Intoduction to algebraic geometry

Exercise 1. (An introduction to algebraic geometry) Let k be an algebraically closed field.

If $S \subseteq k[X_1, \cdots, X_n]$, we set

$$\mathcal{Z}(S) = \{x \in k^n \mid \forall f \in S, f(x) = 0\}$$

Subsets of k^n of the form $\mathcal{Z}(S)$ are called *affine algebraic sets*.

If $T \subseteq k^n$ we set

$$\mathcal{I}(T) = \{f \in k[X_1, \cdots, X_n] \mid \forall x \in T, f(x) = 0\}$$

1. If $S \subseteq k[X_1, \dots, X_n]$, show that $\mathcal{Z}(S) = \mathcal{Z}(\langle S \rangle)$ (where $\langle S \rangle$ is the ideal generated by S). Show that the affine algebraic sets are the closed subset of a topology on k^n. We call this topology the *Zariski topology.*

   If T is a subset of k^n we denote by $\overline{T}$ its closure for the Zariski topology.

2. Show that the maps $S \to \mathcal{Z}(S)$ and $T \mapsto \mathcal{I}(T)$ are decreasing.

3. Show that for all T, $\mathcal{I}(T)$ is a radical ideal of $k[X_1, \cdots, X_n]$ (that is, $\sqrt{\mathcal{I}(T)} = \mathcal{I}(T)$).

4. Show that if T is a subset of k^n, then $\mathcal{Z}(\mathcal{I}(T)) = \overline{T}$.

5. Show that if I is an ideal of $k[X_1, \cdots, X_n]$ then $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$.

6. Deduce that the maps $\mathcal{Z}$ and $\mathcal{I}$ define reciprocal bijections between the set of radical ideals of $k[X_1, \cdots, X_n]$ and the set of closed subsets of k^n. Show that under this bijection, maximal ideals correspond to points, $k[X_1, \cdots, X_n]$ corresponds to the empty set, and $\{0\}$ corresponds to k^n.

7. Let X be a topological space. We say that X is irreducible if for all closed subset F_1 and F_2 of X such that $X = F_1 \cup F_2$, either $X = F_1$ or $X = F_2$. Show that under the bijection of the previous question, prime ideals correspond to irreducible subsets.

8. Let Z be an affine algebraic set. Give a bijection between Z and the set of maximal ideals of $k[X_1, \cdots, X_n]/\mathcal{I}(Z)$.

Exercise 2. (Regular functions) Let k be an algebraically closed field. Let X be a locally closed subset of k^n (for the Zariski topology, i.e., $X = U \cap F$ where F is closed and U is open). We call such a set a "space".

A function f on X is *regular at* $x \in X$ if there is an open subset U of X containing x such that on U, f can be written as a quotient of polynomials.

A function is regular on X if it is regular everywhere. We denote by $\mathcal{O}(X)$ the k-algebra of regular functions on X.

1. Show that any regular function is Zariski-continuous.

2. Show that X is irreducible if and only if $\mathcal{O}(X)$ is integral.

3. Let $X = k^n$. Show that any regular function is represented by a unique rational function $f \in k(X_1, \dots, X_n)$. Deduce that $\mathcal{O}(k^n) = k[X_1, \dots, X_n]$ (Hint: regularity at x means that there is no pole at x).

4. Let $\mathfrak{p}$ be a prime ideal of $k[X_1, \dots, X_n]$. Show that any regular function of $X = \mathcal{Z}(\mathfrak{p})$ is represented by a unique element $f \in \mathrm{Frac}(k[X_1, \dots, X_n]/\mathfrak{p})$. Deduce that $\mathcal{O}(\mathcal{Z}(\mathfrak{p})) = k[X_1, \dots, X_n]/\mathfrak{p}$.

   This can in fact be extended to any radical ideal I (Try to understand why as a bonus; this is hard).

5. Let $f \in k[X_1, \dots, X_n]$, what is $\mathcal{O}(U(f))$, where $U(f) = k^n \setminus \mathcal{Z}(f)$?

6. Let U be an open subset of k^n. Show that any regular function on U is uniquely represented by an element of $k(X_1, \dots, X_n)$. What is $\mathcal{O}(k^2 \setminus \{0\})$?

Exercise 3. (Morphisms) Let k be an algebraically closed field. Let X and Y be spaces (in the sense of the previous exercise). A (continuous) function $f : X \to Y$ is called a morphism if for any regular function $g : Y \to k$, $g \circ f$ is regular. An isomorphism is an invertible morphism. We denote the set of morphisms from X to Y by $\mathrm{Hom}_{alg}(X, Y)$.

1. Show that if $f : X \to Y$ is a morphism, it induces a morphism of rings $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$.

2. Show that a function $k^n \to X \subseteq k^m$ is a morphism if and only if it is polynomial with respect to each coordinate.

3. Same question but replace k^n with an irreducible closed subset.

4. Let $f \in k[X_1, \ldots, X_n]$. Show that $U(f)$ is isomorphic to the closed subset $\mathcal{Z}(fX_0 - 1)$ of k^{n+1}.

5. Let X and Y be irreducible closed spaces. Show that the map

$$\mathcal{O} : \mathrm{Hom}_{alg}(X, Y) \to \mathrm{Hom}_{k-\mathrm{alg}}(\mathcal{O}(Y), \mathcal{O}(X))$$

   is a bijection. (In a fancy formalism, the (contravariant) functor $\mathcal{O}$ is an equivalence of categories between irreducible closed spaces and finite type integral k-algebras). Deduce that $k^2 \setminus \{0\}$ is not an affine algebraic subset of any k^n and another way to compute $\mathcal{O}(U(f))$ for $f \in k[X_1, \ldots, X_n]$.

Exercise 4. (Products) Let k be an algebraically closed field. In this exercise we admit the bonus of question 3 of exercise 2. Let X and Y be closed spaces in k^n and k^m respectively.

1. Let I be an ideal of $k[X_1, \ldots, X_n]$ and J be an ideal of $k[X_{n+1}, \ldots, X_{n+m}]$, show that $k[X_1, \ldots, X_{n+m}]/(I, J) = k[X_1, \ldots, X_n]/I \otimes_k k[X_1, \ldots, X_m]/J$.

2. Let A and B be reduced k-algebras with A finitely generated. We would like to show that $A \otimes_k B$ is reduced.

   (a) Let x be a nilpotent element of $A \otimes_k B$. Show that we can write $x = \sum_{i=1}^{n} a_i \otimes b_i$ with (b_i) linearly independant over k.

   (b) Let $\mathfrak{m}$ be a maximal ideal of A. Show that the a_i are in $\mathfrak{m}$. (Hint: use a consequence of the Nullstellensatz.) Conclude.

3. Deduce that $\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$

Exercise 5. (Spectrum of a ring) Let A be a commutative ring. We would like to assign a "space" X_A to A such that A can be viewed as the ring of regular functions on X_A. For instance we associated the space $\mathcal{Z}(\mathfrak{p})$ to $k[X_1, \ldots, X_n]/\mathfrak{p}$; there is a bijection between the set $\mathrm{Max}(k[X_1, \ldots, X_n]/\mathfrak{p})$ and $\mathcal{Z}(\mathfrak{p})$. However it occurs that when A is not a Jacobson ring (for instance if A is local and is not a field), the set $\mathrm{Max}(A)$ does not capture all the "geometric information" assoxiated to A. Instead, the right space is rather the set $\mathrm{Spec}(A)$ of prime ideals of A.

Let I be an ideal of A. We denote by $V(I)$ the subset $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid I \subseteq \mathfrak{p}\}$.

1. Show that the $V(I)$ are the closed subset of a topology on $\mathrm{Spec}(A)$. This topology is called the *Zariski topology.*

2. Show that if $A = k[X_1, \ldots, X_n]/\mathfrak{p}$, the topology on $\mathrm{Max}(A)$ induced by the Zariski topology on $\mathrm{Spec}(A)$ is the same as the Zariski topology induced by the bijection with $\mathcal{Z}(\mathfrak{p})$.

3. Show that A is a Jacobson ring if and only if $\mathrm{Max}(A)$ is a dense subset of $\mathrm{Spec}(A)$.