Outline

Where did the NP-completeness disappear?

Chaitin et al.

Can each variable be mapped to one of the k registers so that simultaneously-live variables are mapped to different registers?

NP-complete to decide.

SSA-based register allocation

Can the (chordal) interference graph be colored with k colors?

Can be checked in linear time.

So a proof that $P = NP$?

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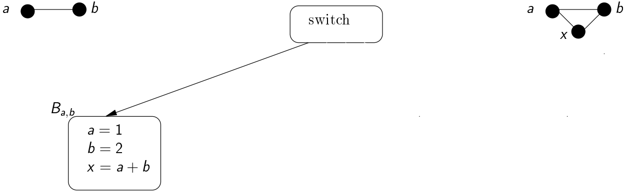
So a proof that P = NP? Of course not. But a new track to analyze register allocation subtleties, in particular the impact of:

- Strictness.
- Live-range splitting.
- Critical edges.
- Parallel copies (e.g., swap).
- Instruction types (ISA).

Interpretation of original proof



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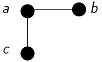

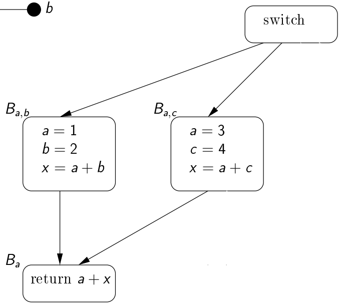

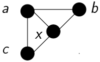

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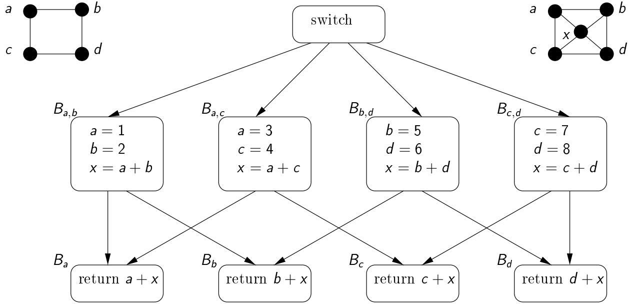
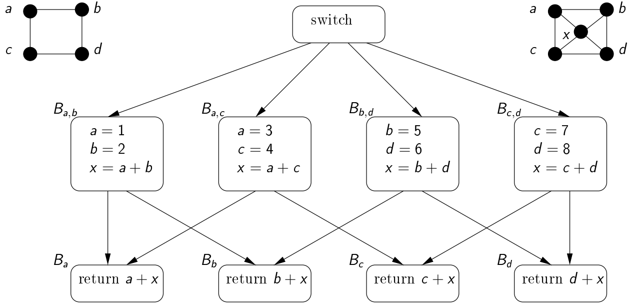


NP-complete if each variable is mapped to a unique register.

☛ But ignore the possibility of using register-to-register moves!

Interpretation of original proof



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Extension 1: NP-complete with *live-range splitting* but *critical edges*.

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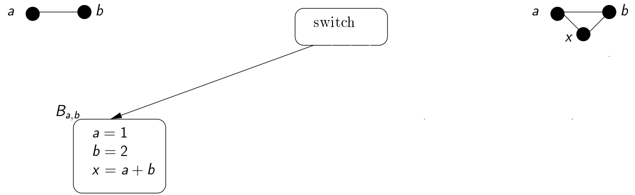

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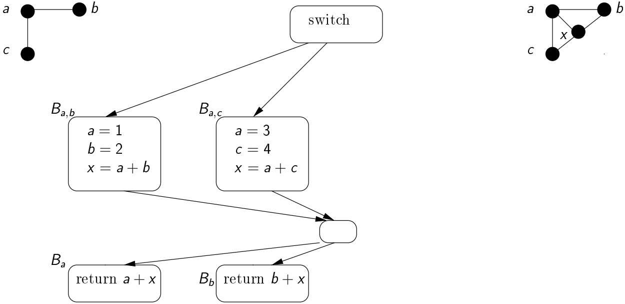

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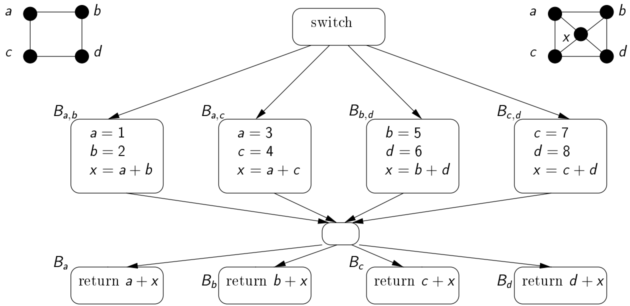
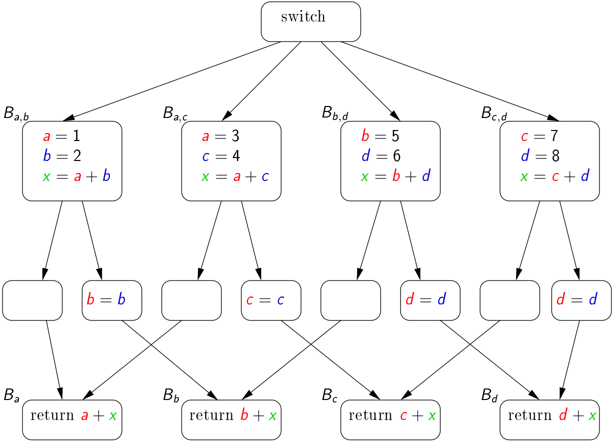


NP-complete if each variable is mapped to a unique register.

Extension 1: NP-complete with *live-range splitting* but *critical edges*.
Extension 2: Same if no critical edge but program is *not strict*.

Note: making a program strict (e.g., with SSA) can increase register pressure.

Useless proof if blocks & moves can be inserted!



Strict program, swaps, and edge splitting allowed

Maxlive = maximal number of distinct variables simultaneously live.

- One needs Maxlive $\leq k$, so spill to get Maxlive $\leq k$.
- Split critical edges (= add basic blocks).
- Color each program point independently with $\leq$ Maxlive colors.
- Use permutations to match colors (thanks to swaps).

☛ correct assignment... but with many many moves.

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More promising approaches:

- Basic block coloring (*interval graph*).
- SSA-like coloring (*chordal graph*).
- Guided live-range/edge splitting + permutation motion.

What if swaps are not available?

Pereira&Palsberg question (fossacs'06)

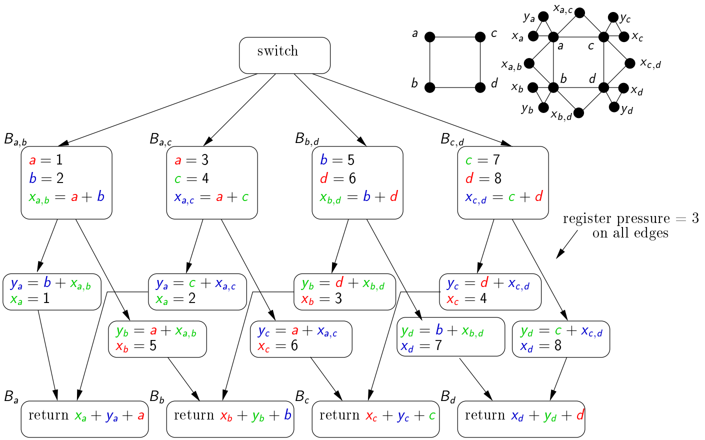
“ Can we do polynomial-time register allocation by first transforming the program to SSA form, then doing linear-time register allocation for the SSA form, and finally doing SSA elimination while maintaining the mapping from temporaries to registers? ”

☛ NP-complete if swaps are not available.

- Reduction from k-coloring circular-arc graph.
- Make sure k variables are live on the back edge (where SSA will split) so that a non-trivial permutation is impossible.

Note: polynomial for a fixed k. (See Garey, Johnson, Miller, Papadimitriou.)

If swaps not available: variant of Chaitin et al.



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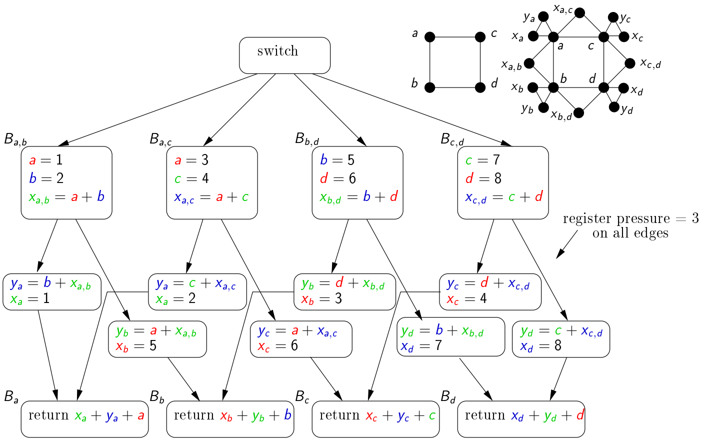

NP-complete if moves on entry/exit of blocks only, even for $k = 3$.

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NP-complete if instructions can define two variables simultaneously.

Proof: change $\boxed{\begin{array}{l} y_a = b + x_{a,b} \\ x_a = 1 \end{array}}$ into $\boxed{(x_a, y_a) = f(b, x_{a,b})}$.

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Polynomial if instructions have only one result!

Proof: greedy traversal (backwards and forwards) along control flow where register pressure $= k$.

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Polynomial if instructions have only one result!

So, NP-completeness did not disappear, it was simply not there! The proof of Chaitin et al. does not say anything about register allocation with live-range splitting and critical edge splitting.

On the complexity of register allocation

☛ If moves are more suitable than loads and stores, it is in general easy to decide if some spilling is necessary or not.

Spill test

Chaitin (degree $\geq k$) → Briggs (potential spill) → Appel-George (iterated) → Biased coloring → Optimal test

But register allocation remains difficult:

- When critical edges cannot be split or code is not strict.
  But compilers often go through strict SSA and almost always split critical edges...
- Because optimal spilling is hard
- Because optimal coalescing is hard

Summary on register allocation complexity

- Complexity has to be considered with care: determining if spilling is necessary is *easier than one can think*.
- Interference graphs of SSA-form programs are chordal.
- Optimal register assignment in linear time (*tree scan*).
- Do not need to construct interference graph.
- Use live-range splitting to handle register constraints.
- Register allocator without iteration (i.e., 2 decoupled phases):

$$\longrightarrow \boxed{\text{Lower pressure}} \longrightarrow \boxed{\text{Color/coalesce}} \longrightarrow \boxed{\Phi\text{-Implem.}} \longrightarrow$$

If moves can be anywhere, the proof is broken.