Finiteness of invariants, Noether's theorem

We recall that, if k is a field, $\mathrm{GL}_n(k)$ acts on $k[X_1, \dots, X_n]$ by k-algebra homomorphisms via $M \cdot P := P(Y_1, \dots, Y_n)$ with

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = M \cdot \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

Exercise 1. Let k be a field of characteristic different from 2.

1. Let
$$\Gamma := \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \mid \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\} \right\},$$
acting naturally on $k[X, Y, Z]$. Determine $k[X, Y, Z]^\Gamma$.

2. Determine $k[X, Y, Z]^{\{\pm I_3\}}$.

Exercise 2.[Molien's theorem] Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$.

1. Show that, for any integer $d \geq 0$, G acts by $\mathbb{C}$-algebra homomorphisms on the space V_d of homogeneous polynomials in $\mathbb{C}[X_1, \dots, X_n]$ of degree d.

2. This defines a representation $\rho : G \to \mathrm{GL}(V_d)$. We let χ_d be its character. Show that $\dim V_d^G = \frac{1}{|G|} \sum_{g \in G} \chi_d(g)$.

3. Compute χ_d. Show that
$$\sum_{d \geq 0} (\dim V_d^G) X^d = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I_n - Xg)}.$$

4. Show that if $\mathbb{C}[X_1, \dots, X_n]^G$ is generated as a $\mathbb{C}$-algebra by algebraically independent polynomials $P_1, \dots, P_r$ of degrees $d_1, \dots, d_r$, then its Molien series is
$$\prod_{i=1}^{r} (1 - X^{d_i})^{-1}.$$

Exercise 3. Let k be a field.

1. Describe $k[X_1, \dots, X_n]^{\mathfrak{S}_n}$. What is its Molien series ?

2. Assume now that the characteristic of k is not 2. Show that any element of $k[X_1, \cdots, X_n]^{\mathfrak{A}_n}$ can be written uniquely as the sum of an element of $k[X_1, \cdots, X_n]^{\mathfrak{S}_n}$, and of an element of $k[X_1, \cdots, X_n]$ which is anti-symmetric (that is, $\sigma(P) = \varepsilon(\sigma)P$ for all $\sigma \in \mathfrak{S}_n$). Show that the set of anti-symmetric polynomials is $k[X_1, \cdots, X_n]^{\mathfrak{S}_n}\Delta$, where $\Delta = \prod_{i<j}(X_j - X_i)$. Give a description of $k[X_1, \cdots, X_n]^{\mathfrak{A}_n}$.

3. Show, using Molien's theorem, that $\mathbb{C}[X, Y, Z]^{\mathfrak{A}_3}$ cannot be generated by algebraically independent polynomials.

Exercise 4. Let k be a field, A a finitely generated k-algebra, G a finite group acting on A by k-algebra homomorphisms and S a multiplicative subset of A such that $g \cdot S \subset S$ for any $g \in G$.

1. Show that S^G is a multiplicative subset of A^G.

2. Show that for any $\frac{a}{s} \in S^{-1}A$, there exist $b \in A$ and $t \in S^G$ such that $\frac{a}{s} = \frac{b}{t}$.

3. Let $\frac{a}{s} \in \left(S^{-1}A\right)^G$ with $s \in S^G$. Show that there exists $u \in S$ such that for every $g \in G$, $us(a - g \cdot a) = 0$, and deduce that one can take $u \in S^G$.

4. Let $b = usa$. Show that $b \in A^G$, and deduce that $\left(S^{-1}A\right)^G \simeq (S^G)^{-1}A^G$.

Exercise 5. Let $k = \mathbb{F}_q$, $f = X^qY - XY^q \in k[X, Y]$, $R = k[X, Y]/(f)$, and let x and y be the images of X and Y in R. Show that R is not finite over $k[x - ay]$ for any $a \in k$ (start with $a = 0$). Deduce that for finite fields we need to use another method to prove the Noether Normalization Theorem.

Exercise 6. Let A be an integral domain and B a finitely generated A-algebra. Show that there exists $f \in A \setminus \{0\}$ and $x_1, \ldots, x_n \in B$ algebraically independant over A such that B_f is finite over A_f.