A dynamical equation for the distribution of a scalar advected by turbulence

Antoine Venaille[a] and Joel Sommeria
Coriolis-LEGI, 21 rue des Martyrs, 38000 Grenoble, France

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A phenomenological model for the dissipation of scalar fluctuations due to the straining by the fluid motion is proposed in this Brief Communication. An explicit equation is obtained for the time evolution of the probability distribution function of a coarse-grained scalar concentration. The model relies on a self-convolution process. We first present this model in the Batchelor regime and then extend empirically our result to the turbulent case. This approach is finally compared with other models. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2472506]

The turbulent transport of tracers such as temperature or salinity is of great importance for many applications.[1] Available models usually provide a set of closed equations for the mean quantities and their variance, and sometimes for the third- and fourth-order moments.[2] For many practical problems, it would, however, be relevant to predict the dynamical evolution of the whole scalar probability distribution. This problem was first addressed in the context of reactive flows,[3] but such a description can also be important to properly model the turbulent mixing of water masses in a stably stratified fluid, in which case the sedimentation under the influence of gravity has an opposite effect for fluid particles heavier or lighter than the surrounding fluid. The case of vorticity in two-dimensional turbulence is also of interest. Indeed, the statistical mechanics of two-dimensional turbulence gives predictions for the final flow organization depending on an initial distribution of vorticity.[4] This theory can be used as a starting point for nonequilibrium transport models,[5] expressed in terms of the local vorticity probability density function (PDF). In the presence of small viscosity, however, turbulent cascades modify this probability distribution by dissipation of fluctuations. For two-dimensional turbulence, this effect leads to modification of the equilibrium state resulting from turbulent mixing.[6] In this Brief Communication, we introduce a simple model for the cascade effects, which can be combined with transport equations in the presence of spatial gradients.

Let us consider a scalar field $\sigma(\mathbf{r},t)$ transported and conserved by the divergence-free turbulent motion of fluid parcels. The effect of molecular diffusion is expected to smooth out the scalar field at Batchelor's diffusive cutoff scale r_d. We prefer, however, to consider the case of a purely advected scalar, with no diffusion, and introduce a local average at a given scale l, $\sigma_l(\mathbf{r},t)=\int G_l(\mathbf{r}-\mathbf{r}')\sigma(\mathbf{r}',t)d\mathbf{r}'$, obtained with a linear filtering operator G_l. This coarse-grained description corresponds to a finite measurement resolution at a scale l larger than r_d. The fine-grained probability distribution of the scalar is preserved in time as the scalar value and the volume of each fluid parcel is conserved in the absence of diffusion. However, the coarse-grained PDF $\rho_l(\sigma,t)$ is not preserved because fluctuations are transferred to scales smaller than the cutoff l. The cutoff may be alternatively provided by the smoothing effect of diffusion.

The problem is then to find a time evolution equation for this PDF. The result should of course depend on the properties of the turbulent field. In the usual Kolmogorov regime, the dissipation of scalar variance is equal to its cascade flux, independent of the cutoff scale. This flux is set by the energy and integral scale of the turbulence, which are described by transport equations in empirical turbulence models (like k-epsilon).

Instead of a Kolmogorov cascade, we shall consider here a random but smooth and persistent straining motion, in which the velocity difference $\mathbf{v}(\mathbf{x}+\mathbf{r},t)-\mathbf{v}(\mathbf{x},t)$ is a linear function of the separation $\mathbf{r}$.[7] This hypothesis holds whenever the kinetic energy spectrum is steeper than k^{-3}. It could be an appropriate model for vorticity in two-dimensional turbulence.

The straining is characterized by the symmetric part of the strain tensor, $\mathbf{\Sigma}_{ij}=\frac{1}{2}(\partial_j u_i+\partial_i u_j)$. As a consequence of the fluid incompressibility, there exists a basis on which this matrix is diagonal with two opposite eigenvalues $\pm s(t)$. The axes of an initial spherical blob corresponding to positive or negative eigenvalues will grow or decrease, respectively. We assume that the angle between the positive eigendirection of $\mathbf{\Sigma}$ and the isoscalar lines evolves slowly compared to the straining rate $s(t)$, i.e., the time for the eigendirections to rotate by $\pi/2$ is longer than the time for the scalar patterns to be strained from the integral scale L to the filtering scale l. With this hypothesis, our problem becomes locally one-dimensional: the fluid is composed of adjacent sheets of fluid uncorrelated with each other in one direction (which can be tilted with respect to the eigendirection of the strain matrix[8]).

After the time $\Delta t_{1/2}$ needed for the width of a strip to be divided by 2, the scalar field filtered at scale l becomes the average of two realizations of the field at the previous time. The probabilities of scalar values in adjacent strips can be assumed independent, as they result from the straining of regions that were initially far apart, at a distance beyond the integral scale of the scalar. Thus the new probability distri-

[a]Electronic mail: venaille@coriolis-legi.org

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bution is the self-convolution of the previous one, describing the sum of the independent random variables, followed by a contraction by a factor 2, $\rho_l(\sigma,t+\Delta t_{1/2})=2\int\rho_l(\sigma',t)\rho_l(2\sigma-\sigma',t)d\sigma'$.

The convolution can be transformed in a product of the Fourier transform of the PDF (characteristic function). If the scalar σ has only positive values [which can be generally obtained by changing σ in $\sigma-\min(\sigma)$], it is more convenient to use the Laplace transform $\widehat{\rho_l}(\kappa)=\int\rho_l(\sigma)e^{-\kappa\sigma}d\sigma$. The Laplace transform of the previous self-convolution relationship leads to $\widehat{\rho_l}(2\kappa,t+\Delta t_{1/2})=[\widehat{\rho_l}(\kappa,t)]^2$. Similarly, calling $\Delta t_{1/n}$ the time to divide the thickness of a sheet of fluid by a factor n, the PDF at time $t+\Delta t_{1/n}$ will be an nth self-convolution. In the spectral representation, this becomes the product of n identical characteristic functions,[9]

$$\widehat{\rho_l}(n\kappa,t+\Delta t_{1/n})=[\widehat{\rho_l}(\kappa,t)]^n. \tag{1}$$

This discrete convolution mechanism was already used in Refs. 3 and 10 and it is also the central key of the Villermaux and Duplat approach of mixing.[11] In order to get a differential equation in time, we now take $n=1+\epsilon$ with $\epsilon=s(t)dt$ in (1). This yields $\widehat{\rho_l}(\kappa+\epsilon\kappa,t+dt)=[\widehat{\rho_l}(\kappa,t)]^{1+\epsilon}$. Taking the limit $\epsilon\rightarrow 0$, we can express $\rho_l(\kappa+\epsilon\kappa,t+dt)$ in terms of partial derivatives with respect to t and κ,

$$\partial_t\widehat{\rho_l}=s(t)[\widehat{\rho_l}\ln\widehat{\rho_l}-\kappa\partial_\kappa\widehat{\rho_l}]. \tag{2}$$

One can check that the normalization $\int\rho_l(\sigma,t)d\sigma=\widehat{\rho_l}(0,t)=1$ is preserved in time by this equation. The mean scalar value $\int\sigma\rho_l(\sigma,t)d\sigma=-\partial_\kappa\widehat{\rho_l}(0,t)$ is also conserved (if it is initially defined).

The right-hand term of Eq. (2) describes the effect of strain on the PDF of the scalar, which could be used with additional terms expressing scalar generation or spatial transport. Equation (2) itself can be analytically solved (using the method of characteristics). For that purpose, we define the integral

$$f(t)=\exp\left(\int_0^t s(t')dt'\right) \tag{3}$$

[such that $s(t)=f'(t)/f(t)$]. We can easily show that a strip with initial width $R(0)$ reaches a width $R(t)=R(0)/f(t)$ at time t, so that $f(t)$ is the reduction factor in the straining process. We can check that the result

$$\widehat{\rho_l}(\kappa,t)=\left[\widehat{\rho_l}\left(\frac{\kappa}{f(t)},0\right)\right]^{f(t)} \tag{4}$$

is a solution of (2). When $f(t)$ is an integer, we recover the expression (1) for the effect of n self-convolutions, in agreement with our initial assumption.

Equation (2) and its solution (4) describe the process of convergence to a Gaussian stated by the central limit theorem: as times goes on, scalar fluctuations initially extending over more and more area become packed by the straining effect below the filtering scale. This can be checked by the convergence to zero of all the cumulants beyond the second-order one. The mth cumulant, defined as $\langle\sigma^m\rangle_c(t)=(-\partial_\kappa)^m\ln[\rho_l(\kappa,t)]|_{\kappa=0}$, is readily obtained from (4),

$$\langle\sigma^m\rangle_c(t)=\frac{\langle\sigma^m\rangle_c(0)}{[f(t)]^{m-1}}. \tag{5}$$

The cumulant of order 2, equal to the variance, $\langle\sigma^2\rangle_c=\langle\sigma^2\rangle-\langle\sigma\rangle^2$, decays as $1/f(t)$. This expresses the decay of the scalar variance by the cascade through the filtering scale. The relative value of the higher-order cumulants is expressed as $\langle\sigma^m\rangle_c/\langle\sigma^2\rangle_c^{m/2}\sim f(t)^{-(m/2-1)}$, so it decays in time, approaching a Gaussian, for which the cumulants with order larger than 2 are strictly equal to 0. Note, however, that if the first or second cumulants are not defined at time $t=0$, the expression (4) converges to a Levy distribution,[9] another form of stable PDF, although the result (5) for the cumulants would not be applicable.

Equation (2) can be extended to the three-dimensional case. The symmetric part of the deformation tensor may have the following:

- One negative eigenvalue $s(t)$: the problem remains one-dimensional, but the fluid is now seen as a succession of adjacent isoscalar planes.
- Two negative eigenvalues $s_1(t),s_2(t)$: filaments are formed instead of sheets and the above description remains valid provided that $s(t)=s_1(t)+s_2(t)$.

In the case of a scalar cascade in usual isotropic turbulence, our hypothesis of a uniform straining rate does not apply, but our approach can still be used in a more empirical way. We have seen indeed that the relative rate of decay of the scalar variance is equal to $s(t)$, and this should be equal to the flux of the scalar variance in the Kolmogorov cascade, independent of the cutoff scale. Then the model equation (2) can be applied to determine the evolution of the whole PDF. This approach ignores the fluctuations of $s(t)$, which are known to generate internal intermittence.[12] The model may still be relevant, but only to describe the conditional PDF [on the value of $f(t)$]. The general PDF should evolve toward a superposition of Gaussians, as in the case of the velocity increments (see Ref. 13). Nevertheless, Eq. (2) can be a good model if the cascade is of limited extent in wave numbers, or if intermittency is dominated by the spatial gradient effects.

Another limitation of the model comes from the hypothesis that the scalar field is not spatially correlated at the coarse-grained scale l. More precisely, the fluctuations of scalar values are expected to be mainly under the coarse-grained scale. In terms of the scalar spectrum $E(k)$, it means that

$$\int_{1/l}^{+\infty}E(k)dk\gg\int_0^{1/l}E(k)dk. \tag{6}$$

It is always justified on long times, in the absence of scalar sources, but some initial conditions are not consistent with this hypothesis. In particular, the initial evolution of a coarse-grained PDF made of two sharp peaks is not correctly described: the model then produces a third central peak, which is exceedingly narrow. This initial condition is indeed associated with the existence of large patches (compare to the coarse-grained scale) with uniform concentrations.

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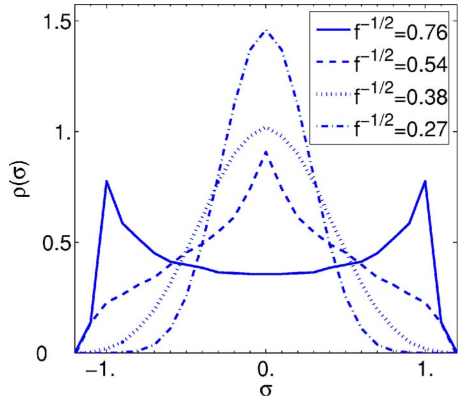

FIG. 1. (Color online) $f=\langle\sigma^2\rangle_0/\langle\sigma^2\rangle$. The scalar concentration PDF for $f^{-1/2}=0.76$ is taken from Ref. 14, Fig. 26. The other PDFs are deduced from our model, and can be compared with the DNS results of Ref. 14, Fig. 26.

The evolution of the PDF has been studied in Ref. 14 from a numerical computation of isotropic turbulence, in which a passive scalar is introduced. The initial PDF was made of two symmetric peaks, and the convergence to a Gaussian PDF with decreasing variance was observed. We compare our model prediction in Fig. 1 with the direct numerical simulation (DNS) result given in Fig. 26 of Ref. 14 by adjusting $f(t)$ to fit the scalar variance decay ($f^{-1/2}=\sqrt{\langle\sigma^2\rangle/\langle\sigma^2\rangle_0}$). We deduce the PDF at time t with our convolution process, knowing the PDF at time t_1 such that $f^{-1/2}(t_1)=0.76$. We do not take the PDF at $t=0$ to exclude an initial condition with sharp initial peaks as mentioned above.

As in DNS,[14] both initial peaks are first translated in the direction of the mean scalar value, their strength decreases, and then they disappear, leaving a single peak around the mean scalar value.

Although our model has no adjustable parameter, except the variance decay, it fits well the DNS results; however, it is too simple to give a precise transient behavior of the scalar distribution: the spurious peak at the center of the inverse bell curve is a consequence of the absence of spatial correlations at the coarse-grained scale.

Various models for the PDF evolution have been compared with this numerical simulation. Most of them are based on the classical "interaction by exchange with the mean" (IEM) model. In its simplest formulation, this model implies that the initial form of the PDF is preserved with time (the PDF normalized with its variance does not change). Two initial peaks will not merge together, and thus there will be no transition toward a Gaussian.[3] A way to overcome this problem is to consider stochastic noise; a new difficulty arises, as scalar values have to remain bounded during the time evolution. Fox proposed a Fokker-Planck closure.[15] The intermediate inverse bell shape is observed, but there is an overestimation of the PDF near scalar bounds in the earlier evolution, as noticed in Ref. 16. Recent contributions[16,17] give ameliorations or extensions of the IEM model, and obtain predictions for the PDF in good qualitative agreement with DNS results.

The evolution of the normalized kurtosis $K=(\langle\sigma^4\rangle/\langle\sigma^2\rangle^2)-3$ with respect to the quantity $1-f^{-1}$ (an increasing function of time from 0 to 1) is presented in Ref. 17. Their model fits well the initial evolution of K in the DNS simulation, whereas our model predicts a linear relationship $K=K_0/f$, which does not allow the change of sign of the kurtosis: the overshooting of a Gaussian is not possible in our model. But in Ref. 17, the evolution toward a Gaussian at later time is not observed, and the shape of the PDF, although better predicted than in the classical IEM model, still differs from the DNS result with the persistence of two peaks for a longer time period than in DNS.

In conclusion, none of the existing models provides good predictions in all cases. Our convolutions model, although not always accurate at short times, has the advantage of conceptual simplicity and consistency with the physics of mixing: in particular, the bounds of the scalar concentrations are preserved.

This self-convolution model is also applicable for nonsymmetric distributions. As an example, we consider experiments performed by locally introducing a dye in a stirred flow.[11] In that case, the initial concentration is 0 everywhere except in the dye streaks, so the corresponding PDF is very skewed. Later, it progressively tends to a Gaussian as stirring proceeds. Villermaux and Duplat[11] have provided a quantitative model of this evolution as an aggregation process of streaks of scalar, which leads to the following kinetic equation:

$$\partial_t\hat{\rho}=s(t)[f(t)[\hat{\rho}^{1+1/f}-\hat{\rho}]-\kappa\partial_\kappa\hat{\rho}], \tag{7}$$

where $f(t)$ is the integrated strain defined by relation (3), as in our model. The first term accounts for the formation of scalar sheets, while the second one describes the decay of the concentration by the competing effects of strain and diffusion in scalar streaks. The solution of (7) approaches a sequence of gamma PDF at large time, whose characteristic function is

$$\hat{\gamma}(\kappa)=\left(\frac{1}{1+\langle\sigma\rangle\kappa/f(t)}\right)^{f(t)}. \tag{8}$$

The average concentration $\langle\sigma\rangle$ is a constant, while the exponent $f(t)$ increases in time. The PDF is very skewed at the beginning and becomes more and more symmetric and narrow at time goes on. The solution (8) has been found to be in good agreement with the experimental results obtained by introducing dye either in a steady stirring motion or in a turbulent pipe flow.

Note that (8) is in the form (4), so that it is also a solution of our dynamical equation (2). Therefore, our model can also account for the experimental results. However in our case, it corresponds to a particular initial condition, an exponential PDF (see Fig. 1). Such an initial condition could result from nonhomogeneous processes occurring near the dye injector. Other forms can be obtained from different initial conditions, although with the same qualitative behavior. By contrast, Villermaux and Duplat found that the family of gamma PDF is an attractive solution,[11] so that it can be approached for a wider class of initial conditions. One can easily check that the dynamical equation (7) becomes identical to (2) for long times, $f\to\infty$. The change of the PDF in

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our model (2) depends only on the straining rate $s(t)$, while in (7) it keeps track of the previous history through $f(t)$, which may not be suitable in the presence of spatial fluxes. Another difference with the model of Villermaux and Duplat[11] is that we consider concentration averaged on a small domain (in the absence of diffusion) instead of pointwise concentration. Distinguishing between the two models would require careful analysis of the experimental data.

The effect of a turbulent diffusion in the presence of a mean scalar gradient has been studied in Ref. 10; the authors show that the PDF develops exponential tails. This provides, therefore, a good rational for an initialization of our cascade model by an exponential, leading then to solutions close to gamma PDF. In a steady regime sustained by a scalar gradient, the index f would then depend on the ratio of the cascade effect to the spatial fluxes, instead of time.

To conclude, the main result of this Brief Communication is Eq. (2), describing a continuous self-convolution process. It accounts for the temporal evolution of the coarse-grained scalar PDF: the probe of width l "sees" structures coming from larger and larger scales, which implies the self-convolution of the PDF. The efficiency of the fluid motion to drive the scalar PDF to a form given by the central limit theorem depends of the rate of strain $s(t)$. This is a simple and efficient way to model the dissipation of scalar fluctuations due to turbulent cascade. A fuller study using this result in the context of statistical mechanics of the mixing of a stably stratified fluid is in preparation and will be reported elsewhere.

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