ENS de Lyon — Master 1 – Algèbre avancée
TD10 — 2020-2021

Localization of modules and homomorphisms; Integral elements, integrally closed rings

Exercise 1. Let $A \subset B$ be an integral extension of integral domains. Show that $A \cap B^\times = A^\times$.

Exercise 2. Let $A \subset B$ be integral domains. We say that $x \in B$ is algebraic over A if there exists some non-zero $P \in A[X]$ with $P(x) = 0$.

1. Let $x \in B$ be algebraic over A. Show that there is some non-zero $a \in A$ such that ax is integral over A.
2. Assume that B is a finitely generated A-algebra and that every element of B is algebraic over A. Show that there exists some $f \in A$ such that $B[1/f]$ is finite over $A[1/f]$.

Exercise 3. Let $A = \mathbb{C}[X,Y]/(Y^2 - X^3)$. Let $B = \mathbb{C}[X,Y]/(Y^2 - X^2(X+1))$

Show that A and B are domains whose fields of fractions are isomorphic to $\mathbb{C}(T)$. Deduce that A and B are not integrally closed and compute their integral closure.

Exercise 4. Let A be an integral domain. Show that if A is integrally closed, then so is $A[X]$.

Exercise 5. Show that the ring of holomorphic functions over $\mathbb{C}$ is integrally closed but is not factorial.

Exercise 6. Let $K/\mathbb{Q}$ be a field extension of dimension n. Let $x \in K$, and let $x_1, \ldots, x_n$ be the roots of the minimal polynomial of x over $\mathbb{Q}$ (taken in an algebraic closure). The endomorphism $m_x : K \to K$ of multiplication by x is a $\mathbb{Q}$-linear map; its trace and its determinant are therefore elements of $\mathbb{Q}$. We define $\mathrm{Tr}_{K/\mathbb{Q}}(x) = \mathrm{Tr}(m_x)$ and $\mathrm{N}_{K/\mathbb{Q}}(x) = \det(m_x)$.

1. Show that $\mathrm{Tr}_{K/\mathbb{Q}}(x) = [K:\mathbb{Q}(x)] \sum_{i=1}^{n} x_i$ and that $\mathrm{N}_{K/\mathbb{Q}}(x) = \left(\prod_{i=1}^{n} x_i\right)^{[K:\mathbb{Q}(x)]}$. Deduce that if x is integral over $\mathbb{Z}$, then $\mathrm{Tr}_{K/\mathbb{Q}}(x)$ and $\mathrm{N}_{K/\mathbb{Q}}(x)$ are integral.
2. Let $x \in K$ be an integer over $\mathbb{Z}$. Show that $1/x$ is integral over $\mathbb{Z}$ is, and only if, $\mathrm{N}_{K/\mathbb{Q}}(x) = \pm 1$.
3. Let p be a prime number. Let $\zeta_p = \exp\left(\frac{2i\pi}{p}\right)$ and $K = \mathbb{Q}(\zeta_p)$.
   (a) Show that $(1-\zeta_p)\mathbb{Z}[\zeta_p] \cap \mathbb{Z} = p\mathbb{Z}$. (Hint: show first that $p = \varepsilon(1-\zeta_p)^{p-1}$, for some $\varepsilon \in \mathbb{Z}[\zeta_p]^\times$)
   (b) Show that if $z = \sum_{i=0}^{p-2} a_i \zeta_p^i$ is integral over $\mathbb{Z}$, then $\mathrm{Tr}_{K/\mathbb{Q}}((1-\zeta_p)z)$ is divisible by p. Deduce that the ring of integers of K is $\mathbb{Z}[\zeta_p]$.
4. We want to prove that the ring $\mathbb{Z}[\sqrt[3]{2}]$ is integrally closed.
   (a) Show that it is integral over $\mathbb{Z}$.
   (b) Let $z = a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \in \mathbb{Q}(\sqrt[3]{2})$ be an integral element over $\mathbb{Z}$. By computing the trace of z, of $\sqrt[3]{2}z$ and of $(\sqrt[3]{2})^2 z$, show that $6z \in \mathbb{Z}[\sqrt[3]{2}]$.
   (c) Show that $6a$, $6b$ and $6c$ are multiple of 6, and conclude.

Exercise 7. Is the algebraic number $\frac{1+\sqrt[3]{3}+3\sqrt[3]{9}}{2}$ integral over $\mathbb{Z}$?

Exercise 8. (Kronecker's theorem) Let $x \in \mathbb{C}$ be integral over $\mathbb{Z}$. Denote by $x_1, \ldots, x_d$ its conjugate.

1. Show that for any integer $n \geqslant 0$, the polynomial $P_n(X) = \prod_{i=1}^{d} (X - x_i^n)$ has integral coefficients.
2. Assume that for any $i \in \{1, \cdots, d\}$, we have $|x_i| \leqslant 1$. Prove that either $x = 0$, or all its conjugate are unit roots (and therefore also is x itself).

3. Deduce that if $P \in \mathbb{Z}[X]$ is a monic polynomial whose all complexes roots are inside the unit disc, then the irreducible factors of P are X and the cyclotomic polynomials.

Exercise 9. (Galois theory and integrally closed rings)

1. Let $A \subseteq B$ be commutative rings, $\mathfrak{p}$ be a prime ideal of A, and $\mathfrak{q}$ be a prime ideal of B. We say that $\mathfrak{q}$ lies over $\mathfrak{p}$ if $\mathfrak{q} \cap A = \mathfrak{p}$.

   (a) Show that if $\mathfrak{q}$ lies over $\mathfrak{p}$, the natural injection $A \hookrightarrow B$ induces an injection $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$, and that if B is integral over A, then $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$.

   (b) Show that if B is integral over A, and that if $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}B \neq B$ (hint: consider the case of a local ring A; and proceed by contradiction thanks to Nakayama's lemma). Deduce the existence of a prime ideal $\mathfrak{q}$ of B lying over $\mathfrak{p}$.

   (c) Show that if B is integral over A, and if $\mathfrak{q}$ is a prime ideal of B lying over a prime ideal $\mathfrak{p}$ of A, then $\mathfrak{q}$ is maximal if, and only if, $\mathfrak{p}$ is maximal.

2. Assume that A is a domain, and let K be its fraction field. Let L/K be a Galois extension,with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{p}$ be a prime ideal of A.

   (a) Show that if $\mathfrak{q}, \mathfrak{q}'$ are two prime ideals of B lying over $\mathfrak{p}$, then there exists $\sigma \in G$ such that $\sigma(\mathfrak{q}) = \mathfrak{q}'$ (hint: by contradiction, provide an element of B contained in all the $\sigma(\mathfrak{q})$ for $\sigma \in G$, but in no-one of the $\sigma(\mathfrak{q}')$ for $\sigma \in G$, and consider its norm, *cf.* exercise **??**).

   (b) Deduce that if A is integrally closed, and if E/K is a separable finite extension, then the set of prime ideals of the integral closure of A in E lying over $\mathfrak{p}$ is finite.

   (c) Let $\mathfrak{q}$ be a prime ideal of B lying over $\mathfrak{p}$. We define $\mathrm{D}_{\mathfrak{q}}$ as the stabilizer of $\mathfrak{q}$ (for the action of G on the prime ideals of B lying over $\mathfrak{p}$). Show that the field of fixed points of $\mathrm{D}_{\mathfrak{q}}$ is the smallest sub-extension E of L/K such that $\mathfrak{q}$ is the unique prime ideal of B lying over $\mathfrak{q} \cap E$. We call this field the totally ramified closure of K in E.