Modeling mixing in two-dimensional turbulence and stratified fluids

Antoine Venaille and Joel Sommeria

Abstract A phenomenological model for turbulent mixing in a stratified fluid is presented. This model describes the evolution of the local probability distribution for the fluid density. It is based on an analogy between the mixing of vorticity in 2D turbulent flows and the mixing of density in (3D) turbulent flows.

1 Introduction

Models of turbulent mixing in stratified fluids are of wide interest in the context of oceanic and atmospheric flows, especially for sub-grid scale parameterization [5]. Although the processes of turbulent density mixing occur at small scales and short time scales, they considerably influence large scale dynamics by controlling water mass properties and the global stratification. It is therefore compulsory to describe carefully these processes.

We propose in this paper a new approach to describe the evolution of the local probability density function (PDF) for the fluid density. The advantage of such a statistical approach is to predict a coarse grained evolution of the system, without describing the complicated fine grained dynamics, but while keeping track of the conserved quantities of this dynamics, which are important physical constraints.

The most commonly used models for small scale density mixing are based on variations around the $k-\varepsilon$ models (see part I and III of [1] for a review). In those approaches, turbulence is represented locally by two parameters for which a dynamical equation based on turbulent diffusivity is proposed: the turbulent kinetic

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energy and either a length or a time scale of the flow. In those models, the effect of density fluctuations are ignored, and their ability to describe properly mixing process strongly relies on parameterization of turbulent diffusion coefficients from the (locally) averaged quantities. Refinement of those models take into account higher order moments of the density (up to the fourth), and in some cases nonlocal effects [14, 5].

In parallel to those approaches, an idealized stochastic model (referred to as one-dimensional turbulence) has been applied for mixing in stratified flows [4]. This model mimics the effects of turbulent cascade, buoyancy and advection on a vertical realization of the density field.

Surprisingly, there has been no attempt to combine the classical modeling of mixing in terms of turbulent diffusion, with a model for the temporal evolution for the probability distribution of density. Beyond the advantage of describing the temporal evolution of the density distribution, such a model can give insight into the role played by density fluctuations in mixing processes. he density in a turbulent stratified flow, as vorticity in a 2D turbulent flow, is a scalar quantity that needs to satisfy conservation laws. Those constraints prevent complete mixing of the scalar. The idea of equilibrium statistical mechanics that are well known for the case of vorticity in 2D flows have been recently applied for stratified fluids in an idealized case [12]. We propose in this paper a phenomenological approach for the (out of equilibrium) turbulent mixing in stratified flows, on the basis of this analogy between the mixing of vorticity in 2D turbulent flows and the mixing of buoyancy in 3D turbulent stratified flows.

The paper is organized as follows: i) We briefly review the statistical theory of 2D flow, and then present the analogy with stratified flows. ii) We propose relaxation equations toward the equilibrium states for stratified flows, based on a work developed previously in the context of 2D flows. Two physical mechanisms are taken into account: turbulent diffusion and buoyancy effects, that tend to drive back the system toward a background “sorted” density profile (which minimizes the potential energy for a given global distribution of density). iii) We incorporate to those relaxation equations a mechanism of dissipation of the density fluctuations, due to turbulent cascade effects that tend to smooth out the density field by transfers from large to small scales iv) we discuss simple limit cases of the previous model v) we explain how the dynamical equations proposed in this paper could be adapted in a more general and realistic context.

2 An analogy between statistical mechanics of 2D flows and density stratified fluids

2.1 Statistical mechanics of 2D flows

The Euler equations can be expressed as a transport equation of the vorticity $\omega(x,y,t)$ in a domain $\mathscr{D}$

$$\partial_t \omega + \mathbf{u}\cdot\nabla\omega = 0 \quad with \quad \mathbf{u} = \left(-\frac{\partial\psi}{\partial y}\ ,\ \frac{\partial\psi}{\partial x}\right) \quad and \quad \omega = \Delta\psi\ , \tag{1}$$

where the (non-divergent) velocity field is expressed in term of a stream function ψ. The transport equation conserves the energy functional $E = \frac{1}{2}\int_{\mathscr{D}}(\nabla\psi)^2\,dxdy = -\frac{1}{2}\int_{\mathscr{D}}\omega\psi dxdy$ and the global distribution of vorticity levels $g(\sigma) = \int_{\mathscr{D}}\delta(\omega-\sigma)dxdy$ (equivalent to the conservation of the infinite number of Casimir functionals $\mathscr{C}_g[q] = \int_{\mathscr{D}} g(q)dxdy$, where g is any continuous function on $\mathscr{D}$).

The Euler equations are known to develop complex vorticity filaments at finer and finer scales as time goes on. Rather than describing the fine-grained structures of the flow, equilibrium statistical theories of two dimensional turbulent flows predict final organization of the flow at a coarse grained level [10, 8]. The *macroscopic state* is given by a field $\rho(x,y,\sigma)$ representing the probability density of finding the vorticity level σ in a small neighborhood of the position (x,y). From this field, one can compute the coarse-grained vorticity field and stream function by inverting the Laplacian with appropriate boundary conditions

$$\overline{\omega} = \int \rho\sigma d\sigma\ , \quad \overline{\omega} = \Delta\overline{\psi} \tag{2}$$

The rationale of the theory is that most accessible microscopic states will approach the macroscopic state which maximizes the mixing entropy $\mathscr{S}[\rho] = -\int_{\mathscr{D}}\int\rho\ln\rho dxdyd\sigma$. Assuming ergodicity, the equilibrium statistical theory provides a variational problem : the most probable (or equilibrium) macroscopic state ρ maximizes the mixing entropy with the constraints provided by the conservation of the energy and the global distribution of fine-grained vorticity levels (both quantities can be theoretically computed from the initial condition):

$$\mathscr{E}[\rho] = \int_{\mathscr{D}}\int\rho\sigma\overline{\psi}dxdyd\sigma \quad \forall\sigma\ ,\ d_\sigma[\rho] = \int_{\mathscr{D}}\rho dxdy. \tag{3}$$

Notice that the energy of the fluctuations are supposed to tend to zero due to the dominance of small scale fluctuations. The variational problem can be summarized as follows:

$$S(E,g) = \max_{\{\rho|N[\rho]=1\}}\{\mathscr{S}[\rho]\ |\ \mathscr{E}[\rho] = E\ \&\ d_\sigma[\rho] = g(\sigma)\} \tag{4}$$

Much effort has been devoted to the study of the equilibrium states of the RSM theory. It has been applied in particular to explain the robustness of the Great Red Spot in the Jovian atmosphere [2].

2.2 Statistical mechanics of stratified fluids

Let us consider now the mixing of the density anomaly $b = g\left(\rho_{fluid} - \rho_0\right)/\rho_0$ in the frame of the Boussinesq approximation (b is the opposite of the buoyancy). This is a tracer advected by a (3D) turbulent non divergent velocity field $\mathbf{v}$:

$$\partial_t b + \mathbf{v}\nabla b = \kappa \Delta b \tag{5}$$

$$\partial_t \mathbf{v} + \mathbf{v}\nabla\mathbf{v} = -\nabla P - b\mathbf{k} + \nu\Delta\mathbf{v} + \mathbf{F} \tag{6}$$

where $\mathbf{F}$ is a mechanical forcing and $\mathbf{k}$ is the vertical unit vector.

In the absence of forcing and dissipation ($\mathbf{F} = 0$, $\kappa = \nu = 0$), the total energy of the flow $E = \int_{\mathscr{V}} \left(\frac{1}{2}\mathbf{v}^2 + bz\right) dxdydz$ and the global distribution of density levels $g(\sigma) = \frac{1}{|\mathscr{V}|}\int_{\mathscr{V}} \delta(b-\sigma) dxdydz$ are conserved. We suppose in addition that the mean value of the velocity field at any location is zero: there is no mean flow.

We define a *microscopic state* as a given fine grained density field $b(x,y,z)$ and velocity field $\mathbf{v}(x,y,z)$. From the knowledge of a microscopic configuration, one can compute the conserved quantities $E, g(\sigma)$. The problem is assumed to be statistically homogeneous on the horizontal $\mathscr{P}_z$ (parallel to Oxy). A integration over the directions x and y will be considered as an ensemble average, and denoted by an upper bar : $\overline{b}(z) = \frac{1}{|\mathscr{P}_z|}\int_{\mathscr{P}_z} b dxdy$.

A *macroscopic state* of the system is given by the field $\rho(z,\sigma,\mathbf{v})$ that describes the probability to measure a given scalar and velocity value at height z. As for the mixing of the vorticity in 2D flows, the most probable macroscopic state is the one that maximize the mixing entropy $\mathscr{S} = -\int_{[0\ H]}\int_{]-\infty\ +\infty[^3}\int_{[\sigma_{min}\ \sigma_{max}]}\int \rho \ln \rho d\sigma d\mathbf{v} dz$ (the bounds of integration will be dropped for simplicity) among all the states that satisfy the constraints of the problem, namely the energy conservation

$$\mathscr{E}[\rho] = \mathscr{E}_c[\rho] + \mathscr{E}_p[\rho] = \int\int\int \rho \left(\frac{\mathbf{v}^2}{2} + \sigma z\right) dz d\sigma d\mathbf{v} = E \tag{7}$$

and the conservation of the global scalar distribution:

$$\mathscr{H}_\sigma[\rho] = \int\int \rho dz d\mathbf{v} = g(\sigma) \tag{8}$$

where H is the total height of the domain. We make at this point the strong assumption that each microscopic state is accessible, and compute the most probable macroscopic state satisfying the constraints of the problem, as in the case of vorticity in 2D flows.

In order to compute critical points of the variational problem, we introduce the Lagrange multipliers β and $\gamma(\sigma)$ associated respectively with the energy (7) and with the constraints of the global vorticity distribution (8), and then compute first variations with respect to ρ:

$$\delta\mathscr{S} - \beta\delta\mathscr{E} + \int \gamma(\sigma)\delta\mathscr{H}_\sigma d\sigma = 0\ . \tag{9}$$

This gives $\int\int\int\left(-1-\ln(\rho)-\beta\mathbf{v}^2-\beta z\sigma+\gamma\right)\delta\rho d\mathbf{v}d\sigma dz = 0$. Since this equality holds for any variation $\delta\rho$, we obtain

$$\rho = A\exp\left(-\beta\mathbf{v}^2/2 - \beta\sigma z + \gamma(\sigma)\right) \tag{10}$$

The value of the Lagrange multipliers β and $\gamma(\sigma)$ are determined by the expression of the constraints $\mathscr{E}[\rho] = E$ and $\mathscr{H}_\sigma[\rho] = g(\sigma)$, and A is a normalization factor.

Notice that the PDF (10) of the statistical equilibrium can be expressed as a product of a PDF for density and velocity, which means that b and $\mathbf{v}$ are two independent quantities. The predicted velocity distribution is Gaussian, and is isotropic. The predicted isotropy is not likely to be observed in a real flow, in which vertical motion is inhibited by stratification. However, a careful examination of the flow structure at the interface of two turbulent layers of different density shows that mixing occurs mainly by the occurrence of intermittent (both in time and space) turbulent patches that break the interface, stir and mix the density of the patches [6]. At the early stage of those mixing events, the distinction between vertical and horizontal velocities is not obvious.

The predicted velocity profile does not depend on z; the kinetic energy profile $e(z)$ is therefore constant along the vertical axis, with

$$e = \frac{1}{2}\int\int \mathbf{v}^2\rho d\sigma d\mathbf{v} = \frac{3}{2}\beta^{-1}. \tag{11}$$

The inverse of β (a "temperature" of the turbulent field) is thus proportional to the variance of the velocity fluctuations. This implies that $\beta > 0$. In the following, we shall focus on the density distribution $\rho(\sigma,z)$ (and the associated moments), ignoring the independent distribution in velocity.

$$\rho(\sigma,z) = B\exp\left(-\beta\sigma z + \gamma(\sigma)\right)\ , \quad \overline{b^n} = \int \sigma^n\rho d\sigma \tag{12}$$

We can then find another expression for β, which links this quantity with a form of potential energy, related to density fluctuations:

$$\beta^{-1} = \frac{\int_0^H\left(\overline{b^2}-\overline{b}^2\right)dz}{-\int_0^H \partial_z\overline{b}dz} = \frac{\int_0^H\left(\overline{b^2}-\overline{b}^2\right)dz}{\overline{b}(0)-\overline{b}(H)}\ . \tag{13}$$

Let us consider as an example an initial state composed of two density values ($b = 0$ and $b = \sigma_0$). Let us first assume that both values are in equal proportions. According to equation (12), the probability p to measure $b = \sigma_0$ at height z is

$$p(z) = \frac{e^{-\beta\sigma_0(z-H/2)}}{1+e^{-\beta\sigma_0(z-H/2)}} \quad (14)$$

The vertical profile of the mean density at statistical equilibrium is then $\overline{b}(z) = \sigma_0 p(z)$, a Fermi-Dirac distribution, represented in figure (1). This expression has been proposed recently by [12], using similar arguments.

Assuming ergodicity, such an equilibrium state is expected to be reached if the inertial time scale $\tau \sim \beta^{1/2}H$ is smaller than time scales of forcing and dissipating mechanisms. Let us notice that in the limit of infinit boundaries ($H \to +\infty$), the inertial time scale tends to infinity, and one can not expect to reach the equilibrium state. More generally, real flows are out of equilibrium systems, and the computation of the equilibrium states is only a starting point before more complex approaches.

Another case of interest is the dilute limit, for which the global probability $\int pdz$ to measure the level σ_0 tend to zero while keeping constant $\sigma_0 \int pdz$. This would correspond to the case of a sediment suspension, for which the Boussinesq approximation done earlier is no more valid, but it would be straightforward to generalize this result to a non Boussinesq flow. In this limit, we recover the standard expression $p(z) \simeq \exp -\beta\sigma z$ for a gas in a uniform gravity field.

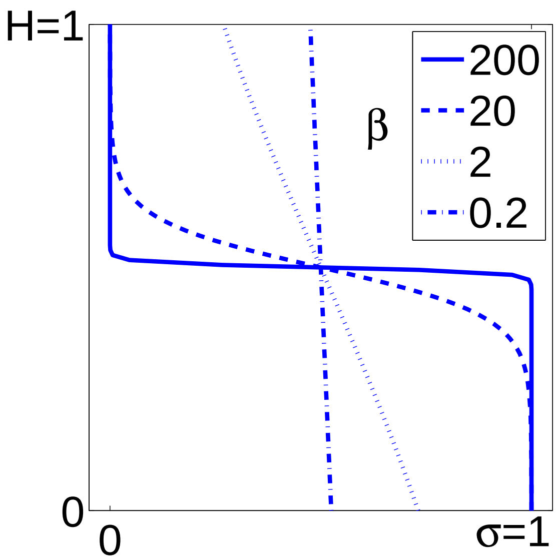

Fig. 1 Equilibrium profile for a two level system. We represent $\overline{b}(z) = \sigma p(z,t)$ where p is given by equation (14), for three different values of β.

3 Relaxation toward statistical equilibrium

We propose in this section an equation describing the relaxation toward the statistical equilibrium state. The general idea is that the system will evolve with increasing mixing entropy while preserving its conserved quantities.

We introduce the turbulent flux of probability $J(\sigma,z,t)$ (directed along z) and still consider that there are neither sinks nor sources for the density σ. The temporal evolution of the PDF $\rho(\sigma,z,t)$ thus satisfies the general conservation law

$$\partial_t \rho + \partial_z J = 0 \, , \tag{15}$$

with $J = 0$ at lower and upper boundaries. This equation conserves the global density distribution, since $d_t \left(\int_0^H \rho dz \right) = 0$. A convenient way to obtain an equation for the relaxation toward an entropy maximum is to assume that J maximizes the entropy production at fixed energy (with a condition of bounded fluxes). A similar approach has been previously applied to 2D and geostrophic turbulence [9, 3].

The entropy production reads $\dot{S} = -\int J(\partial_z \rho/\rho) d\sigma dz$, and the time derivative of the energy reads

$$\dot{\mathscr{E}} = d_t \left(\int_0^H e dz \right) + \int_0^H \sigma z \partial_t \rho d\sigma dz = d_t \left(\int_0^H e dz \right) + \int_0^H \sigma J d\sigma dz \tag{16}$$

We assume in addition that the flux J is bounded at each location, $\int (J^2/2\rho) d\sigma < C(z)$ (the quantity J^2/ρ can be considered as the square of a diffusion velocity, a natural quantity to bound). In order to ensure the conservation of the norm $\int \rho d\sigma = 1$, we impose the additional constraint $\int J d\sigma = 0$ at any height z. Then the first variation (with respect to the flux J) of the entropy production with the constraints of the problem gives

$$\delta \dot{S} - \beta \delta \dot{\mathscr{E}} - \int \frac{1}{D} \frac{J}{\rho} \delta J d\sigma dz - \int \zeta(z) \delta J d\sigma dz = 0, \tag{17}$$

where β, $\zeta(z)$ and $-1/D(z)$ are Lagrange parameters associated with the different constraints. A direct computation of those critical points gives

$$J = -D \left(\partial_z \rho + \beta \left(\sigma - \overline{b} \right) \rho \right) , \tag{18}$$

where $\zeta(z)$ has been determined by using $\int J d\sigma = 0$. The coefficient D must be positive for the entropy production to be positive. We expect this diffusion coefficient to be related to the turbulent kinetic energy and a characteristic turbulent length scale l, $D \sim l e^{1/2}$.

We assume at this stage that l (hence D) is constant, and we make the strong assumption that velocity reaches its equilibrium distribution much faster that density, which means that the kinetic energy does not depend on z, with $e = \frac{3}{2}\beta^{-1}$. These hypothesis will be relaxed later on.

We distinguish two contributions to J in (18): a “down-gradient” diffusion term and a sedimentation term, which tends to drive back a fluid particle with density σ to its equilibrium position, where $\overline{b} = \sigma$.

When $J = 0$, turbulent diffusion and sedimentation cancel each other, yielding $\partial_z \rho = -\beta \rho \left(\sigma - \overline{b}\right)$, whose solution is the vertical profile (12) of the statistical equilibrium.

We use the energy conservation $\dot{\mathscr{E}} = 0$ and equation (16) to compute the kinetic energy :

$$e = \frac{3}{2}\beta^{-1} \ , \quad \frac{H}{D} d_t e = \frac{3}{2e} \int \left(\overline{b^2} - \overline{b}^2\right) dz - \left(\overline{b}(0) - \overline{b}(H)\right) \ . \tag{19}$$

At equilibrium ($d_t e = 0$), we recover equation (13) that links the kinetic energy e to the fluctuations of density. Since $e > 0$, we see that this equilibrium results from a competition between density fluctuations that tend to increase the kinetic energy and the (vertically integrated) stratification $\left(\overline{b}(0) - \overline{b}(H)\right)$ that tends to decrease the kinetic energy e if the profile is stable ($\overline{b}(0) > \overline{b}(H)$). Let us also notice that an unstable profile ($\overline{b}(0) < \overline{b}(H)$) cannot correspond to a stationary state, since the term $\left(\overline{b}(0) - \overline{b}(H)\right)$ acts then as a source of kinetic energy.

4 Dissipation of density fluctuations by turbulent cascade

The existence of a turbulent cascade implies that the global distribution of $g(\sigma)$ is actually not conserved: fluctuations are transferred to smaller and smaller scales, until molecular diffusive effects occur. For instance, in a system initially composed of two levels $\{0, \sigma_0\}$, this will create a third level $\frac{1}{2}\sigma_0$, and so on...

This effect has to be taken into account in relaxation equations toward equilibrium, by adding a dissipation term in the dynamical equation (15) of the density distribution:

$$\partial_t \rho + \partial_z J = s \mathscr{D}_c[\rho] \ , \tag{20}$$

where $s(z)$ is a straining rate depending mainly on the velocity field properties in each horizontal plane. At a given height z, the term $\mathscr{D}_c(\sigma)$ depends on the whole PDF $\rho(\cdot, z)$. This operator must conserve the norm and the mean of the distribution ($\int \mathscr{D}_c d\sigma = 0$ and $\int \mathscr{D}_c \sigma d\sigma = 0$), and should dissipate the fluctuations at a rate $s \sim e^{1/2}/l$ depending on the local strain of the flow. One should have $\partial_t \left(\overline{b^2} - \overline{b}^2\right) = -s\left(\overline{b^2} - \overline{b}^2\right)$ in the absence of other processes.

To estimate $\mathscr{D}_c$, several models have been developed in the context of mixing of reactive flows [11]. We choose here a simple model based on a self-convolution process :

$$\widehat{\mathscr{D}}_c(\kappa) = (\widehat{\rho} \ln \widehat{\rho} - \kappa \partial_\kappa \widehat{\rho}) \quad \widehat{\mathscr{D}}_c(\kappa) = \int e^{-\sigma\kappa} \mathscr{D}_c(\sigma) d\sigma \ , \tag{21}$$

where $\widehat{\rho}$ and $\widehat{\mathscr{D}}_c$ are Laplace transform of ρ and $\mathscr{D}_c$. In the absence of other processes, the model predict that an initial PDF will evolve by a succession of self convolutions of the PDF, corresponding to the addition of concentrations of independent scalar sheets becoming adjacent due to random turbulent motion, and simultaneously elongated by straining (see [13] for a more detailed presentation and discussion of the model).

Let us discuss the consequence of the addition of such a dissipation term $\mathscr{D}_c$, whatever its explicit form. The total energy $E = \frac{3}{2}H\beta^{-1} + \int \overline{b}zdz$ is still a conserved quantity in the presence of this dissipation term. Then equation (19) is still valid. If the initial condition is an equilibrium state, the dissipation will lower the contribution of the fluctuation term $\left(\overline{b^2} - \overline{b}^2\right)$ in equation (19), which will imply a decrease of the kinetic energy, and thus an increase of the potential energy of the system.

5 A simple example: mixing of a two layer stratified fluid

To illustrate the mechanisms presented in previous sections, we consider an idealized situation for which the kinetic energy (hence β) is fixed and study the time evolution of the PDF ρ by equations (18), (20) and (21). The unrealistic hypothesis of a fixed kinetic energy will be relaxed later on.

Since the diffusion coefficient D is assumed to be constant, the time unit can be always chosen such that $D = 1$, then the dynamical equation for the PDF is

$$\partial_t \rho = \partial_{zz}\rho + \beta \partial_z \left((\sigma - \overline{\sigma}) \rho \right) + \tau_{diss}^{-1} \mathscr{D}_c \ , \tag{22}$$

where $\tau_{diss}^{-1} = s$. There are two independent parameters, namely β, linked to the imposed kinetic energy, and the time scale of the dissipation process τ_{diss}. The parameter β can be expressed as a sedimentation time scale $\tau_{sedim} = L/\beta\sigma_0$, where σ_0 is the density of the unmixed dense fluid, and L is a characteristic scale of the mean profile $\overline{b}(z)$. The behavior depends on the values of τ_{diss} and τ_{sedim} with respect to the diffusion time scale $\tau_{diff} = L^2$. We distinguish four limit cases:

- **i)** Diffusion and sedimentation dominate dissipation ($\tau_{diff} \sim \tau_{sedim} \ll \tau_{diss}$: The system relaxes toward the statistical equilibrium state corresponding to the (fixed) value of β. On a longer time scale, new density levels are created or destroyed by the dissipation mechanism. The system then goes through a sequence of equilibrium states until it reaches the homogeneous (fully mixed) state.
- **ii)** Dissipation dominates sedimentation and diffusion ($\tau_{diss} \ll \tau_{diff} \sim \tau_{sedim}$): The fluctuations of the initial state are first dissipated. Then the mean profile evolves through the diffusive mechanism $\partial_t \overline{b} = \partial_{zz}\overline{b}$ until complete mixing is achieved.

- **iii)** Diffusion dominates dissipation which dominates sedimentation ($\tau_{diff} \ll \tau_{diss} \ll \tau_{sedim}$). The mean profile evolves mainly by the diffusive process $\partial_t \overline{b} = \partial_{zz} \overline{b}$, until reaching an homogeneous mean vertical profile. Fluctuations around the mean are then dissipated.
- **iv)** Sedimentation dominates dissipation, which dominates diffusion ($\tau_{sedim} \ll \tau_{diss} \ll \tau_{diff}$). The system relaxes first toward the sorted profile (the potential energy minimum for a given global distribution of density levels). There are no more fluctuations in this state, and the vertical profile evolves by the diffusive process $\partial_t \overline{b} = \partial_{zz} \overline{b}$ until complete homogenization is achieved.

In cases **ii)** and **iii),** taking into account fluctuations around the mean profile is not particularly relevant, since the evolution of the mean vertical profile does not depends on these fluctuations. In case **iv),** the fluctuations are important in the early evolution of the flow, but the knowledge of the initial condition is sufficient to determine the sorted profile, and then the evolution equation does not imply any fluctuations.

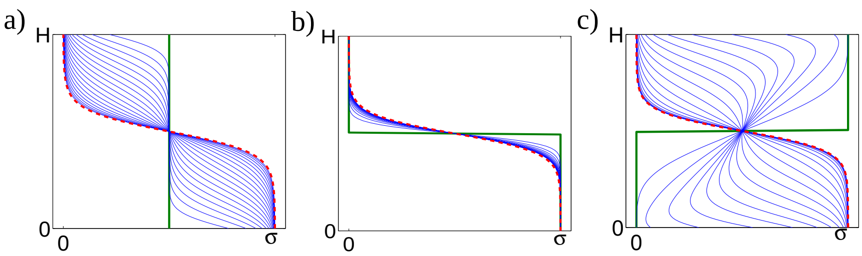

Fig. 2 Relaxation toward the equilibrium state (dashed red line) for a two level system, with $\beta = 20$, $H = 1$, $\sigma_0 = 1$, for three different initial profiles represented in bold plain green line. The thin plain blue curve represent the density profile at successive times with constant time interval.

Let us then consider case **i)**. We first display in figure 2 the relaxation toward an equilibrium state for a two level system ($b = 0$ or $b = \sigma_0 = 1$ in equal proportions), in a case without dissipation. The value of $\beta = 2/(3e)$ is still supposed to be fixed. Three different initial conditions are considered: a) the initial state is completely mixed ($\forall z,\ \rho(0,z) = \rho(\sigma_0, z) = 1/2$); b) the initial state is the sorted profile (the dense fluid is at the bottom) c) the initial state corresponds to the highest possible potential energy (the dense fluid is at the top, which is an instable initial condition).

Let us now consider the evolution of the density profile when the dissipation term is taken into account, figure 3. The initial condition is the equilibrium profile of the two level system for $\beta = 200$. The time scale for dissipation is $\tau_{diss} \sim 1000$, which is much greater than the characteristic time for relaxation toward equilibrium, of order one. The temporal evolution of the mean profile is represented in figure 3-a. Far from the interface (figure 3-b), the PDF is a sharp peak, there is almost no fluctuation, but the density of the peak decreases little by little. Closer to the interface (figure 3-c),

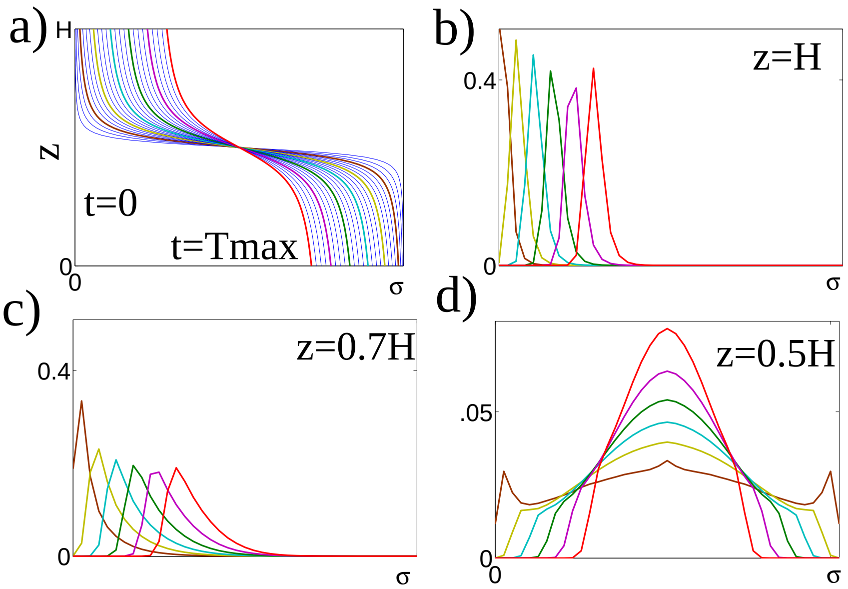

Fig. 3 a) Temporal evolution of the mean profile between $t = 0$ and $T_{max} = 1000$. The initial condition is the equilibrium profile of a two-level system characterized by $\beta = 200$, $\tau_{diss} = 1000$ b-c-d) Temporal evolution of the PDF at different altitudes z.

the PDF is asymmetric, with important fluctuations. Finally, at the middle of the interface (figure 3-d), the PDF is symmetric. The extreme values σ_{min} and σ_{max} of density progressively diminish.

Let us stress that the evolution of the vertical profile is not given by a classic turbulent diffusion: at leading order, diffusion is compensated by sedimentation. The temporal evolution is driven by the dissipation term that creates intermediate density levels, changing little by little the equilibrium profile.

6 Coupling the model with an equation for the kinetic energy

In the general case, the kinetic energy e (and then β) is not uniform, but should satisfy itself a diffusion equation. We assume that its diffusivity has a similar form as for density fluctuations, namely $D \sim le^{1/2}$. We furthermore introduce a dissipation term $a_4 l^{-1} e^{3/2}$ (the usual Kolmogoroff scaling for a turbulent cascade), a production term $\mathscr{P} = \mathbf{F}.\mathbf{v}$ and take into account the exchange with potential energy due to the buoyancy flux. This yields the energy equation

$$\partial_t e = a_3 \partial_z \left(le^{1/2} \partial_z e \right) - \int \sigma J d\sigma - a_4 l^{-1} e^{3/2} + \mathscr{P} \,. \tag{23}$$

One can check that in the absence of production and dissipation, the total energy $E = \int_0^H (\overline{\sigma} z + e)\, dz$ is indeed conserved by equations (15) and (23).

Now that the kinetic energy e varies in space and time, let us assume that the kinetic energy still satisfies locally the link (11) with the inverse temperature β obtained at equilibrium: $e(z,t) = (3/2)\beta^{-1}(z,t)$.

Since the diffusion coefficient D in the buoyancy flux (18) depends on the kinetic energy, $D = a_1 l e^{1/2}$, it is also time and space dependent. The mean density flux is then

$$\int \sigma J d\sigma = -a_1 \left(l e^{1/2} \left(\partial_z \overline{b}\right) - \frac{3l}{2e^{1/2}} \left(\overline{b^2} - \overline{b}^2\right) \right). \tag{24}$$

Notice that this buoyancy term has the same form as in the case of the "level 3 configuration" in the hierarchy of models by Mellor and Yamada [7].

Let us summarize the full model for the temporal evolution of the kinetic energy $e(z,t)$ and the density PDF $\rho(z,\sigma,t)$:

$$\partial_t \rho = -\partial_z J + a_2 l^{-1} e^{1/2} \mathscr{D}_c \tag{25}$$

$$J = -a_1 \left(l e^{1/2} \partial_z \rho + 3 e^{-1/2} \rho \left(\sigma - \overline{\sigma}\right)/2 \right) \tag{26}$$

$$\widehat{\mathscr{D}} = \left(\widehat{\rho} \ln \widehat{\rho} - \kappa \partial_\kappa \widehat{\rho}\right) \quad \widehat{\mathscr{D}}_c = \int e^{-\sigma\kappa} \mathscr{D}(\sigma) d\sigma \tag{27}$$

$$\partial_t e = a_3 \partial_z \left(l e^{1/2} \partial_z e \right) + \int J \sigma d\sigma - a_4 l^{-1} e^{3/2} + \mathscr{P} \tag{28}$$

There are four intrinsic non-dimensional constants of order unity, : i) a_1 and a_3 quantify the turbulent diffusivity for density fluctuations and kinetic energy (velocity fluctuations) respectively. ii) a_2 and a_4 quantify the rate of cascade of density and velocity fluctuations respectively. An additional relation should be given to determine the turbulent scale l. This could be done in the spirit of the k-ε model or mixing length theories. Finally the forcing term $\mathscr{P}$ could represent energy injection by external effects, like oscillating grid. Extension to shear driven turbulence should be proposed for more general cases.

7 Conclusion and perspectives

We have proposed a model for turbulent mixing in a stratified fluid. While most turbulence models deal with the mean and variance of fluctuating quantities, this model predicts the whole probability distribution of density fluctuations. It can deal with highly non-Gaussian distributions. The structure of the model is derived from conservation laws and general principles of entropy production maximization. It can account for re-stratification by gravity. Tests are needed for the validation in more realistic configurations.

References

1. Simpson J.H. Sndermann J. Baumert, H.Z. *Marine Turbulence: theories, observations, models.* Cambridge University Press, 1998.
2. F. Bouchet and J. Sommeria. Emergence of intense jets and jupiter's great red spot as maximum entropy structures. *J. Fluid. Mech.*, 464:465–207, 2002.
3. E. Kazantsev, J. Sommeria, and J. Verron. Subgrid-scale eddy parameterization by statistical mechanics in a barotropic ocean model. *Journal of Physical Oceanography*, 28:1017–1042, 1998.
4. A. R. Kerstein. One-dimensional turbulence: model formulation and application to homogeneous turbulence, shear flows, and buoyant stratified flows. *Journal of Fluid Mechanics*, 392:277–334, August 1999.
5. W. G. Large. Modeling and parameterizing vertical mixing. In E. Chassignet and J. Verron, editors, *Ocean modeling and parameterization*, 1998.
6. J. L. McGrath, H. J. S. Fernando, and J. C. R. Hunt. Turbulence, waves and mixing at shear-free density interfaces. Part 2. Laboratory experiments. *Journal of Fluid Mechanics*, 347:235–261, September 1997.
7. G. L. Mellor and T. Yamada. A hierarchy of turbulence closure models for planetary boundary layers. *Journal of Atmospheric Sciences*, 31:1791–1806, 1974.
8. Jonathan Miller. Statistical mechanics of euler equations in two dimensions. *Phys. Rev. Lett.*, 65(17):2137–2140, Oct 1990.
9. R. Robert, J. Sommeria. Relaxation towards a statistical equilibrium state in two-dimensional perfect fluid dynamics. *Phys. Rev. Lett.*, 69(19):2776, 1992.
10. R. Robert and J. Sommeria. Statistical equilibrium states for two-dimensional flows. *J. Fluid Mech.*, 229:291–310, August 1991.
11. S. B. Pope. An improved turbulent mixing model. *Prog. Energy Combust. Sci.*, 28:131, 1982.
12. E. Tabak and F. Tal. Mixing in simple models for turbulent diffusion. *Comm. Pure. Appl. Math.*, 57:563–589, 2004.
13. A. Venaille and J. Sommeria. Is Turbulent Mixing a Self-Convolution Process? *Physical Review Letters*, 100(23):234506–+, June 2008.
14. Y Cheng, V.M. Canuto, A Howard . Nonlocal convective pbl model based on new third- and fourth-order moments. *J. Atmos. Sci.*, 62:2189–2204, 2005.