ENS de Lyon — Master 1 – Introduction à la Théorie des Nombres

TD5 — 2020-2021

Dedekind rings

Exercise 1. [Ideals of Dedekind rings]
Let A be a Dedekind ring.

1. Let I be a non-zero ideal of A. Prove that for every non-zero prime ideal $\mathfrak{p}$ of A, $v_{\mathfrak{p}}(I)$ is the largest integer n such that $I \subset \mathfrak{p}^n$.
2. Let I, J be non-zero ideals of A. Compute $v_{\mathfrak{p}}(I+J), v_{\mathfrak{p}}(IJ)$ and $v_{\mathfrak{p}}(I \cap J)$. Deduce that $IJ = (I+J)(I \cap J)$.
3. Does the above formula hold in general (commutative integral) rings ?
4. In this question, assume A has characteristic 0. Let $\mathfrak{p}$ be a non-zero prime ideal of A. Prove that there exists a prime number p below $\mathfrak{p}$, *i.e.* $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$, and show that $A/\mathfrak{p}$ is a finite extension of $\mathbb{F}_p$.
5. Prove the approximation lemma : for every distinct non-zero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of A, every $x_1, \ldots, x_r \in A$ and every $n_1, \ldots, n_r \in \mathbb{N}$, there exists $x \in A$ such that $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$ for $1 \leq i \leq r$.
6. Let $\mathfrak{a} = \prod_{i=1}^r \mathfrak{p}_i^{n_i}$ be a a non-zero ideal of A. We will show that $\mathfrak{a}$ can be generated by two elements.
   (a) Prove that there exists $x \in \mathfrak{a}$ such that $v_{\mathfrak{p}_i}(x) = n_i$ for $1 \leq i \leq r$.
   (b) Write $xA = \prod_{i=1}^s \mathfrak{p}_i^{n_i}$ (with $s \geq r$). Show that there exists $y \in \mathfrak{a}$ such that $v_{\mathfrak{p}_i}(y) = n_i + 1$ for $1 \leq i \leq r$ and $v_{\mathfrak{p}_i}(y) = 0$ for $r < i \leq s$.
   (c) Show that $\mathfrak{a} = (x, y)$.
7. Show that if A has finitely many prime ideals, then A is principal. Give an example of such a ring.
8. Show that any ideal of A is projective. *(Hint : Use $II^{-1} = A$.)*

Exercise 2. [Local characterization of Dedekind rings]
Show that an integral noetherian ring A is a Dedekind ring if and only if $A_{\mathfrak{p}}$ is a principal ideal domain for every non-zero prime ideal $\mathfrak{p}$ of A.

Exercise 3. [Decomposition of primes in quadratic number fields] Let $d \neq 0, 1$ be a squarefree integer, $K = \mathbb{Q}(\sqrt{d})$ and p an prime number. Show that :

1. If $p \mid d$ then $p\mathcal{O}_K = (p, \sqrt{d})^2$ and $(p, \sqrt{d})$ is prime.
2. If $\left(\frac{d}{p}\right) = 1$ then $p\mathcal{O}_K = (p, x+\sqrt{d})(p, x-\sqrt{d})$, where $x \in \mathbb{Z}$ is such that $x^2 \equiv d \bmod p$, and $(p, x+\sqrt{d})$ and $(p, x-\sqrt{d})$ are both prime.
3. If $\left(\frac{d}{p}\right) = -1$ then $p\mathcal{O}_K$ is prime.
4. Find the decomposition of $2\mathcal{O}_K$. *(Hint : When $d \equiv 1 \bmod 4$, argue according to the value of $d \bmod 8$.)*