Compilation avancée et optimisation de programmes

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Multi-dimensional polyhedral optimizations
Outline

1. The polyhedral model
2. Systems of uniform recurrence equations
3. Multi-dimensional scheduling and applications
Outline

1. The polyhedral model
   - Paul Feautrier’s static control programs
   - Analyses, optimizations, and tools
   - The polyhedral model is... a model

2. Systems of uniform recurrence equations

3. Multi-dimensional scheduling and applications
Affine bounds and affine array access functions

**Fortran DO loops:**

```
DO i=1, N
  DO j=1, N
    a(i,j) = c(i,j-1)
    c(i,j) = a(i,j) + a(i-1,N)
  ENDDO
ENDDO
```

- Nested loops, static control.
- Iteration domain and vector.
- Loop increment = 1.
- Affine bounds of surrounding counters & parameters.
- Multi-dimensional arrays, same restriction for access functions.
Affine bounds and affine array access functions

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🎉 Polyhedral model: the “all-affine” world, with exact analysis

- Iteration domain = polytope.
- Sequential order $\leq_{\text{seq}}$.
- Data = images of polytopes by affine functions.
Affine bounds and affine array access functions

Fortran DO loops:

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\text{DO } & i=1, N \\
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\text{ENDDO} \\
\text{ENDDO}
\end{align*}
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 )); Typical criticism: such codes do not exist.
The polyhedral model
Systems of uniform recurrence equations
Multi-dimensional scheduling and applications

Paul Feautrier’s static control programs
Analyses, optimizations, and tools
The polyhedral model is... a model

(Parametric) analysis, transformations, optimizations

Data-flow array analysis
- Array expansion.
- Single assignment.
- Liveness array analysis.
- Data reuse.

Mapping computations & data
- Systolic arrays design.
- Data distribution.
- Communication opt.

And many more...

Loop transformations
- Automatic parallelization.
- Transformations framework.
- Code generation (with loops or with automaton).

Counting & Ehrhart polynomials
- Cache misses.
- Memory size computations.
- Latency computations.
Many languages fit in the polyhedral model

**C for loops:**

```c
for (i=1, i<=N, i++) {
    for (j=1, j<=N, j++) {
        a[i][j] = c[i][j-1];
        c[i][j] = a[i][j] + a[i-1][N];
    }
}
```

**C while loops:**

```c
y = 0; x = 0;
while (x <= N && y <= N) {
    if (?) {
        x=x+1;
        while (y >= 0 && ?) y=y-1;
    }
    y=y+1;
}
```

**Uniform recurrence equations**

∀(i,j) such that 1 ≤ i, j ≤ N

\[
\begin{align*}
    a(i,j) &= c(i,j - 1) \\
    b(i,j) &= a(i - 1,j) + b(i,j + 1) \\
    c(i,j) &= a(i,j) + b(i,j)
\end{align*}
\]

**FAUST: audio processing**

```c
random = +(12345) ~ *(1103515);
noise = random/2147483.0;
process = random/2 : @(10);
```

and more: Matlab, Fortran90, StreamIt, HPF, C for HLS, ...
Many tools and a recent revival

**PIP**  Parametric integer programming.
**POLYLIB**  Polyhedra manipulations.
**FADALIB**  Fuzzy array data-flow analysis.
**CLOOG**  Code generation, from polytopes to loops.
**EHRHART & BARVINOK**  Counting tools.
**CL@K**  Critical and admissible lattices.
**PIPS**  Automatic parallelizer & code transformation framework.
**PLUTO**  Automatic parallelizer & locality optimizer for multicores.
**GRAPHITE**  High-level memory optimizations framework in GCC.
**R-STREAM**  High-level compiler of Reservoir Labs.

...
But still, how to deal with non-static control programs?

Polyhedral model.
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Polyhedral model.

Real life.
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Extensions.
- Non-affine constraints.
- Handling of while loops.
- Recursive programs.
- Beyond induction variables.
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Approximations.
- Dependences, lifetime, data & iteration domains, etc.
- Do not assume exact information is available.
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Think conservative!
Apparent dependence graph and parallelism detection

Is there some loop parallelism (i.e., parallel loop iterations) in the following two codes? What is their degree of parallelism?

\[
\begin{align*}
\text{DO } i &= 1, N \\
\text{DO } j &= 1, N \\
&\quad a(i,j) = c(i,j-1) \\
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\text{ENDDO}
\end{align*}
\]

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Is there some **loop parallelism** (i.e., parallel loop iterations) in the following two codes? What is their **degree of parallelism**?

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ENDDO

DO i=1, N  
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    a(i,j) = c(i,j-1)  
    c(i,j) = a(i,j) + a(i-1,j)  
  ENDDO  
ENDDO

![Apparent dependence graph and parallelism detection](image)
Does this program terminate? 
If yes, how many steps in the worst case? Useful for WCET.

```c
y = 0; x = 0;
while (x <= N && y <= N) {
    if (y) {
        x=x+1;
        while (y >= 0 && y) y=y-1;
    }
    y=y+1;
}
```

Terminates in at most $N^2 + 3N + 2 = O(N^2)$ steps.

Note: a single while loop can generate quadratic (or more) WCCC.

Surprisingly, similar to parallel detection in Fortran DO loops.
Apparent evolution of variables and program termination

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If yes, how many steps in the worst case? Useful for WCET.

\[
y = 0; \quad x = 0;
\]
\[
\text{while } (x \leq N \&\& y \leq N) \{
\text{if } (?) \{
\text{x=x+1;}
\text{while } (y \geq 0 \&\& ?) y = y - 1;
\}
\text{y=y+1;}
\}
\]

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Note: a single while loop can generate quadratic (or more) WCCC. Surprisingly, similar to parallel detection in Fortran DO loops.
Outline

1. The polyhedral model

2. Systems of uniform recurrence equations
   - Model and problems
   - Computability of a system
   - Scheduling of a system

3. Multi-dimensional scheduling and applications
SURE: system of uniform recurrence equations (1967)


\[ \forall p \in \mathcal{P} = \{ p = (i, j) \mid 1 \leq i, j \leq N \} \]

\[
\begin{align*}
    a(i, j) &= c(i, j - 1) \\
    b(i, j) &= a(i - 1, j) + b(i, j + 1) \\
    c(i, j) &= a(i, j) + b(i, j)
\end{align*}
\]

Semantics:

- **RDG** (reduced dependence graph) \( G = (V, E, w) \).
- Explicit dependences & iteration domain \( \mathcal{P} \), implicit schedule.
- \( e = (u, v) \Leftrightarrow v(p) \text{ depends on } u(p - w(e)) \), i.e., must be computed after. If \( p - w(e) \notin \mathcal{P} \), it is an input.
- **EDG** (expanded dep. graph): vertices \( V \times \mathcal{P} = \) unrolled RDG.
Two main problems: computability & scheduling

Computability
Can we compute $a(p)$ in a finite number of steps?
- $a(p)$ is computable iff no infinite path in the EDG to $(a, p)$. 
Two main problems: computability & scheduling

Computability

Can we compute $a(p)$ in a finite number of steps?

- $a(p)$ is computable iff no infinite path in the EDG to $(a, p)$.
- If $\mathcal{P} = (\mathbb{N})^n$, computable for all $p \in \mathcal{P}$ if and only the RDG has no cycle $C$ with $w(C) \leq 0$ (component-wise).
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- A SURE is computable for all bounded domains $\mathcal{P}$ if and only if the RDG has no cycle $C$ with $w(C) = 0$. 
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Computability

Can we compute \( a(p) \) in a finite number of steps?

- \( a(p) \) is computable iff no infinite path in the EDG to \((a, p)\).
- If \( \mathcal{P} = (\mathbb{N})^n \), computable for all \( p \in \mathcal{P} \) if and only the RDG has no cycle \( C \) with \( w(C) \leq 0 \) (component-wise).
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Scheduling

For a computable SURE:

- How to compute or evaluate the minimal number of steps to compute \( a(p) \) (free schedule = ASAP schedule)?
- How to evaluate the potential for parallelism?
- How to find an explicit schedule? With guaranteed latency?
Related models

Nested DO loops in imperative languages

- Explicit iteration domain given by loop bounds.
- Explicit (sequential) order = lexicographic on counters + text.
- Implicit dependences: needs powerful program analysis.
- No computability problem but more general dependences.
Related models

Nested DO loops in imperative languages

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Petri nets: transitions, places, accessibility problems.
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\[
\begin{array}{c}
\text{a} \quad \text{b} \quad \text{c} \\
|0, 0| \quad |0, 1| \quad |0, 0| \\
|1, 0| \quad |0, -1| \\
\end{array}
\]

\[
\begin{array}{c}
\text{0} \quad \text{1} \quad \text{0} \\
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\text{a} & \text{b} & \text{c} \\
| & 0 & 0 \\
| & 0 & 1 \\
| & 1 & 0 \\
| & 0 & 0 \\
\end{array} \]

\[ \begin{array}{c}
\text{a} & \text{1} & \text{c} \\
\text{0} & \text{2} & \text{0} \\
\text{b} & \text{0} & \text{0} \\
\text{b} & \text{0} & \text{0} \\
\end{array} \]
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Circuits: retiming, cyclic scheduling, clock and registers.
General programs: flowchart automaton, evolution of variables.
Looking for zero-weight cycles

**Computability:** Can we compute $a(p)$ in a finite number of steps?  
**Scheduling:** If yes, how to find an explicit and “good” schedule?

**Lemma 1**

A SURE is computable for all bounded domains $P$ if and only if the RDG has *no cycle* $C$ with $w(C) = 0$. 

Looking for zero-weight cycles

Computability: Can we compute $a(p)$ in a finite number of steps?
Scheduling: If yes, how to find an explicit and “good” schedule?

Lemma 1

A SURE is computable for all bounded domains $\mathcal{P}$ if and only if the RDG has no cycle $C$ with $w(C) = 0$.

Key structure: the subgraph $G'$ induced by all edges that belong to a multi-cycle (i.e., union of cycles) of zero weight.
Three elementary key lemmas.

**Lemma 2**

*A zero-weight cycle is a zero-weight multi-cycle.*

» *Look in $G'$ only.*
Key properties

Three elementary key lemmas.

**Lemma 2**

A zero-weight cycle is a zero-weight multi-cycle.  
- Look in $G'$ only.

**Lemma 3**

A zero-weight cycle belongs to a strongly connected component.  
- Look in each strongly connected component (SCC) separately.
Key properties

Three elementary key lemmas.

Lemma 2

A zero-weight cycle is a zero-weight multi-cycle.

Look in $G'$ only.

Lemma 3

A zero-weight cycle belongs to a strongly connected component.

Look in each strongly connected component (SCC) separately.

Lemma 4

If $G'$ is strongly connected, there is a zero-weight cycle.

Terminating case.
Lemma 4

If $G'$ is strongly connected, there is a zero-weight cycle.

- $\sum_i e_i$ cycle that visits all vertices.
- $e_i$ in multi-cycle $C_i$, with $w(C_i) = 0$.
- $C_i = e_i + P_i + C_i'$. 
- Follow the $e_i$, then the $P_i$ and, on the way, plug the $C_i'$. 
Karp, Miller, and Winograd’s decomposition

Boolean KMW(G):

- Build $G'$ the subgraph of zero-weight multicycles of $G$.
- Compute $G'_1, \ldots, G'_s$, the $s$ SCCs of $G'$.
  - If $s = 0$, $G'$ is empty, return TRUE.
  - If $s = 1$, $G'$ is strongly connected, return FALSE.
  - Otherwise return $\land_i \text{KMW}(G'_i)$ (logical AND).

Then, $G$ is computable iff KMW($G$) returns TRUE.
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Then, $G$ is computable iff KMW($G$) returns TRUE.

Depth $d$ of the decomposition
$d = 0$ if $G$ is acyclic, $d = 1$ if all SCCs have an empty $G'$, etc.

Theorem 1 (Depth of the decomposition)

If $G$ is computable, $d \leq n$, otherwise, $d \leq n + 1$.

($n$ is the dimension of the problem, i.e., the dimension of $P$.)
Theorem 2 (Longest dependence path)

If \( \mathcal{P} \) contains a \( n \)-dimensional cube of size \( \Omega(N) \), there exists a dependence path of length \( \Omega(N^d) \).

Subtlety: needs to make sure that the path stays in the EDG.
But how to compute $G'$? Primal and dual programs.

$e \in G'$ iff $v_e = 0$ in any optimal solution of the linear program:

$$\min \left\{ \sum_e v_e \mid q \geq 0, \ v \geq 0, \ q + v \geq 1, \ Cq = 0, \ Wq = 0 \right\}$$

✓ A single (rational) linear program.
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A single (rational) linear program.

Always interesting to take a look at the dual program:

$$\max \{ \sum_e z_e \mid 0 \leq z \leq 1, \ X.w(e) + \rho_v - \rho_u \geq z_e, \ \forall e = (u, v) \in E \}$$

Additional property, for any optimal solution:

- $e \in G' \iff X.w(e) + \rho_v - \rho_u = 0$.
- $e \notin G' \iff X.w(e) + \rho_v - \rho_u \geq 1$. 
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Additional property, for any optimal solution:

- $e \in G' \iff X.w(e) + \rho_v - \rho_u = 0$.
- $e \notin G' \iff X.w(e) + \rho_v - \rho_u \geq 1$.

Schedule $\sigma : V \times \mathcal{P} \rightarrow \mathbb{N}$, with $\sigma(u, p) = X.p + \rho_u$, is valid if:

$$\sigma(v, p) \geq \sigma(u, p - w(e)) + 1$$

$\iff X.p + \rho_v \geq X.(p - w(e)) + \rho_u + 1$

$\iff X.w(e) + \rho_v - \rho_u \geq 1$
The polyhedral model
Systems of uniform recurrence equations
Multi-dimensional scheduling and applications

Scheduling: dual of computability.

- \( e \in G' \iff X.w(e) + \rho_v - \rho_u = 0. \)
- \( e \notin G' \iff X.w(e) + \rho_v - \rho_u \geq 1. \)

Multi-dimensional scheduling: hours, minutes, seconds, etc.

- \( e \notin G' \): \( u \) & \( v \) computed at different hours.
  - Different iterations of the outer loop = loop-carried.
- \( e \in G' \): \( u \) & \( v \) same hour, constraints pushed to inner dimensions.
  - Same iteration of outer loop = loop-independent.

Special form of schedule: affine, same linear part in a SCC of \( G' \).
Scheduling: dual of computability.

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$e \in G'$: $u \& v$ same hour, constraints pushed to inner dimensions.
Same iteration of outer loop = loop-independent.

Special form of schedule: affine, same linear part in a SCC of $G'$.

$X_1.(0, 1) = 0$
$X_1.(1, 1) \geq 2$ \implies \begin{align*}
X_1 &= (2, 0), \quad \rho_a = 1 \\
\rho_b &= 0, \quad \rho_c = 1
\end{align*}

Final schedule \begin{align*}
\sigma_a(i, j) &= (2i + 1, 2j) \\
\sigma_b(i, j) &= (2i, -j) \\
\sigma_c(i, j) &= (2i + 1, 2j + 1)
\end{align*}
Performance of schedules for computable equations

**Theorem 3 (Optimality of multi-dimensional schedules)**

If $P$ contains a $n$-dim. cube of size $\theta(N)$, there is a dependence path of length $\Omega(N^d)$ and a schedule of latency $O(N^d)$.

**Theorem 4 (Case of one-dimensional schedules)**

If $d = 1$, the best affine schedule is $\sim \lambda N$, for some $\lambda > 0$, and so is the maximal dependence length.

**Theorem 5 (Case of a single equation)**

For one equation, $d = 0$ or $d = 1$. Moreover, if $d = 1$, the best linear schedule is optimal up to a constant.

**Theorem 6 (Link with tiling)**

The maximal number of permutable loops is linked to the dimension of the vector space $\text{Vect} \{ w(C) \mid C \text{ cycle of } G' \}$.