Approx-SVP in Ideal lattices with Pre-Processing

Alice Pellet-Mary, Guillaume Hanrot and Damien Stehlé

LIP, ENS de Lyon

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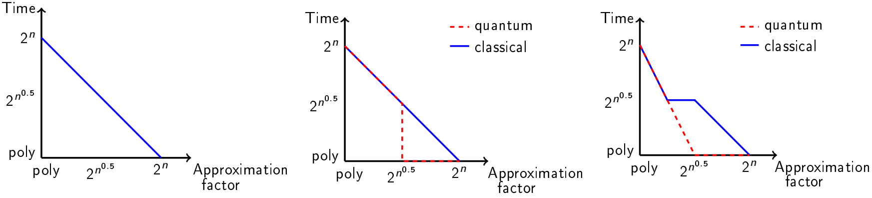
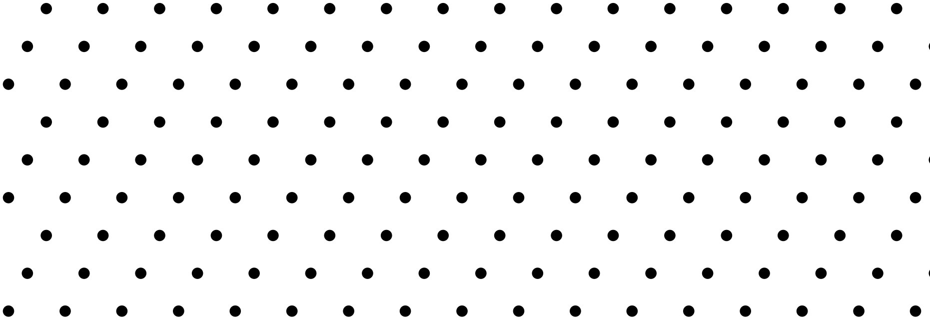
What is this talk about

Time/Approximation factor trade-off for SVP in ideal lattices:



BKZ algorithm | [CDPR16,CDW17] | This work (with $2^{O(n)}$ pre-processing)

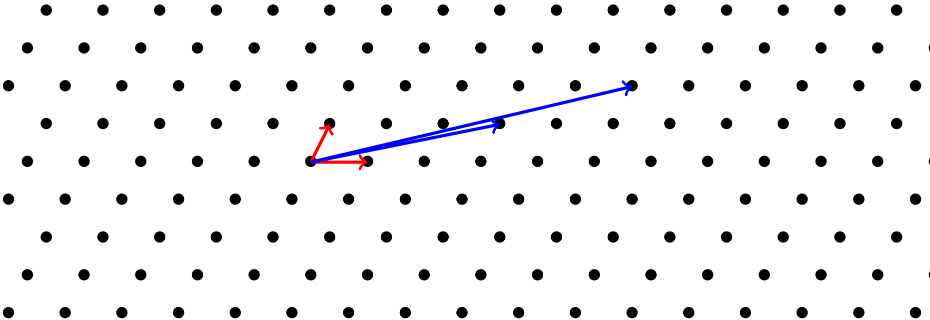
Lattices



Lattice

A lattice L is a discrete 'vector space' over $\mathbb{Z}$.

Lattices

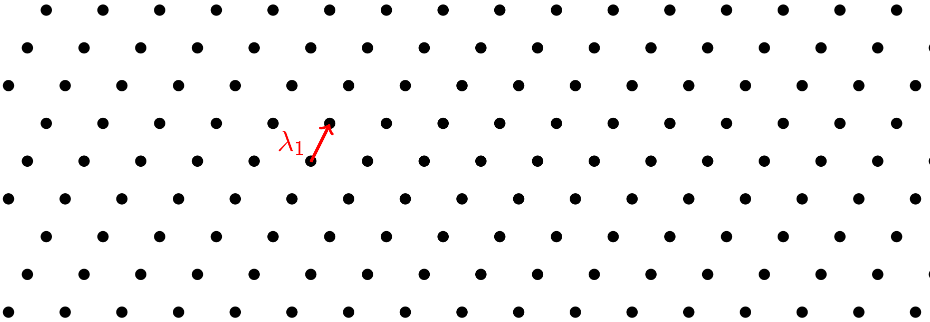


Lattice

A lattice L is a discrete 'vector space' over $\mathbb{Z}$.
A basis of L is an invertible matrix B such that $L = \{Bx \mid x \in \mathbb{Z}^n\}$.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 11 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

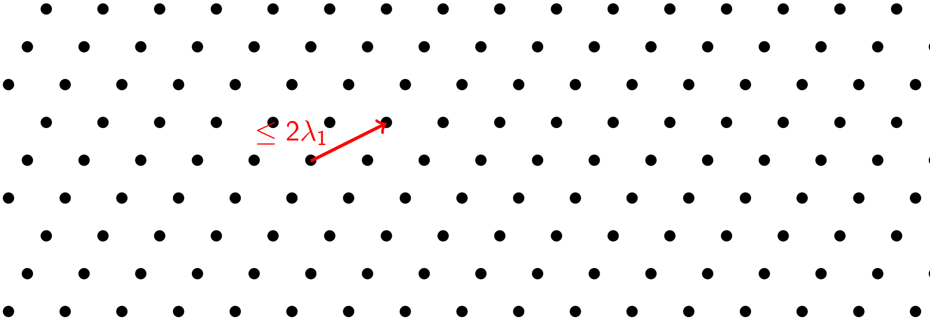
Lattices



Shortest Vector Problem (SVP)

Find a shortest (in Euclidean norm) non-zero vector.
Its Euclidean norm is denoted λ_1.

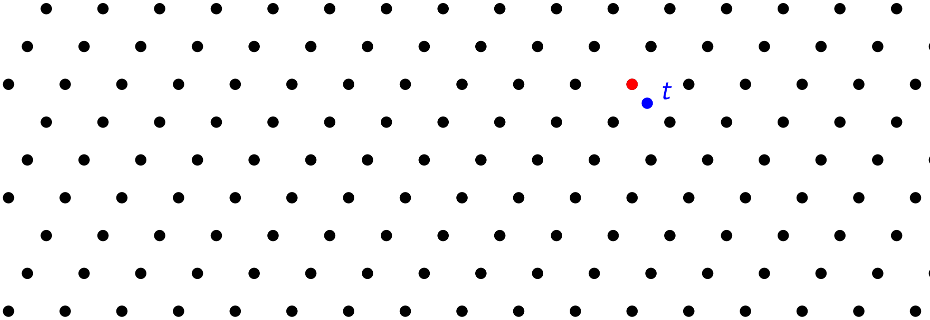
Lattices



Approximate Shortest Vector Problem (approx-SVP)

Find a short (in Euclidean norm) non-zero vector.
(e.g. of norm $\leq 2\lambda_1$).

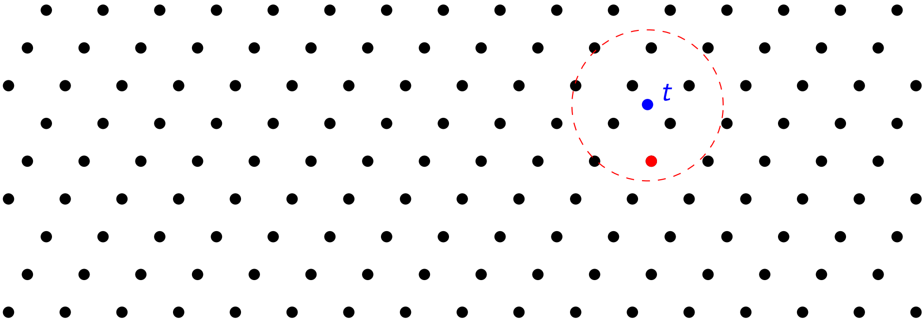
Lattices



Closest Vector Problem (CVP)

Given a target point t, find a point of the lattice closest to t.

Lattices



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Complexity of SVP/CVP

Applications

SVP and CVP in general lattices are conjectured to be hard to solve both quantumly and classically $\Rightarrow$ used in cryptography

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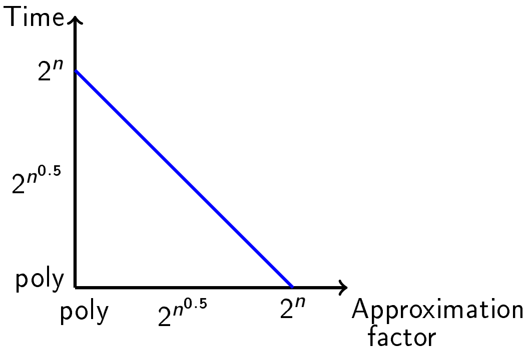
Best Time/Approximation trade-off for general lattices: BKZ algorithm



Structured lattices

Improve efficiency of lattice-based crypto using structured lattices.
$\Rightarrow$ E.g. ideal lattices

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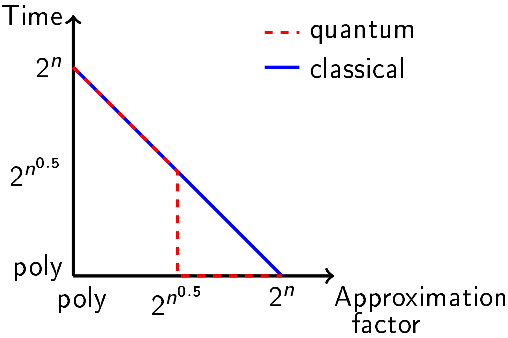
Is approx-SVP still hard when restricted to ideal lattices?

SVP in ideal lattices

[CDPR16,CDW17]: Better than BKZ in the quantum setting

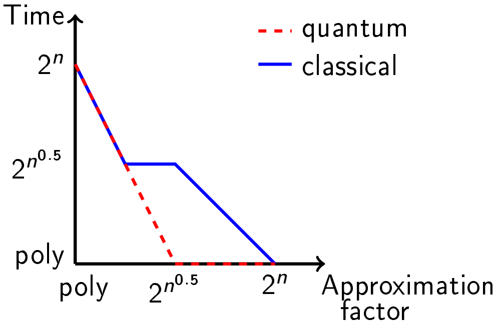


- Heuristic
- For prime power cyclotomic fields

[CDPR16] R. Cramer, L. Ducas, C. Peikert and O. Regev. Recovering Short Generators of Principal Ideals in Cyclotomic Rings, Eurocrypt.

[CDW17] R. Cramer, L. Ducas, B. Wesolowski. Short Stickelberger Class Relations and Application to Ideal-SVP, Eurocrypt.

This work



- Heuristic
- Pre-processing $2^{O(n)}$, independent of the choice of the ideal (non-uniform algorithm).

Outline of the talk

1. Definitions and objective
2. The CDPR algorithm
3. This work

First definitions

Notation

$R = \mathbb{Z}[X]/(X^n + 1)$ for $n = 2^k$

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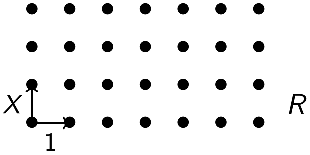
- Units: $R^\times = \{a \in R \mid \exists b \in R, ab = 1\}$
  - e.g. $\mathbb{Z}^\times = \{-1, 1\}$
- Principal ideals: $\langle g \rangle = \{gr \mid r \in R\}$ (i.e. all multiples of g)
  - e.g. $\langle 2 \rangle = \{\text{even numbers}\}$ in $\mathbb{Z}$
  - g is called a generator of $\langle g \rangle$
  - The generators of $\langle g \rangle$ are exactly the ug for $u \in R^\times$

Why is $\langle g \rangle$ a lattice?

$R \simeq \mathbb{Z}^n$

$$R = \mathbb{Z}[X]/(X^n + 1) \to \mathbb{Z}^n$$
$$r = r_0 + r_1 X + \cdots + r_{n-1} X^{n-1} \mapsto (r_0, r_1, \ldots, r_{n-1})$$



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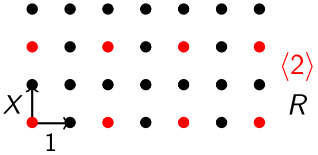
$\langle g \rangle \subseteq R \simeq \mathbb{Z}^n$ + stable by '+' and '-' $\Rightarrow$ lattice



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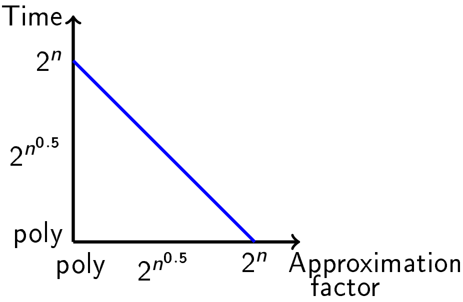
BKZ algorithm can do it in time $2^{O(n^{1-\alpha})}$, can we do better?



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Main idea of the CDPR algorithm (on an idea of [CGS14])

Idea

Maybe g is a somehow small element of $\langle g \rangle$

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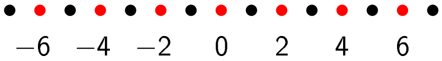

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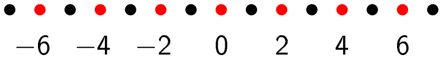

For larger n: one of the generators is somehow small

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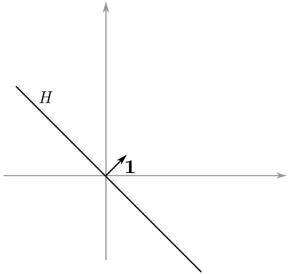
The Log space

Log : $R \to \mathbb{R}^n$ (somehow generalising log to R)

Let $\mathbf{1} = (1, \cdots, 1)$ and $H = \mathbf{1}^{\perp}$.



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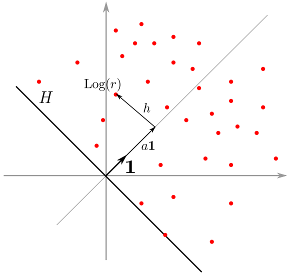
$\text{Log}\, r = h + a\mathbf{1}$, with $h \in H$

- $a \geq 0$



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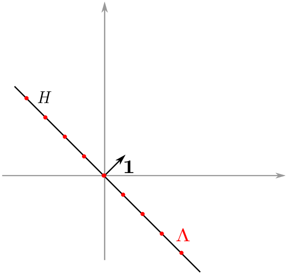

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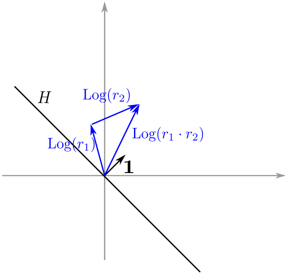

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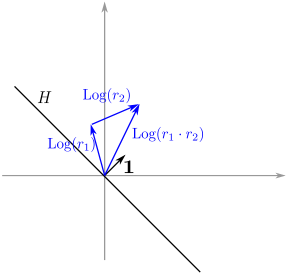
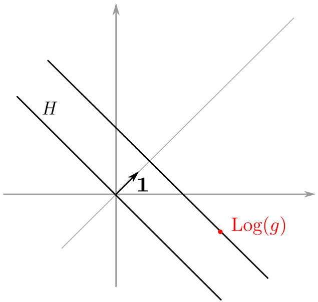
- $\|r\| \simeq 2^{\|\text{Log}\, r\|_\infty}$



The CDPR algorithm

What does $\mathrm{Log}\langle g \rangle$ look like?



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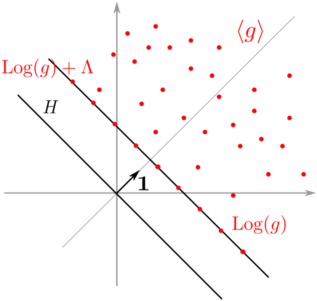

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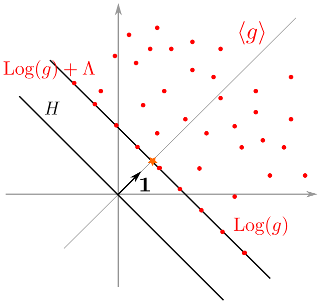

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The CDPR Algorithm:

- Find a generator g_1 of $\langle g \rangle$.
  - [BS16]: quantum time $\mathrm{poly}(n)$
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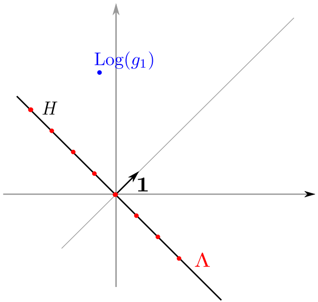

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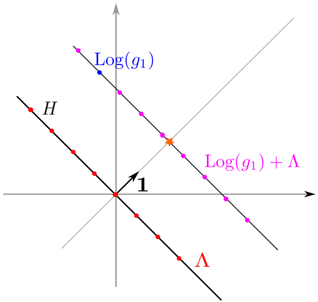

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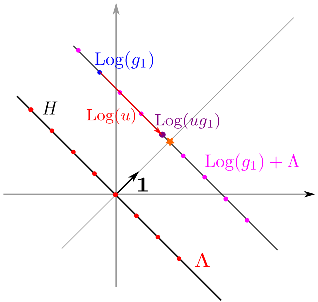

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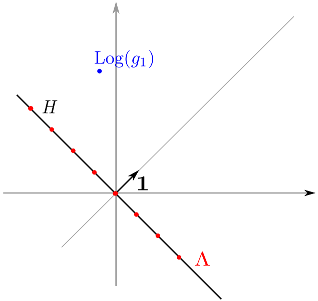

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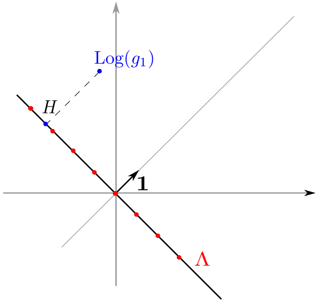

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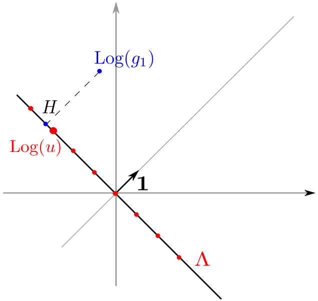

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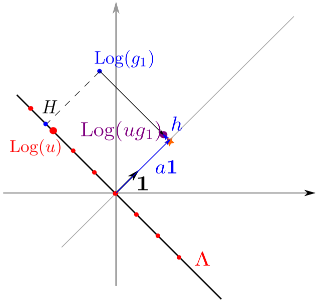

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    $\Rightarrow$ CVP in poly time
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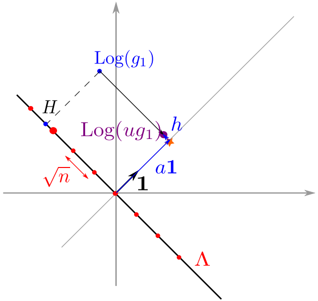

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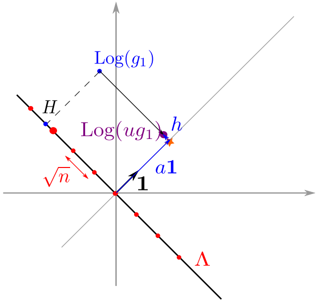

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Outline of the talk

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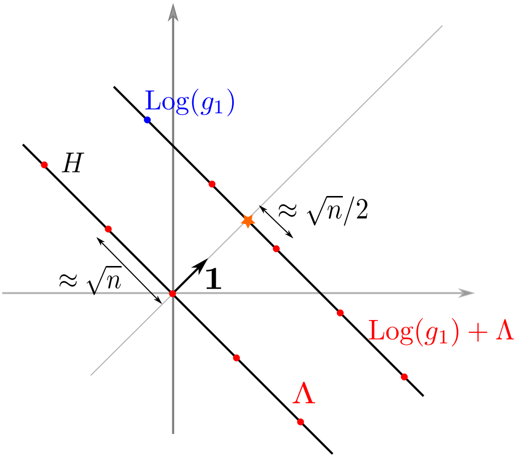
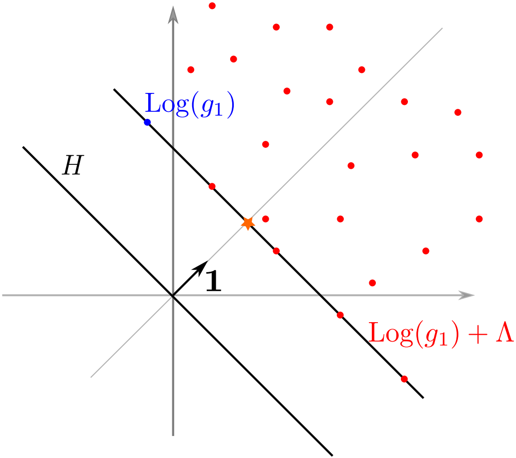


$\mathrm{Log}(g_1)$
H
$\approx \sqrt{n}/2$
$\approx \sqrt{n}$
$\mathbf{1}$
$\mathrm{Log}(g_1) + \Lambda$
Λ

Idea



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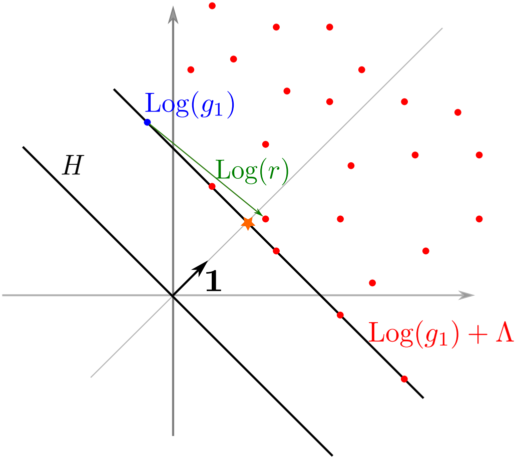

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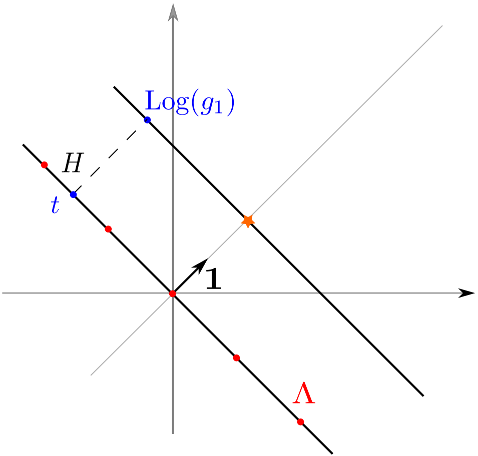

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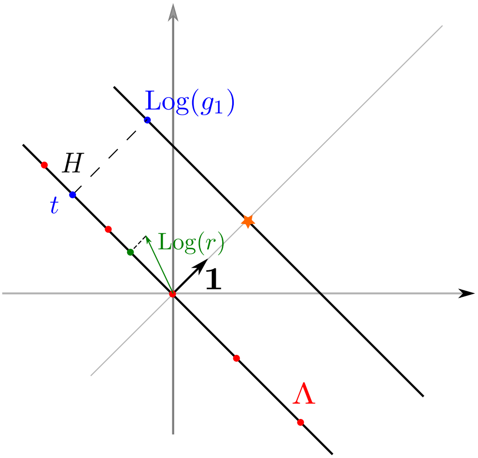

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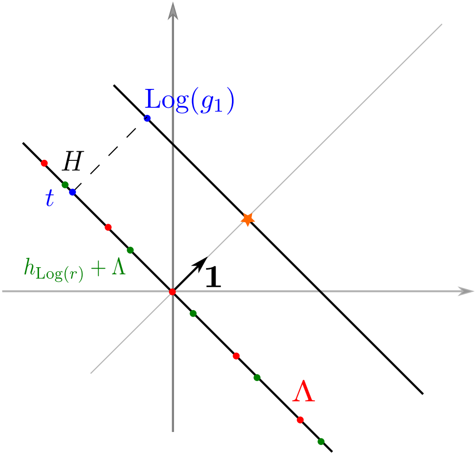
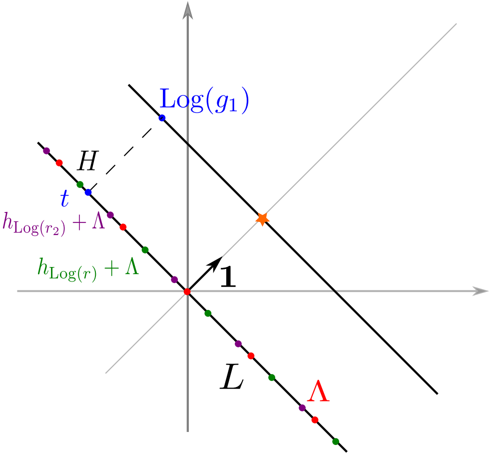


Idea



$\mathrm{Log}(g_1)$
H
t
$h_{\mathrm{Log}(r_2)} + \Lambda$
$h_{\mathrm{Log}(r)} + \Lambda$
$\mathbf{1}$
L
Λ

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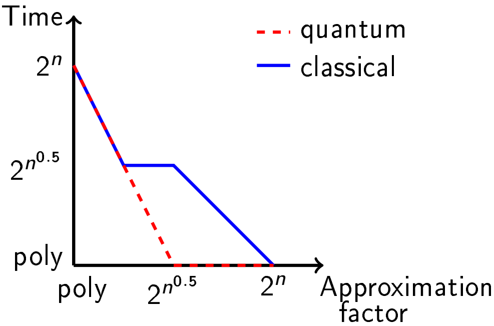
[Laa16]:
- Find $s \in L$ such that $\|s - t\| = \widetilde{O}(n^{\alpha})$
- Time: $2^{\widetilde{O}(n^{1-2\alpha})}$ (query) $+ 2^{O(n)}$ (pre-processing)

[Laa16] T. Laarhoven. Finding closest lattice vectors using approximate Voronoi cells. SAC.

Conclusion

Approximation	Query time	Pre-processing
$2^{\widetilde{O}(n^\alpha)}$	$2^{\widetilde{O}(n^{1-2\alpha})} + (\text{poly}(n)$ or $2^{\widetilde{O}(\sqrt{n})})$	$2^{O(n)}$



$+2^{O(n)}$ Pre-processing / Non-uniform algorithm

Extensions

- Non principal ideals ✓
- Generalization to other number fields ✓
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Questions?