Exact sequences, projective modules and a bit of tensor products

All rings are assumed to be commutative.

Exercise 1. Let $M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3$ be an exact sequence of A-modules. Show that the following properties are equivalent:

1. the sequence splits, that is, there is an isomorphism $\phi : M_2 \to M_1 \oplus M_3$ such that $\phi \circ \alpha$ is the canonical injection and $\beta \circ \phi^{-1}$ is the canonical projection.
2. β is surjective, and there exists a retraction of α, that is, a linear map $\rho : M_2 \to M_1$ such that $\rho \circ \alpha = \mathrm{Id}_{M_1}$.
3. α is injective, and there exists a section of β, that is, a linear map $\sigma : M_3 \to M_2$ such that $\beta \circ \sigma = \mathrm{Id}_{M_3}$.

Exercise 2. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.

Exercise 3. Let M be a finitely generated A-module and $u : F \twoheadrightarrow M$ be a surjective linear map, where F is a free A-module of finite rank. Show that M is finitely presented (that is if there is an exact sequence $A^s \to A^s \to M \to 0$ with $r, s \in \mathbb{N}$) if and only if $\ker(u)$ is finitely generated. Hint : use the snake lemma.

Exercise 4. Let A be a local ring, that is having a single maximal ideal $\mathfrak{m}$.

1. Show that $\mathfrak{m}$ is exactly the set of non-invertible elements of A.
2. **(Nakayama's lemma)** Let M be a finitely generated A-module such that $M \subset \mathfrak{m}M$. By using the determinant trick, show that $M = \{0\}$.
3. Let M be a finitely generated A-module. Show that $M/\mathfrak{m}M$ is a finite dimensional $A/\mathfrak{m}$-vector space.
4. Let $(\overline{u_1}, \dots, \overline{u_n})$ be a basis of $M/\mathfrak{m}M$.
   (a) Show that $(u_1, \dots, u_n)$ is a generating family of M. Hint : apply Nakayama's lemma to a well-chosen A-module.
   (b) Assume moreover that M is a projective A-module. Show that $(u_1, \dots, u_n)$ is a basis of M.

In conclusion, finitely generated projective modules over local rings are free.

Remark : This holds generally for projective modules over local rings.

Exercise 5. Let A and B be rings, and $C = A \times B$.

1. Show that $I_A = \{0\} \times B$ and $I_B = A \times \{0\}$ are ideals of C and that $C/I_A \simeq A, C/I_B \simeq B$.
2. Show that if M is a A-module and N is a B-module then $M \times N$ is naturally a C-module, and that conversely any C-module is of this form.
3. Show that if M_1, M_2 are A-modules and N_1, N_2 are B-modules then $\mathrm{Hom}_C(M_1 \times N_1, M_2 \times N_2) \simeq \mathrm{Hom}_A(M_1, M_2) \times \mathrm{Hom}_B(M_2, N_2)$.
4. Show that if M is a projective A-module and N is a projective B-module then $M \times N$ is a projective C-module.
5. Show that if M is a free A-module and N is a free B-module then $M \times N$ is not necessarily a free C-module. Give a sufficient condition for it to be a free C-module.

Exercise 6. Let A be the ring of continuous π-periodic functions from $\mathbb{R}$ to $\mathbb{R}$, P be the set of continuous π-antiperiodic functions from $\mathbb{R}$ to $\mathbb{R}$ (that is $f(x+\pi) = -f(x)$ for every $x \in \mathbb{R}$). Show that P is a finitely generated projective A-module that is not free, and that $P \oplus P \simeq A^2$. Hint : use $\cos^2 x + \sin^2 x = 1$.

Exercise 7. Let P be a finitely generated projective A-module and M a finitely generated A-module. Show that $\operatorname{Hom}_A(P, M)$ is finitely generated. Show that if M is finitely presented then $\operatorname{Hom}_A(P, M)$ is finitely presented.

Exercise 8. Compute $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$ where $m, n \in \mathbb{N}^*$.

Exercise 9. Give an example of modules M, N with respective submodules M', N' such that there exist $x \in M', y \in N'$ satisfying $x \otimes y \neq 0$ in $M' \otimes N'$ yet $x \otimes y = 0$ in $M \otimes N$.