

Quantitative realizability

A biorthogonality-based framework for space (and time) complexity

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Dal Lago & Hofmann : a realizability framework for time complexity

- Uniform proofs of soundness for various type systems w.r.t some time complexity classes.
e.g : **EAL**, **LFPL**, **SAL**, ...
- Time bounds are represented by elements of a **resource monoid**.

Krivine : classical realizability

- based on orthogonality between programs and environments.
- reveal the computational meaning of various axioms (e.g : ZF + dependant choice axiom)
- has been recently used to generalize Cohen's forcing
- modular

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We describe here a realizability framework :

- Based on the KAM, orthogonality, and resource monoids
- That allows to prove space (and time) complexity properties of programs
- We apply it to STA_B (Gaboardi, Marion & Ronchi della Rocca), and prove its soundness w.r.t **PSPACE**.

Summary

- Define the space measure on λ -terms.
- Recall the notion of resource monoid
- Build a realizability model of STA_B :

① IMLL

② We add the modalities

③ We add the additive *if* rule

KAM (1/2)

(Terms)	$t, u ::= x \mid tu \mid \lambda x.t \mid \boxtimes$
(Closures)	$c ::= (t E)$
(Environments)	$E, F ::= [] \mid E[x := c]$
(Stacks)	$\pi ::= \diamond \mid c.\pi$
(Processes)	$S ::= c \star \pi$

The set of closures is denoted by Λ and the set of stacks is denoted by Π .

(Push)	$(tu E) \star \pi \succ (t E_t) \star (u E_u).\pi$
(Grab)	$(\lambda x.t E) \star c.\pi \succ (t E[x := c]) \star \pi$
(Var)	$(x E) \star \pi \succ E(x) \star \pi$

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KAM (2/2)

Subterm property

If t is a closed term, if $(t|[])\star\diamond\asymp^* S$, then every term u appearing in S is a subterm of t .

→ pointer representation of the processes.

$\#S$ is the number of pointers in S , i.e the number of closures.

$$\text{Space}(S) = \max\{\#S' | S \asymp^* S'\}$$

If t normalizes, then its normal form is computable in space $O(|t|\text{Space}(t))$.

Resource monoid (Dal Lago, Hofmann)

We introduce a slight generalization of resource monoid.

A structure $\mathcal{M} = (M, +, 0, \leq, \|\cdot\|)$ is a **quantitative monoid** if :

- $(M, +, 0, \leq)$ is a preordered commutative monoid.
- For all $p, q \in M$ then $\|p\| + \|q\| \leq \|p + q\|$.
- If $p \leq q$ then $\|p\| \leq \|q\|$.
- For every $n \in \mathbb{N}$, there is an element $\mathbf{n} \in \mathcal{M}$ such that $n \leq \|\mathbf{n}\|$.

e.g : $(\mathbb{N}, +, 0, \leq_{\mathbb{N}}, n \mapsto n)$.

Each resource monoid is a quantitative monoid (by taking $\|p\| = \mathcal{D}(0, p)$).

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Each resource monoid is a quantitative monoid (by taking $\|p\| = \mathcal{D}(0, p)$).

$$A, B ::= \quad X \quad | \quad A \multimap B \quad | \quad \forall X A$$

$$\frac{}{x : A \vdash x : A} (ax)$$

$$\frac{\Gamma \vdash t : A}{\Gamma, y : C \vdash t : A} (W)$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x. t : A \multimap B} (\multimap_i) \quad \frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash tu : B} (\multimap_e)$$

+ 2nd order...

Realizability : definitions

A **quantitative monoid** : $\mathcal{M} = (\mathcal{M}, +, 0, \leq, \|\cdot\|)$

A choice of an observable : $\mathbb{I} \subseteq (\Lambda \star \Pi) \times \mathcal{M}$

e.g : $\mathbb{I} = \{ (S, p) \mid \text{Space}(S) \leq \|p\| \}$

Orthogonality :

- We write $(c, p) \perp (\pi, q)$ whenever $(c \star \pi, p + q) \in \mathbb{I}$
- $X^\perp = \{ (\pi, q) \mid \forall (c, p) \in X, (c, p) \perp (\pi, q) \}$

As usual :

- $X \subseteq X^{\perp\perp}$
- $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$
- $X^{\perp\perp\perp} = X^\perp$

X is a **behaviour** iff $X^{\perp\perp} = X$.

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Realizability : interpretation

We choose a set D of behaviours and a valuation Φ from atomic formulae to D .

To each formula A , we associate an element $|A|$ of D .

- $|A|_\phi = \phi(A)$ if A is atomic.
- $|A \multimap B|_\phi = \{ (c.\pi, p + q) \mid (c, p) \in |A|_\phi, (\pi, q) \in |B|_\phi^\perp \}^\perp$
- $|\forall X A|_\phi = \bigcap_{F \in D} |A|_{\phi[X \mapsto F]}$

The realizability relation : $(c, p) \Vdash A$ means $(c, p) \in |A|$.

$(c, p) \Vdash A$ is equivalent to $\forall(\pi, q) \in |A|^\perp, (c, p) \star (\pi, q) \in \perp$.

Realizability : interpretation

If we know that \perp is :

- \leq -saturated : if $(S, p) \in \perp$ and $p \leq q$, then $(S, q) \in \perp$.
- \succ -saturated : if there is a $k \in \mathbb{N}$, such that if $S \succ S'$ and $(S', p) \in \perp$, then $(S, p + k) \in \perp$.

Adequacy lemma

If $x_1 : A_1, \dots, x_n : A_n \vdash t : B$, there is some p such that whenever $(c_i, q_i) \Vdash A_i$, we have $(t[x_1 := c_1, \dots, x_n := c_n], p_t + \sum_i q_i) \Vdash B$.
With $p_t \leq |t|k$.

Remark :

- We don't need the norm $\|.\|$ to obtain adequacy.
- It does not break if you add new instructions and reduction rules.

Reducibility candidates

Space observable

- $\mathbb{L}_S = \{ (S, p) \mid \text{Space}(S) \leq \|p\| \}$
- $r_G = r_P = 0$ and $r_V = 1$

A behaviour X is a reducibility candidate iff

$$(\mathbf{\Xi}, \mathbf{1}) \in X \subseteq \{(\diamond, \mathbf{0})\}^\perp = \{ (c, p) \mid (c \star \diamond, p) \in \mathbb{L} \}$$

If X is a reducibility candidate then

$$(\diamond, \mathbf{0}) \in X^\perp \subseteq \{(\mathbf{\Xi}, \mathbf{1})\}^\perp = \{ (\pi, q) \mid (\mathbf{\Xi} \star \pi, q + \mathbf{1}) \in \mathbb{L} \}$$

Reducibility candidates are stable by \multimap and \forall .

Finally,

- ➊ We can now instantiate the monoid : we choose $(\mathbb{N}, +, 0, \leq_{\mathbb{N}}, n \mapsto n)$.
- ➋ By adequacy, $\vdash t : A \Rightarrow (t, |t|) \in |A|$.
- ➌ $|A|$ is a reducibility candidate, so $\text{Space}((t|\square) \star \diamond) \leq |t|$.
- ➍ The normal form of t is computable in space $O(|t|^2)$.

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The modalities

$$\frac{\Gamma, x_1 : A, \dots, x_n : A \vdash t : B}{\Gamma, x : !A \vdash t[x/x_1, \dots, x/x_n] : B} (Contr) \quad \frac{\Gamma \vdash t : B}{! \Gamma \vdash t : !B} (!)$$

A quantitative monoid \mathcal{M} together with $! : \mathcal{M} \rightarrow \mathcal{M}$.

We interpret $!$ as

$$|!A| = \{ (t, !p) \mid (t, p) \in |A| \}^{\perp\perp}$$

To obtain adequacy we need the following properties :

- $!(p + q) \leq !p + !q + r_{Prom}$
- $np \leq !p + r_{contr}^n$

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Soft monoid

Elements of the monoids are (n, f) where $n \in \mathbb{N}$ and f is a polynomial.

- $(n, f) + (m, g) = (\max(n, m), f + g)$
- $(n, f) \leq (m, g)$ iff
 - ▶ $n \leq m$
 - ▶ $\forall x, m \leq x \Rightarrow f(x) \leq g(x)$
 - ▶ $g - f$ is non-decreasing for $m \leq x$
- $\|(n, f)\| = f(n)$
- $!(n, f) = (n, x \mapsto (x + 1)f(x))$

The problem of !

Let's try to prove adequacy for contraction...

We know that for every $(d, q) \Vdash A$ and $(d', q') \Vdash A$,
 $((t[x := c, y := d]), p + q + q') \Vdash B$.

We take $(c, q) \Vdash !A$. We want to prove $((t[z/x, z/y])[z := c]), p + q \Vdash B$.
But we know nothing about the shape of (c, q) , because of the $(.)^{\perp\perp}$.

In particular, we would like to know that it is sufficient to take
 $(c, q) \in \{ (d, !r) \mid (d, r) \in |A| \} \dots$

The problem of !

Conclusion :

We need generalized contexts, (e.g : $(t|x := \bullet) \star \pi$), so we add them to the set of tests...

But...

It is easier to prove adequacy for $(\neg o_i)$ if we only need to check orthogonality with head contexts.

So what do we do ?

The problem of !

Conclusion :

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But...

It is easier to prove adequacy for (\multimap_i) if we only need to check orthogonality with head contexts.

So what do we do ?

The problem of ! : head interaction is enough

We now have two different notions of orthogonality \perp_h and \perp .

Context lemma

If $\{(c, p)\}^{\perp_h} \subseteq \{(d, q)\}^{\perp_h}$ then $\{(\boxtimes, \mathbf{1}), (c, p)\}^{\perp} \subseteq \{(\boxtimes, \mathbf{1}), (d, q)\}^{\perp}$

Useful because for every reducibility candidate X ,

$$X = \bigcup_{(t, p) \in X} \{(\boxtimes, \mathbf{1}), (t, p)\}^{\perp\perp}$$

We can then prove adequacy to typing.

Adequacy theorem

If $x_1 : A_1, \dots, x_n : A_n \vdash t : B$, there is some p such that whenever $(c_i, q_i) \Vdash A_i$, we have $(t|[x_1 := c_1, \dots, x_n := c_n], p + \sum_i q_i) \Vdash B$.

Moreover, we know the shape of the majorizer p :

$$\frac{}{x \triangleright k} (ax)$$

$$\frac{t \triangleright p}{t \triangleright p} (W)$$

$$\frac{t \triangleright p}{\lambda x. t \triangleright p + k} (\neg o_i)$$

$$\frac{t \triangleright p \quad u \triangleright q}{tu \triangleright p + q + k} (\neg o_e)$$

$$\frac{t \triangleright p}{t \triangleright !p + r_{Prom}} (Prom)$$

$$\frac{t \triangleright p}{t \triangleright p + r_{contr}^n} (Contr_n)$$

Additive rules

$t, u ::= \dots \quad | \quad \text{if } t \text{ then } u_0 \text{ else } u_1 \quad | \quad \text{true} \quad | \quad \text{false}$

$$\frac{}{\vdash \text{true} : \text{Bool}} (\text{true}) \quad \frac{}{\vdash \text{false} : \text{Bool}} (\text{false})$$

$$\frac{\Gamma \vdash t : \text{Bool} \quad \Gamma \vdash u_0 : A \quad \Gamma \vdash u_1 : A}{\Gamma \vdash \text{if } t \text{ then } u_0 \text{ else } u_1 : A} (\text{if})$$

An imaginary additive rule

$$\frac{x : C \vdash t : A \quad x : C \vdash u : A}{x : C \vdash \langle t, u \rangle : A \oplus B}$$

To obtain adequacy, we need an operation $\oplus : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ such that $p \oplus q$ is the l.u.b of p and q .

So $(\mathcal{M}, \leq, \oplus)$ will be a join-semilattice.

In fact : all common resource and quantitative monoids are join-semilattices.

An imaginary additive rule

$$\frac{(c, q) \Vdash C \quad (t|[x := c], \mathbf{p}_t + \mathbf{q}) \Vdash A \quad (u|[x := c], \mathbf{p}_u + \mathbf{q}) \Vdash B}{x : C \vdash \langle t, u \rangle : A \oplus B}$$

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We add a trackback mechanism.

$$\begin{array}{lllllll} (\text{Push}) & (tu|E) & \star & (\pi, S) & \succ & (t|E) & \star & ((u|E).\pi, S) \\ (\text{Grab}) & (\lambda x.t|E) & \star & (c.\pi, S) & \succ & (t|E[x := c]) & \star & (\pi, S) \\ (\text{Var}) & (x|E) & \star & (\pi, S) & \succ & (E + S)(x) & \star & (\pi, S) \end{array}$$

We add a trackback mechanism.

$$(\text{If}) \ (\text{if } t \text{ then } u_0 \text{ else } u_1 | E) \star (\pi, S) \succ (t | []) \star (\diamond, (u_0, u_1, E, \pi) :: S)$$

$$\begin{array}{lll} (\text{true}) & (\text{true}|E') & \star (\diamond, (u_0, u_1, E, \pi) :: S) \succ (u_1|E) \star (\pi, S) \\ (\text{false}) & (\text{false}|E') & \star (\diamond, (u_0, u_1, E, \pi) :: S) \succ (u_0|E) \star (\pi, S) \end{array}$$

Suppose that we are working with the space observable \mathbb{I}_S and :

- X is a reducibility candidate
- $(u_0|E, q_0), (u_1|E, q_1) \in X$
- $(t|E)$ reduces to an element of $\{\boxtimes, \text{true}, \text{false}\}$ and $\text{Space}(t|E) \leq \|p\|$.

Then (if t then u_0 else $u_1|E, p \oplus q_0 \oplus q_1) \in X$

A possible interpretation :

$$|Bool| = \{(\text{true}, 1), (\text{false}, 1)\}^{\perp\perp}$$

We have for free the adequacy for the two rules (**true**) and (**false**).

In what case do we have the adequacy for

$$\frac{\Gamma \vdash t : Bool \quad \Gamma \vdash u_0 : A \quad \Gamma \vdash u_1 : A}{\Gamma \vdash \text{if } t \text{ then } u_0 \text{ else } u_1 : A} (\text{if})$$

We need to know what is $|Bool|$.

Suppose we don't allow in \perp_S the normal forms of the shape

$$(\lambda x.t|E) \star (\diamond, S) \quad \text{with } S \neq []$$

e.g : $\perp = \{ (C, p) \mid \text{Space}(C) \leq \|p\| \text{ and the normal form} \in \{\mathbb{X}, \text{true}, \text{false}\} \}$

Now, we pose $\pi = (\diamond, (\mathbb{X}, \mathbb{X}, [], \diamond))$. Then we have for some q such that $(c, p) \Vdash \text{Bool}$ implies

$$(c \star \pi, p + q) \in \perp$$

So c reduces to an element of $\{\mathbb{X}, \text{true}, \text{false}\}$.

- The soft monoid is a join-semilattice

$$(n, f) \oplus (m, g) = (\max(n, m), \max(f, g))$$

- Elements of $|Bool|$ reduce to elements of $\{\text{true}, \text{false}, \times\}$.
- Executing $(\text{if } t \text{ then } u_0 \text{ else } u_1)$ in head position uses at most $\max(\text{Space}(t), \text{Space}(u_0), \text{Space}(u_1))$ space.

Thanks to these three points we can prove adequacy of the *if* rule for the **space observable**.

$$\frac{t \triangleright p \quad u_0 \triangleright q_0 \quad u_1 \triangleright q_1}{\text{if } t \text{ then } u_0 \text{ else } u_1 \triangleright p \oplus q_0 \oplus q_1} (\text{if})$$

Conclusion

We have presented :

- A tool to prove space complexity properties, successfully applied to STA_B .
- Another application of resource monoids

Still a lot of work :

- Time complexity
- Why does it look like realizability algebras ?
- Can we do a better treatment of the modality ?