ENS de Lyon — Master 1 – Algèbre avancée
TD2 — 2020-2021

Free, finitely generated and Noetherian modules

Exercise 1. Let A be a ring, M be an A-module and $(e_i)_{i\in I}$ a family of elements of M. Show that (e_i) is a basis of M if and only if $M = \oplus_{i\in I} Ae_i$ and the e_i are not torsion.

Exercise 2. Let A be a ring

1. Show that any A-module has a generating family. Deduce that any A-module is the quotient of some free A-module.
2. Let M be an A-module and N a submodule. Assume that N and M/N are free. Show that M is free.
3. Show that any direct sum and any finite product of free A-modules is free.
4. Let $A = \mathbb{Z}/6\mathbb{Z}$. Show that $2A$ and $3A$ are not free A-modules, but $2A \oplus 3A$ is a free A-module.

Exercise 3. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module.

1. Show that any finitely generated submodule of $\mathbb{Q}$ is free of rank 0 or 1.
2. Deduce that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module. Show that $\mathbb{Q}/\mathbb{Z}$ is not a finitely generated $\mathbb{Z}$-module.
3. Let $(e_i)_{i\in I}$ be a generating family of $\mathbb{Q}$, and let $j \in I$. Show that the family $(e_i)_{i\in I\setminus\{j\}}$ is still a generating family of $\mathbb{Q}$.
4. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Exercise 4. Let A be a ring

1. If I is an ideal of A, show that A/I is a noetherian A-module if and only if A/I is a noetherian ring.

   Now, let M be an A-module and I be the annihilator of M, that is, $I = \mathrm{Ann}(M) = \{a \in A, ax = 0\ \forall x \in M\}$.
2. Show that I is an ideal of A, and that M is naturally an A/I-module.
3. Assume that M is finitely generated, and let $(x_1, \dots, x_n)$ be a generating family. Let $u : A \to M^n$, $a \mapsto (ax_1, \dots, ax_n)$. Show that $I = \ker(u)$.
4. Assume that M is a noetherian A-module. Show that A/I is a noetherian ring, and that M is a finitely generated A/I-module (but A itself is not necessarily noetherian).

Exercise 5. Let A be a ring and let I be an ideal of A. We say that I is nilpotent if $I^n = 0$ for some $n \in \mathbb{N}$.

1. Show that every element of a nilpotent ideal is nilpotent.
2. Suppose that A is noetherian, prove that an ideal I is nilpotent if and only if it is generated by nilpotent elements.
3. Give an example of a non-nilpotent ideal generated by nilpotent elements.

Exercise 6. Let A be a ring and consider the polynomial ring in two variables $A[X, Y]$. Let

$$B = A[X] \bigoplus_{n=1}^{\infty} A[X]XY^n.$$

1. Show that B is a subring of $A[X,Y]$.
2. Is B a noetherian ring? (Hint: consider the ideal generated by the XY^n).

Exercise 7. Let A be a ring. Let I be an ideal of A. Let M be a finitely generated A-module and ϕ an endomorphism of M such that $\phi(M) \subset IM$.

1. Show that ϕ naturally induces a structure of $A[X]$-module on M.
2. Assume ϕ is surjective. Show that ϕ is an isomorphism by using the determinant trick for the identity map of M, seen as a $A[X]$-linear map. Find a counterexample when M is not finitely generated.
3. Is an injective endomorphism of a finitely generated A-module surjective ?
4. Let M be a free A-module of rank n and $(f_1, \ldots, f_n)$ be a generating family of M. Show that $(f_1, \ldots, f_n)$ is a basis of M.
5. If $(f_1, \ldots, f_n)$ is a linearly independent family of a rank n A-module M, is it a basis of M ?

Exercise 8. (On exact sequences) Let A be a ring.

1. Find A-modules N_2, N_3 and M and a surjective map $v : N_2 \to N_3$ such that the corresponding map $v_* : \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_3)$ is not surjective.
2. Find A-modules N_1, N_2 and M and an injective map $u : N_1 \to N_2$ such that the corresponding map $u^* : \mathrm{Hom}_A(N_2, M) \to \mathrm{Hom}_A(N_1, M)$ is not surjective.
3. Show that if A is a field and N_1, N_2, N_3 and M are finite dimensional over A then for any exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, the corresponding sequences

$$0 \to \mathrm{Hom}_A(M, N_1) \to \mathrm{Hom}_A(M, N_2) \to \mathrm{Hom}_A(M, N_3) \to 0$$

and

$$0 \to \mathrm{Hom}_A(N_3, M) \to \mathrm{Hom}_A(N_2, M) \to \mathrm{Hom}_A(N_1, M) \to 0$$

are exact.

Exercise 9. (Chains and artinian modules) Let A be a ring. Let M be an A-module. We define the length of M as the supremum of the lengths of chains of submodules of M, that is

$$l(M) = \sup\{n \in \mathbb{N} \mid \exists \{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M\}$$

1. Find a noetherian module of infinite length.
2. If $l(M) = 1$ we say that M is simple. Show that, if M is simple, M is noetherian and is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A.
3. Let N be a submodule of M. Show that

$$l(M) = l(N) + l(M/N).$$

Deduce that if $\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ is a maximal chain of submodules of M, then M_{i+1}/M_i is simple for all $i \in \{0, \ldots, n-1\}$.
4. Show that if M is of finite length, there exists $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ maximal ideals of A such that $\mathfrak{m}_1 \cdots \mathfrak{m}_r M = 0$.
5. An A-module M is said to be artinian if any decreasing chain of submodules of M is stationary. Show that artinian noetherian module are exactly modules of finite length.

Remark. Any artinian ring is in fact noetherian.