TUTORIAL 9

1 Binary splitting on the arctangent series

We aim at computing, with N bits of accuracy (i.e., with error less than 2^{-N}),

$$\arctan\left(\frac{1}{q}\right) = \frac{1}{q} - \frac{1}{3q^3} + \frac{1}{5q^5} - \cdots = \sum_{k\geqslant 0} \frac{(-1)^k}{(2k+1)\cdot q^{2k+1}}. \tag{1}$$

We assume that $q > 0$ is a "small" integer (so that its size can be considered constant in a complexity analysis), and $M(p)$ is the time required to multiply two p-bit numbers.

1. Define, for $a < b$,

$$\begin{aligned} R(a,b) &= (2a+3)(2a+5)(2a+7)\cdots(2b+1), \\ Q(a,b) &= (2a+3)(2a+5)(2a+7)\cdots(2b+1)\cdot q^{2(b-a)}, \end{aligned}$$

   and

$$P(a,b) = (-1)^{a+1}\frac{R(a,b)}{2a+3}q^{2(b-a-1)} + (-1)^{a+2}\frac{R(a,b)}{2a+5}q^{2(b-a-2)} + \cdots + (-1)^{b}\frac{R(a,b)}{2b+1}q^{0}.$$

   Show that the sum of the first $K+1$ terms of the series (1) is equal to

$$\frac{1}{q}\cdot\left(1 + \frac{P(0,K)}{Q(0,K)}\right).$$

2. We obviously have $R(a,b) = R(a,m)\cdot R(m,b)$ and $Q(a,b) = Q(a,m)\cdot Q(m,b)$. Express $P(a,b)$ as a function of $P(a,m)$, $R(a,m)$, $P(m,b)$, and $Q(m,b)$.

3. Show that the sizes (in number of bits) of the integers $Q(0,K)$ and $R(0,K)$ are $O(K\log K)$. What can be said about the size of $P(0,K)$?

4. Give a quasi-linear time algorithm to compute $P(0,K)$ and $Q(0,K)$.

5. Deduce the time required to evaluate the series (1) with error less that 2^{-N} – you may have to treat the case $q = 1$ separately.

2 Directly computing binary or hexadecimal digits of π

Plouffe's Formula (or the BBP formula[1]) for π is the following:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left[\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right]. \tag{2}$$

[1] Plouffe is from Quebec while Bailey and Borwein are American and Canadian. This may explain why the French community only kept the name of Plouffe...

The objective of this exercise is to use (2) for directly computing the zillionth hexadecimal digit of π, without having to compute the previous ones. In the following, $\{u\}$ is the fractional part of u (i.e. $\{u\} = u - \lfloor u \rfloor$), and for $j = 1, 4, 5, 6$, we define

$$S_j = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}.$$

We (almost) straightforwardly have:

$$\{16^d\pi\} = \left\{4\{16^d S_1\} - 2\{16^d S_4\} - \{16^d S_5\} - \{16^d S_6\}\right\}.$$

1. We wish to evaluate, with error less than 16^{-p} (p is a small integer), the number

   $$\{16^d S_j\} = \left\{\left\{\sum_{k=0}^{d} \frac{16^{d-k}}{8k+j}\right\} + \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+j}\right\},$$

   suggest an algorithm for that.

2. Give an algorithm that returns the $(d+1)$-th hexadecimal digit of π.

3. Give a rough complexity analysis.

3 Plouffe's Formula for π (or BBP formula in English)

In this exercise, we prove Plouffe's formula (2), used in the previous exercise.

1. Let $k \in \mathbb{N}, k \geq 1$. Show that

   $$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8} dx = \frac{1}{\sqrt{2}^k} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}.$$

2. Consider the following sum:

   $$\mathcal{S} = \sum_{i=0}^{\infty} \frac{1}{16^i} \left[\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right].$$

   Show that

   $$\mathcal{S} = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} dx,$$

   and deduce that

   $$\mathcal{S} = \int_0^1 \frac{16y - 16}{y^4 - 2y^3 + 4y - 4} dy.$$

3. Show that $\mathcal{S} = \pi$, which gives Plouffe's formula:

   $$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left[\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right]. \tag{3}$$