)\le M[\int_{0\le t_1\le t_2\le t}|C-\widetilde{C}|(t_2,t_1)dt_1dt_2] \tag{4.13}$$

- *The final fixed point argument*
  We now consider

$$D(s):=\int_0^s|C-\widetilde{C}|(s,t)dt,$$

noting that (4.9) and (4.13) imply that

$$|R(s,t)-\widetilde{R}(s,t)|\le M\int_{0\le t_2\le t_1\le s}|C(t_1,t_2)-\widetilde{C}(t_1,t_2)|dt_1dt_2=M\int_{0\le t_1\le s}D(t_1)dt_1, \tag{4.14}$$

and

$$|K-\widetilde{K}|(t)\le M[\int_{0\le t_2\le t}D(t_2)dt_2]. \tag{4.15}$$

Thus, integrating (4.10) with respect to t and observing that

$$\begin{aligned}
\int_0^s(I_1(s,t)+I_2(s,t))dt &\le M\int_0^s D(u)du \text{ by (4.15)},\\
\int_0^s(I_4(s,t)+I_8(s,t))dt &\le M\int_0^s dt\int_t^s du\int_0^u dv|C-\widetilde{C}|(u,v)\le M\int_0^s D(u)du \text{ and}\\
\int_0^s(I_9(s,t)+I_5(s,t)+I_3(s,t))dt &\le M\int_0^s D(u)du \quad \text{by definition of D and Fubini},\\
\int_0^s(I_6(s,t)+I_7(s,t)+I_{10}(s,t))dt &\le M\int_0^s dt\int_t^s du\int_0^u dv|R-\widetilde{R}|(u,v)\le M\int_0^s D(u)du \text{ by (4.14)},
\end{aligned}$$

we obtain from (4.10) that

$$D(s)\le M\int_0^s D(u)du\,.$$

Recall that D is non-negative and non-decreasing, so by the preceding Gronwall argument, now with $g=0$ we conclude that $D(s)=0$ for all $s\in[0,T]$. This in turn implies by (4.14) and (4.15) that

$$K(t)=\widetilde{K}(t),\qquad R(s,t)=\widetilde{R}(s,t)\quad \text{for all } (t,s)\in\Delta_T$$

and $C(s,t) = C(s,t)$ for almost all $t \le s$ and all $s \le T$. Either by (4.10) or directly by the continuity of the covariances we conclude that

$$C(s,t) = \widetilde{C}(s,t) \quad \text{for all } (t,s) \in \Delta_T$$

which finishes the proof. $\square$

We conclude this section with the,

Proof of Theorem 1.2: Recall Proposition 2.3 that we have pre-compactness of $(A_N^a, F_N^a, \chi_N^a, C_N^a) : [0,T]^2 \to \mathbb{R}^4$, in the topology of uniform convergence on $[0,T]^2$. This implies the existence of limit points for this sequence. By Proposition 1.3 every such limit point is a solution of the integral equations (1.17)–(1.20) with the stated symmetry and boundary conditions. By Lemma 4.1 each such solution results with C and χ (i.e. R) that satisfy the integro-differential equations (1.12)–(1.14). In view of Proposition 4.2 the latter system admits at most one solution per given boundary conditions. Hence, we conclude that the sequence (χ_N^a, C_N^a) converges uniformly in $[0,T]^2$ to the unique solution of (1.12)–(1.14) subject to the appropriate boundary conditions. Further, by (2.19) of Proposition 2.4 both $C_N - C_N^a$ and $\chi_N - \chi_N^a$ converge uniformly to zero, almost surely. Thus, the solution of (1.12)–(1.14) is also the unique almost sure uniform (in s,t) limit of (χ_N, C_N), as stated in Theorem 1.2. The L_p convergence then follows by the uniform bounds on moments of C_N and χ_N (see Proposition 2.3), thus completing the proof of the theorem. $\square$

Appendix A. Itô's calculus

Let $\{\{x_t^i, y_t^i, z_t^i, w_t^i\}_{t\ge 0}, i \in \mathbb{N}\}$ be semi-martingales such that,

$$d\langle r^i, p^j\rangle_t = \delta_{i=j}\, dt$$

for any $p, r \in \{x,y,z,w\}$. Denoting, for $p,r \in \{x,y,z,w\}$, $s,t \ge 0$, $N \in \mathbb{N}$,

$$K_{p,r}^N(s,t) := \frac{1}{N}\sum_{i=1}^N p_s^i r_t^i\,,$$

we already made use of the following simple stochastic calculus lemma.

Lemma A.1. *For any polynomials P, Q, and any $\sigma, \theta, v \ge 0$,*

$$\begin{aligned} P(K_{x,y}^N(\sigma,\theta))Q(K_{z,w}^N(\sigma,v)) &= P(K_{x,y}^N(0,\theta))Q(K_{z,w}^N(0,v)) \\ &+ \int_0^\sigma P'(K_{x,y}^N(u,\theta))Q(K_{z,w}^N(u,v))d_u K_{x,y}^N(u,\theta) \\ &+ \int_0^\sigma P(K_{x,y}^N(u,\theta))Q'(K_{z,w}^N(u,v))d_u K_{z,w}^N(u,v) \\ &+ \frac{1}{2N}K_{y,y}^N(\theta,\theta)\int_0^\sigma P''(K_{x,y}^N(u,\theta))Q(K_{z,w}^N(u,v))du \\ &+ \frac{1}{2N}K_{w,w}^N(v,v)\int_0^\sigma P(K_{x,y}^N(u,\theta))Q''(K_{z,w}^N(u,v))du \\ &+ \frac{1}{N}K_{y,w}^N(\theta,v)\int_0^\sigma P'(K_{x,y}^N(u,\theta))Q'(K_{z,w}^N(u,v))du \end{aligned}$$

where

$$d_u K_{z,w}^N(u,v) := \frac{1}{N}\sum_{i=1}^N w_v^i dz_u^i,$$

and all the stochastic integrals are defined via our convention (of putting terms such as $y_\theta^{i_1}\cdots y_\theta^{i_a} w_v^{j_1}\cdots w_v^{j_b}$ outside the integral).

Proof: By the bi-linearity of the formula given, it is enough to prove the lemma for $P(x) = x^a$ and $Q(x) = x^b$. In this case, writing

$$(K^N_{x,y}(\sigma,\theta))^a (K^N_{z,w}(\sigma,v))^b = N^{-(a+b)} \sum_{i_1,\cdots i_a} \sum_{j_1,\cdots j_b} y^{i_1}_\theta \cdots y^{i_a}_\theta w^{j_1}_v \cdots w^{j_b}_v \left(x^{i_1}_\sigma \cdots x^{i_a}_\sigma z^{j_1}_\sigma \cdots z^{j_b}_\sigma \right) ,$$

and using Itô's formula for $x^{i_1}_\sigma \cdots x^{i_a}_\sigma z^{j_1}_\sigma \cdots z^{j_b}_\sigma$ gives the stated result. □

Appendix B. Supremum of Gaussian processes indexed on large dimensional spheres.

In this section we prove the bound (2.12) which is a direct consequence of the following general lemma about supremum of Gaussian processes indexed on large dimensional balls. The outline for a direct proof of such a result by a chaining argument was kindly communicated to us by A. Bovier, whom we thank gratefully. This chaining argument can be adapted rather straightforwardly from [9]. Anton Bovier also mentioned that this result should be the consequence of a more general one. We have indeed found the proper way to see it as a consequence of classical and well known facts on Gaussian processes, and to give simple references.

Let $(X_N(\mathbf{x}))$ be a sequence of real valued, centered Gaussian processes indexed by $\mathbf{x} \in \mathbb{R}^N$. Consider, for every $\rho > 0$, the closed Euclidean ball $B_N(0,\rho)$ in $\mathbb{R}^N$, and define

$$X^*_N(\rho) = \sup_{\mathbf{x}\in B_N(0,\rho)} \frac{|X_N(\mathbf{x})|}{\sqrt{N}}$$

We will also introduce the usual metric on $\mathbb{R}^N$ associated to the process X_N,

$$d_X(\mathbf{x},\mathbf{y}) = \mathbb{E}[|X_N(\mathbf{x}) - X_N(\mathbf{y})|^2]^{1/2}$$

We denote by $||\cdot||$ the Euclidean norm and by $(\mathbf{x},\mathbf{y})_N$ the corresponding inner product on $\mathbb{R}^N$.

Lemma B.1. *Suppose that*

$$\sup_N \mathbb{E}[X_N(0)^2] < \infty \tag{B.1}$$

and that

$$\sup_N \sup_{\mathbf{x},\mathbf{y}\in B_N(0,\rho)} \frac{d_X(\mathbf{x},\mathbf{y})}{||\mathbf{x}-\mathbf{y}||} < \infty\,. \tag{B.2}$$

Then, for every $k \in \mathbb{N}$

$$\sup_N \mathbb{E}[X^*_N(\rho)^k] < \infty\,. \tag{B.3}$$

Moreover, there exists a constant $\kappa < \infty$ such that for all N and every $t > 0$,

$$\mathbb{P}[X^*_N(\rho) \geq \kappa + t] \leq \exp(-Nt^2/\kappa)\,. \tag{B.4}$$

Proof: This result is a direct consequence of Dudley's theorem ([13]). Indeed, the assumption (B.2) implies that for any N and $\epsilon > 0$ one can cover $B_N(0,\rho)$ by the union of certain $C(\rho)\epsilon^{-N}$ balls of radius ϵ, in the metric d_X, where the constant $C(\rho)$ depends on ρ but not on the dimension N. Thus, Dudley's theorem (see also [18, Theorem 11.17]) shows that

$$\mathbb{E}[\sup_{\mathbf{x}\in B_N(0,\rho)} X_N(\mathbf{x})] \leq C'(\rho)\sqrt{N}\,,$$

where the constant $C'(\rho)$ again depends only on ρ and not on N. Using the obvious fact that

$$\begin{aligned}\mathbb{E}[\sup_{\mathbf{x}\in B_N(0,\rho)} |X_N(\mathbf{x})|] - \mathbb{E}[|X_N(0)|] &\leq \mathbb{E}[\sup_{\mathbf{x},\mathbf{y}\in B_N(0,\rho)} |X_N(\mathbf{x}) - X_N(\mathbf{y})|] \\ &= \mathbb{E}[\sup_{\mathbf{x},\mathbf{y}\in B_N(0,\rho)} \{X_N(\mathbf{x}) - X_N(\mathbf{y})\}] = 2\mathbb{E}[\sup_{\mathbf{x}\in B_N(0,\rho)} X_N(\mathbf{x})]\,,\end{aligned}$$

and the assumption (B.1), we see that the conclusion (B.3) holds for $k=1$. Thus, X_N admits a version with almost all sample paths bounded and uniformly continuous on $B_N(0,\rho)$. One can then consider X_N as an (infinite dimensional) Gaussian vector in the space of continuous functions on the ball $B_N(0,\rho)$, equipped with the supremum norm. It is also a well known fact that, for such a Gaussian vector, all moments of the norm are controlled by the first (e.g. see the last statement of [18, Corollary 3.2]). This is thus enough to ensure that (B.3) holds for every $k\in\mathbb{N}$.

The tail estimate (B.4) is also classical in the Gaussian context. For instance, the assumptions (B.1) and (B.2) immediately imply that the weak variance

$$\sigma(X_N)=\sup_{\mathbf{x}\in B_N(0,\rho)}\left\{\mathbb{E}[X_N(\mathbf{x})^2]^{1/2}\right\}$$

of [18, page 56] is bounded in N. Hence, by [18, estimate (3.2), page 57] it is easy to see that there exists a finite constant $\kappa>\sup_N\mathbb{E}[X_N^*(\rho)]$ for which (B.4) applies. $\square$

We proceed to apply Lemma B.1 to the situation of interest here. To this end, fixing $p\in\mathbb{N}$ consider the Gaussian process defined on $(\mathbb{R}^N)^p$ by

$$X_{N,p}(\mathbf{x})=\sum_{1\le i_j\le N,1\le j\le p}G_{\{i_1,\cdots,i_p\}}x^1_{i_1}x^2_{i_2}x^3_{i_3}\cdots x^p_{i_p}$$

with independent, centered Gaussian variables $G_{\{i_1,\cdots,i_p\}}$ of variances $c(\{i_1,\ldots,i_p\})$ of (1.3). Considering in $(\mathbb{R}^N)^p$ the Cartesian product

$$B(\rho)=\prod_{i=1}^p\{\mathbf{x}^i\in\mathbb{R}^N,\|\mathbf{x}^i\|\le\rho\}$$

of p Euclidean balls, we wish to estimate the moments and tail of $\sup_{\mathbf{x}\in B(\rho)}|X_{N,p}(\mathbf{x})|$. To this end, note that since $X_{N,p}$ is a symmetric p-linear form on $\mathbb{R}^N$ (i.e., $X_{N,p}(\mathbf{x})$ is invariant to permutations of the vectors $\mathbf{x}^1,\ldots,\mathbf{x}^p$), we have by polarization that

$$\sup_{\mathbf{x}\in B(\rho)}|X_{N,p}(\mathbf{x})|\le C(p)\sup_{\|\mathbf{u}\|\le p\rho}|X_N(\mathbf{u})|\,, \tag{B.5}$$

where one defines $X_N(\mathbf{u})$ for $\mathbf{u}\in\mathbb{R}^N$ by

$$X_N(\mathbf{u})=X_{N,p}(\mathbf{u},\ldots,\mathbf{u})=\sum_{1\le i_j\le N}G_{\{i_1,\cdots,i_p\}}u_{i_1}u_{i_2}u_{i_3}\cdots u_{i_p}\,.$$

The Gaussian process $X_N(\mathbf{u})$ obviously satisfies (B.1) since $X_N(0)=0$ for all N. Turning to check that (B.2) is satisfied as well, note that

$$X_N(\mathbf{u})=N^{(p-1)/2}\sum_{i=1}^N G^i(\mathbf{u})u_i\,,$$

for the case where $\nu'(r)=(p-1)!r^{p-1}$. Thus, by (3.8) and (3.9), the covariance of the process X_N is

$$\mathbb{E}[X_N(\mathbf{u})X_N(\mathbf{v})]=p!(\mathbf{u},\mathbf{v})_N^p\,,$$

so that

$$d_X(\mathbf{u},\mathbf{v})^2=p!\left[(\mathbf{u},\mathbf{u})_N^p+(\mathbf{v},\mathbf{v})_N^p-2(\mathbf{u},\mathbf{v})_N^p\right].$$

It is then easy to check that for all $\mathbf{u}$ and $\mathbf{v}$,

$$d_X(\mathbf{u},\mathbf{v})\le C(p)\|\mathbf{u}-\mathbf{v}\|\,,$$

yielding that (B.2) holds. Hence, by Lemma B.1, for every positive integers p and k,

$$\sup_N\mathbb{E}[\sup_{\mathbf{x}\in B(\rho)}(\frac{|X_{N,p}(\mathbf{x})|}{\sqrt{N}})^k]\le\sup_N\mathbb{E}[X_N^*(\rho)^k]<\infty\,. \tag{B.6}$$

In view of (2.1), combining (B.6) for $p = 1, \ldots, m$ then proves the bound (2.12). We further have by (B.4) and (B.5) that for some $\widetilde{\kappa} < \infty$, any $p \leq m$, all N and every $t > 0$,

$$\mathbb{P}[\sup_{\mathbf{x} \in B(\rho)} (\frac{|X_{N,p}(\mathbf{x})|}{\sqrt{N}}) \geq \widetilde{\kappa} + t] \leq \exp(-Nt^2/\widetilde{\kappa}) \,,$$

from which it follows that,

$$\mathbb{P}[\|\mathbf{J}\|_\infty^N \geq \widetilde{\kappa} + t] \leq m \exp(-Nt^2/\widetilde{\kappa}) \,. \tag{B.7}$$

Appendix C. Gaussian change of measure identities

The change of measure that is key to the proof of Lemma 3.3 is a special case of the following "standard" Gaussian computation (compare (3.16) and (C.1)).

Proposition C.1. *Suppose under the law $\mathbb{P}$ we have a finite collection $\mathbf{J} = \{J_\alpha\}_\alpha$ of non-degenerate, independent, centered Gaussian random variables, and $G_s^i = \sum_\alpha J_\alpha L_s^i(\alpha)$ for all $s \in [0,\tau]$ and $i \leq N$, where for each α the coefficients L_s^i which are independent of $\mathbf{J}$ are in $L_2(\{1,\ldots,N\} \times [0,\tau])$. Suppose further that U_s^i is a continuous semi-martingale, independent of $\mathbf{J}$ and such that for each α the stochastic integral*

$$\mu_\alpha := \sum_{i=1}^N \int_0^\tau L_u^i(\alpha) dU_u^i \,,$$

is well defined and almost surely finite. Let $\mathbb{P}^*$ denote the law of $\mathbf{J}$ such that $\mathbb{P}^* = \Lambda_\tau / \mathbb{E}(\Lambda_\tau)\mathbb{P}$, where

$$\Lambda_\tau = \exp\Big\{ \sum_{i=1}^N \int_0^\tau G_s^i dU_s^i - \frac{1}{2} \sum_{i=1}^N \int_0^\tau (G_s^i)^2 ds \Big\} \,. \tag{C.1}$$

Let $k_{ts}^{ij} = \mathbb{E}(G_t^i G_s^j)$, $V_s^i = \mathbb{E}^*(G_s^i)$ and $\Gamma_{ts}^{ij} = \mathbb{E}^*[(G_t^i - V_t^i)(G_s^j - V_s^j)]$. Then, for any $s \leq \tau$ and $i \leq N$,

$$V_s^i + [k_\tau V]_s^i = [k_\tau \circ dU]_s^i \,, \tag{C.2}$$

and for any $s,t \leq \tau$ and $i,l \leq N$,

$$\sum_{j=1}^N \int_0^\tau k_{su}^{ij} \Gamma_{ut}^{jl} du + \Gamma_{st}^{il} = k_{st}^{il} \,. \tag{C.3}$$

Proof: Let $v_\alpha = \mathbb{E}(J_\alpha^2) > 0$ denote the variance of J_α and

$$R_{\alpha\gamma} := \sum_{i=1}^N \int_0^\tau L_u^i(\alpha) L_u^i(\gamma) du \,, \tag{C.4}$$

observing that

$$\Lambda_\tau = \exp\Big\{ \sum_\alpha J_\alpha \mu_\alpha - \frac{1}{2} \sum_{\alpha,\gamma} J_\alpha J_\gamma R_{\alpha\gamma} \Big\} \,.$$

With $\mathbf{D} = \mathrm{diag}(v_\alpha)$ a positive definite matrix and $\mathbf{R} = \{R_{\alpha\gamma}\}$ positive semi-definite, it follows from this representation of Λ_τ that under $\mathbb{P}^*$ the random vector $\mathbf{J}$ has a Gaussian law with covariance matrix $(\mathbf{D}^{-1} + \mathbf{R})^{-1}$ and mean vector $\mathbf{q} = \{q_\alpha\} = (\mathbf{D}^{-1} + \mathbf{R})^{-1} \boldsymbol{\mu}$. Hence, for any α,

$$q_\alpha + v_\alpha \sum_\gamma R_{\alpha\gamma} q_\gamma = v_\alpha \mu_\alpha \,. \tag{C.5}$$

As $k_{su}^{ij} = \sum_\alpha L_s^i(\alpha) v_\alpha L_u^j(\alpha)$, it is not hard to check that

$$[k_\tau \circ dU]_s^i := \sum_{j=1}^N \int_0^\tau k_{su}^{ij} dU_u^j = \sum_\alpha L_s^i(\alpha) v_\alpha \mu_\alpha \,.$$

Obviously, $V_s^i = \sum_\alpha L_s^i(\alpha) q_\alpha$ so we get (C.2) out of (C.5) upon verifying that

$$[k_\tau V]_s^i := \sum_{j=1}^N \int_0^\tau k_{su}^{ij} V_u^j du = \sum_{\alpha,\gamma} L_s^i(\alpha) v_\alpha q_\gamma \sum_{j=1}^N \int_0^\tau L_u^j(\alpha) L_u^j(\gamma) du = \sum_{\alpha,\gamma} L_s^i(\alpha) v_\alpha R_{\alpha\gamma} q_\gamma ,$$

with the last identity due to (C.4).

Turning to prove (C.3), since Γ_{ut}^{jl} is the covariance of G_u^j and G_t^l under the tilted law $\mathbb{P}^*$, we have that

$$\Gamma_{ut}^{jl} = \sum_{\alpha,\gamma} L_u^j(\alpha) [(\mathbf{D}^{-1} + \mathbf{R})^{-1}]_{\alpha\gamma} L_t^l(\gamma) ,$$

and hence by (C.4) we see that

$$\sum_{j=1}^N \int_0^\tau k_{su}^{ij} \Gamma_{ut}^{jl} du = \sum_{\alpha,\gamma} L_s^i(\alpha) v_\alpha [\mathbf{R}(\mathbf{D}^{-1} + \mathbf{R})^{-1}]_{\alpha\gamma} L_t^l(\gamma) .$$

With $\mathbf{D} = \text{diag}(v_\alpha)$ we easily get (C.3) out of the matrix identity $(\mathbf{I} + \mathbf{DR})(\mathbf{D}^{-1} + \mathbf{R})^{-1} = \mathbf{D}$. □

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