ÉCOLE NORMALE SUPÉRIEURE DE LYON

Laboratoire de Physique, UMR 5672

HABILITATION À DIRIGER DES RECHERCHES

présentée et soutenue publiquement le 13 décembre 2017
Antoine VENAILLE

Waves, turbulence and invariants in geophysical fluids

devant la commission d'examen formée de

DAVID CARPENTIER	ENS de Lyon	Examinateur
COLM-CILLE CAULFIELD	Cambridge University	Rapporteur
STEPHAN FAUVE	LPS ENS Paris	Rapporteur
GUILLAUME LAPEYRE	LMD ENS Paris	Examinateur
LEO MAAS	Utrecht University	Rapporteur
ANNE-MARIE TREGUIER	LOPS Brest	Examinatrice

Acknowledgements

I warmly thank Com-cille Caulfield, Stephan Fauve and Leo Maas for having accepted to write a report on the habilitation thesis; David Carpentier, Guillaume Lapeyre and Anne-Marie Treguier for their participation to the committee, and Thierry Dauxois for his encouragements to write this habilitation thesis. The results and projects presented in this manuscript owe much to a large number of collaborators, and it is a pleasure to thank them for these fruitful interactions. Finally, I thank colleagues of the laboratoire de physique, a very pleasant place to work.

Contents

Chapter 1

Foreword

This habilitation thesis summarizes my work over the last 8 years (2009-2017), which has been mostly revolving around waves and turbulence in geophysical fluid dynamics, motivated by phenomena taking place in atmospheres and oceans. This period of time starts at the end of my PhD, which was entitled "A statistical mechanics approach of mixing and oceanic circulation". Since then I spent half a year in a group devoted to realistic modeling in oceanography (MEOM team in LEGI, now at LGGE Grenoble), two years as a post-doc in Princeton University, at the Geophysical Fluid Dynamics Laboratory, before going to ENS de Lyon, first thanks to a salutary post-doctoral position offered in 2011 by T. Dauxois and S. Ruffo (who was there as an invited professor), then as a CNRS researcher in 2012. From 2012 to 2017, I have been collaborating with the internal wave group and the climate and statistical mechanics group at ENS de Lyon, and with the stratified mixing group of Ecole Centrale Lyon, LMFA, and I started to work on the topology of geophysical waves.

The introduction gives a personal view on the interest of studying atmospheric and oceanic dynamics in a physics lab, including a short part more specifically devoted to theoretical approaches in geophysical fluid dynamics. It also provides a short account of my scientific itinerary. Then, each chapter presents a different subject, summarizing my contributions to specific problems, with an introduction putting them into a broader context, including additional unpublished discussions. I have not included reprints of my papers, but I provide the relevant references and a link to the pdf files at the end of each subsection. The last chapter is a research project based on a new activity started at the end of 2016 with condensed matter physicists. A summary of the main results is given in the conclusion, and a short CV is provided in an Appendix. These studies are the result of fruitful collaborations with several researchers:

- Chapter 3: G. Vallis, S. Griffies, L.-P. Nadeau, J. LeSommer, B. Barnier, T. Dauxois, S. Ruffo
- Chapter 4: T. Dauxois, S. Joubaud, P. Odier, G. Bordes, F. Bouchet, A. Renaud (PhD defense in 2018)
- Chapter 5: L. Gostiaux, A. Delache, D. Micard, E. Horne, J. Sommeria
- Chapter 6: P. Delplace, B. Marston

I have initiated most of the work presented here, and I will give justice to collaborators when it is not the case. In addition to these collaborations I would also like to mention Yuki Yasuda (Tokyo), who visited Lyon a few months in 2015 during his PhD supervised by Prof. Sato in Japan, to work with Freddy Bouchet and myself. Yuki proposed a very nice application of equilibrium statistical mechanics to sudden stratospheric warming event. Because the project was initiated by Yuki himself, I decided not to present this in this manuscript. I would also like to mention Kaushal Gianchandani (NISER India) who spent a summer internship in Lyon in 2016, and did his master thesis under my supervision in 2016-2017 on baroclinic turbulence.

Chapter 2

Introduction

2.1 Atmospheric and oceanic dynamics in a physics laboratory

Over the second half the twentieth century, geophysical fluid dynamics has thrived, with impressive progress due to the concomitant development of observations and numerical computing. Observations from satellites have revealed incessant motion of waves turbulence and flows taking place over a huge range of scales in Earth oceans and planetary atmospheres. Computers have made possible numerical simulations of geophysical fluids, from toy models to comprehensive general circulation models, making true the "Weather factories" envisioned by Richardson in 1922. It is however fair to say that a number of basic phenomena occurring in atmospheres and oceans are out of reach for numerical computing, and that the complexity of the whole system continues to resist to physicists.

Our ability to simulate our climate and to understand its variability is faced with many challenges, some of them involving parameterizations of physical processes occurring at scales smaller than those resolved in the models, some of them being related to the chaotic nature of the climate system, together with the huge number of degrees of freedom involved in the problem. We may also expect that future observations of our own climate system, especially the interior of the oceans which are poorly known, or the discovery of other planetary atmospheres, will bring surprises and new problems to be solved. To take only one example: altimetry revealed in the 90ties that oceans are full of coherent vortices, which was a surprise for many oceanographers, and these observations have motivated a number of theoretical questions on what sets the shape and vertical structure of these eddies [see chapter 3]. The same surprise arguably just occurred a coupled of weeks ago with the Juno mission revealing literally "a sea of coherent vortices" in the polar region of Jupiter (figure 2.1), contrasting with the hexagonal polar vortex in Saturn. Oceans and atmospheres are, and surely will be, a vast playground for physicists [112, 208].

It is worth stressing that theoretical tools developed in geophysical context can be useful for other scientific communities. The most striking example is perhaps given by Lorenz work on predictability in atmospheric dynamics, that triggered or reinvigorated a strong interest in nonlinear dynamics across many fields of sciences, from mathematics to biology. This is also the case of stochastic resonance, that was initially proposed to account for the periodic recurrence of Earth ice ages. In turn, progresses in other fields of physics or mathematics have made possible substantial advances to the understanding of geophysical flows. This mutual benefit of gathering different scientific communities that share a common interest has for instance been well illustrated through the development of the physics of long-range interacting systems [34]. This concerns geophysical fluid dynamics, as long-range interacting systems include quasi-geostrophic models, that are relevant to describe extra-tropical vortices and jets [see chapter 3].

Finally, particular processes occurring in atmospheres and oceans provide new "model systems" with a physical interest on their own, independently from geophysical applications. For instance, gravity waves that propagate in stratified fluids are known to play an essential role in redistributing momentum and energy in geophysical flows, but also have unusual properties with respect to other waves encountered in physics textbooks: their group and phase velocities are perpendicular, their reflection laws are completely different from the usual Snell-Descartes laws, among other peculiarities, which continue to be discovered and explored in idealized experiments [see chapter 4].

Pioneers of modern meteorology were simultaneously interested into practical questions on weather predictions, and on the mathematical foundation of geophysical fluid dynamics. For instance, Rossby work on mid-latitude planetary wave seen in weather maps was a crucial step towards the introduction of the central concept of potential vorticity, and Charney's work on the instabilities that give rise to mid-latitude weather has been concomitant to his derivation of quasi-geostrophic equations. Over the last decades, numerical modeling of comprehensive climate models and the theory of geophysical flows have been much more specialized. It has become extremely difficult to know simultaneously how is built a comprehensive general circulation models, while being aware of the development of dynamical system theory, statistical mechanics approaches, or other tools developed in different branches of physics. Each sub-field has created its own jargon, and has been involved with its own scientific problems, not necessarily related to the others. As far as our understanding of geophysical phenomena are concerned, the danger of this independent evolution of theory and the development of numerical modeling is that physicists may spend a lot of efforts on problems that are not directly motived by questions of current interest in geophysical context, given state of the art observations and numerical models. On the other side, numerical modelers can be confronted with models as complex as real flows, without being always aware of existing tools that may allow to approach with a fresh view some aspects of these complex systems.

The importance of bridging the gap between our understanding of relatively simple models and the actual climate system (or numerical simulations of it) has been emphasized many times, by advocating the need for a hierarchy of numerical models, from the “fruit fly" of climate to comprehensive general circulation models, e.g. [73, 40, 183]. For this reason, I have tried to have a foot in each community, from my post-doc in a laboratory devoted to climate until now, in a physics laboratory.

I would also like to stress the interest of laboratory experiments as a tool to study particular physical processes of geophysical relevance. Such experiments were for instance central in Rossby work on planetary wave dynamics. First, experiments deal with real fluids. Second, experiments have often an immediate aesthetic aspect, and the direct visualization of flows can reveal unexpected phenomena. For instance, the generation of strong vortical flows due to the three-dimensional structure of internal wave beams in stratified fluids has been observed recently in laboratory experiments (Lyon and Grenoble) as a side effect that was not expected. This observation then motivated further numerical and theoretical work, thus reinvigorating the interest into wave-mean flow interactions in stratified fluids [Chapter 4]. Finally, experiments still make possible extremely long observations that would not be reachable with current numerical capabilities. These long time observations are for instance useful in wave turbulence. This said, one should also keep in mind that typical Reynolds and aspect ratio numbers of geophysical flows are hardly possible in a rotating tank, and that numerical models have the great advantage of simulating a self-consistent, simplified set of equations obtained through an asymptotic expansion, taking thus advantage of the existence of small parameters in the actual geophysical problem. My impression is that laboratory experiments will continue to play an important role to understand particular physical processes, but that the understanding of large scale atmospheric and oceanic dynamics requires the use of computers.

Clearly, a relevant question for climate dynamics is not necessarily one that will lead to a phenomenon easily isolated in a lab and the most intriguing physical processes are not necessarily those that will lead to an efficient parameterization in a state of the art climate model. Being in a physics laboratory gives the luxury to be driven by curiosity rather than by those applications, and the contact with colleagues in different areas of physics can bring original point of views [Chapter 6]. However, I believe that being aware of the advancement of realistic numerical modeling and observations also provides a tremendous source of fundamental questions feeding this curiosity with practical problems.

2.2 On the role of symmetries and invariants in geophysical fluids

Robust phenomena. Despite the seemingly erratic motion taking place over a variety of temporal and spatial scales, there is some large scale order and surprisingly robust features emerging from the dynamics of oceanic and atmospheric flows. By robust, we mean that the phenomena exist despite (and sometimes thanks to) the presence of turbulence or other sources of disorder. Robust phenomena can be observed in different geophysical flow systems (stars, atmospheres, oceans), reproduced in models with different complexity, or even in laboratory experiments with only few salient ingredients. Most of my work has been motivated by the understanding of some of the physical mechanisms underlying this robustness.

A first example of robust phenomena is the self-organization of extra-tropical turbulent flow into coherent large scale structures: sharp eastward jets and long lived circular vortices are observed in ocean (as in the Kuroshio region

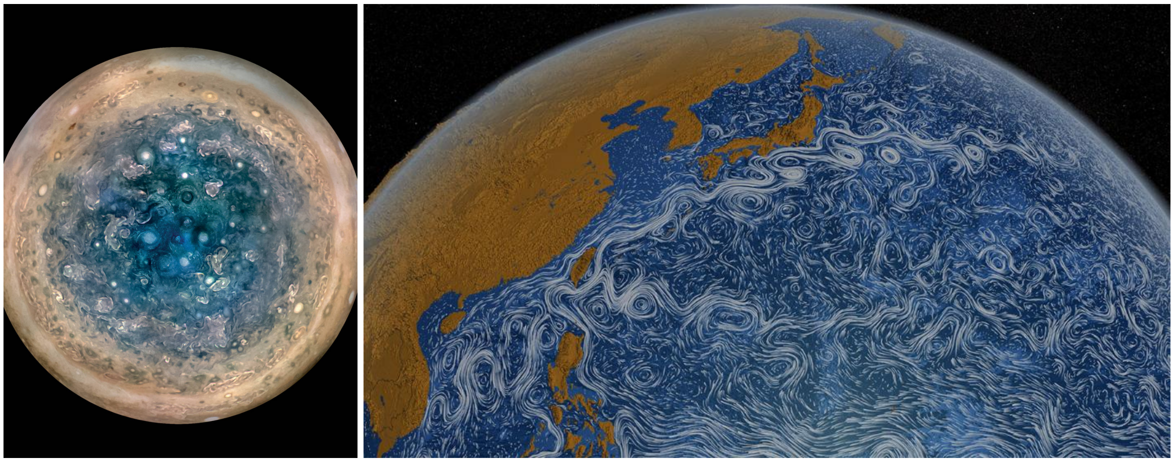

Figure 2.1: **Left:** Jovian vortices observed during the JUNO mission in 2017 [18]. **Right:** mesoscale oceanic vortices and jets from a synthesis of a numerical model with observational data. Credits: NASA/SVS. I have been interested on what sets the vertical structure of such vortices, see [Chapter 3].

shown in figure 2.1), in the Earth atmosphere (the Jet stream), and in other planetary flows. A second example of robust phenomena is the remarkable periodicity of large scale flow reversals in the equatorial atmosphere. This quasi-biennial oscillation is the archetype of wave-mean flow interaction problem; it occurs in the stratosphere (between 10 km and 50 km), a region that supports the propagation of internal gravity waves, which drives these oscillations. A third class of robust phenomena is the propagation over great distances of edge waves, without much backscattering of their energy into other waves. This is arguably the case of equatorial Kelvin waves, that play a crucial role in El Nino phenomenon, see figure 2.2.

These three examples are emblematic geophysical problems that have motivated more or less directly a large part of my work over the last few years, when studying self-organization of geostrophic turbulence [chapter 3], wave-mean flow interactions in stratified fluids [chapter 4], and topological properties of geophysical waves [chapter 6]. I have been primarily interested into the study of minimal models that could reveal basic principles explaining some aspects of these robust features. It is worth remarking that the three examples above are concerned with different classes of dynamical systems: turbulent flows are inherently strongly nonlinear systems, the quasi-biennial oscillation can be addressed in a quasi-linear framework, while the robustness of equatorial edge waves to disorder can be phrased in a linear framework. Clearly, different theoretical tools must be considered, depending on the problem at hand. But these different theoretical tools share a common goal: to predict macroscopic or global quantities without describing the detail of the microscopic dynamics or of the local properties of the underlying flow model.

Symmetries. As usual in physics, important insights on robust geophysical phenomena can be obtained by exploiting symmetries of the problem. Considerations based on symmetries are for instance extremely useful to derive amplitude equations describing the emergence of large scale flow patterns [54, 61]. Another example of the importance of symmetries in fluids is given by the conundrum of locomotion at low Reynolds number: a mean motion is possible only through non-reciprocal change of the body shape, breaking time-reversal symmetry. Similarly, and at much larger Reynolds numbers, the emergence of wave-driven mean motion at geophysical scale can be understood generically as a the result of symmetry breaking mechanism [107]. I will present in section 4.1 a mechanism for the wave-driven mean flows where time-reversal symmetry is broken by viscous terms, playing a crucial role in the mean-flow generation.

One can distinguish symmetries that are related to the boundary conditions, forcing and dissipation terms, from symmetries that are intrinsic to the inviscid, unforced dynamical equations. Such internal symmetries are for instance the invariance freely evolving flows by time translation and particle relabelling. These symmetries are related through Noether theorem to dynamical invariants, namely energy, and potential vorticity conservation (in 2D) or helicity (in

3D) [154, 165]. These invariants have deep physical consequence, as they strongly constrain the macroscopic flow behavior. For instance, the existence of multiple steady states and of an inverse energy cascade in two-dimensional turbulence is related to the conservation of energy and enstrophy (the fluctuations if potential vorticity). I have been particularly interested on how such global contraints affect the properties of geostrophic turbulence [chapter 3], shallow-water turbulence [section 4.2], and turbulent mixing in stratified fluids [chapter 5]. More recently, I started to work on the physical consequence of discrete symmetries, using tools from topology.

Topology. Topology describes global properties that are invariant under smooth local deformations of the system. This is another powerful tool to understand robust features of a physical system. It has for instance played a central role in dynamical system theory, to describe the behavior of systems with few degrees of freedom; it has also been proven very useful in fluids to describe knots and linkages of three-dimensional vorticity field, which are related to the helicity invariant [8, 126]. It is now well understood that changing the topology of an oceanic basin for a given wind forcing drastically changes the patterns of oceanic currents: in closed basins, oceanic currents are always intensified along the western boundaries (Gulf-Stream, Kuroshio, North Brazil currents, etc), in total contrast with circumpolar currents observed in the Antarctic ocean, where there is no obstruction from continents. In that case topology plays a role through the boundary condition. Topological properties can also be intrinsic to a dynamical equations describing a physical system.

A striking and unexpected manifestation of intrinsic topological properties in physics came from condensed matter in 1982 when Thouless and co-workers explained the robust quantization of the Hall effect as an intrinsic topological properties of electronic states in two dimensions [179], by introducing a topological invariant known in mathematics as the Chern number. In the 90ties Hatsugai explained how these topological bulk properties are related to the existence of unidirectional edge sates propagating without backscattering of their energy in the presence of boundaries [71]. This subject has been reinvigorated one decade ago by the discovery of other states of matter with different topological properties, together with a classification of these properties depending on the discrete symmetries of the system. Over the last few years, these ideas have been applied to a number of other fields of physics, including mechanics [77]. Waves, boundaries and external fields breaking discrete symmetries are ubiquitous in geophysical and astrophysical flows, but topology has so far play little role in that context. Our project in chapter 6 aims at filling this gap.

In brief, symmetries and topology are extremely useful to describe the robustness of geophysical fluid dynamics, whatever the theoretical approach used to describe a given phenomenon. Let us now give a short account of the different theoretical tools available.

From linear to nonlinear dynamics. Although most aspects of fluid dynamics are genuinely nonlinear, a fair amount of understanding in atmospheric and oceanic flows can actually be obtained with linear dynamics as a first step before more complicated approaches. For instance, the generic western intensification of midlatitude oceanic currents in closed basin can be understood qualitatively by dropping nonlinear terms from the equations. Similarly the typical time scales and length scales of our weather system are well captured by baroclinic instability, a purely linear mechanism. This instability corresponds to the growth of meanders breaking the zonal (east-west) symmetry of the eastward jets in oceans and atmospheres. The size of the most unstable meanders correspond roughly to the size of midlatitudes anticyclones and cyclones.

It is worth stressing that a number of observed and simulated phenomena continue to be interpreted using linear analysis [167]. Perhaps more surprisingly, some aspect of linear waves such as their topological properties remain largely unexplored. Finally, even if the linear approach is a useful first step, it can not describe the saturation of the baroclinic instability, it can not account for the low-frequency variability at planetary scales in the climate system, and it can not account for the spontaneous emergence of coherent structures or for the interactions between waves. For those phenomena, nonlinearities are essentials, and one must therefore use the tools of nonlinear physics and statistical mechanics. This view is actually expressed in the conclusion section of the seminal paper of Eady on baroclinic instability in 1949 [51]:

The above is no more than a prelude to the rather formidable task facing theoretical meteorology - that of discovering the nature of and determining quantitatively all the forecastable regularities of a "permanently unstable" (i.e., permanently turbulent) system. We can be certain that these regularities are necessarily statistical and to this extent our technique must resemble statistical mechanics.

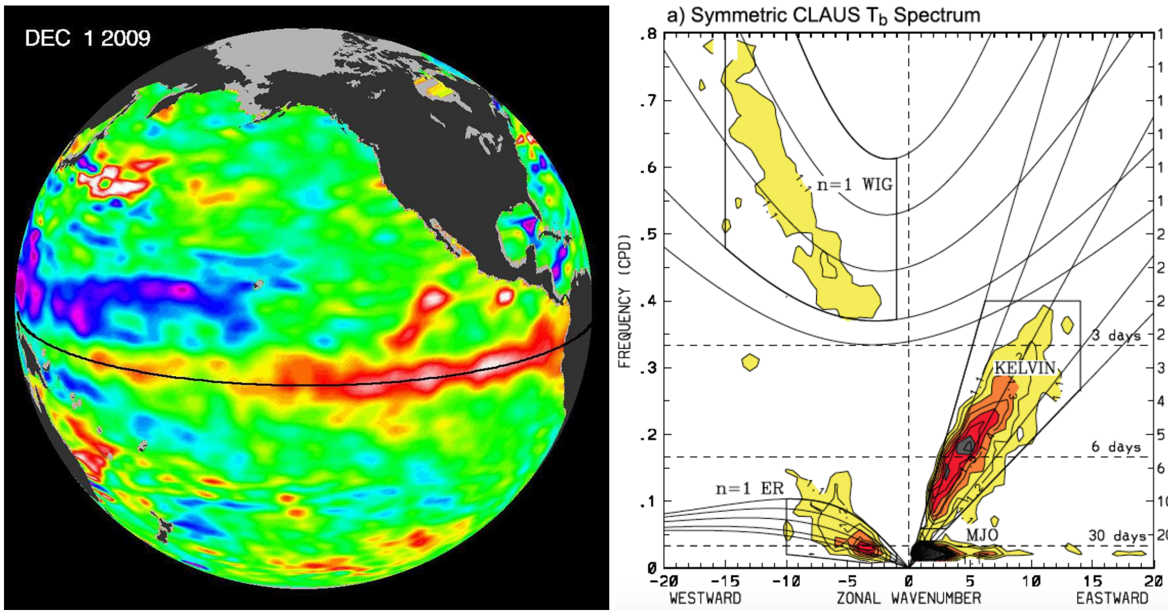

Figure 2.2: Kelvin waves in the Pacific Ocean and the Earth atmosphere. **Left:** Temperature anomaly in the Pacific ocean, revealing a Kelvin wave (in red) trapped at the equator, and propagating energy eastward. Credits: NASA. **Right:** Frequency-wavenumber spectrum of the symmetric part (with respect to the equator) of the temperature field in the atmosphere, from [88]. Despite the existence of disorder (topography, eddies,...), the robust signature of a Kelvin wave is striking. We have shown that such edge waves have an origin in topology, see Ref. [44] and the project in [Chapter 6].

Asymptotic expansions. Before presenting the statistical mechanics approach, suited for systems with a large number of degrees of freedom, it is worth pausing to mention the importance of asymptotic expansions in geophysical fluid dynamics, which allows us to reduce the primitive set of equations to simpler dynamical models, taking advantage of small parameters in the problem (such as the aspect ratio of the flow domain, among others). This approach has for instance been essential to formulate in the simplest possible setting the baroclinic instability problem mentioned above, though the introduction of quasi-geostrophic dynamics in the 40ties. I have heavily used this quasi-geostrophic system to describe phenomena taking place in mid-latitude oceans and atmospheres. There remain probably other interesting reduced models beyond quasi-geostrophy to be found, for instance to describe the interplay between stratification and geostrophic turbulence.

When the dynamical fields are characterized by different length and time scales, or by particular spatial (say, zonal) symmetries, a variety of tools such as quasi-linear approaches and kinetic theory allow to simplify even more the whole dynamical system, which has long been used for wave-mean flow interaction problem [1]. Even if such procedure rarely lead to a systems with only few degrees of freedom, it provides set of equations considerably simpler than in the original problem. I have used this approach to describe the generation of mean flow by waves in stratified fluids [section 4.1].

Depending on the final simplified system of equations and on the problem at hand, one can then use tools from dynamical system theory and statistical mechanics to describe the consequences of nonlinearities.

The dynamical system approach. The physics of nonlinear systems owe much to atmospheric dynamics. The issue of predictability led to beautiful pieces of physics in the sixties: using a toy model with three degrees of freedom, Lorenz illustrated and popularized the sensibility to initial condition in weather predictions (the butterfly effect) and showed the existence of an order underlying seemingly erratic signals in deterministic systems (the strange attractor). These low dimensional models are often far from our climate system involving a huge number of degrees of freedom, but had the double advantage to permit numerical experiments with limited computational capabilities, and to provide a dynamical system tractable for theoretical physicists and mathematicians. Since then, a number of phenomena, such as the El Nino Southern Oscillation, have been understood or interpreted as the result of an intrinsic chaotic variability of the climate system, contrasting with (and complementary to) other approaches that interpret the observed variability as the linear responses of the system to an external forcing. One drawback of this dynamical systems approach is that it most often relies on ad hoc models reducing the whole system down to a set of equations with few degrees of freedom, even if important understanding or guide for these reduced models can be obtained through symmetry arguments [54, 61]. How to derive such models from the actual flow dynamics is in general a formidable task. Important progress on the understanding of the intrinsic chaotic variability of systems as complex as the wind-driven oceanic circulation have however been possible thanks to continuation algorithms [47, 62]. The idea of this approach is to start from highly dissipative cases, that involve few degrees of freedom, to decrease these dissipative terms towards more realistic values, and to describe bifurcations occurring along the way.

Statistical mechanics approaches. Another route towards an understanding of turbulent geophysical phenomena is to start by considering the inviscid limit, and then to add dissipative terms. Given the huge number of degrees of freedom coupled nonlinearly in the system, this inviscid limit has to be addressed with the tools of statistical mechanics, in order to predict macroscopic quantities such as a large flow patterns or the local flow statistics, without describing the detail of the microscopic dynamics. Equilibrium statistical mechanics for Euler and quasi-geostrophic flows has been developed in the 90ties by Robert-Sommeria and Miller, building on previous contributions by Onsager, Kraichnan and others (see e.g. the reviews [170, 110, 24]). The theory relies on the strong, but single assumption, that the system evenly explores phase space. It then predicts the large scale flow as the most probable outcome of turbulent stirring, with constraints given by dynamical invariants. The interest of this approach is that one can build phase diagrams and thus describe bifurcations in the macroscopic configurations (for instance the transition from a surface intensified to a bottom-trapped flow) of the system when external parameters are varied.

The equilibrium theory is not without drawback either. First, the ergodic hypothesis is not satisfied in many practical cases [see the last section of Chapter 3]. Second, actual geophysical flows are forced and dissipated, while the equilibrium theory applies to inviscid, isolated dynamics. It can therefore be only a first step before more comprehensive studies discussing the effect of forcing and dissipation. In this habilitation thesis, I would like to convey through several examples the idea that equilibrium statistical mechanics remains a natural first step that allows us to quantify the combined effect of turbulence and dynamical invariants in a variety of geophysical problems. When correctly interpreted, this brings new understanding to the physical behavior of the system, and it can possibly be used as a guide for subgrid-scale parameterizations [87, 75]. The work on mixing in stratified fluids presented in chapter 5 could be a step in this direction.

Because many geophysical phenomena are far from equilibrium, a large amount of theoretical efforts are now devoted to the development of out-of equilibrium statistical mechanics theories [21]. The emergence and dynamics of multiple zonal jets are currently addressed with direct statistical simulations, and kinetic approaches are used to close equations for the statistics [1]. Linear response theory is applied to general circulation models in order to predict possible patterns of climate change [104]. Tools from large deviations are used to propose efficient algorithm speeding up the computation of rare events such as heat waves [22]. There have also been important progresses in experimental technics to measure (intrinsically out-of-equilibrium) energy cascade in rotating stratified fluids, with the hope to better characterize elusive wave turbulence regimes [31, 215].

Among the different tools presented above, I have been particularly involved into equilibrium statistical mechanics theory, and, to a less extent, to quasi-linear approaches. I used these tools to discuss different aspects of energy transfers in geophysical flows.

Where does the energy go? Equilibrium statistical mechanics was at the heart of one of the rare breakthrough in turbulence theory, namely the understanding of the fundamental difference between two and three dimensional turbulence. Onsager explained in 1949 how same sign point vortices in two dimensions tend to clump together, forming long-lived structures, as a tendency of maximizing entropy [134]; two decades later, Kraichnan computed absolute equilibria of Galerkin-truncated models of two-dimensional and three-dimensional turbulence to interpret the direction of the energy cascade as a tendency to reach these equilibria [92]. In two dimensions, energy piles up at the domain scale, forming large scale structures, sometimes called condensates, while energy in three dimensional flows is transferred towards small scales and ultimately get dissipated by molecular effects. Real flows are three-dimensional, but the large scale dynamics of oceans and atmospheres may be considered as quasi two-dimensional owing to rotation, stable stratification, and their small aspect ratio. When forcing injects energy at a given scale, will this energy be transferred to larger scales as in two-dimensional flows, or will it be transferred towards small scales as in three-dimensional flows?

Routes towards dissipation in geophysical flow. The energy cycle of oceans and atmospheres is actually one of the important unsolved problem of oceanic and atmospheric dynamics [211, 55]. The source of energy are well identified (the sun heats the planet, winds blow over the oceans, and the tides move water masses back and forth), but much less is known about the dissipative mechanisms. Answering this question amounts to understand how waves and turbulence shape the large scale structure of planetary flows by redistributing energy, momentum and other tracers over the globe. Energy transfers from the forcing scales to the dissipative scale involve a zoology of physical processes [118], and my contribution has been to study some of these processes. More precisely, I have been interested into the following questions, each of them involving a combination of theory, numerical or laboratory experiments:

- [Chapter 3] How do the energy of surface intensified coherent vortices can be dissipated to the bottom?
- [Chapter 4] How can the energy of internal waves can be sent back to large scale through the generation of vortical flows?
- [Chapter 5] How much of the energy released by a breaking internal wave can effectively be used to irreversibly mix the background stratification?

2.3 Scientific itinerary

I did my undergraduate studies at ENS de Lyon. The close connection between the physics laboratory (for reseearch) and the physics department (teaching) has played an important role in my subsequent scientific choices. I wanted initially to work on neuronal sciences, and I approached briefly this field during a very interesting summer internship with Misha Rabinovich and Pablo Varona at UCSD in 2004 [199]. Before this, I had a first contact with atmospheric dynamics by working on low ozone event during a summer intern with Raf Toumi at Imperial College in 2003. A kind of scientific determinism then brought me back to turbulence, one of speciality of the physics lab at ENS de Lyon: thanks to the judicious advice of Thierry Dauxois, I did my PhD with Joel Sommeria in LEGI, Grenoble and with Freddy Bouchet that was at INLN (Sophia Antipolis). This was a good opportunity to be at the same time close to experiments (literally speaking, my office was just near the Coriolis turnable), while being confronted to theoretical aspects of geophysical flows in close contact with physicists working on nonlinear dynamics (Eric Simonnet at INLN) or statistical mechanics. I also enjoyed the stimulating atmosphere of the doctoral school "Terre-Univers-Environnement" in Grenoble gathering fluid dynamicists, astrophysicists and geologists.

At the end of my PhD I had three main directions of research: i) phenomenological approaches of mixing guided by statistical mechanics ideas, related to experiments in stratified turbulence [195, 196]; ii) the interpretations of mesoscale oceanic turbulence features with equilibrium statistical mechanics [188]; iii) the physics of long-rangeinteracting systems [187, 189]. This last subject was rather academic, but this work led recently to an interesting application to stratospheric dynamics in collaboration with Yuki Yasuda [216].

After my PhD, I wanted to work with physical oceanographers; I hoped to use in more realistic settings some of the statistical mechanics ideas developed in idealized context. I benefited from a few month of additional funding after my PhD defense, and used them to start a collaboration with the MEOM team in LEGI, who are specialist of realistic numerical modeling of the general circulation of oceanic flows. I was hosted by Bernard Barnier and Julien LeSommer, who proposed me to work on the Zapiola anticyclone, a huge and deep oceanic vortex trapped above a giant sediment dome in the Argentine basin, with a mass transport as large as the Gulf Stream. The MEOM team has been famous for having discovered this vortex in their numerical model simultaneously to *in situ* observation in the 90ties. Models and observations also revealed fluctuations of the mass transport as large as the transport itself. I have proposed an idealized model to interpret these fluctuations as driven by external eddy fluxes, in good agreement with outputs from the realistic numerical models [193], [see section 3.2].

I was then offered in 2009 a post-doctoral fellowship in Princeton University, hosted by Geoff Vallis at the Geophysical Fluid Dynamics Laboratory, to work on geostrophic turbulence, which describes large scale oceanic and atmospheric flows. The general context was the current interest of physical oceanographers on what sets the vertical and the horizontal structure of mesoscale turbulence (from 50km to 500km). This question was raised by altimetry observations a decade before: the observation showed a limited inverse energy cascade on the horizontal, with a lot of incertitude on the vertical structure underneath the observed surface flows. In addition to this main motivation, I found the questions on the vertical structure particularly interesting for two reasons: i) the results of statistical mechanics concerning the horizontal shape of large scale coherent structures obtained during my PhD also depended strongly on the vertical structure that we postulated; ii) this issue was related to the mystery of the emergence of the Zapiola anticyclone whose deep vertical structure contrasts with the surrounding surface-intensified vortices [see Chapter 3].

Switching from a physics lab (in Nice) and a fluid dynamics lab (in Grenoble) to a lab devoted to climate in the US has been an abrupt transition, with new jargon, new scientific questions, new ways of working, even if the phenomena studied were after all not so different. A posteriori, these post-doctoral years at Princeton have been extremely fruitful: most of my current knowledge of ocean and atmospheric dynamics comes from there, and numerous discussions with Isaac Held and Geoff Vallis shaped my view on this discipline, and remain after a few years a source of current research projects. This period has also been a unique opportunity to meet and begin collaborations with other post-doctoral researchers in North America. My ongoing collaboration with Louis-Philippe Nadeau in physical oceanography started here. The proximity between NYU, GFDL, MIT and others labs seemed to favor the circulations of ideas from one place to another. This environment has also been very useful to put into a geophysical context some of my work performed during my PhD, including the interpretation of oceanic rings and jets as statistical equilibria with Freddy Bouchet [188, 24, 25].

My first contribution to the vertical structure of geostrophic turbulence has been to use linear stability analysis and idealized simulations in doubly-periodic geometries of geostrophic turbulence to interpret data from altimetry and output from primitive equation models [197]. I have been confronted during this work to a zoology of phenomena that motivated subsequently a number of more idealized studies.

My second contribution has been to show how planetary vorticity gradients and bottom topography shape the vertical structure of geostrophic turbulence in decaying configurations, just by changing the vertical enstrophy profile of the flow. I generalized for that purpose the statistical mechanics approach usually discussed in one or two-layer quasi-geostrophic models to cases with continuous stratification. The computation of equilibrium states allowed me to quantify the combined effect of turbulence and dynamical constraints [198], [see section 3.1]. This studies shed new light on barotropization processes (tendency to become depth independent). It also provided an explanation on how surface intensified eddies can be transformed into a bottom-trapped flow above a topographic anomaly (such as the Zapiola anticyclone), thus providing a new route towards dissipation [186], [see section 3.2].

My third contribution has been to address the role of bottom friction in shaping the vertical structures of geostrophic turbulence, in collaboration with Louis-Philippe Nadeau who performed all the simulations with his quasi-geostrophic code. The remarkable result in this studies is that the limit of strong bottom friction of a two-layer model leads to quasi-inviscid states of a one-layer model. The spontaneous emergence of meandering sharp jets (ribbons) separating regions of homogenized vorticity that can then be interpreted as a competition between a tendency to reach an equilibrium state of the one-layer model, and baroclinic states [194] [see section 3.3]. The similarities between these jets with the Gulf stream and the Jet stream is remarkable. I continue to work on these problems, focusing on how such states can be recovered without having to use an artificially high bottom friction (internship and master thesis of Kaushal Gianchandani).

These previous studies have shown the interest of using a statistical mechanics approach as a guide to interpret phenomena in idealized numerical simulations. However, these numerical studies also revealed a number of regimes for which the statistical mechanics approach is not suited. When I came back in Lyon in 2011 to work with Stefano Ruffo and Thierry Dauxois, I decided to address in more detail the role of the interactions between fluid particles during self-organization, by considering alpha-turbulence models. This is a family of two-dimensional flow models with a parameter that changes the interactions between fluid particles, including two-dimensional Euler dynamics and surface quasi-geostrophic dynamics. This project also stemmed from discussions with Isaac Held who introduced (with collaborators) these flow models to study non-locality in turbulent cascades [137]. Using these flow models, we have shown the essential role of the range of interaction in selecting the final state of self-organisation when the scale of energy injection is prescribed [190] [see section 3.4].

During that time, I started to work on waves in stratified fluids. Guilhem Bordes was finishing his PhD with Thierry Dauxois, and reported the surprising observation of a mean flow generated by an internal wave beam in laboratory experiments, something that was also seen by Nicolas Grisouard and collaborators in Grenoble, using a different experimental setting. The experiment of Guilhem Bordes reminded me wave-mean flow interaction problems familiar to meteorologists, and I started to work with the internal wave team of ENS de Lyon on the explanation of this phenomenon, using asymptotic analysis [19, 42], [see section 4.1]. Since then, I have been working on the interplay between boundaries, waves and mean flows in stratified fluids with Antoine Renaud during his PhD [144]. Our long term aim is to address with such tools the problem of the quasi-biennial oscillation, and interactions between geostrophic turbulence and internal gravity waves mediated by bottom topography.

The interaction between waves and vortical flows is a problem that can also be addressed with statistical mechanics. The simplest flow model to consider for that purpose is the shallow-water model. I co-supervized with Freddy Bouchet the one year research internship of Antoine Renaud on this problem. Using large deviation theory, we generalized to the shallow-water model the equilibrium statistical mechanics of two-dimensional flows. In addition to the spontaneous emergence of large scale vortical flows, this model supports the presence of surface waves, which are the equivalent of sound waves in two-dimensional compressible turbulence. These waves make the derivation of equilibrium theory very difficult, because they can carry energy to small scales, and they can a priori induce correlations with the vortical flow that are extremely difficult to handle theoretically. We have derived such an equilibrium theory based only a on few assumptions controlled well. The theory predicts the partition of the energy between a large scale vortical flow and small scale fluctuations carried by Poincaré waves [145] [See section 4.2].

In parallel to these studies, I continued the work initiated during my PhD with Joel Sommeria on turbulent mixing in stratified fluids, in close relation with experiments, using equilibrium statistical mechanics as a guide to interpret experiments. Since 2012, I collaborate with researchers of LMFA, Centrale Lyon, on this problem, as part of a research project STRATIMIX held by Louis Gostiaux. The aim of the project is to quantify how much irreversible mixing of buoyancy occurs when a given amount of turbulent kinetic energy is injected into a stratified fluid. My contribution to the project has been to propose and develop the use of statistical mechanics ideas to make such predictions [192]. I have also proposed possible tests of the theory in numerical simulations performed by Alexandre Delache and Ernesto Horne [76], or experiments by Diane Micard and Louis Gostiaux [123]. This is an ongoing project, and my longer term aim is to propose parameterizations based on these ideas, as well as to connect these results with my work on internal waves and geostrophic turbulence, to understand the interplay between these different processes [see chapter 5].

The work on the shallow-water model has indirectly been very useful on a completely unrelated and unexpected subject, relating the physics of topological insulators to geophysical waves. This started with the arrival of Pierre Delplace at the physics lab as a CNRS researcher in the condensed matter team. He was willing to interact with physicists in other fields to see if possible connections with topological insulators could be made, and I was curious about what we could learn in geophysics from this point of view. Meanwhile, Brad Marston, an invited professor at ENS Lyon that has over the last decade strongly advocated the fruitful connections between condensed matter and climate physics, started to work on the ribbon phase that I describe in section 3.3. At some point he suggested to see if the robustness of these states could be related to topology. That was exactly the kind of problems Pierre Delplace and I were looking for. After some work to overcome a barrier of language, we eventually found the problem of ribbons very interesting but too complicated; but I realized that the rotating shallow-water model contained all the ingredients to exhibit properties similar to topological insulators, namely the existence of broken discrete symmetries, together with unidirectional edge states filling a frequency gap between different wave bands. Pierre Delplace has then computed non-trivial bulk topological invariants in that case, I have been rephrasing these results in terms of equatorial waves using bulk-boundary correspondence, and direct numerical simulations by Brad Marston confirmed the analogy with topological insulators [44]. This work unveil the topological origin of Kelvin waves shown in figure 2.2. This also opens a number of fundamental questions: i) what are the other physical manifestations of these properties; ii) can we find other topological invariants related to other discrete symmetries; iii) could we build fluid analogues to the quantized Hall effect. This will be my main research project in the coming years [see chapter 6].

Since my PhD thesis, I have been involved in teaching activities on different forms. Apart from occasionally lecturing elementary mathematics in New Jersey prisons, I did not had much teaching activity during my post-doc in the US. Since I am back in ENS de Lyon, I have been teaching tutorials on continuum mechanics, rewriting a number of them, and I am currently preparing lecture notes on geophysical fluid dynamics for master student (starting in 2018). I have also been regularly involved in actions popularizing science, be it for "fête de la science", "pint of science", or related events. In particular, I built in Grenoble and Lyon a small rotating turntable that can be installed on a retro-projector, and used it to explain basics facts on planetary-scale atmospheric dynamics.

Chapter 3

What sets the vertical structure of geostrophic turbulence?

I describe in this chapter my work related to geostrophic turbulence, motivated by phenomena taking place in mid-latitude oceans and mid-latitude-atmospheres, namely the tendency of these flows to become depth-independent [section 3.1], the formation and dynamics of bottom-trapped recirculation above topographic anomalies [section 3.2], and the emergence of surface intensified jets [section 3.3]. I have used equilibrium statistical mechanics to quantify the effect of dynamical invariants on the outcome of turbulent stirring in freely evolving cases related to these different phenomena. The theory has been used to rationalize output from numerical experiments, and I have more specifically investigated how and when freely evolving flows relax towards the actual equilibrium states in two-dimensional alpha-turbulence models [section 3.4]. Important concepts and scientific questions related to geostrophic turbulence and equilibrium statistical mechanics are briefly presented below.

Geostrophic balance. Geostrophic turbulence describes oceanic motion at mesoscales, from 50 to 500 km, and atmospheric motion at synoptic scales, from 500km to 5000km, i.e. at the scale of our weather system, with meandering jets and long lived vortices. These large scale flows are dominated by rotation: the order of magnitude of nonlinear advection terms in horizontal momentum equations are much smaller than the Coriolis force, hence the name geostrophic. The importance of the Coriolis force with respect to advection terms in the momentum equations is quantified by the Rossby number $Ro = UL/f$, with U and L typical velocity and horizontal scale of the flow, and $f = 2\Omega \sin\theta$ the Coriolis parameter, that depends on Earth rotation rate Ω and the latitude θ. At lowest order in this parameter, there is a balance between horizontal pressure forces and the Coriolis term: streamlines are iso-pressure lines in weather-maps. A second small parameter of the problem is provided by the ratio of vertical scales to horizontal scales: the ocean are 4km deep and the troposphere (where takes place our weather system) is 10km deep, which means that both fluids are as thin as a sheet of paper. This tiny aspect ratio implies hydrostatic balance. Taking advantage of these small parameters, one can derive from primitive equations (mass and momentum conservation, thermodynamic equation, equation of state...) a reduced set of equations describing geostrophic turbulence. This reduced set of equation is the quasi-geostrophic model.

Quasi-geostrophic dynamics. Quasi-geostrophic equations are much simpler than primitive (Naviers-Stokes) equations, but it is not a toy model: early numerical weather predictions were performed with a quasi-geostrophic model and these practical applications certainly motivated the derivation of this reduced set of equations by Charney in 1948 [35]. In addition, a large part of our understanding of the large scale dynamical patterns in atmospheric and oceanic flows can be phrased in this framework. From a theoretical point of view, the quasi-geostrophic has the nice property to retain key symmetries of the primitive equations, which allows to discuss the consequence of these symmetries in a simplified framework. This said, the price to pay by using this simplified model is that it can not account for possible feedbacks between geostrophic turbulence and the background stratification, which is prescribed once for all in the quasi-geostrophic world. In addition, quasi-geostrophic dynamics filters out inertia-gravity wave modes, and therefore does not account for their possible interactions with geostrophic turbulence. To address these questions, one must use shallow-water models, or Boussinesq flow models. In this section, we focus on basic physical mechanisms that can be understood in a quasi-geostrophic framework.

Scientific context. Altimetry measurements have revealed in the nineties a sea of eddies (waves, jets and vortices) carrying most of the kinetic energy in oceanic currents. These eddies are the equivalent of an oceanic weather system, redistributing heat and momentum in the oceans. The factors that determine the shape and the size of these eddies remain poorly understood. In this context, the question on the vertical structure of geostrophic turbulence is central. First, knowing this vertical structure is essential to interpret correctly altimetry measurements that only give the dynamics of the upper layer flows [162, 94, 93]. Second, the global energy cycle of oceanic mesoscale eddies strongly depend on the vertical energy transfers, that may under some circumstances bring the energy of surface intensified eddies into bottom-trapped flows, where interactions with topography and Ekman layers efficiently remove energy from the system [55]. My post-doctoral work at GFDL-Princeton was initially motivated by these questions. This section gives a summary of my contributions related to this subject.

Baroclinic instability. Geostrophic turbulence in atmospheres and oceans is primarily fed by baroclinic instability of mean flow patterns. These mean flow patterns are driven by external forcing: mostly winds (and, to a less extent, surface heating) at the surface of the oceans, and pole-to-equator differential heating in the atmosphere. These forcing terms induce vertically sheared mean flows, that are associated with tilted iso-density lines through the combination of hydrostatic and geostrophic balance (thermal wind balance). This means that large scale flow patterns correspond to a huge reservoir of potential energy. Baroclinic instability releases some of this potential energy into smaller scale disturbances called eddies. The instability occurs from a subtle interplay between rotation (Coriolis parameter f) and stratification (buoyancy frequency $N = \sqrt{-g\partial_z \rho_0/\rho_0}$, related to the vertical density profile $\rho_0(z)$ in a fluid of depth H). The horizontal wavelength of the most unstable modes usually scales as the internal Rossby radius of deformation NH/f, an intrinsic length scale of physical system resulting from a competition between rotation and stratification. This scale is of order $50km$ in oceans and $500km$ in the Earth atmosphere. The vertical structure of the most unstable modes strongly depends on the problem parameters, but these modes are usually intensified at the surface of the oceans, and in the upper layers of the troposphere.

Objectives and methods. A natural question is then to determine wether the shape and the size of atmospheric and oceanic eddies (defined as deviations from mean currents) can be interpreted as being close to the most unstable modes of this instability, or if the subsequent nonlinear evolution of these eddies completely changes their vertical and horizontal structure, with possible feedback on the large scale flow. This question was the starting point of my post-doctoral project at GFDL Princeton with Geoff Vallis. I initially compared linear instability computations with output from primitive equation models and idealized quasi-geostrophic simulations, to test the locality hypothesis, stating that the properties of oceanic eddies at a given location can be deduced from mean flow properties [197]. This preliminary work has then motivated more idealized studies on geostrophic turbulence to determine how the nonlinear evolution of these eddies is affected by external parameter such as planetary vorticity gradients [section 3.1], bottom topography [section 3.2] and bottom friction [section 3.3]. In all these studies, the numerical results were rationalized with statistical mechanics arguments in a quasi-geostrophic framework. The idea of this approach is to predict the outcome of turbulent stirring as the most probable state among those that satisfy the constraints of the dynamics (provided by global invariants). The theory allows to compute those states, and to describe changes in their structure when key parameters are varied, thus bypassing the actual fine-grained dynamics. These arguments have some limitations as they rely on the strong hypothesis of ergodicity, but there can be seen as a poor's man approach to describe qualitative features of geostrophic turbulence in freely evolving flows.

Potential vorticity. The quasi-geostrophic dynamics can be expressed as the advection of a tracer q by a non-divergent horizontal velocity field, the potential vorticity, which is central to geophysical fluids:

$$\partial_t q + \mathbf{u} \cdot \nabla q = F + D, \quad \mathbf{u} = (\partial_x \psi, -\partial_y \psi), \quad q = \nabla^2 \psi + \mathcal{Q}[\psi] \tag{3.1}$$

where F and D stand for forcing and dissipative terms, and where q is related to the streamfunction ψ through a linear operator $\nabla^2 + \mathcal{Q}$, with $\nabla^2 = \partial_{xx} + \partial_{yy}$ the two-dimensional Laplacian operator. This family of models include one-layer models, in which case q and ψ are two-dimensional fields, but also multi-layer models, or continuously stratified model, in which case q and ψ are three dimensional fields. The formal analogy of Eq. (3.1) with two-dimensional

Euler flows is appealing, as all the theoretical tools developed in the Euler context can rather easily been generalized to quasi-geostrophic models. The cornerstone of two-dimensional turbulence is the inverse energy cascade, with self-organization into large scale coherent structures. Less is known about the vertical organization of the energy in stratified quasi-geostrophic models.

Initial value problem. Even if the full understanding of oceanic eddy properties eventually requires the inclusion of forcing and dissipation, it is natural to consider first an initial value problem to describe their energy cycle, in order to focus on the salient effects of geostrophic turbulence, without forcing. One can for instance consider the decay of a large scale flow pattern close to the observed one, to see if eddies similar to those observed spontaneously emerge [this is the approach followed in section 3.3]. Or one can take a different point of view, by considering an initial field of eddies whose horizontal size and vertical shape have initially the spatial structure of the fastest growing mode in the linear baroclinic instability problem, and to study their nonlinear evolution in the absence of forcing [this is the approach followed in sections 3.1 and 3.2]. These initial value problems provide useful insights on a general tendency for the nonlinear evolution of the system, even if this is only a first step before more comprehensive forced-dissipative studies. The main caveat of the approach is a certain degree or arbitrariness in the choice of the initial condition.

Equilibrium theory. The equilibrium theory is well suited to describe the final state organization of a freely evolving, inviscid two-dimensional and geophysical flows, for two reasons: i) the free evolution ensures that the isolated system is conservative (so that equilibria can be computed in the microcanonical ensemble) ii) the hope is that turbulent stirring arguably help the system to evenly explore phase space (ergodicity). Equilibrium theory is a useful tool to describe actual decaying experiments provided that dissipative time scales are much larger than inertial (eddy-turnover) time scales of the problem, and provided that there is sufficiently mixing in phase space. I will heavily rely on the statistical mechanics approach developed in the nineties by Miller-Robert-Sommeria in this chapter. During my PhD, I contributed to the explicit computation of statistical equilibria predicted by this approach in a variety of one-layer quasi-geostrophic models. During my post-doc, I extended the approach to multilayer and continuously stratified models, using the equilibrium theory as a guide to understand how the parameters entering the expression of the potential vorticity q can affect the vertical structure of geostrophic turbulence. The equilibrium theory quantifies the combined effect of turbulent stirring and material conservation of potential vorticity, which strongly constrain the vertical structure of geostrophic turbulence described by Eq. (3.1), without forcing and dissipation ($F + D = 0$). The fundamental postulate of the equilibrium theory is equiprobability of microscopic configuration, which are described by the potential vorticity field. One then need to define relevant macroscopic observable and to compute the most probable value of these observables among all the microscopic configurations satisfying the constraints of the dynamics. In the case of an initial value problem, the constraints are set by the initial condition, and the system is attracted towards the equilibrium state if the dynamics is ergodic, i.e. if the system evenly explores phase space. This a strong, but a single and simple assumption from which interesting physical insight can be obtained.

Macrostates. A peculiar property of systems described by continuous fields is that there is a lot of room in phase space for microscopic configurations characterized by wild small scale fluctuations, as shown in figure 3.1. Such states are well described at a macroscopic level by introducing the probability $\rho(\sigma, \mathbf{x})$ to measure a given potential vorticity level $q = \sigma$ in the vicinity of a given point $\mathbf{x}$. This probability field is obtained through a coarse-graining procedure, but the final result does not depend on this procedure. It is then possible to count the number of microscopic configurations associated with a given field ρ, and to show that there is an overwhelming number of microscopic configurations concentrated close to the most probable macrostate ρ of the system. This most probable state is computed by maximizing a mixing entropy $\mathcal{S} = -\int \mathrm{d}\sigma \mathrm{d}\mathbf{r}\ \rho \log \rho$, while satisfying the constraints provided by dynamical invariants, namely the total energy and the global distribution of potential vorticity in each layer. An essential point of the whole approach is that the constraints can be expressed as functional of the macroscopic state ρ. By contrast with many classical systems encountered in physics, this mean-field treatment is not an approximation in the case of two-dimensional Euler and quasi-geostrophic flow models, owing to the long-rangeinteraction potential between fluid particles.

What can we learn from phase diagrams ? The interest of using equilibrium statistical mechanics is to build phase diagrams for the large flow structure of the equilibrium state, shedding some light on the effect of varying parameters such as bottom friction, planetary vorticity gradients and bottom topography. These phase diagrams can

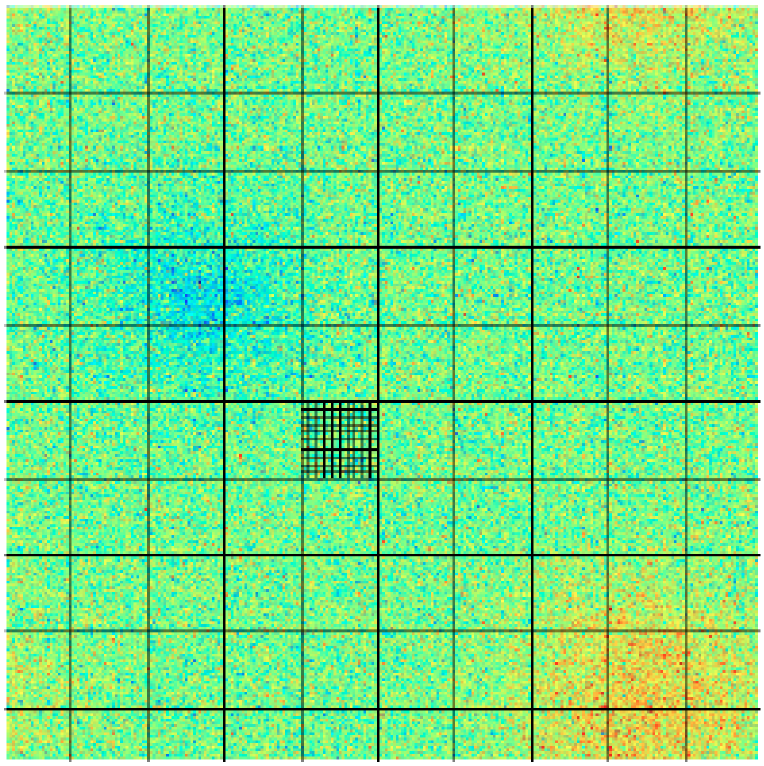

Figure 3.1: Snapshot of a typical potential vorticity field at large time, for a numerical simulation of Galerkin-truncated Euler dynamics, described by Eq. (3.1) with $\mathcal{Q} = 0$, in the absence of forcing and dissipation ($F + D = 0$). Blue and red color correspond to negative and positive values, respectively. We clearly see a scale separation between a large scale flow and wild small scale fluctuations looking like dots on a newspaper. A uniform coarse-grained grid and a fine grained grid are introduced. Within the framework of the equilibrium theory, each cell of this fine-grained grid correspond to a fluid particle carrying a given value of potential vorticity. A macrostate is defined on the coarse-grained grid, by computing histograms of potential vorticity levels within each cell of the coarse-grained grid (macrocell). The macrostates of the continuous dynamics are recovered by considering first the limit of an infinite number of fluid particles per macrocell, and then the limit of an infinite number of macrocells. After this coarse-graining procedure, the theory describes the probability to measure a given potential vorticity level in the vicinity of each point. A crucial point is that the fluctuations of potential vorticity within a macrocell do not contribute to the total energy after this coarse-graining procedure.

then be used as a guide to interpret observations and to rationalize idealized numerical simulations.

3.1 Barotropization processes [JFM 2012a]

Barotropization. Barotropization refers to the tendency of a quasi-geostrophic flow to reach a depth-independent configuration [155]. The term was coined by P. Rhines when discussing numerical simulations of two-layered quasi-geostrophic equations [147]. Prior to these numerical studies, Charney used an analogy between two-dimensional Euler and three-dimensional quasi-geostrophic dynamics to predict this tendency as the result of an inverse cascade in the vertical direction, similar to the inverse energy cascade in the horizontal directions [36]. In the particular case of an isotropic initial condition, i.e. when there is no vertical variation of the enstrophy (fluctuations of potential vorticity), this view is supported by numerical simulations [119, 159]. By contrast, barotropization can not be complete in the case of surface quasi-geostrophic dynamics [93], i.e. when enstrophy is confined to an upper layer with vanishing potential vorticity in the interior. This suggests that the vertical profile of enstrophy plays a central role in barotropization processes. Building on previous statistical mechanics studies of the continuously stratified quasi-geostrophic model, my contribution in collaboration with Steve Griffies and Geof Vallis during my post-doc at GFDL/Princeton University, has been to quantify the efficiency of barotropization in freely evolving turbulent flows, by computing the most probable state, arguably the attractor of the system if the flow is turbulent, by taking the vertical enstrophy profile as an input.

Beta plane. The equilibrium statistical mechanics results that we present below provide a physical interpretation for the enhanced barotropization of surface-intensified oceanic eddies in the presence of planetary vorticity gradients. The simplest setting that takes into account these gradients is the beta plane configuration: the flow is assumed to take place on a plane tangent to the sphere. The only effect of the Earth's sphericity retained in the flow model is the variation of the Coriolis parameter in the (South-North) y-direction. The Coriolis parameter is twice the projection of the Earth rotation vector along the the local vertical axis. Within the beta plane approximation, only linear variations are retained, so that the Coriolis parameter is $f + \beta y$. The fact that barotropization of surface-intensified eddies is favored in the presence of a beta pane was observed in previous numerical simulations but remained unexplained [168].

The dynamical system. Continuously stratified quasi-geostrophic flows take place in three dimensions, but their dynamics is quasi two-dimensional because the non-divergent advecting velocity field has only horizontal components, which can be described by a streamfunction $\psi(x, y, z, t)$. Such flows are stably stratified with a prescribed buoyancy profile $N(z) = \sqrt{-g\partial_z \rho/\rho_0}$, and are strongly rotating, with Coriolis parameter f. In the absence of forcing and dissipation, the dynamics in the domain bulk (interior) is expressed as the advection of potential vorticity q, as in Eq. (3.1), with

$$q = \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial}{\partial z}\psi\right) + \beta y \ , \tag{3.2}$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$ is the horizontal Laplacian, βy the planetary vorticity term. We readily see that for a given field q, the ratio f/N controls the vertical structure of the velocity field: in the large rotation limit, the velocity is depth independent, which is consistent with the intuition that strong rotation favors vertical columnar structures for the velocity while strong stratification favors pancakes structures for the velocity. The boundary condition at the bottom $z = -H$ and at the surface $z = 0$ are given by

$$\partial_z \psi\Big|_{z=-H} = 0, \quad f\partial_z \psi\Big|_{z=0} = b_s \ , \quad \partial_t b_s + \mathbf{v}|_{z=0} \cdot \nabla b_s = 0 \ . \tag{3.3}$$

The upper boundary condition describes the advection of buoyancy at the surface. In the particular case $q = 0$, the whole dynamics is governed by this surface advection. This is the surface quasi-geostrophic model. These boundary conditions can be formally replaced by the condition of no buoyancy variation ($\partial_z \psi = 0$ at $z = 0$), provided that surface buoyancy anomalies are interpreted as a thin sheet of potential vorticity just below the rigid lid [30]. For this reason, and without loss of generality, we assume $b_s = 0$ in the following. To simplify the discussion, we restrict ourselves to the case of a square doubly-periodic domain $\mathcal{D}$. We adimensionalize length such that the domain length is 2π, so that the streamfunction is 2π-periodic in the x, y directions.

Invariants. Dynamical invariants include the total (kinetic plus potential) energy

$$E_0 = \frac{1}{2}\int_{-H}^{0} \mathrm{d}z \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y \left[(\nabla\psi)^2 + \frac{f^2}{N^2}(\partial_z\psi)^2 \right], \tag{3.4}$$

and the Casimir functionals $\mathcal{C}_g(z)[q] = \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y\ g(q)$ where g is any continuous function. The key point is that the energy constraint is a global one, while the potential vorticity field is constrained to evolve in horizontal planes, which provides a set of Casimir functional at each altitude $z = cst$. As we shall see in the following, these Casimir conservation laws within each layers have far reaching consequences on the vertical structure of geostrophic turbulence. We will address in particular the role of the vertical enstrophy profile

$$Z_0(z) = \frac{1}{2}\int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y\ q^2 \tag{3.5}$$

in shaping the vertical structure of geostrophic turbulence.

Computation of equilibrium states in a low energy limit. We recall that a state picked up at random for a given set of constraints is characterized by wild small scale fluctuations of potential vorticity q. Denoting $\overline{q}$ the coarse-grained potential vorticity field (smoothing out local potential vorticity fluctuations), and considering a low energy limit for analytical convenience, we have shown in Ref. [198] that the calculation of Miller-Robert-Sommeria equilibrium states amounts to finding the minimizer $\overline{q}_{min}$ of the "total macroscopic enstrophy"

$$\mathcal{Z}_{cg}^{tot}[\overline{q}] = \frac{1}{2}\int_{-H}^{0} \mathrm{d}z \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y\ \frac{\overline{q}^2}{Z_0} \tag{3.6}$$

among all the fields $\overline{q}$ satisfying the energy constraint, where ψ and $\overline{q}$ are related through Eq. (3.2). This variational problem can be seen as a generalization to the stratified case of the phenomenological minimum enstrophy principle or selective decay hypothesis [29].

Critical states of this variational problem are computed by introducing Lagrange multiplier β_t associated with the energy constraint, and by solving $\delta\mathcal{Z}_{cg}^{total} + \beta_t\delta\mathcal{E} = 0$, which leads to the linear relation $\overline{q} = \beta_t Z_0 \psi$. The next step is to find which of these critical states are actual minimizers of the macroscopic enstrophy for a given energy. We have performed these computations in various cases of geophysical interest in Ref. [198].

Surface quasi-geostrophy. Simple and yet important insights are obtained by considering first the case of surface quasi-geostrophy, for which the potential vorticity is zero everywhere excepted into a thin layer at the surface, together with $\beta = 0$. The role of planetary vorticity gradients will be discussed later. In that case, the relative vorticity in the bulk $\nabla^2\psi$ is compensated by the stretching term $\partial_z\left(\frac{f^2}{N^2}\partial_z\psi\right)$. This shows that horizontal flow structures of length L_{flow} are related to the penetration of the vertical velocity field over the depth $h = fL_{flow}/N$. We see that an horizontal energy cascade (an increase of L_{flow}) is necessarily associated with a vertical energy cascade (an increase of h), and we recover that increasing f/N leads to more barotropic flow. One can compute explicitly the equilibrium states and show that L_{flow} is given by the domain size, which provides an upper limit for the penetration depth $h_{max} = Lf/N$, with L the domain size. Barotropization is clearly incomplete when $h_{max} < H$.

Role fo the vertical enstrophy profile. In the previous case, the reason for incomplete barotropization was that potential vorticity (and hence enstrophy) vanished in the bulk. We show in figure 3.2 the more general case of a two-step microscopic enstrophy profile

$$Z_0 = Z_{\mathrm{surf}}\Theta(z + H_1) + Z_{\mathrm{int}}\Theta(-z - H_1), \quad H_1 \ll H, \tag{3.7}$$

where Θ is the Heaviside function, and where $Z_{\mathrm{int}}/Z_{\mathrm{surf}}$ varies between 0 and 1. We find in that case that the minimum macroscopic enstrophy state are always characterized by the smallest horizontal mode on the horizontal ($K = 1 Wave - meanflow$). As for the vertical structure, we observe figure 3.2 a tendency towards more barotropic flows when the ratio $Z_{\mathrm{int}}/Z_{\mathrm{surf}}$ tends to one, while we recover the surface quasi-geostrophic case when $Z_{\mathrm{int}} \ll Z_{\mathrm{surf}}$.

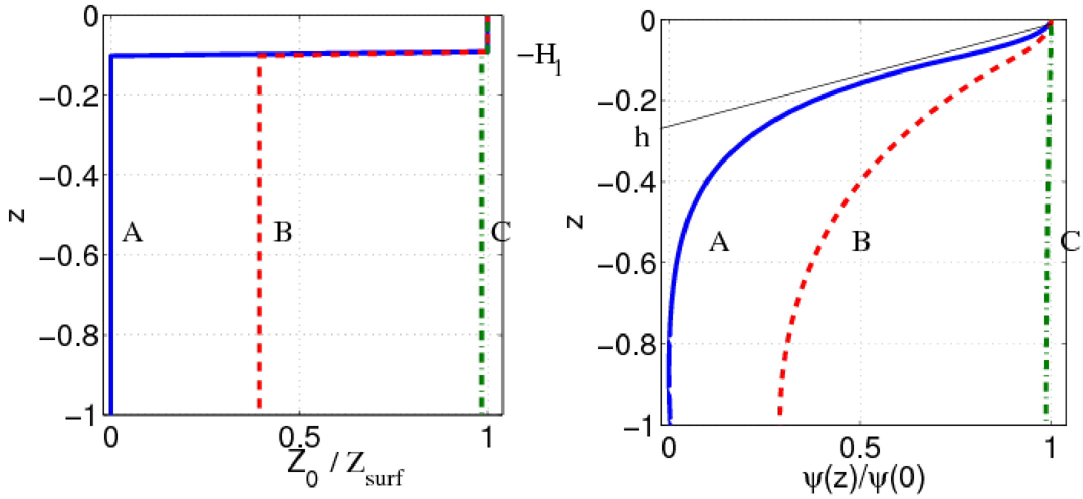

Figure 3.2: **Left panel:** three different vertical profiles of the microscopic enstrophy defined in Eq. (3.7). **Right panel**: corresponding vertical structure of statistical equilibrium states, in the case of constant stratification ($f^2/N^2 = 0.1$). $\psi(z)$ is the amplitude of the gravest horizontal Laplacian eigenmode. The e-folding depth in case A is $h = f/NK$ ($K = 1$ for the equilibrium state).

These examples show the importance of the conservation of microscopic enstrophy $Z_0(z)$ to the vertical structure of the equilibrium state.

The role of beta effect in barotropization process. For a given initial condition $\psi_0(x, y, z)$, increasing β increases the contribution of the (depth independent) available potential enstrophy defined as $Z_p = \beta^2 \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y \; y^2$ to the total microscopic enstrophy profile $Z_0(z) = \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y \; q_0^2$, where q_0 is the initial potential vorticity field that can be computed by injecting ψ_0 in (3.2). For sufficiently large values of β, the potential vorticity field is dominated by the beta effect ($q_0 \approx \beta y$), Z_0 therefore tends to Z_p and becomes depth independent. Because statistical equilibria are fully barotropic when the microscopic enstrophy Z_0 is depth-independent, we expect a tendency toward barotropization by increasing β, provided that the flow is attracted towards the equilibrium state.

Consistency with Held-Larichev scaling in two-layer beta plane baroclinic turbulence. The previous results are consistent with numerical simulations of Smith and Vallis [168], but may at first sight seem to be in contradiction with Held-Larichev scaling for geostrophic turbulent eddies in two-layer baroclinic turbulence [74]. Indeed, their scaling is such that the ratio of barotropic to baroclinic kinetic energy varies as $1/\beta^2$: in that case, increasing planetary vorticity gradients decreases the relative contribution of the barotropic mode. A closer look at the problem in the weak beta limit shows that there is no contradiction. Indeed, the problem considered by Larichev and Held can be recast as an initial value problem in a channel, with an homogeneous eastward flow in the upper layer and no flow at the bottom (just as in the section on ribbon turbulence in this chapter). Without beta effect, and assuming two layers of equal depth, the initial potential vorticity field in the upper layer and lower layer are respectively $q_1^0 = (U/R^2)y$ and $q_2^0 = -(U/R^2)y$, with U the initial velocity in the upper layer, and R the Rossby radius of deformation. We see that the potential enstrophy is the same in both layers, and one may expect complete barotropization if the flow becomes turbulent. Now, if one add planetary vorticity gradients, i.e. a term βy in each layer, then we wee that the potential enstrophy becomes larger in the upper layer than in the lower layer. According to the previous arguments, the flow should then become less barotropic as the contribution from the beta term increases. To conclude, one should distinguish the effect of planetary vorticity gradients on surface intensified small scale eddies in continuously stratified (or multilayer) models, and on large scale eastward baroclinic mean flows in two layer models [paper in preparation with K. Gianchandani].

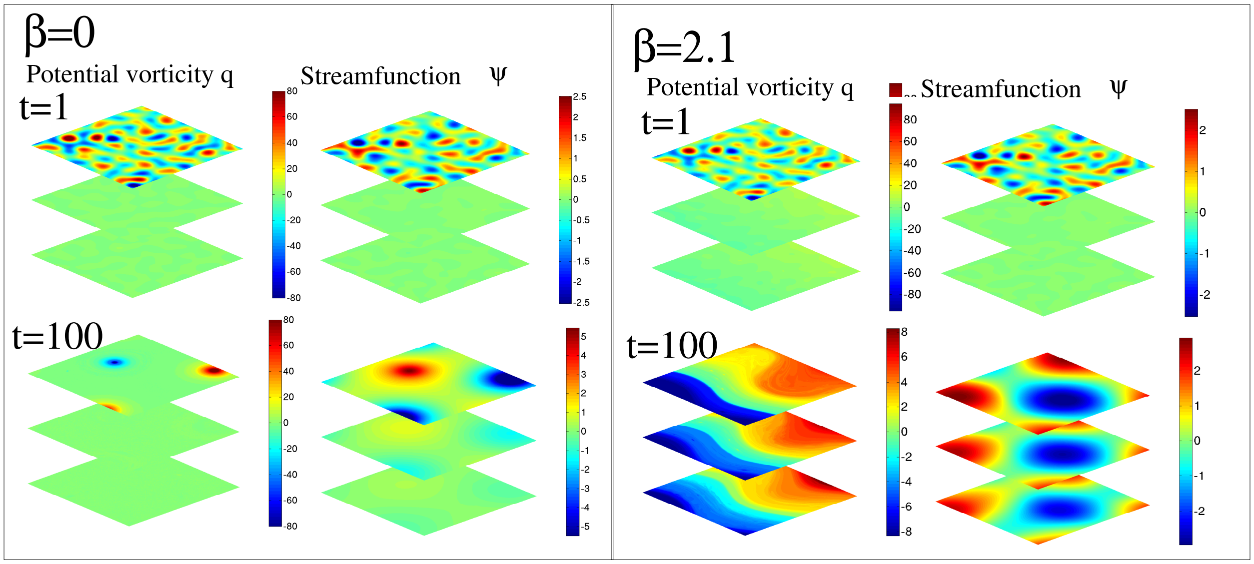

Figure 3.3: **Left panel**: case without beta plane. Only the fields in upper, middle and lower layer are shown. The end-state is surface-trapped. **Right panel**: case with a beta plane; the initial streamfunction is the same as on the left. The end-state is almost depth-independent. The interior beta plane is not clearly visible in the upper panel of potential vorticity because the color scale is different than in the lower panel.

Numerical experiments. We describe in this paragraph numerical results on the large time organization of an initial flow of small scale surface intensified eddies, varying the values of β. The initial potential vorticity field is $q_0 = q_{\text{surf}}(x, y)\Theta(z + H_1) + \beta y$, such that $q_0 = \beta y$ in the interior $(-H < z < -H_1)$ and $q_0 \approx q_{\text{surf}}$ in a surface layer $z > -H_1$. The surface potential vorticity $q_{\text{surf}}(x, y)$ is a random field with random phases in spectral space, and a Gaussian power spectrum peaked at wavenumber $K_0 = 5$, with variance $\delta K_0 = 2$, and normalized such that the total energy is equal to one ($E_0 = 1$).

We performed simulations of the dynamics by considering the linearly stratified case, with a vertical discretization into 10 layers of equal depth, in a case where the first baroclinic Rossby radius is of the order of the domain $Lf/HN = 1$. The initial condition were such that the enstrophy was zero in each layer when $\beta = 0$, excepted in the upper layer. The case with $\beta = 0$ is presented in the left panel of figure 3.3. An inverse cascade in the horizontal leads to flow structures increasing in size, associated with a tendency toward barotropization.

We now switch on the beta effect, with the same initial surface-intensified streamfunction $\psi_0(x, y, z)$. As a consequence, the contribution of the depth independent part of the microscopic enstrophy increases, which means a tendency toward a more barotropic equilibrium, according to the previous subsection. This is what is actually observed in the final state organization of figure 3.3 in the presence of beta effect. This result reflects the fact that in physical space, the initial surface-intensified flow stirs the interior potential vorticity field (initially a beta plane), which in turn induces an interior flow, which stirs even more the interior potential vorticity field, and so on. We conclude that in this regime, the beta effect is a catalyst of barotropization, as predicted by statistical mechanics. For larger values of β, the result is not as good, and the dynamics leads to states different than those predicted by the equilibrium theory (see figure 3.4). The reason is that the flow dynamics becomes dominated by waves for sufficiently large planetary vorticity gradients, and we are far form the ergodicity assumption underlying the equilibrium theory.

Figure 3.4 may be used to interpret the efficiency of barotropization depending on the latitude. Indeed, the beta term is an increasing function of the co-latitude, from the North pole to the equator. The latitudinal variations of barotropization in the atmosphere reported in figure 2-f of Ref. [38] is similar to the dependence on beta in our numerical simulations, shown in figure 3.4. Given the complexity of atmospheric turbulence, with much more param-

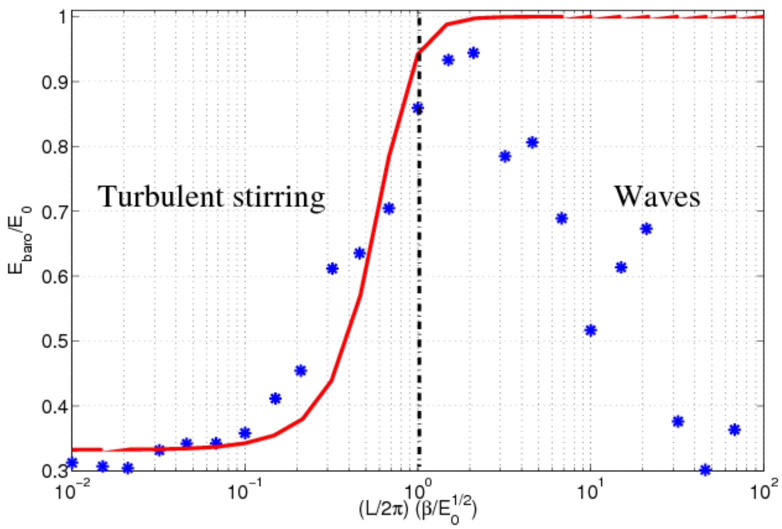

Figure 3.4: Ratio of barotropic kinetic energy to the total kinetic energy. The red curve is a fit with Statistical mechanics predictions (the horizontal length of the equilibrium states is taken as a parameter). The equilibrium theory account for the increase of barotropization favored by planetary vorticity gradients when β is small. For large β, the dynamics is dominated by waves and statistical mechanics predictions fail. This figure is reproduced from [198]. The increase in β can be interpreted as an increase in co-latitude. In that case, the numerical results are surprisingly similar to observations in the atmosphere, see figure 2-f in Ref. [38].

eters than in our idealized simulations, further work will be necessary to determine wether this similarity is a pure coincidence, or if it can truly be interpreted as the consequences of beta varying with latitude. In addition, previous simulations in the presence of wind forcing have shown that the phenomenology of beta plane turbulence with continuous stratification can be qualitatively different in the forced-dissipative case than this the freely evolving case [181, 169]. The relevance of equilibrium statistical mechanics for some of those regimes remain to be addressed, and other theoretical tools more suited for out-of-equilibrium problems will have to be considered in general.

Reference (with link to the paper):

Venaille, A., Vallis, G. K., and Griffies, S. M. (2012). The catalytic role of the beta effect in barotropization processes. Journal of Fluid Mechanics, 709, 490-515.

3.2 Bottom-trapped recirculations above topographic anomalies [GRL 11, JFM 12b]

Two puzzles in the Argentine basin. The Zapiola anticyclone is a strong recirculation about 500 km wide that takes place above a sedimentary bump in the Argentine Basin, where bottom-intensified velocities larger than 0.1 m.s^{-1} have been reported simultaneously from *in situ* measurements [157] and numerical models [43]. The mass transport of this 4km-deep vortex is as large as any other major currents, such as the Gulf Stream or the Kuroshio (around 100 Sv, i.e. 10^8 m^3.s^{-1}). It is also widely believed that such bottom-trapped anticyclonic flows take place above other seamounts, even if the deep circulation of oceanic currents remains poorly observed. The Zapiola anticyclone points to an interesting routes for the dissipation of geostrophic turbulence: the energy of surface intensified eddies is transformed into a bottom-trapped mean flow, which is eventually dissipated by Ekman damping, or through different kinds of interactions with bottom topography, such as the radiation of internal gravity waves. This leads to the following question:

Q1 *How does geostrophic turbulence transfer energy from the top to the bottom of the oceans?*

The second question posed by this anticyclonic recirculation comes from altimetry and realistic numerical ocean models, that have revealed that the fluctuations of the mass transport are as large as the transport itself:

Q2 *What drives the huge fluctuations of the mass transport above topographic anomalies?*

An eddy-driven mean flow. My first work on the Zapiola anticyclone was motivated by the second question (Q2), posed by oceanographers of the MEOM team at LEGI, that actually discovered the existence of this recirculation in their numerical ocean model in the nineties, almost prior to its observation in the actual ocean. I proposed an idealized model for the fluctuations of the mass transport, described by a Langevin equation, in the framework of a barotropic model, assuming that the flow is depth-independent. The model predictions were in good agreement with output from a comprehensive general circulation model. The main idea of this work was based on an analogy between the anticyclonic recirculation above topography anomalies and the generation of a westward flow away from a stirring region of enhanced eddy activity around a polar cap (variations of the Coriolis parameters are equivalent to variations of bottom topography in the barotropic quasi-geostrophic model). The total mass transport of the anticyclone is then interpreted as the result of a competition between dissipation (Ekman drag) and eddy forcing at the boundary of the region of eddy activity. To fix the ideas, let assume an axisymmetric topographic bump. Averaging the fields along the azimuthal direction, and denoting $u' = u - \overline{u}$ fluctuations around the mean flow, the dynamics can be written as

$$\partial_t \overline{u} = -\frac{1}{r}\partial_r \left(r\overline{u'v'} \right) - \gamma \overline{u}, \tag{3.8}$$

Using phenomenological arguments, the eddy forcing term $\partial_r r\overline{u'v'}$ can be decomposed into constant term whose sign depends on the topography slope and whose amplitude depends on the eddy activity, plus an additional white noise contribution, both of them being diagnosed in numerical simulations. This provides a mechanism for internal stochastic variability of mean flows above topographic anomalies in the ocean, consistent with output from an ocean general circulation model [193].

Potential vorticity mixing and anticyclonic flows. The formation of anticyclonic flows above topography was actually already well known, either using quasi-linear theory (as in figure 3.5), potential vorticity homogenization ideas, selective decay hypothesis [29], or statistical mechanics ideas [156], and confirmed by idealized numerical simulations [180]. The common point between these approaches is the idea that potential vorticity mixing is essential to the explanation of the emergence of the vortex [155]. However, none of these approaches explained how surface intensified eddies lead to bottom-trapped flows (see Q1 above).

Bottom intensification as the most probable outcome of turbulent stirring. The main point I would like to make in the following is that an understanding of the tendency for bottom intensification can be obtained by counting states, taking advantage of our ignorance of the complex dynamics. The idea is that that turbulent stirring may drive the system towards the most probable macroscopic state, which is then observed just because it is the most probable one. Arguments based on potential vorticity homogenization for the formation of bottom-trapped flows were previously given by Dewar [45] in a forced-dissipated case. A complementary point of view was given by Merryfield [122], who computed critical states of equilibrium statistical mechanics for truncated dynamics. He observed that some of those states were bottom intensified in the presence of topography. My contribution has been to find the equilibrium states among these critical states and to show how they depend on the initial microscopic enstrophy profile[186].

Topography is a source of potential enstrophy. We saw in the previous section that planetary vorticity gradients (beta effect) provide a depth-independent source of microscopic enstrophy that favors barotropization. We have used similar ideas to address the effect of bottom topography. In the framework of the freely evolving continuously stratified model introduced in previous section, topography $h_b(x, y)$ is taken into account by changing the bottom boundary condition into

$$\frac{f}{N^2}\partial_z \psi \bigg|_{z=-H} = -h_b \ , \tag{3.9}$$

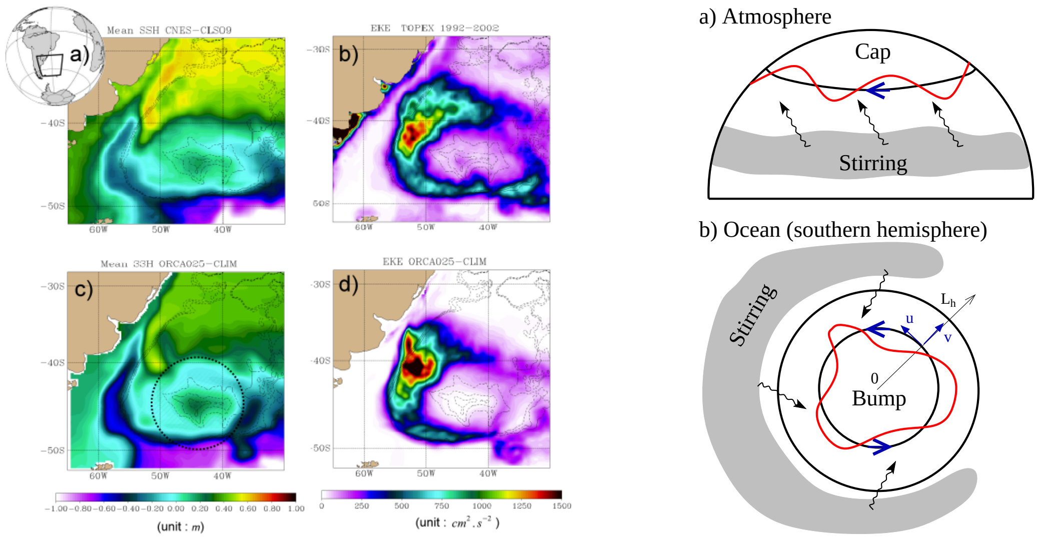

Figure 3.5: **Left panel**: a) Observation of mean sea surface height, related to streamfunction in the upper layer. Dashed line corresponds to the bathymetry. Note the mean flow around the Zapiola drift. b) Observation of the fluctuations of sea surface height, related to surface eddy kinetic energy. The Zapiola anticyclone corresponds to a minimum of eddy kinetic energy. c,d) idem in a comprehensive primitive equations numerical ocean model. **Right panels** a) classical explanation for the generation of westward flows around a polar cap away from the eddying region, using Kelvin circulation theorem, adapted form Vallis book. B) case of anticyclonic recirculation above a topography anomaly.

By integrating the potential vorticity over a thin layer of fluid at the bottom, assuming an initial flow at rest, and using this bounndary condition, we see that bottom topography provides a source of available potential enstrophy localized at the bottom. According to the discussion of the previous chapter, this should favor the spontaneous emergence of a bottom-trapped flow.

Let us consider the configuration of figure 3.6-a, with two thin layers of constant density of thickness h at the top and the bottom, and a linear stratification with buoyancy N in the domain bulk. In the homogeneous density layers, the streamfunction is depth independent, denoted by $\psi^{top}(x,y,t)$ and $\psi^{bot}(x,y,t)$, respectively, and the dynamics is then fully described by the advection of the vertical average of the potential vorticity fields, denoted by $q^{top}(x,y,t)$ and $q^{bot}(x,y,t)$. The interior potential vorticity field is denoted by $q^{int}(x,y,z,t)$. For a given field $q^{top}, q^{int}, q^{bot}$, the streamfunction is obtained by inverting the following equations:

$$q^{top} = \nabla^2\psi^{top} - \frac{f^2}{hN^2}\frac{\partial}{\partial z}\psi\Big|_{z=-h} , \tag{3.10}$$

$$q^{bot} = \nabla^2\psi^{bot} + \frac{f^2}{hN^2}\frac{\partial}{\partial z}\psi\Big|_{z=h-H} + f\frac{h_b}{h} , \tag{3.11}$$

$$0 = \nabla^2\psi + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2}\psi \quad \text{in the domain bulk} . \tag{3.12}$$

Equations (3.10-3.11) are obtained by averaging Eq. (3.2) in the vertical direction in the upper and the lower layers, respectively, and by using the boundary condition (3.9). In the following, the initial condition is a surface-intensified velocity field induced by a perturbation of the potential vorticity field confined in the upper layer:

$$q_0^{top} = q_0^{pert} , \quad q_0^{bot} = \frac{f}{h}h_b, \tag{3.13}$$

together with $q_0^{top} \ll q_0^{bot}$. The potential vorticity fields are therefore associated with microscopic enstrophies $Z_0^{top} \ll Z_0^{bot}$. One can then show that for a fixed topography, in the *low energy limit*, the equilibrium streamfunction is a

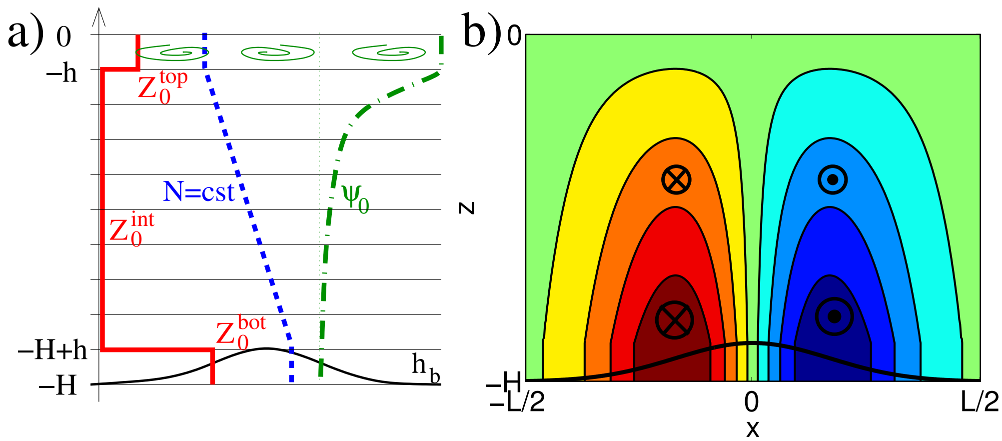

Figure 3.6: a) Sketch of the flow configuration. The continuous red line represents the initial microscopic enstrophy profile (here $Z_0^{int} = 0$). The dashed blue line represents the density profile, and the dashed-dotted green line represents the streamfunction amplitude shape, which is initially surface-intensified. The thick continuous black line represents bottom topography. b) Vertical slice of the meridional velocity field v of the statistical equilibrium state in the low energy limit.

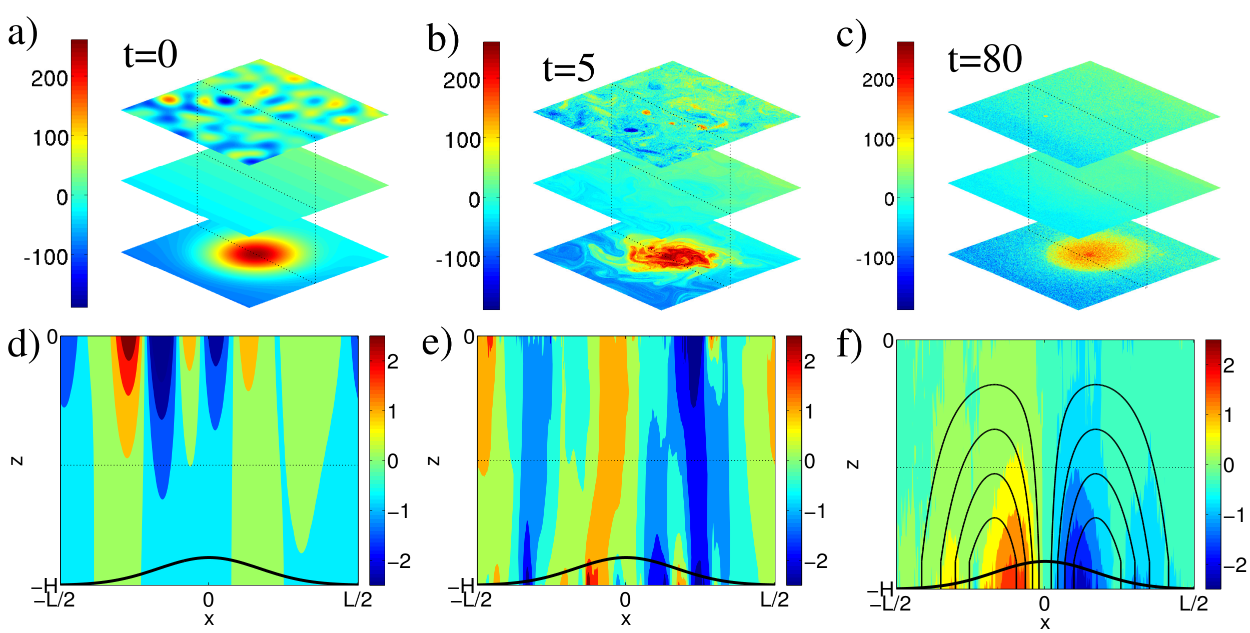

Figure 3.7: a),b),c) Potential vorticity field at three successive times (adimensionalized by initial eddy turnover time). In each panel, only layers 1 (top), 5 (middle) and 10 (bottom) are represented. b),d),f) Vertical slices of the meridional velocity fields v taken at the center of the domain ($y = 0$), and associated with the potential vorticity fields given in panels d),e),f), respectively. The bold continuous dark line represents bottom topography. The continuous black contours of panel f) give the structure of the statistical equilibrium state in the low energy (or large topography) limit, corresponding to figure 3.6-b. Contour intervals are the same as those between the different shades.

bottom-intensified quasi-geostrophic flow such that bottom streamlines follow contours of topography with positive correlations, see figure 3.6-b.

The initial condition of figure 3.7-d is the same surface-intensified velocity field as the one used in figure 3.3 when discussion the role of beta effect. After a few eddy turnover times, the enstrophy of the upper layer has cascaded towards small scales as shown by numerous filaments in figure 3.7-c, concomitantly with an increase of the horizontal energy length scale. As in the case without topography, this inverse energy cascade on the horizontal leads therefore to a deeper penetration of the velocity field, shown in figure 3.7-d. When this velocity field reaches the bottom layer, it starts to stir the bottom potential vorticity field. This induces a bottom-intensified flow, which then stirs the surface potential vorticity field, and so on. The corresponding flow is shown in figure 3.7-e, which clearly represents a bottom-intensified anticyclonic flow above the topographic anomaly, qualitatively similar to the one predicted by statistical mechanics in the low energy limit.

We conclude that topographic anomalies make possible the transfer of surface-intensified eddy kinetic energy into bottom-trapped mean kinetic energy, which would eventually be dissipated in the presence of bottom friction, as for instance in the case of the Zapiola anticyclone. In the case of the Zapiola anticyclone, the dissipation time scale is of the order of a few eddy turnover-time [45, 193]. It is therefore not *a priori* obvious that the results obtained in a freely evolving configuration may apply to this case. In fact, depending on its scale, amplitude, and on the presence of wind forcing, bottom topography can have a very different effect, if not opposite, to the vertical structure of geostrophic turbulence [182]. One can hope to build upon the statistical mechanics results to address the role of forcing and dissipation in vertical energy transfers above topographic anomalies in some of these more realistic cases. I should stress that as usual, the range of validity of statistical mechanics predictions is also limited due to the strong assumption of ergodicity. This hypothesis is certainly broken when the topography amplitude is too high (the flow is then dominated by waves), and when stratification is too strong (thus screening the interactions between the upper layer flow and the bottom topography).

References (with link to the papers):

-Venaille, A., Le Sommer, J., Molines, J. M., and Barnier, B. (2011). Stochastic variability of oceanic flows above topography anomalies. Geophysical Research Letters, 38(16).

-Venaille, A. (2012). Bottom-trapped currents as statistical equilibrium states above topographic anomalies. Journal of Fluid Mechanics, 699, 500-510.

Outlook. An important generalization of this work will be to take into account internal wave generation above bottom topography by the bottom-trapped recirculation, using a primitive equation model. These waves could feedback on the mean flow, or on the stratification. This is the object of an ongoing project with L.-P. Nadeau and A. Renaud. A second generalization is to study forced dissipative configurations in the multilayer case. A third generalization would be to understand the effect of rough topography: in a quasi-geostrophic setting, roughness would inject potential enstrophy in the bottom layer, must one may wonder if this roughness could really be dynamically efficient to transfer energy from the top to the bottom. Finally, a longstanding chicken-and-the-egg question on the Zapiola anticyclone is to quantify the feedback between the sedimentary bump and the anticyclonic recirculation [207].

3.3 From vortices to ribbons: the role of bottom friction [PoF 2014]

Motivation. When discussing the vertical structure of geostrophic turbulence in the previous section, we have considered purely inertial effects. In earth oceans and atmospheres, one can not get rid of a rigid bottom, where velocity should vanish. This implies the existence of a frictional (Ekman) layer. The effect of this boundary layer is usually modeled in quasi-geostrophic model by adding a linear friction term acting in the lower layer, for the sake of simplicity. One may then wonder what is the effect of this bottom friction on the vertical structure of geostrophic turbulence. This is the main motivation of this section. I initiated this project during my post-doc in Princeton GFDL with G. Vallis. This was also the starting point of an ongoing collaboration with L.-P. Nadeau (who was then at NYU, now at Rimouski), who performed all the numerical simulations presented here.

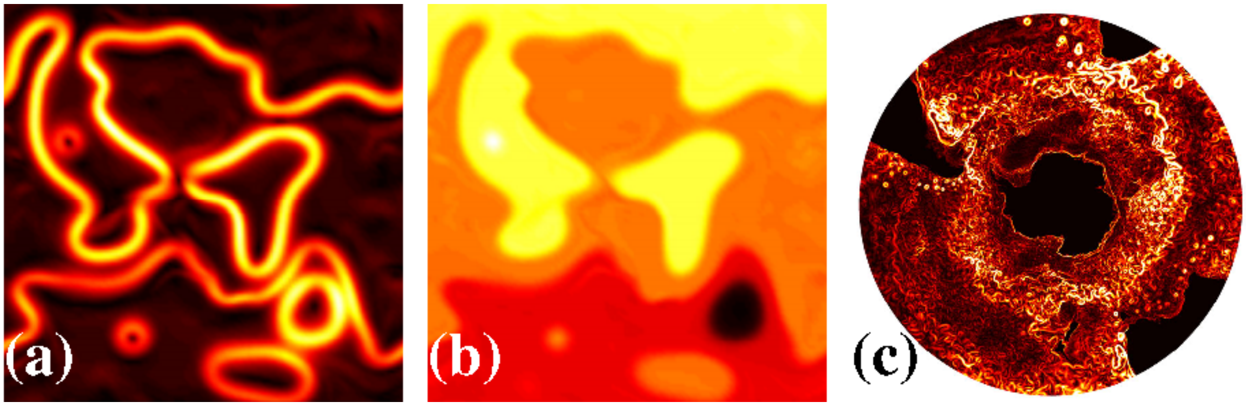

Figure 3.8: a) Snapshot of the kinetic energy in the upper layer of a two-layer quasi-geostrophic model with very large bottom friction ($rR/U \gg 1$) in a doubly-periodic domain of size $L \gg R$, with R the internal Rossby radius of deformation. The energy is concentrated into ribbons whose width is given by R b) Corresponding upper layer potential vorticity field. We see that the ribbons correspond to interfaces between regions where the potential vorticity is homogenized. c) Snapshot of the surface eddy kinetic energy in Southern oceans (reanalysis from ECCO2). These oceans are full of ribbons.

Ribbons. One can readily guess that the effect of large bottom friction is to damp any bottom flow, which goes against barotropization. Our contribution has been to quantity this effect, and to explore the consequences of a change of vertical structure on the horizontal flow patterns. In fact, a two-layer model with strong bottom friction is the simplest flow model where one can reproduce the spontaneous organization of geostrophic turbulence into sharp meandering jets such as the jet stream in the upper troposphere, the Gulf stream in the Atlantic ocean, or the multiple jets of the antarctic circumpolar currents. The common point between all these self-organized structures is their surface intensification, their localization at the interface between regions of homogenized potential vorticity, their width and their meanders that scale as the internal Rossby radius of deformation R (see figure 3.8). Because the kinetic energy of such flows is dominated by the meandering jets (or rings that pinch off the jets), they look like ribbons interacting in a seemingly erratic way, through a sequence of coalescence-dispersion processes, hence the name ribbon turbulence [4]. This condensation of the energy into ribbons contrasts with the usual phenomenology of two-dimensional turbulence, where kinetic energy is condensed into a vortex at the domain scale.

The key non-dimensional parameter. To describe the spontaneous emergence of ribbons, we have considered the decay of an initial homogeneous eastward flow in the upper layer of a two-layer quasi-geostrophic model. This initial value problem may be considered as the most simple configuration relevant to describe atmospheric and oceanic geostrophic turbulence. Indeed, the initial state is close to a configuration that would correspond either to the effect of wind forcing (in the oceans) or differential heating (in the atmosphere). Baroclinic instability can not occur in a one-layer model. In a two-layer model on an f-plane without bottom friction (the Coriolis parameter is assumed to be constant), the Eady time scale for the instability is R/U. In the presence of linear friction, there is another intrinsic time scale given by the inverse of the bottom friction coefficient r. The non-dimensional parameter rR/U comparing these two time scale is central to discriminate the different flow regimes in the problem [5, 178].

Two-layer quasi-geostrophic model. The two-layer quasi-geostrophic model is described by the advection of potential vorticity in each layer with an incompressible velocity field $\mathbf{u}_i = (-\partial_y \psi_i, \partial_x \psi_i)$:

$$\begin{aligned} \partial_t q_1 + \mathbf{u}_1 \cdot \nabla q_1 &= 0, \qquad & q_1 = \nabla^2 \psi_1 + \frac{\psi_2 - \psi_1}{R^2}, \\ \partial_t q_2 + \mathbf{u}_2 \cdot \nabla q_2 &= -\frac{rR}{U} \nabla^2 \psi_2, \quad & q_2 = \nabla^2 \psi_2 + \frac{\psi_1 - \psi_2}{R^2}\ . \end{aligned} \tag{3.14}$$

Both layers are coupled via the stretching terms proportional to $\psi_2 - \psi_1$, with a coefficient related to the stratification through the internal Rossby radius of deformation R.

We considered the case of an initial state $\psi_2 = 0$, $\psi_1 = -Uy$, which corresponds to an eastward flow homogeneous in the upper layer. Such states are always baroclinically unstable, and we studied the relaxation of this turbulent flow towards a state of rest.

Phase diagrams of large scale flow patterns. We have used a combination of statistical mechanics arguments, cascade phenomenology and numerical simulations to describe the different regimes of decay, depending on the parameter rR/U. These regimes were already reported in previous numerical studies, but the corresponding patterns of large scale organization remained largely unexplained:

1. In the weak friction limit, there is self-organization at the domain scale with a vertical structure almost completely depth independent (complete barotropization). In that case, the initial two-layer model tends to a final state well described by a (one-layer) barotropic model.
2. In the intermediate friction limit, friction time scale is of the order of the instability time scale, which prevents the growth of the eddies at scales larger than the initial instability wavelength, even if the dynamics becomes strongly nonlinear, with the formation of surface intensified point vortices
3. In the large friction limit, the system evolves towards the ribbon turbulence regime.

We have revisited the barotropization problem in this two-layer model (case 1), with statistical mechanics arguments relying on rearrangements of the potential vorticity field, which offers a complementary point of view to the classical cascade picture described in R. Salmon's book [155]. Second, we have described in detail the phenomenology of the ribbon-turbulence regime (case 2).

Spontaneous emergence of the ribbons. We have shown that at lowest order in the parameter rR/U, the dynamics is described by a $1\frac{1}{2}$ layer quasi-geostrophic model (which amounts to assume $\psi_2 = 0$ in the dynamical system above). Equilibrium theory predicts in that case the spontaneous emergence of regions with homogeneous potential vorticity, separated by strong jets (the ribbons), whose length should be minimal, just as bubbles in usual thermodynamics [23], and I had applied this theory to oceanic jets and vortices during my PhD thesis [188]. One qualitative argument is that the potential vorticity field can be fully mixed almost everywhere in the limit of small Rossby radius R, but energy conservation prevents complete mixing, which requires the formation of different regions of homogenized potential vorticity. This is very similar to phase coexistence in usual thermodynamics; here, each phase corresponds to a region with a different value of potential vorticity. The parameter R plays a crucial role in this problem: it is a screening length scale for the interactions between fluid particles, which are then much more local than in the Euler case. This parameter also corresponds to the width of the jets at the interface between the regions of homogenized potential vorticity, a result that stems from the inversion of the potential vorticity $q_1 = \nabla^2\psi_1 - \psi_1/R^2$. There were so far no numerical confirmation showing that a wide set of initial condition actually converges towards such solutions. It was in fact unexpected that a strongly damped two-layer model model would lead to states so close to the predictions obtained in the framework of a $1\frac{1}{2}$ quasi-geostrophic model.

Meanders and baroclinic instability. We have then shown that the meandering ribbons result from a competition between the tendency of the two-layer model to reach an equilibrium state of the $1\frac{1}{2}$ model, and the baroclinic instability of the jets (with typical length scales of meanders being of a order of few internal Rossby radius R). It is worth stressing that having the presence of at least two-layers is essential for baroclinic instability to occur. The occurrence of this instability through the whole decay process permits a slow but continuous mixing of potential vorticity blobs from one region to another in the upper layer. Qualitatively, the presence of this instability favors the exploration of phase space and therefore convergence towards the equilibrium state, just as the addition of a small noise usually favors ergodicity in a dynamical system. This may explain why we could observe the convergence towards equilibria of the $1\frac{1}{2}$ layer models in a two-layer model with large friction in the bottom layer much more easily than in freely decaying simulations of $1\frac{1}{2}$ quasi-geostrophic turbulence.

Cascade phenomenology. Finally, we have proposed cascade arguments complementary to the statistical mechanics approach for the $1\frac{1}{2}$ layer quasi-geostrophic model. At scales much larger than the internal Rossby radius R in this model, the dynamics is described at lowest order by the advection of the streamfunction $\psi 1$ by the vorticity field $\nabla^2\psi_1$. These two quantities play a role that is opposite than in the two-dimensional Euler case. A direct consequence is that kinetic energy goes to small scales. By contrast, at scales much smaller than R, the $1\frac{1}{2}$ layer dynamics is described at lowest order by two-dimensional Euler equations, and kinetic energy goes to large scale. This means that kinetic energy should pile up at scale R. The emergence of the ribbons of width R is then explained by the fact that the potential energy (proportional to the quadratic norm of the streamfunction ψ_1) goes to large scales, a process that occurs by expelling gradients of streamfunction at the boundaries.

Time scales for dissipation: analogy with a damped pendulum. This study has also shed new light on how the time scale for energy dissipation depends on the bottom friction coefficient in a freely evolving unstable mean flow. In the weak friction limit, the typical time for dissipation varies as the inverse of the friction coefficient ($\tau \sim 1/r$), as expected in a purely barotropic case: the smaller the friction, the longer the relaxation time towards a state of rest. In contrast, this relaxation time is proportional to the friction coefficient ($\tau \sim rR^2/U^2$) in the large friction limit: the decay of ribbons slows down with increasing bottom friction! This problem is somewhat equivalent to a pendulum in a viscous fluid: in a weak viscosity limit, the relaxation time decreases as the viscosity decreases (oscillations are damped more slowly). in the large viscosity limit, there is a change of dynamical regime (no more oscillations), and the relaxation time scale towards a state of rest increases when viscosity increases.

Reference (with link to the paper):

Venaille, A., Nadeau, L. P., and Vallis, G. (2014). Ribbon turbulence. Physics of Fluids, 26(12), 126605.

Outlook Observations suggests that the bottom friction values required to reach the ribbon regime are far from those usually considered in general circulation models of atmospheres and oceans. However, the ressemblance between the ribbons and oceanic or atmospheric jets (in the upper troposphere) is striking. Future work will be needed to explain how additional parameters such as additional layers, beta effect or a slopping bottom may lead to the ribbon regime without artificially high bottom friction coefficients. The role of the beta effect in this context has been investigated by Kaushal Gianchiandani (NISER, India) during a summer internship and a master thesis under my supervision, and I have ongoing collaborations with Brad Marston (for the atmospheric case) and L.-P. Nadeau (for the oceanic case) on the emergence of ribbons in forced-dissipated configurations.

3.4 Violent relaxation in alpha-turbulence models [PRE 15]

In most of the previous subsections, we have been using equilibrium statistical mechanics as a guide to interpret numerical experiments on geostrophic turbulence. Here we present a numerical work discussing in more detail under which circumstances a flow model actually relaxes towards an equilibrium state. We consider for that purpose a class of models that make possible a discussion on the role of long-range interactions during the relaxation towards equilibrium. The term "violent relaxation" is taken from Lynden-Bell [106], who used it to describe the rapid self-organization of self-gravitating systems into a quasi-stationary state described by continuous fields, followed by a much slower relaxation towards the equilibrium state of the underlying discrete models.

Alpha-turbulence models for two-dimensional flows. We have mentioned in section 3.1 the surface quasi-geostrophic model [93]. This model is derived from the continuously stratified quasi-geostrophic equations, assuming vanishing potential vorticity in the interior. Even if each horizontal component of the velocity field is a three dimensional field, it can be considered as a two-dimensional model since the dynamics is fully governed by the advection of buoyancy at the surface. The essential difference between this model and more familiar two-dimensional Euler equations is the interaction potential between fluid particles, which is logarithmic for two-dimensional Euler dynamics, and inversely proportional to the distance between fluid particles in the surface quasi-geostrophic model. Those flow models are actually two special cases of the more general family of two-dimensional alpha-turbulence models introduced in the nineties to understand local and non-local properties of turbulent cascade [137]. These models are described by the advection of a potential vorticity scalar q by a non-divergent velocity velocity field $\mathbf{v} = (-\partial_y\psi,\ \partial_x\psi)$, just as in

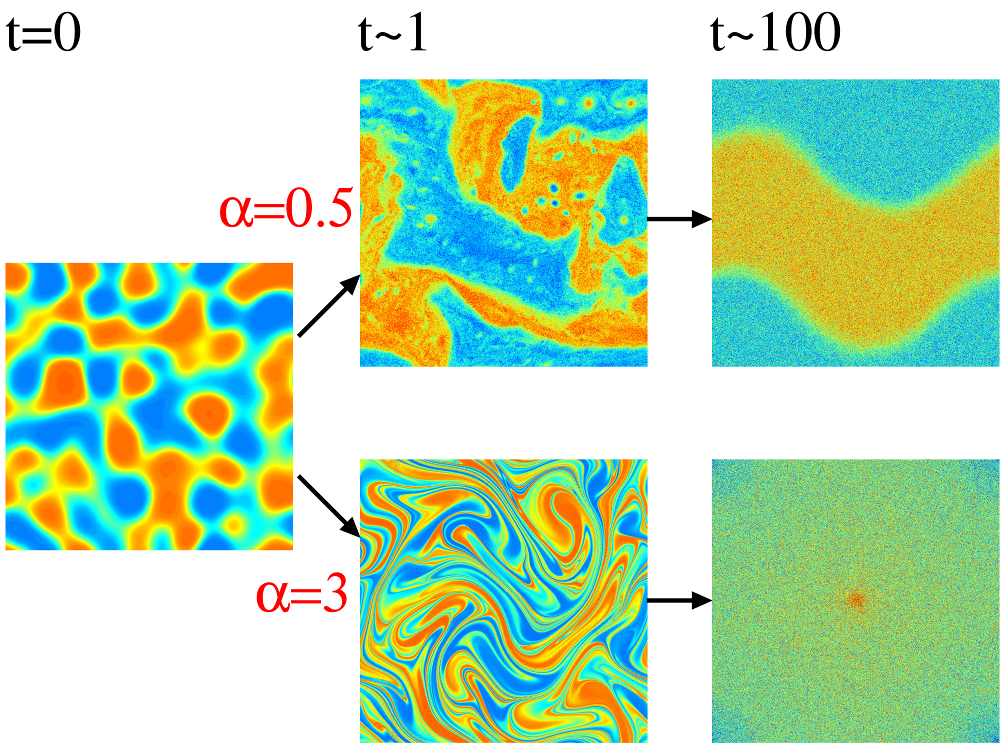

Figure 3.9: Numerical simulation of violent relaxation of an initial condition through Galerkin-truncated dynamics of two-dimensional alpha-turbulence models. The initial alpha vorticity field is the same for both cases, but the phenomenology of their temporal evolution is drastically different, and lead to different final states. The dipolar state is favored by increasing the range of interactions, or by decreasing the typical length scale of initial alpha-vorticity structures. Taken from Ref. [190] .

Eq. (3.1):

$$\partial_t q + \mathbf{v} \cdot \nabla q = 0, \quad q = -(-\Delta)^{\frac{\alpha}{2}} \psi \,, \tag{3.15}$$

where $(-\Delta)^{\frac{\alpha}{2}}$ fractional Laplacian operator (defined in Fourier space as $q_k = -|k|^\alpha \psi_k$). The alpha-energy can be written as $E = \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \; q(\mathbf{r})V(|\mathbf{r}-\mathbf{r}'|)q(\mathbf{r}')$, with the interaction potential $V \sim r^{\alpha-2}$. The case $\alpha = 1$ corresponds to the surface quasi-geostrophic model, while the case $\alpha = 2$ corresponds to the Euler equations.

long-range interacting systems. Alpha flow models belong to the family of long-range interacting systems. These systems are characterized by a two-body potential on the form $V \sim r^{\alpha-d}$, with $\alpha > 0$ and d their spatial dimension [34]. Other examples of such systems are non-neutral plasma and self-gravitating systems. Their energy is non-additive, which gives them remarkable common dynamical and thermodynamical properties such as spontaneous self-organization into large scale structures or negative specific heat. The interest of the alpha-turbulence model in this context is that the parameter $\alpha > 0$ allows us to vary the interaction potential between fluid particles from marginally long-range interactions ($\alpha = 0$) to extremely long-rangeinteractions ($\alpha \to +\infty$).

Relaxation of improbable initial states. The equilibrium statistical mechanics theory of long-ranged interacting systems is now fairly understood, but how and when a system actually relaxes towards equilibrium remains highly debated [97]. The major issue is to determine wether a given initial condition will or will not be attracted towards the equilibrium state. A naive (and correct) answer is that an overwhelming number of initial conditions are actually concentrated close to the equilibrium state, by definition of this state. One should keep in mind, however, that these typical states are most often characterized by wild small scale fluctuations, as in figure 3.1. In practice, one usually consider numerical experiments with initial states corresponding to a smooth potential vorticity fields. In other words, we are interested in the relaxation of an improbable initial condition. Improbable means that the probability to pick up such state for a given set of constraints will be close to zero.

Multiple steady states. One of the principal difficulty concerning the relaxation towards equilibrium in flow models is that the ensemble of invariant measures is in general much larger than the ensemble of microcanonical measure computed in the equilibrium framework [20]. Consequently, improbable initial states can be attracted towards a different state than the one predicted by equilibrium statistical mechanics. This issue is related to a general property of flow models: given the huge number of conserved quantities (Casimir and energy), there exists an infinite number of stable steady states, obtained for instance by looking for extrema of energy-Casimir functionals. The equilibrium states are only one small subset of these stable steady states.

Numerical studies of relaxation towards equilibrium. In fact, previous comparisons of statistical mechanics predictions and numerical simulations have lead to mitigated results either in the case of self-gravitating systems [97], or in the case of two-dimensional flow models [176, 50]. In these studies, the range of interactions are prescribed, and the relaxation towards equilibrium was studied by varying the initial condition. We have proposed a different approach by prescribing the initial condition, and varying the range of interactions in flow models. The numerical model relied on a discretized version of the flow models. As far as statistical mechanics predictions are concerned, a part of the difficulty is that the equilibrium states of the discretized model are in general different than the equilibrium states of the continuous dynamics for this kind of systems (one can not switch infinite time limit and the continuum limit). We considered more particularly an initial state such that the equilibrium theory predicts a tanh-like relation for between coarse-grained alpha-vorticity and streamfunction.

Main results. We have shown that the discrete system relaxes as expected rapidly towards a quasi-stationary state close to (or oscillating around) a steady state of the continuous dynamics. By contrast, the wild small scale fluctuations of vorticity were best described by the equilibrium statistical mechanics of the discrete systems, with equipartition of enstrophy at small scales, and local Gaussian fluctuations of α-vorticity, thus confirming previous theoretical predictions [131, 20]. For $\alpha \to 0$ (weak long-range interactions), a coarsening process led the emergence of two phases of homogenized α-vorticity, see the case $\alpha = 0.5$ is shown figure 3.9. This state were predicted by the equilibrium statistical theory, but the theory could not explain large scale oscillations of the interface observed in the simulations. For $\alpha > 2$ (strong long-range interactions), the system is irreversibly trapped through a filamentation process into a dipolar state with isolated vortices, different than flow structure predicted by the equilibrium theory, see $\alpha = 3$ on figure 3.9. We have propose a non-dimensional parameter quantifying the ability of the initial condition to reach equilibrium, depending on the range of interactions, and on the initial injection length scale.

We have thus shown that in contrast with equilibrium statistical mechanics expectations, the range of interactions plays a central role in selecting the final state of organization on two-dimensional turbulent flow models.

Reference (with link to the paper):

Venaille, A., Dauxois, T., and Ruffo, S. (2015). Violent relaxation in two-dimensional flows with varying interaction range. Physical Review E, 92(1), 011001.

3.5 Outlook: the interplay between geostrophic turbulence ant stratification at planetary scale

In all the studies presented in the previous sections, the stratification was prescribed, and in most of the problems, we considered a decaying flow. Yet planetary flows are forced-dissipative systems, and their global stratification results from a complex interplay between stratification and geostrophic turbulence in response to a large-scale thermal forcing. Indeed, geostrophic motion is thought to play a key role in shaping the stratification of midlatitude planetary flows, as it redistributes a significant amount of heat from the equator to the pole, hence reducing the meridional (pole to equator) temperature gradient set by large scale radiative forcing. As we have seen previously the primary source of geostrophic turbulence is baroclinic instability, which feeds on the potential energy provided by the horizontal temperature gradients. On the one hand, geostrophic turbulence is fairly well described by quasi-geostrophic models, and a large amount of numerical and theoretical efforts have been devoted to the understanding of peculiar dynamical properties in this model. On the other hand, the quasi-geostrophic model requires a prescribed stratification that does not vary with time. One can compute vertical and meridional geostrophic heat fluxes, but one can not take into account the feedback of those fluxes on the global stratification. There is therefore a gap between our understanding of the dynamical properties of geostrophic turbulence and its feedbacks on the density stratification. In future work on geostrophic turbulence, my objective will be to fill this gap.

Ideally, one would need to start from primitive (Boussinesq) equation, and to find asymptotic regimes for which one can derive a simpler model describing the two-way coupling between geostrophic turbulence and a slowly evolving background stratification. Previous attempts to derive such models have been faced with the following difficulties : i/ the models are derived from primitive equations using asymptotic analysis but break fundamental conservations laws of the original flow model ii/ or the models have good conservation laws but rely on ad hoc hypothesis. Promising results have been obtained with multiple scale analysis to describe coupling between planetary scale flows and local geostrophic turbulence, see e.g. [67], but one needs to assume a spatial scale separation that will not be satisfied whenever geostrophic turbulence lead to self-organization at the domain scale. We will build upon those previous work to explore regimes where one does not need to assume this spatial scale separation.

Given the difficulty of this task, one could develop in parallel a complementary approach, by devising an algorithm to compute stratification profiles consistent with both quasi-geostrophic dynamics and large scale forcing (surface heat flux and wind in the ocean, radiative balance in the atmosphere). The idea will be to use the output from quasi-geostrophic simulations to obtain a lateral geostrophic heat flux that will be included in another model for the vertical stratification. This second objective will allow us to go one step beyond classical geostrophic turbulence studies that consider the stratification independently from the forcing mechanisms. Both the asymptotic model from the first objective or the algorithm will make possible the computation of phase diagrams for the thermal structure and the flow patterns that emerge in response to a given external forcing. In particular, we propose to apply those tools to build phase diagrams of the thermal structure of oceanic currents in channel geometries (as the antarctic circumpolar current) in response to surface winds and thermal forcing, and to build phase diagrams of midlatitude tropospheres in response to radiative forcing. This approach will allow to obtain scaling laws for the thermal structure of those flows depending on forcing parameters. Those predictions will be tested against direct numerical simulations of primitive equations in idealized settings (Boussinesq hydrostatic equations in channel geometry).

A central question related to the interplay between quasi-geostrophic dynamics and large scale stratification is the possibility for baroclinic adjustment proposed by Stone in the seventies [172]. He argued that planetary flows tend to reach a state of marginal baroclinic instability as a result of this adjustment. This view has been challenged by the fact that observed or simulated flow can sometimes be supercritical with respect to baroclinic instability. A simplified model coupling geostrophic turbulence to stratification would help to explore different regimes of baroclinic adjustment. We intend for that purpose to consider the free evolution of an initially baroclinically unstable flow in numerical and laboratory experiment in rotating annulus.

Chapter 4

Emergence of vortical mean flows in the presence of surface or internal gravity waves

We have presented in the previous chapter results on self-organization of geostrophic turbulence into large scale potential-vortical flows in a quasi-geostrophic framework. The quasi-geostrophic model filters out inertia-gravity waves, taking advantage of a frequency gap between these waves and potential vortical modes. This is convenient for numerical computations and theoretical studies specifically devoted to the understanding of geostrophic turbulence properties. The drawback is that interactions between such inertia-gravity waves and potential-vortical flows can not be described in this framework. Yet wave-mean flow interactions are important in several ways. First, large scale vortices can spontaneously radiate their energy into of inertia-gravity waves, thus providing one route toward energy dissipation for geostrophic turbulence [185]. Second, nonlinear interactions between inertia-gravity waves can generate large scale vortical flows [32].

I have started to work on wave-mean flow interactions when I arrived at the physics lab ENS de Lyon in 2011. In collaboration with Thierry Dauxois and the internal wave team, I used weakly-nonlinear analysis to address the generation of a vortical mean flow through nonlinear interactions between internal waves in (non-rotating) stratified fluid [section 4.1]. Our main result was to show the importance of both viscous effects and three dimensional structure of the wave-beam. Since then, I have continued to work on wave-mean flow interaction problems, as part of the PhD thesis of Antoine Renaud. We have been focusing on the effect of streaming at boundaries [144]. Boundary streaming has long been known in acoustics [143], but had so far not been discussed in stratified fluids. In parallel, I also worked on the wave-mean flow interaction problem with the point of view of statistical mechanics, by deriving the equilibrium statistical mechanics theory for shallow-water equations, a two-dimensional flow model that support the presence of inertia-gravity (Poincaré) waves, in collaboration with A. Renaud and F. Bouchet. The theory predicts the energy partition between a large scale vortical flow and inertia gravity waves for a freely evolving, inviscid fluid [section 4.2]. Although the theoretical tools used in sections 4.1 and 4.2 are drastically different, both approaches describe the possible emergence of a large scale flow from nonlinear interactions between waves, and both approaches point to the important role of boundaries (bottom topography in the shallow-water case, an undulating wall in experiments).

As mentioned in the introduction, the importance of breaking discrete symmetries to induce motion in fluids has long been realized in the context of locomotion at low Reynolds numbers. In that case, only non-reciprocal deformation breaking both time reversal and reflexion symmetry of a body can lead to its unidirectional motion in the limit of zero Reynolds number [142, 164]. However, the case of motion at low Reynolds is peculiar as the fluid motion is slaved on the body shape. In geophysical context, the effect of breaking simultaneously time-reversal and mirror symmetry as been addressed by Leo Maas in the case of a rotating fluid in a close basin with a slopping boundary [107]: rotation breaks time-reversal symmetry, and make possible to propagation of inertial waves; the slopping boundary breaks reflectional symmetry, and makes possible the focusing of waves into an attractor.

I present in this chapter two different mechanisms of mean flow generation, that involve two different way of breaking time-reversal symmetry. In the case of internal wave streaming, time reversal symmetry is broken by the presence of viscous dissipation. In the case of vortex formation in shallow-water equilibria, we found that a large scale flow could emerge at equilibrium only in the presence of rotation, that also breaks time-reversal symmetry in the system.

4.1 Streaming instability by internal waves [PoF 2012]

This part is the result of a collaboration with T. Dauxois, P. Odier and S. Joubaud during the PhD thesis of G. Bordes [19]. It is adapted from the section 3.2 of a review to be published in 2018 [42]. I did not participate to the experiments, which were built, performed and largely analyzed by G. Bordes, with initially a different objective than the study of mean flow generation; my contribution has been to propose a model for this unexpected phenomenon, and I participated to data analysis and writing of the results. I currently work on wave-mean flow interactions in collaboration with A. Renaud, as part of his PhD thesis.

Internal gravity waves beams. Internal gravity waves play a primary role in geophysical fluids [174]: they contribute significantly to mixing in the ocean [211] and they redistribute energy and momentum in the middle atmosphere [59]. At a linear level, the peculiar properties of these waves are well understood, but their interactions through nonlinearities continue to yield surprises for physicists. One of the striking physical manifestation of nonlinear interactions between waves is the possible generation of strong mean flows, as described below.

Accoustic streaming. Internal gravity wave beams share several properties with acoustic wave beams [99, 3]. In particular, both kinds of waves may be subject to streaming in the presence of dissipative effects. Streaming refers here to the emergence of a slowly evolving, non-oscillating, Eulerian flow forced by nonlinear interactions of the oscillating wave-beam with itself [133]. As reviewed in [148], streaming occurs actually in a variety of flow models; it remains an active field of research for both theoretical [212] and experimental [129] points of view.

Non-acceleration theorem. The fact that dissipative effects are required to generate irreversibly a mean flow through the nonlinear interactions of a wave beam with itself can be thought of as a direct consequence of "non-acceleration" arguments that came up in the geophysical fluid dynamics context fifty years ago, see e.g. [184]. These ideas have played a crucial role in the understanding of the quasi-biennial oscillation, and have led to a now famous experimental idealization of this phenomenon by Plumb and Mc Ewan [139]. The emergence oscillations require more than one wave beam, but [138] discussed first how a single wave beam propagating in a vertical plane could generate a mean flow. He predicted the vertical shape of this mean flow, emphasizing the important role played by the wave attenuation through dissipative effects. The experiment by [139] may be thought of as the first quantitative observation of streaming in stratified fluids, and has recently been revisited by Semin and collaborators [163].

Generation of wave-driven vortical flows. Those examples correspond, however, to a very peculiar instance of streaming, with no production of vertical vorticity. By contrast, most applications of acoustic streaming involve the production of vorticity by an irrotational wave [133]. However, experimental observation of the emergence of a vortical flow in stratified fluids through this mechanism remained elusive until recently. While studying the internal wave generation process via a tidal flow over seamounts in a stratified fluid in three dimensions, [219] observed a strong flow in the plane perpendicular to the oscillating tidal flow, stressed the importance of nonlinearities for this process, without further explanations. A few years later, studying the reflection of an internal wave beam on a sloping bottom, Grisouard and his collaborators have also discovered this mean-flow generation in experiments of internal wave reflexion on a slopping bottom [64, 96, 66]. Comparing two- and three-dimensional numerical simulations, they showed that this mean flow is of dissipative origin.

Theoretical approaches. The complete and theoretical understanding of the generation of a slowly evolving vortical flow by an internal gravity wave beam was possible using an even simpler set-up that we describe in the following section. [19] reported observations of a strong mean flow accompanying a time-harmonic internal gravity beam, freely propagating in a tank significantly wider than the beam. We describe below in detail the experimental set-up and the observations, together with the theory by [19], which has been further developed by [86]. Those approaches bear strong similarities with the result obtained by [65], who used generalized Lagrangian mean theory, in order to describe

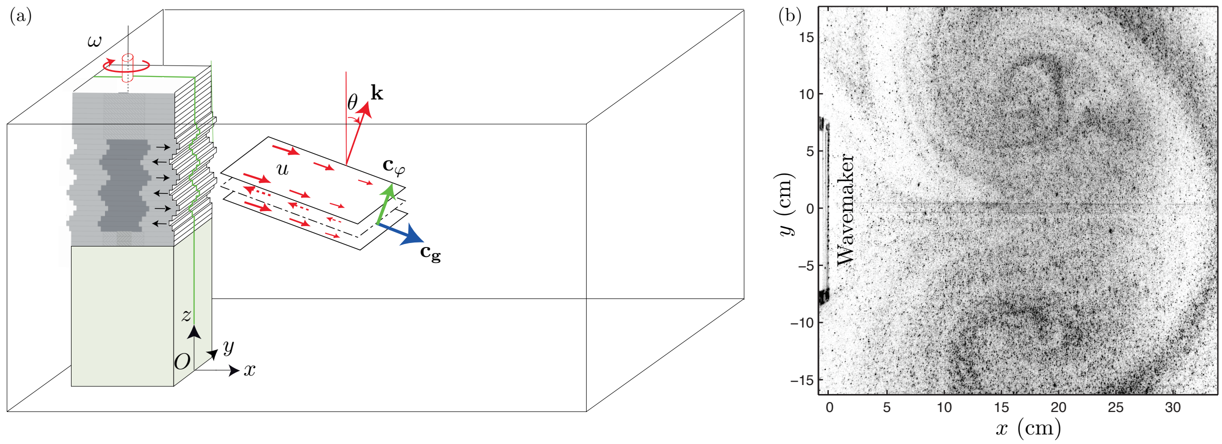

Figure 4.1: (a) Schematic representation of the experimental set-up with the generator on the left of the tank. The tank is filled with a linearly density stratified fluid. (b) Top view of the particle flow in a horizontal plane at intermediate depth.

the emergence of a vortical flow in the presence of an oscillating flow of a barotropic tide above topography variations.

Experimental observations. The experiment of [19] featured an internal gravity wave beam of limited lateral extent propagating along a significantly wider stratified fluid tank. Previously, most experimental studies that were using the same internal wave generator [121] were quasi-two-dimensional (beam and tank of equal width) and therefore without significant transversal variations.

Figure 4.1a presents a schematic view of the experimental set-up in which one can see the generator, the tank and the representation of the internal wave beam generated. The direct inspection of the flow field shows an unexpected and spontaneously generated pair of vortices, emphasized in **Figure 4.1b** by the tracer particles dispersed in the tank to visualize the flow field using particle image velocimetry. This structure is actually a consequence of the generation of a strong mean flow. This experiment provides therefore an excellent set-up to carefully study the mean-flow generation and to propose a theoretical understanding that explains the salient features of the experimental observations.

These observations are summarized in **Figure 4.2**, which shows side and top views not only of the generated internal wave beam but also of its associated mean flow. One sees that the wave part of the flow is monochromatic, propagating at an angle θ and with an amplitude varying slowly in space compared to the wavelength λ. These waves are accompanied by a mean flow with a jet-like structure, in the direction of the horizontal propagation of waves, together with a weak recirculation outside the wave beam. Initially produced inside the wave beam, this dipolar structure corresponds to the spontaneously generated vortex shown in **Figure 4.1b**. Moreover, the feedback of the mean-flow on the wave leads to a transverse bending of wave beam crests that is apparent in **Figures 4.2e** and **4.2f**.

The dynamical system. We consider an incompressible non rotating stratified Boussinesq fluid in Cartesian coordinates ($\boldsymbol{e}_x$,$\boldsymbol{e}_y$,$\boldsymbol{e}_z$) where $\boldsymbol{e}_z$ is the direction opposite to gravity. The Boussinesq approximation amounts to neglecting density variations with respect to a constant reference density $\rho_{\rm ref}$, except when those variations are associated with the gravity term g. The relevant field to describe the effect of density variations is then the buoyancy field $b_{\rm tot} = g\left(\rho_{\rm ref} - \rho\right)/\rho_{\rm ref}$, with $\rho(\mathbf{r}, t)$ the full density field, $\boldsymbol{r}$=(x,y,z) the space coordinates and t the time coordinate. Let us call $b_0(z)$ the buoyancy of the flow at rest. We assume for the sake of simplicity that the buoyancy profile is linear, with buoyancy frequency $N = (\partial_z b_0)^{1/2}$. The equations of motion can be written as a dynamical system for

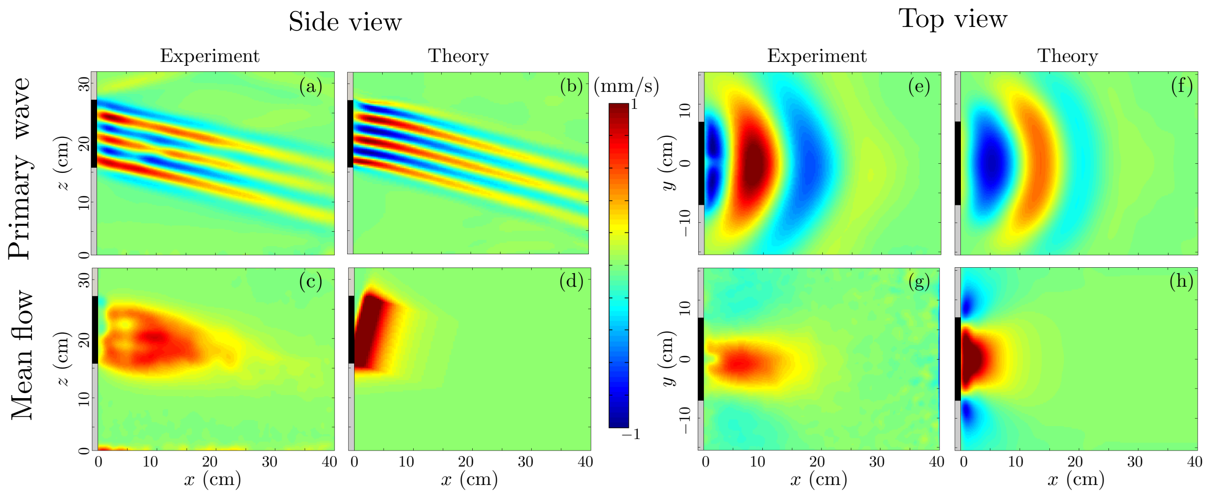

Figure 4.2: Experimental (a,c,e,g) and theoretical (b,d,f,h) horizontal velocity fields u_x for the primary wave (top plots, obtained by filtering the velocity field at the forcing frequency) and the mean flow (bottom plots, obtained by low-pass filtering the velocity field) as reported respectively in [19] and [86]. The four left panels present the side view, while the right ones show the top view. The wave generator is represented in grey with its moving part in black. Future work will be needed to know if the wave pattern is primarily due to diffraction (as suggested by preliminary results), or to the feedback of the mean flow on the waves.

the perturbed buoyancy field $b = b_{\text{tot}} - b_0$ and the three components of the velocity field $\boldsymbol{u} = (u_x, u_y, u_z)$:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + b\boldsymbol{e}_z + \nu \nabla^2 \boldsymbol{u}, \tag{4.2}$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + u_z N^2 = 0. \tag{4.3}$$

with $p(\boldsymbol{r}, t)$ the renormalized pressure variation with respect to the hydrostatic equilibrium pressure, and ν the kinematic viscosity. We have neglected the molecular diffusivity, a common assumption for laboratory experiments using salt as the stratifying element.

A preliminary multiple scale analysis. Taking advantage of the physical insights provided by the experiments, we have proposed an approximate description that uses a time-harmonic wave flow with a slowly varying amplitude in space. The problem contains two key non-dimensional numbers, the Froude number $U/\lambda N$ and the ratio $\nu/\lambda^2 N$ between the wavelength λ and the attenuation length scale of the wave beam due to viscosity, $\lambda^3 N/\nu$. For analytical convenience, we considered a distinguished limit with the small parameter $\varepsilon = Fr^{1/3}$, together with the scaling $\nu/\lambda^2 N = \varepsilon/\lambda_\nu$ where $\lambda_\nu \sim 1$. As usual, the appropriate scaling in the small parameter ε is deduced from a mix of physical intuition and analytical handling of the calculations. In our case, we were looking for a regime with small nonlinearity and dissipation. In terms of the velocity components (u_x, u_y, u_z), the buoyancy b and the vertical vorticity $\Omega = \partial_x u_y - \partial_y u_x$, the governing dimensionless equations in this three-dimensional setting read now

$$\nabla_H \cdot \boldsymbol{u}_H = -\partial_z u_z, \tag{4.4}$$

$$\partial_t b + \varepsilon^3 (\boldsymbol{u} \cdot \nabla b) + u_z = 0, \tag{4.5}$$

$$\partial_t \Omega + \varepsilon^3 \left(\boldsymbol{u}_H \cdot \nabla_H \Omega + (\nabla_H \cdot \boldsymbol{u}_H)\, \Omega + \partial_x (u_z \partial_z u_y) - \partial_y (u_z \partial_z u_x) \right) = \varepsilon \lambda_\nu^{-1} \nabla^2 \Omega, \tag{4.6}$$

$$\nabla^2\partial_{tt}u_z + \nabla_H^2 u_z = \varepsilon\lambda_\nu^{-1}\nabla^4\partial_t u_z - \varepsilon^3\left(\partial_t\left(\nabla_H^2\left(\boldsymbol{u}\cdot\nabla u_z\right) - \partial_z\nabla_H\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}_H\right)\right) + \nabla_H^2\left(\boldsymbol{u}\cdot\nabla b\right)\right), \tag{4.7}$$

in which the index H in $\boldsymbol{u}_H$ and ∇_H reduces to the horizontal velocity field, gradient or Laplacian operator.

Introducing rescaled spatial and time coordinates, a multiple scale analysis is now at hand. Looking for a flow field in perturbation series $u_r = u_r^0 + \varepsilon u_r^1 + o(\varepsilon)$ for $r = x, y$ or z with a priori $u_y^0 = 0$ as suggested by the structure of the beam, together with the vertical vorticity field $\Omega = \varepsilon^2\Omega_2 + \varepsilon^4\Omega_4 + o(\varepsilon^4)$, a tedious but straightforward application of the multiple scale framework (with $x_i = \varepsilon^i x$ and $t_i = \varepsilon^i t$) gives then, to the first three orders, the structure of the beam: the first order ε^0 provides the expressions for u_x^0 and u_z^0, the second order ε^1 gives the expression for u_y^1 and finally order ε^2 shows that u_z^0 does not depend on the slow timescale t_2.

Nonlinear terms contribute a priori for the first time to order ε^3, but one interestingly finds that they vanish to this order, which is a remarkable properties of internal gravity wave already noticed in different contexts. To order ε^4, one obtains that the term independent of the slow time t_0 vanishes and, thus, nonlinear terms do not induce a mean flow to this order either. It is only to order ε^5 that nonlinear terms directly contribute to the mean-flow generation. The governing equation of the vortical flow induced by the mean flow is then given in the original dimensional units by

$$\partial_t\overline{\Omega} = \frac{\partial_{xy}\ ,\mathcal{U}^2}{(2\cos\theta)^2} + \nu\nabla^2\overline{\Omega} \tag{4.8}$$

where the overline stands for the filtering over one period and $\mathcal{U}(x, y)$ is the amplitude of the wave envelope. This equations leads to the following conclusion concerning streaming by internal waves:

- i) As emphasized by the first term on the right-hand side, nonlinear terms are crucial as a source of vertical vorticity.
- ii) The variations of the wave field in the y-direction (implying $\partial_y \neq 0$) are necessary for nonlinearities to be a source of vertical vorticity. This illuminates why three-dimensional effects are crucial and therefore why no mean-flow generation was noticed in two dimensions.
- iii) Finally, the viscous attenuation of the wave field is also necessary to produce vertical vorticity.

A self-consistent approach. One drawback of the above approach, however, is that it does not describe the feedback of the mean flow on the waves. This effect has been taking into account through a careful asymptotic expansion by [86], building on our previous work. Their analysis has been successfully compared to our experimental results, as seen in figure 4.2. It is argued in [86] that taking into account the feedback of the mean flow on the waves explains the bended pattern reported Guilhem Bordes experiments. Based on preliminary computations taking into account the finite size of the wave beam (and hence diffraction effect), I think that a complementary explanation for these patterns can also be obtained in a purely linear framework. This remains to be addressed in future work.

Boundary streaming. None of the previous approaches have discussed the particular role of boundaries. This is surprising, given that the generation of waves usually involve an undulating wall. While the role of viscous boundary layers in acoustic streaming has thoroughly been addressed, this remains largely unexplored in the case of internal waves. During his PhD thesis, A. Renaud has computed the mean-flow generated close to an undulating wall that emits internal waves in a viscous, linearly stratified two-dimensional Boussinesq fluid. Using a quasi-linear approach, we have shown that the mean-flow behavior depends strongly on the boundary conditions, and found good agreement with numerical simulations. We have applied these computations to an idealized model for the quasi-biennial oscillation, and found that the presence of boundary layers have a significant impact on the period of flow reversals within the domain bulk [144].

References (with link to the papers):

Bordes, G., Venaille, A., Joubaud, S., Odier, P., and Dauxois, T. (2012). Experimental observation of a strong mean flow induced by internal gravity waves. Physics of Fluids, 24(8), 086602.

Renaud, A., Venaille, A. (2017). Boundary streaming by internal waves, arXiv:1708.00068.

Dauxois, T., Joubaud, S., Odier, P., Venaille, A. (2018). Instabilities of internal wave beams, Accepted for Annual Review of Fluid Mechanics

4.2 Energy partition in the shallow-water model [JSP 2016]

We have seen in the previous subsection that weak nonlinear interactions between gravity waves in the presence of viscosity can lead to the emergence of vortical flows. Here we ask wether the spontaneous emergence of a large scale flow can also result from the inviscid dynamics of waves interacting together. We also ask about the possibility of the opposite phenomenon, namely the spontaneous radiation of waves from a large scale vortical flow. These issues have been addressed in a variety of contexts, from laboratory experiments to observations, and are usually tackled in the framework of weakly nonlinear dynamics. Here we address these problems using a novel point of view provided by equilibrium statistical mechanics. Useful physical insights can be obtained in this simplified framework, even if the equilibrium theory applies only to situations without forcing and dissipation, such as initial value problems, under the assumption of ergodicity.

In the third chapter, we used a generalization to continuously stratified quasi-geostrophic flows of the statistical mechanics theory initially developed by Miller-Robert-Sommeria for two-dimensional incompressible flows. From a fundamental point of view, such generalizations were relatively straightforward, since the quasi-geostrophic dynamics is expressed as the advection of a tracer, the potential vorticity, by an incompressible velocity field and the energy in these model goes to large scales, just as in two-dimensional Euler flows. By contrast, it has long been known that equilibrium states of three-dimensional turbulent flows correspond to a situation where all the energy lost at small scales, consistently with the observed direct energy cascade in three-dimensional turbulent flows. It is then natural to ask wether the Miller-Robert-Sommeria approach can be derived in models that are intermediate between two-dimensional turbulence and three-dimensional turbulence. This was for instance done by Thalabard and collaborators for axisymmetric Euler equations [177]. Our contribution has been to derive the equilibrium theory of for the shallow-water model using large deviation tools. I started to work on this subject in collaboration with F. Bouchet in 2012, and I co-supervised with him the one-year internship of A. Renaud in 2014 on this problem.

Shallow water equations. The shallow-water model (équations de Saint-Venant) describes the dynamics of a thin layer of fluid with height $h(x, y)$, and a two-dimensional velocity field $\mathbf{u}(x, y) = (u, v)$ depending only on the horizontal coordinate (x, y):

$$\partial_t h + \nabla(\mathbf{u}h) = 0$$

$$\partial_t \mathbf{u} + \mathbf{u}\nabla\mathbf{u} + f\mathbf{k} \times \mathbf{u} = -g\nabla\left(h - h_b\right),$$

where g is gravity and f is the Coriolis parameter (twice the rotation rate). It is the simplest geophysical flow model sharing common properties with two-dimensional turbulence (conservation of Casimir Functionals, related to Lagrangian conservation of potential vorticity), and properties of three-dimensional turbulence (possible energy cascade towards small scales and shocks due to the presence of inertia-gravity waves, in contrast with quasi-geostrophic models).

Potential vorticity, inertia-gravity waves and dynamical invariants. When linearized around a state of rest, this flow model admits propagating waves called inertia-gravity waves (or Poincaré) waves, and zero frequency modes called potential vortical modes (see chapter 6 for more details on these linear modes). These modes are convenient to discuss low energy solutions, but are not appropriate to treat the general case, where more relevant decompositions and combinations of the different fields can be found. However, small scale fluctuations of energy can always be interpreted as a combination of inertia-gravity wave modes, and I will use this term for convenience when referring to these fluctuations. The relevant field describing the large scale flow is the potential vorticity $q = (\nabla \times \mathbf{u} + f)/h$. This is Lagrangian tracer: each fluid particle (of constant volume) carry a prescribed value of potential vorticity. Thus the global volume distribution of potential vorticity is conserved. This conservation law is equivalent to the conservation of Casimir functionals. The other crucial conserved quantity is the total energy, defined as the sum of kinetic and potential energy.

Objective. The first aim of our work was to compute the most probable macrostate (defined through a coarse-graining procedure) of this dynamical system for a given set of constraints provided by dynamical invariants, assuming equiprobability of all microscopic states satisfying these constraints. The microscopic configuration are described by a triplet of field, for instance the height field and the two velocity components, or the potential vorticity field, the divergence field and the height field. If the dynamics were ergodic (presumably a consequence of its turbulent evolution), then any initial condition satisfying the constraint of the problem would be attracted towards this most probable macrostate. The second aim of our work was to compute how much energy of the equilibrium states is carried by a large scale vortical flow, the other part being carried by (sub-grid) small fluctuations corresponding to inertia-gravity waves. In that respect, the theory can predict spontaneous inertia-gravity wave radiation from an initial large scale flow. Previous attempts to compute equilibria of the shallow-water model failed to answer to this second question (unless in very specific limiting cases), mostly because of strong theoretical obstacles.

Difficulty. The main difficulty was due to the fact that microscopic configurations are described by continuous fields, one of them being a space variable (the height h), which makes impossible a discretization of the other fields on a uniform horizontal grid (such procedure would break volume conservation of fluid elements). There is no such problem for two-dimensional Euler or quasi-geostrophic models, in which case microscopic configurations are fully described by the potential vorticity, which can be discretized on a uniform grid, each grid corresponding to a fluid particle. A second difficulty was how to handle the expression of the total energy. In two-dimensional or quasi-geostrophic models, all the energy is carried by a large scale flow, and the mean-field approach neglecting the effect of potential vorticity fluctuations on this energy is exact. Because of the presence of inertia-gravity waves that can carry energy at small scale, the problem is much more difficult for the shallow-water model. A third difficulty, which is related to the others, was to find the relevant phase space variables, that must satisfy a Liouville theorem (otherwise the assumption of equiprobability at one time has no reason to be valid at another time), and that must be such that the dynamical constraints are easily expressed in terms of these quantities. For instance, the potential vorticity is a natural variable to consider as Casimir functionals are easily expressed in terms of this quantity, but the energy is not simply expressed in term of this potential vorticity field.

Method. Guided by the previous considerations, we have built a discrete version of the continuous shallow-water fields, by describing these fields in terms of N fluid particles, by expressing the constraints in terms of these discretized fields, and by defining a coarse-graining procedure. Once the discretized model was introduced, the computation of the microcanonical measure (and hence of the equilibrium states) was performed by considering the thermodynamics limit, using large deviation tools, without the need for any further assumptions. The only obscure point that will need to be clarified in future work is

- i) the fact that we were not able to conserve simultaneously the notion of fluid particle of equal volume and a discretization consistent with the Liouville theorem
- ii) the fact that we postulated a mean-field form for the energy. Both points could be only technical details if different discretization procedures would lead to the same result, but this is not the case (see the discussion on Weichman approach below). We only can justify these hypothesis a posteriori, as we have recovered known limiting cases once the thermodynamics limit has been taken.

Comparison to previous contributions. Prior to our work, they were only a very limited number of results on statistical equilibrium states of the shallow-water system. Warn studied the equilibrium states "a la Kraichnan" in Galerkin-truncated shallow-water models, focusing to a weak flow limit in the energy-enstrophy ensemble [204]. He discussed a low energy limit, and gave a number of illuminating physical consequences of these computations, one of them being that no mean flow could be sustained at equilibrium. We have generalized this result, and shown that when rotation and bottom topography are taken into account, a large scale flow can be obtained as equilibrium. Chavanis and Sommeria proposed a generalization to the shallow-water case of the variational problem given by the Miller-Robert-Sommeria theory for two-dimensional Euler flows, but without deriving their result from a microscopic model, and they did not take into account the presence of small scale fluctuations [37]. Our study confirm the form of the variational problem proposed by Chavanis and Sommeria, but restrict its validity to a certain range of parameters.

An competing theory by P. Weichman. Our approach takes a different point of view than [206], which has ben revisited in [205] after publication of our own work. In both papers, the authors compute a class of statistical equilibrium states of the shallow-water system, starting from a grand canonical distribution. The main difference with our approach is that P. Weichman considered a discretized model consistent with Liouville theorem (in the second paper), but inconsistent with volume conservation of fluid particles, while we have made the opposite choice. None of the approach are fully satisfactory. We stress, however, that our approach has the interest of yielding simple results, such as a self-consistent mean-field treatment of the equilibria, with large scale flows that are stationary at equilibrium. This contrats with Weichman's theory, that yields at the end to an equilibrium state that can not be described through a mean-field approach. In that case, there is no decoupling between the different dynamical fields at small scale, and correlations between these fields are such that the coarse-grained flow is not a stationary state of the shallow-water system, which mean that the equilibrium state is not stable to coarse-graining.

Phase diagrams for the energy partition in the presence of bottom topography. We show in figure 4.3 an exemple of phase diagram for the energy partition between fluctuation E_{fluc} (inertia gravity wave modes) and mean flow E_{mf} (vortical modes), in the presence of both rotation and bottom topography. Panel a shows isolines of E_{fluc} as dashed lines in (Z_2, E_{mf}) space, with Z_2 the potential enstrophy of the microscopic flow.

Let us focus on the case $Z_2 = Z_b$, which corresponds to the potential enstrophy of an initial condition without potential vortical flow. One can vary the energy of such an initial condition by exciting inertia-gravity wave modes only. We see on panel (b) that in a low energy limit, there is one fifth of the total energy that is carried by the mean vortical flow once the system has reached equilibrium. This result is remarkable, as it shows the spontaneous generation of a mean large scale flow from an initial conditions containing only surface waves.

Let us now consider an initial condition with energy much larger than E_{mix}, whatever the value of the initial potential enstrophy Z_2. We see that most of the energy of this initial condition is radiated into inertia-gravity waves (fluctuations) once the equilibrium is reached. This shows, as in Chapter 3, the crucial importance of the enstrophy in selecting the final repartition of the energy in the system.

Can we observe shallow-water equilibria? One of the interest of this work is to revisit the foundations of equilibrium statistical mechanics for flow models in exotic configurations. In addition, the results provide a framework that may be useful for phenomenological parameterizations (for instance in the context of adjustment towards geostrophic equilibrium). However, and as already envisioned in the work of Warn [204], followed by numerical results by Farge and Sadourny [53], the applicability of the equilibrium theory to actual shallow-water flows will be limited. First, the formation of shocks breaks conservation laws, so a theory based on these conservation laws should be taken with care: the equilibrium theory provides in that case at best a tendency for the flow evolution. Second, the equilibrium results are by essence independent from the dynamical properties of the flow model. By contrast, the time scale required to reach equilibrium depends on the dynamics. This means that weak coupling between vortical modes and inertia-gravity wave mode can lead to relaxation time scale towards equilibrium that are way too large to observe relaxation towards equilibrium in practical situations [53].

Quasi-equilibrium states in weakly coupled flows. Qualitatively, and beyond the particular case of the shallow-water model, increasing the complexity of a flow model by adding additional dynamical fields weakly coupled to the original one, increase phase space volume, but the new available part of phase space may not be easily explored if it is almost disconnected from the part of phase space where is located the initial condition. Other instance of weakly coupled fields is given by a two-layer quasi-geostrophic in the presence of bottom topography, with a large internal Rossby radius (strong stratification). In the absence of boundaries, equilibrium states of the upper layer flow isolated from the lower layer are necessarily negative temperature states. By contrast, in a low energy limit, the equilibrium state of the global system is a positive temperature state (a Fofonoff mode with the velocity following topography

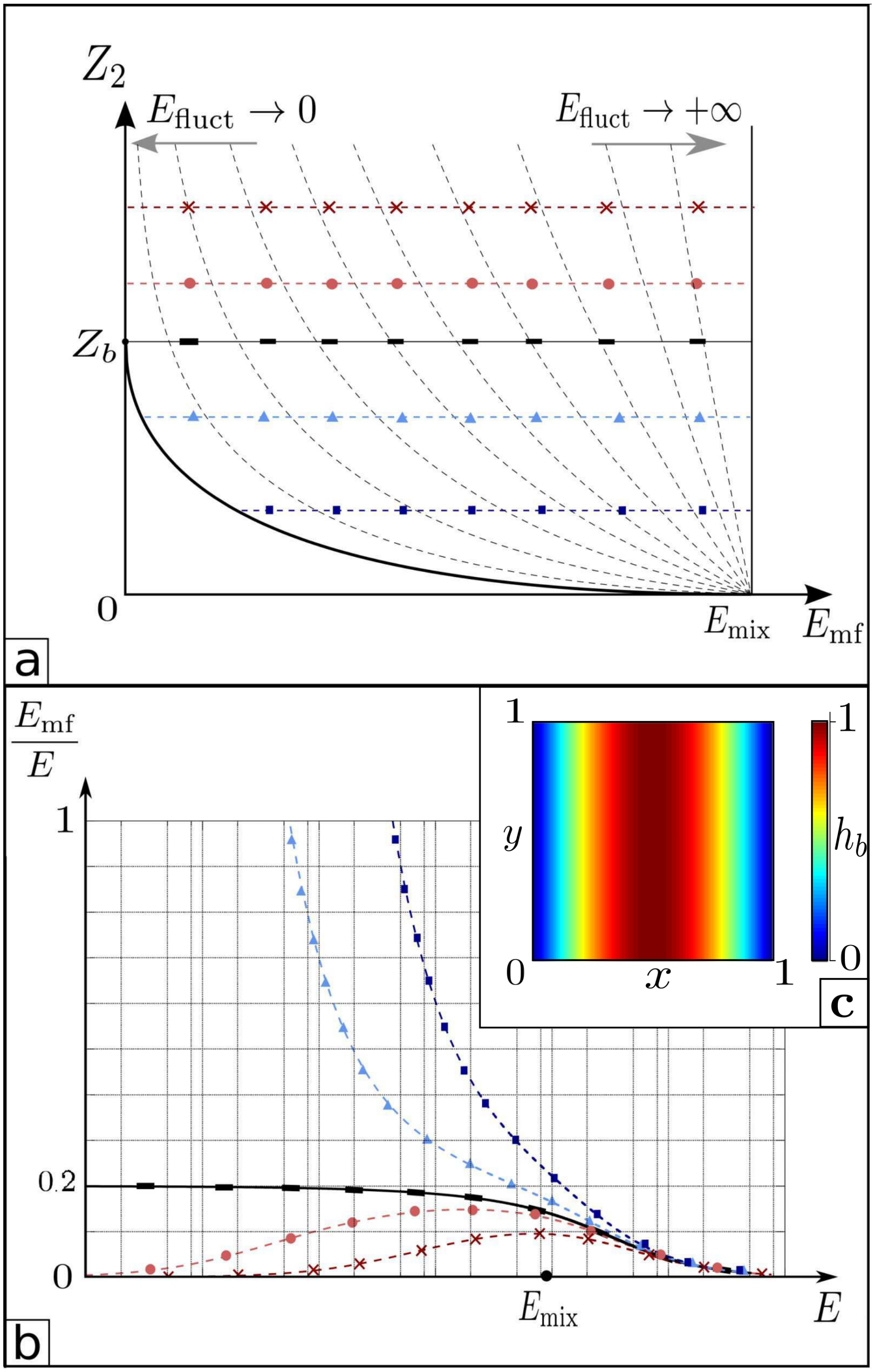

Figure 4.3: Phase diagram of the energy-enstrophy ensemble for the shallow-water model in a small Rossby (large rotation) limit. a) (Z_2 is the microscopic enstrophy, $Z_b$the quadratic norm of topography, corresponding to the microscopic enstrophy when there is no mean flow, E_{mf} is the energy of the mean flow, E_{fluc} is the energy of fluctuations (inertia-gravity waves), E_{mix} is the maximum reachable energy for the mean-flow. The dashed lines corresponds to isotherms ($E_{fluct} = cst$). b) Energy partition at equilibrium with $E = E_{mf} + E_{fluct}$ the total energy. Each curves correspond to a given value of Z_2. c) $h_b = \sin(2\pi x/L_x)$. Reproduced from [145]

contours). More generally I think that a study on the dynamics of weakly coupled sub-systems that are initially in quasi-equilibrium states with different temperatures deserves further studies in the context of geostrophic turbulence.

Reference (with link to the paper):

Renaud, A., Venaille, A., and Bouchet, F. (2016). Equilibrium statistical mechanics and energy partition for the shallow-water model. Journal of Statistical Physics, 163(4), 784-843.

4.3 Outlook

A common point between the different approaches presented in this chapter is that topography (in the ocean) or undulating boundaries (in laboratory experiments) play a central role either to generate waves or to make possible the emergence of large scale flow structures. Our long term goals are fourfolds:

- First, there exist experimental observations of vortical mean flows generated in a direction opposite to the one observed in Guilhem Bordes experiments (unpublished experimental results by [Lasbleis Joubaud, Odier] and [Horne, Joubaud, Odier]). Recently Felix Beckebanze went further in the analysis of internal wave streaming mechanism to explain these experimental observations. It will be interesting to see if mean-flow reversal can be also related to spontaneous instability of the primary wave beam, as reported recently in the case of surface gravity waves [141].
- Second, we would like to address the role of the dynamics close to the boundaries in model experiments of the quasi-biennial oscillations [139, 163], which describes cyclic reversal of a large scale horizontal flow in the presence of an oscillating bottom boundary (project with T. Dauxois and T. Akylas).
- Third, we would like to address the interplay between internal waves and geostrophic flows above random bottom topography which are thought to be important in the antarctic circumpolar current. This is the subject of an ongoing collaboration with A. Renaud and L. P. Nadeau (Rimouski, Quebec), using direct numerical simulations of primitive equations).
- Fourth, the role of symmetry breaking in the problem of boundary streaming through internal waves generation bears strong similarities with phenomena encountered in acoustic on the one hand, and with locomotion at low Reynolds number on the other hand. There could be interesting connections to be established found in future work.

Chapter 5

What sets the mixing efficiency of stratified turbulence?

Internal waves and mixing. In the previous chapter, we have described how and when nonlinear interactions between internal or surface gravity waves give rise to strong mean flows. This phenomenon corresponds to a transfer of energy from small scales to large scales. As far as geophysical applications are concerned, another important aspect of internal waves in stratified fluids is their ability to provide a source of energy for small scale mixing when they break [2]. The flow dynamics after a wave-breaking event is fully turbulent, and a natural question is to determine how much of the energy released by the wave will effectively be used to irreversibly mix the stratification [136, 80]. This is the longstanding problem of mixing efficiency in stratified turbulence, which goes beyond the particular case of internal wave breaking. Whatever the source of turbulence, these processes usually occur at scale much smaller than current model resolution for large scale circulation. They must therefore be parameterized, and their is an increasingly body of evidence that different parameterizations for these small scale processes can have important impact on the large scale flow structure. For instance, small scale mixing is thought to be an important ingredient of the lower (deep) cell of the meridional overturning circulation [113]. In this context, a crucial question is to determine how mixing efficiency varies with key external (macroscopic) parameters of the problem are changed [114].

A statistical mechanics approach. I started to work on the problem of mixing in stratified fluid as a master student in 2005, with Joel Sommeria, who proposed to use tools of statistical mechanics approach for such problems. I have chosen to present this work in this habilitation thesis because it is only over the last few years that I reformulated our initial results (together with those of E. Tabak and F. Tal, also based on statistical mechanics ideas) in order to quantify mixing efficiency. This work benefited from a fruitful collaboration with LMFA group around Louis Gostiaux at Ecole Centrale Lyon, from 2012 to now. My contribution in this collaboration has been to propose possible tests of the statistical mechanics predictions in numerical experiments. The numerical simulations have been performed and analysed by Ernesto Horne (post-doc), mostly supervized by Alexandre Delache and Louis Gostiaux. I have also proposed a novel interpretation of mixing efficiency in lock-exchange laboratory experiments, guided by statistical mechanics (ongoing work). The experiments have been designed, performed and analyzed by Diane Micard during her PhD supervized by Louis Gostiaux.

5.1 A statistical mechanics approach to mixing in stratified fluids [JFM 2016]

This part is adapted from a conference proceeding [191] and summarizes the work presented in Ref. [192].

Objective and methods. Predicting how much mixing occurs when a given amount of energy is injected into a Boussinesq fluid is a longstanding problem in stratified turbulence. Here we address this problem with the point of view of equilibrium statistical mechanics. Assuming random evolution through turbulent stirring, the theory predicts that the unforced, inviscid, adiabatic dynamics is attracted towards a state characterized by wild small scale velocity fluctuations carrying kinetic energy, and by a smooth buoyancy profile superimposed with wild small scale buoyancy fluctuations. It is then possible to compute how much of the injected energy has been irreversibly lost into small scale kinetic energy, the remaining part being used to irreversibly raise the potential energy of the system. This yields to quantitative predictions for a global, cumulative mixing efficiency in freely evolving configurations. The central idea of this work is that one does not need to consider molecular effects to describe irreversible mixing, which occurs here at a statistical level.

Mixing efficiency in decaying experiments. The traditional approach to estimate the efficiency of mixing in stratified turbulence involves direct analyses of the diffusive destruction of small scale buoyancy variance, that may be represented by the time derivative $\mathcal{M}$ of base-state potential energy plus a small correction due to the action of molecular diffusion on the initial stratification, a correction that become negligible in the limit of high Reynolds number. The time dependent efficiency of turbulent mixing may then be computed from direct numerical or laboratory experiments as $\eta_{inst} = \mathcal{M}/(\mathcal{M}+\epsilon)$ where ϵ is the viscous dissipation of kinetic energy in the flow; see e.g. [136]. This definition of mixing efficiency is global in space since the computation of the base-state potential energy requires a rearrangement of the fluid particle at the domain scale. Using a number of additional assumptions, it may be related to a local mixing efficiency that is often used in oceanography to model an effective diffusivity for diapycnal mixing [135]. In decaying experiments, one can define a cumulative mixing efficiency $\eta = \int_0^{+\infty} \mathrm{d}t\mathcal{M}/\int_0^{+\infty} \mathrm{d}t\,(\mathcal{M}+\epsilon)$, which measures how much of the total injected energy has been used to irreversibly raise the potential energy of the flow in the experiment. Our point is to use statistical mechanics to predict this global, cumulative mixing efficiency in the limit of large Reynolds and Péclet numbers (weak molecular effects).

The dynamical system. We consider an inviscid incompressible Boussinesq fluid that takes place in a three-dimensional domain $\mathcal{V}_{\mathbf{x}}$ of volume V. Spatial coordinates are denoted $\mathbf{x} = (x, y, z)$, with $\mathbf{e}_z$ the vertical unit vector pointing in the upward direction. At each time t the system is described by the buoyancy field $b(\mathbf{x}, t)$ and by the velocity field $\mathbf{u}(\mathbf{x}, t) = (u, v, w)$, which is non-divergent. The unforced, inviscid, adiabatic dynamics is given by

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\varrho_0}\nabla P + b\mathbf{e}_z\ , \quad \nabla\cdot\mathbf{u} = 0, \tag{5.1}$$

$$\partial_t b + \mathbf{u}\cdot\nabla b = 0\ . \tag{5.2}$$

Equilibrium theory. The first step before computing equilibrium states of this dynamical system is to define what is a *microscopic configuration* of the system, which requires to identify the relevant phase space that satisfy a Liouville theorem. This ensures that the flow in phase-space is non-divergent. Consequently, if all microscopic states are equiprobable at a given time, they remain equiprobable through the flow evolution. The quadruplet of fields $b, \mathbf{u}$ satisfies such a Liouville theorem. The second step is to identify relevant dynamical invariants, which are here the total energy and the global distribution of buoyancy levels. The third step is to describe the system at a macroscopic level, which involve a coarse-graining procedure depicted in figure 5.1: the fluid is discretized at a microscopic level into a uniform fine-grained grid, and at a macroscopic level into another uniform grid, with a grid mesh much larger than the one of the fine grained grid, so that each macro-cell contains a huge number of micro-cells. We take first the continuum limit for the fine-grained grid, and second the continuum limit for the coarse-grained grid. The final result does not depend on the coarse-graining procedure. Each microscopic state $(b(\mathbf{x}), \mathbf{u}(\mathbf{x}))$ is described at a macroscopic level by the pdf $\rho(\mathbf{x}, \sigma, \mathbf{v})$, which can be interpreted as the local volume proportion of fluid particles carrying the buoyancy level σ and velocity level $\mathbf{v}$ within each macro-cell of figure 5.1. Several useful macroscopic fields can be deduced from ρ, such as the macroscopic buoyancy field $\overline{b}(\mathbf{x}) = \int_{\mathcal{V}_\sigma} \mathrm{d}\sigma \int_{\mathcal{V}_\mathbf{v}} \mathrm{d}\mathbf{v}\ \rho\sigma$ and the local eddy kinetic energy field $\overline{\mathbf{u}^2}/2 = \int_{\mathcal{V}_\sigma} \mathrm{d}\sigma \int_{\mathcal{V}_\mathbf{v}} \mathrm{d}\mathbf{v}\ \rho\mathbf{v}^2/2$.

A counting argument. The interest of considering the probability field ρ for a macroscopic description of the system is that global constraints provided by dynamical invariants can be expressed in term of this quantity. This field describes wild small scale velocity and buoyancy fluctuations, superimposed to large scale variations. We stress that in a real fluid, these small scale fluctuations would be eventually dissipated by molecular effects. One may therefore interpret the small scale velocity fluctuations at equilibrium described by the probability field ρ as the total amount of

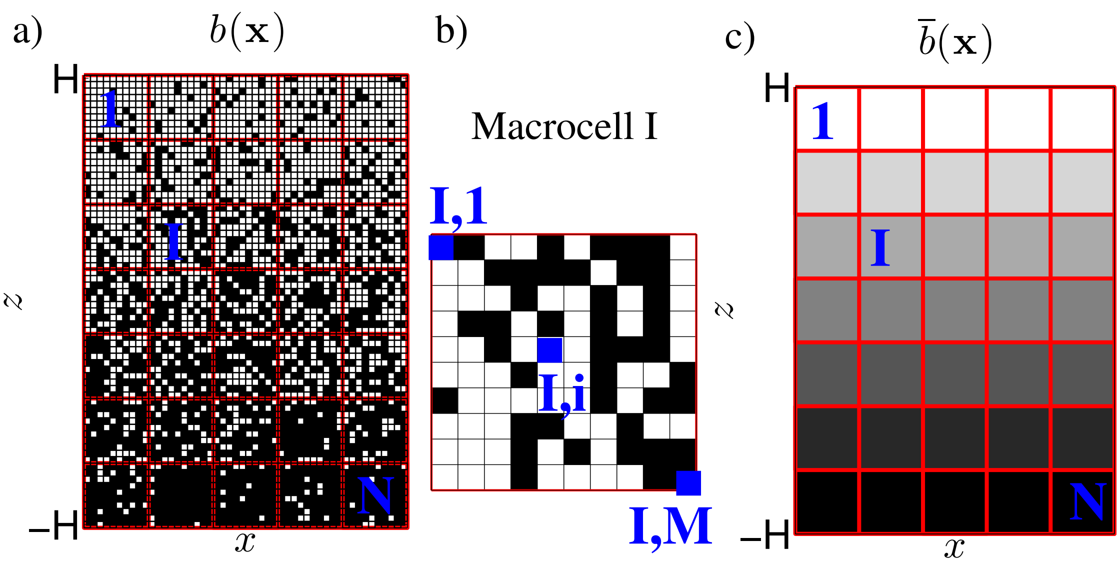

Figure 5.1: a) A microscopic configuration of the discretized buoyancy field $b(\mathbf{x})$, defined on a uniform *fine-grained* grid containing. b) Zoom on a single macrocell, containing M fluid particles. c) The macroscopic buoyancy field $\bar{b}(x)$ is defined on the uniform *coarse-grained* grid (red colour), and is computed by averaging the microscopic buoyancy field within each macrocell. The macroscopic state is fully characterized by the distribution ρ of buoyancy levels within each macrocell.

kinetic energy dissipated during the flow evolution. The last step is to count how many microscopic configurations are associated with a given macrostate ρ, and to show that there is a concentration of an overwhelming number of microscopic states close to the most probable macrostate in a given microcanonical ensemble (i.e. for a given set of constraints). It is shown in [145] that this most probable ρ state maximises a mixing entropy:

$$\mathcal{S} = -\int_{\mathcal{V}_{\mathbf{x}}} \mathrm{d}\mathbf{x} \int_{\mathcal{V}_{\mathbf{v}}} \mathrm{d}\mathbf{v} \int_{\mathcal{V}_{\sigma}} \mathrm{d}\sigma \, \rho \ln \rho \, , \tag{5.3}$$

while satisfying the constraints of the problem given by

$$\mathcal{E}[\rho] = \int_{\mathcal{V}_{\mathbf{x}}} \mathrm{d}\mathbf{x} \int_{\mathcal{V}_{\mathbf{v}}} \mathrm{d}\mathbf{v} \int_{\mathcal{V}_{\sigma}} \mathrm{d}\sigma \, \rho \left(\frac{\mathbf{v}^2}{2} - \sigma z \right) \, , \mathcal{G}_{\sigma}[\rho] = \int_{\mathcal{V}_{\mathbf{x}}} \mathrm{d}\mathbf{x} \int_{\mathcal{V}_{\mathbf{v}}} \mathrm{d}\mathbf{v} \, \rho \, . \tag{5.4}$$

together with a normalization constraint. The derivation of this variational problem is a direct application of Miller-Robert-Sommeria theory. This generalizes previous results by [175], who derived a similar expression, but without taking into account the degrees of freedom associated with the velocity field, which was interpreted directly as a thermostat. We stress, however, that the present approach is not fully satisfactory, as we have not taken into account constraints given by potential vorticity conservation. In the absence of rotation, there is little doubt that adding these constraints would not change the results (since they can not prevent a direct energy cascade of the kinetic energy), but this remained to be shown properly.

Properties of the equilibria. The equilibrium states are characterized by several important properties:

- The distributions of buoyancy levels b and velocity levels $\mathbf{u}$ are independent.
- The variance of velocity fluctuations and the variance of buoyancy fluctuations are proportional, and their ratio varies linearly with the local mean buoyancy gradient.

- The predicted velocity distribution is Gaussian, isotropic and homogeneous in space. It is therefore fully characterized by its variance, namely the local eddy kinetic energy $e_c \equiv \overline{\mathbf{u}^2}/2$.

Irreversible mixing in adiabatic fluids. In order to introduce a relevant definition of mixing efficiency in the context of an inviscid, adiabatic flow model, we need to consider two essential results stemming from the equilibrium theory. First, the most probable macrostate is an attractor for the dynamics, according to statistical mechanics predictions: convergence of microscopic configurations towards the equilibrium state is irreversible. We stress that this irreversibility is entirely due to the inertial dynamics, not to molecular processes. Second, the most probable macrostate has a peculiar structure: its buoyancy field is characterized by a smooth buoyancy profile $\overline{b}(z)$ superimposed with wild small scale buoyancy and velocity fluctuations.

Because all the kinetic energy of the equilibrium state is carried by small scale velocity fluctuations, the kinetic energy is literally lost irreversibly at subgrid scale. Indeed, once the equilibrium is reached, the energy of those small scales fluctuations can not be used to overturn the coarse-grained buoyancy field. Similarly, the small scale buoyancy fluctuations can not be used to modify the mean buoyancy profile $\overline{b}$ once the equilibrium state is reached. In that respect, there is irreversible mixing of the buoyancy field at a coarse-grained level.

Global cumulative adiabatic mixing efficiency. Let us assume that a given amount of energy denoted E_{inj} is injected into a fluid initially at rest, characterized by a sorted (or background, or base) buoyancy profile $b_s(z)$. Turbulent stirring implies rearrangements of fluid parcels, which changes the initial sorted buoyancy field b_s into another buoyancy field b. Such rearrangements are necessarily associated with an increase of potential energy $E_p = -\int_{\mathcal{V}_\mathbf{x}} \mathrm{d}\mathbf{x}\ (b - b_s)\, z$. At equilibrium, this quantity can be computed using

$$E_p = \frac{V}{2H} \int_{-H}^{+H} \mathrm{d}z\ \left(\overline{b} - b_s\right) z\ . \tag{5.5}$$

Since the local kinetic energy e_c is uniform in space once the equilibrium is reached, the total kinetic energy is $E_c = V e_c$, and the conservation of energy leads to :

$$E_p + E_c = E_{inj}. \tag{5.6}$$

We define the (statistically irreversible) mixing efficiency and the Richardson number as

$$\eta \equiv \frac{E_p}{E_{inj}}, \quad Ri = \frac{H\left(b_s(H) - b_s(-H)\right)}{E_c/V}\ . \tag{5.7}$$

Phase diagrams. We show in figure 5.2 how the mixing efficiency η varies with the global Richardson number for two different initial sorted buoyancy profiles b_s,. Case (a) in figure 5.2 presents the statistical mechanics predictions for mixing efficiency in a two buoyancy level configuration,corresponding to a sorted profile with two homogeneous layers, for which an analytical solution exits; case (b) is the case of a initial linear sorted buoyancy profile. We see in this figure that whatever the sorted buoyancy profile, the equilibrium buoyancy profile $\overline{b}$ can be considered as almost completely homogenised in the low Richardson number limit ($Ri \ll 1$). In that case, most of the injected energy is lost in small-scale velocity fluctuations with $E_c = V e_c \simeq E_{inj}$, and the mixing efficiency is inversely proportional to the Richardson number, with a prefactor that depends on the initial buoyancy profile. The large Richardson behavior of the mixing efficiency depends drastically on the initial sorted buoyancy profile b_s: the mixing efficiency decreases to zero with increasing Richardson numbers in the two-level case of figure 5.2-a, while it increases to an asymptotic value close to 0.25 in the linearly stratified case of figure 5.2-b. One can also show analytically this asymptotic value of $\eta = 0.25$ is indeed expected in a low energy limit, as a consequence of energy equipartition, provided that the stratification of the sorted profile is always strictly positive.

Main conclusion. We have shown that several predictions for the cumulative, global mixing efficiency η can be obtained within the framework of the equilibrium statistical mechanics theory:

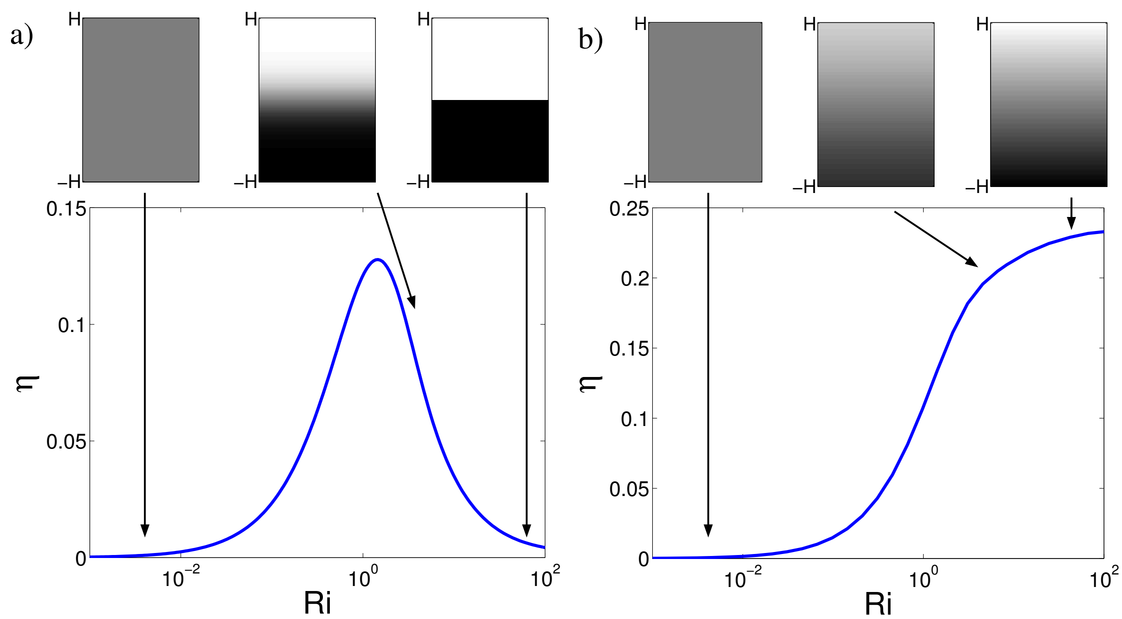

Figure 5.2: Variation of the mixing efficiency (ratio of irreversible gain of potential energy to total energy injected) with the Richardson number. a) initial buoyancy profile with two homogeneous layers; b) initial linear buoyancy profile. The three insets show the equilibrium buoyancy field $\overline{b}$ for different values of Ri.

1. The cumulative mixing efficiency varies as $\eta \sim 1/Ri$ in the limit of small Richardson numbers, whatever the initial buoyancy profile, which is consistent with scaling arguments given by [109] in a forced-dissipative case.

2. The cumulative mixing efficiency tends to $\eta = 0.25$ in the limit of infinite Richardson numbers, provided that the initial buoyancy profile is sufficiently smooth. This value is a consequence of energy equipartition, and it supports previous purely kinematic arguments by [116, 115].

3. The shape of the curve $\eta(Ri)$ depends strongly on the initial buoyancy profile, and can be non-monotonic. In the particular case of a fluid with two homogeneous layers of different buoyancy, the theory predicts a bell-shape for the cumulative mixing efficiency as a function of the bulk Richardson number with a maximum $\eta = 0.15$, just as observed experimentally in [100], but the universality of this phenomenon remains under debate.

4. When the initial buoyancy profile is linear, the curve $\eta(Ri)$ is monotonic. This is consistent with previous studies on mixing in decaying experiments. In addition, the shape of the curve predicted by the equilibrium theory is consistent with empirical parameterizations for the variations of the flux Richardson number with the gradient Richardson number, see e.g.[120, 85, 200].

Caveats of the approach. To the best of our knowledge, there is so far no other theoretical results that provide such predictions in a unified framework. There remain, however, several caveats for the application of the statistical mechanics theory:

1. The ergodic hypothesis underlying the theory is a very strong assumption that may often be broken. Indeed, there are many experimental and numerical evidence showing that the efficiency of mixing often depends strongly on the energy injection mechanism, while the theory predicts that the result does not depend on how the energy is injected.

2. The theory applies to fluids in the limit of infinite Reynolds and Péclet number, while existing laboratory and numerical experiments are usually carried in intermediate regimes where mixing efficiency can be affected by finite values of molecular viscosity and diffusion, see e.g. [102, 26, 152] and references therein.

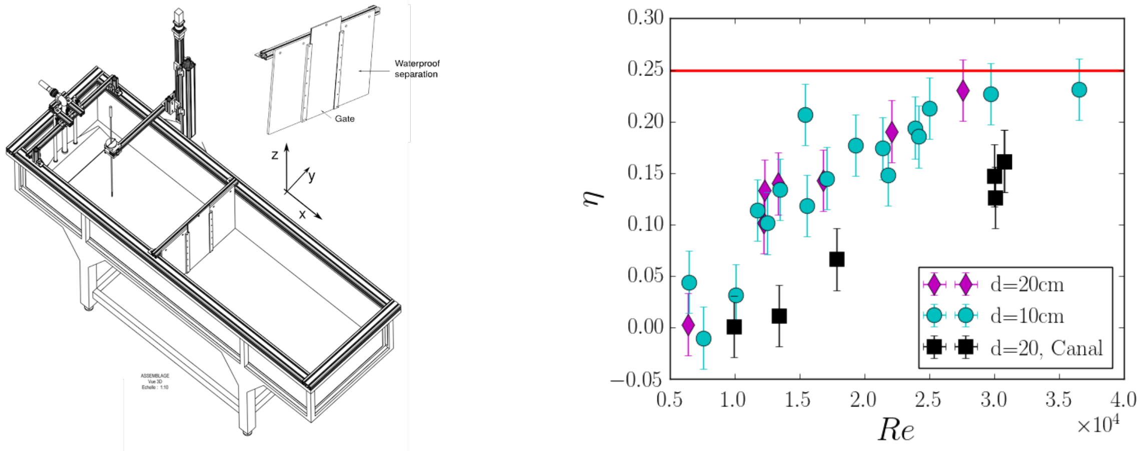

Figure 5.3: **Left panel**: experimental set up of the lock-exchange experiment by D. Micard and L. Gostiaux. **Right panel**: cumulative, global mixing efficiency η as a function as a function of the Reynolds number $Re = \sqrt{gH}\ H/\nu$.

3. Finally, the equilibrium theory does not predict how the system converges towards equilibrium, or what would be the energy fluxes in a forced-dissipative case. In those cases, equilibrium theory only provide a hint for the tendency of the system to be more or less efficient in mixing the buoyancy field, and other theoretical tools will be needed to model those important out-of-equilibrium features.

Despite those limitations, we believe that the statistical mechanics approach is a useful tool for the understanding of mixing in stratified turbulence, and we hope that the present work will motivate further studies in those directions.

Reference (with link to the paper):

Venaille, A., Gostiaux, L., and Sommeria, J. (2017). A statistical mechanics approach to mixing in stratified fluids. Journal of Fluid Mechanics, 810, 554-583.

5.2 Outlook: tests of the statistical mechanics approach

Applications of the equilibrium statistical mechanics theory to decaying turbulent stratified fluids rely on strong assumptions. It is therefore essential to judge *a posteriori* the interest of this approach, by comparing the relevance of the underlying hypothesis and its predictions to numerical and laboratory experiments.

Lock-exchange experiments (preliminary results). One of the classical decaying experiment in stratified turbulence is the lock-exchange set-up shown in the left panel of figure 5.3. A rectangular tank is separated into two identical reservoirs, each of them being filled with an homogeneous fluid of different density. When a gate separating the two reservoirs in opened, the available potential energy of the system is released, and one measures how much of this energy has been used to irreversibly mix the buoyancy once the system has relaxed to a state of rest. This makes possible a direct measurement of a cumulative, global mixing efficiency defined as the ratio of the potential energy irreversibly gained by the system to the initial available potential energy.

The global Richardson number $Ri = \Delta b H/E_c^0$ in this experiment is not a control parameter, since its value can not be changed (The total energy released into the system E_c^0 is proportional to $H\Delta b$). Assuming that the available potential energy is initially converted into kinetic energy, one can estimate $Ri \approx 1$. Panel (b) shows the evolution of the mixing efficiency with the Reynolds number, as measured by D. Micard, who finds an asymptotic value of $\eta = 0.25$. This value is larger than those obtained previously in elongated (channel) geometries [140, 79], but we conjecture that the previous experiments will converge to 0.25 at higher Reynolds numbers. In any case, these observations confirm that mixing efficiency does not rely on viscous effect in the large Reynolds limit. Similarly, one may expect that the results are independent from diffusive effects in the large Péclet limit. Thus, we are left in the experiment with a set of relevant external parameters that are of a purely geometrical nature: the tank aspect ratio and the adimensionalized width of the central gate.

Given that the initial buoyancy distribution is a two level distribution, and that the global Richardson number is $Ri \approx 1$, the value of mixing efficiency predicted the equilibrium theory is $\eta \approx 0.15$, as seen on figure 5.2. This contradicts the asymptotic value of 0.25. To understand the convergence towards this value in a statistical mechanics framework, one has to make additional phenomenological assumptions, using information coming from the dynamics.

It is intriguing that this value 0.25 is actually a robust prediction of the statistical mechanics theory for any mixing event characterized by large Richardson numbers and sufficiently smooth background buoyancy profile. Let us attempt to justify the relevance of these two assumptions to the lock-exchange experiments. The key idea is the existence of a time scale separation between the injection of large scale kinetic energy through the formation of gravity currents in the upper and lower parts of the central gate, and small scale turbulent mixing occurring in these gravity currents. After a transient state, the stratification in regions close to these gravity currents is much smoother than the initial two level distribution. In addition, the slow injection of kinetic energy (compared to turbulent mixing processes) means that one can interpret the total experiment characterized by $Ri \simeq 1$ as a succession of mixing events characterized by $Ri \gg 1$ (each mixing event corresponds to the injection of an infinitesimal amount of kinetic energy). These arguments remain to be quantified on more solid ground, but they could provide some useful physical interpretation on the ubiquitiy of measurements for mixing efficiency close to $\eta = 0.25$.

Numerical simulations (preliminary results). A number of numerical studies have been devoted to the computation of mixing efficiency, especially in the case of shear-stratified fluids, see. e.g. [109, 153] and references therein. Only few of them investigated the decay of a fully three-dimensional turbulent flow. A. Delache and E. Horne performed a set of Direct Numerical Simulations (DNS) of three-dimensional decaying turbulent stratified flow by solving Navier-Stokes equation under Boussinesq approximation in a box with periodic boundary conditions in horizontal directions and no fluxes conditions at the bottom and at the top of the box. The energy was injected at $t = 0$ by perturbing the initially stable buoyancy profile, using an initial turbulent velocity field obtained from previous direct simulations of homogeneous turbulence. The simulation was performed over ~ 48 initial overturning times L_t/U_t, with L_t and U_t integral length scale and rms of the initial turbulent flow respectively. This initial condition is characterized by a geometrical parameter L_t/H. While this parameter plays no role in the statistical mechanics theory, it has to be taken into account to properly interpret the numerical results (which already shows some limitation of the theory).

Typical snapshots of the buoyancy fields after one eddy turnover time are shown on the top panels of figure 5.4, in the case of a linearly stratified fluid with buoyancy frequency N. The simulation parameters are chosen to fix the Reynolds number $Re = U_t L_t/\nu \simeq 1000$, while spanning a large range of global Richardson number $Ri_g = N^2 L_t^2/U_t^2$ (each Ri_g characterizes one numerical experiment). We take into account one information from the dynamics, by computing the Thorpe's scale $L_{th}(t)$ that quantifies the average length scale of overturning events. We define at each time a cumulative mixing efficiency η (as before), and a Richardson number $Ri = N^2 L_{th}^2/e_c$, where e_c is the cumulated amount of specific kinetic energy that has been dissipated at a given time. The parameter Ri varies with time, and is plotted for different numerical experiments (different Ri_g) in figure 5.4, superimposed with the statistical mechanics predictions $\eta = f(Ri)$. Because of molecular dissipative effects, the Thorpe's scale and then the Richardson number Ri tend to zero in the large time limit. We argue that the statistical mechanics theory is relevant to describe qualitatively the variation of cumulative mixing efficiency with the Richardson number, provided that the Richardson number considered is defined as $Ri = N^2 \max {L_{th}}^2/e_c$. The (preliminary) comparison between numerical simulations and statistical mechanics theory shown in figure 5.4 is qualitatively correct, but further work is needed to explain why L_{th} and not H (as in the theory) is used in the definition of the Richardson number. In addition, the theory is based on the existence of turbulent stirring, and therefore fails for initial conditions characterized by large Ri, as the dynamics is then dominated by waves. We see that just as in the laboratory experiments, the statistical mechanics theory must be supplemented with additional ingredients to yields useful results concerning the numerical simulations.

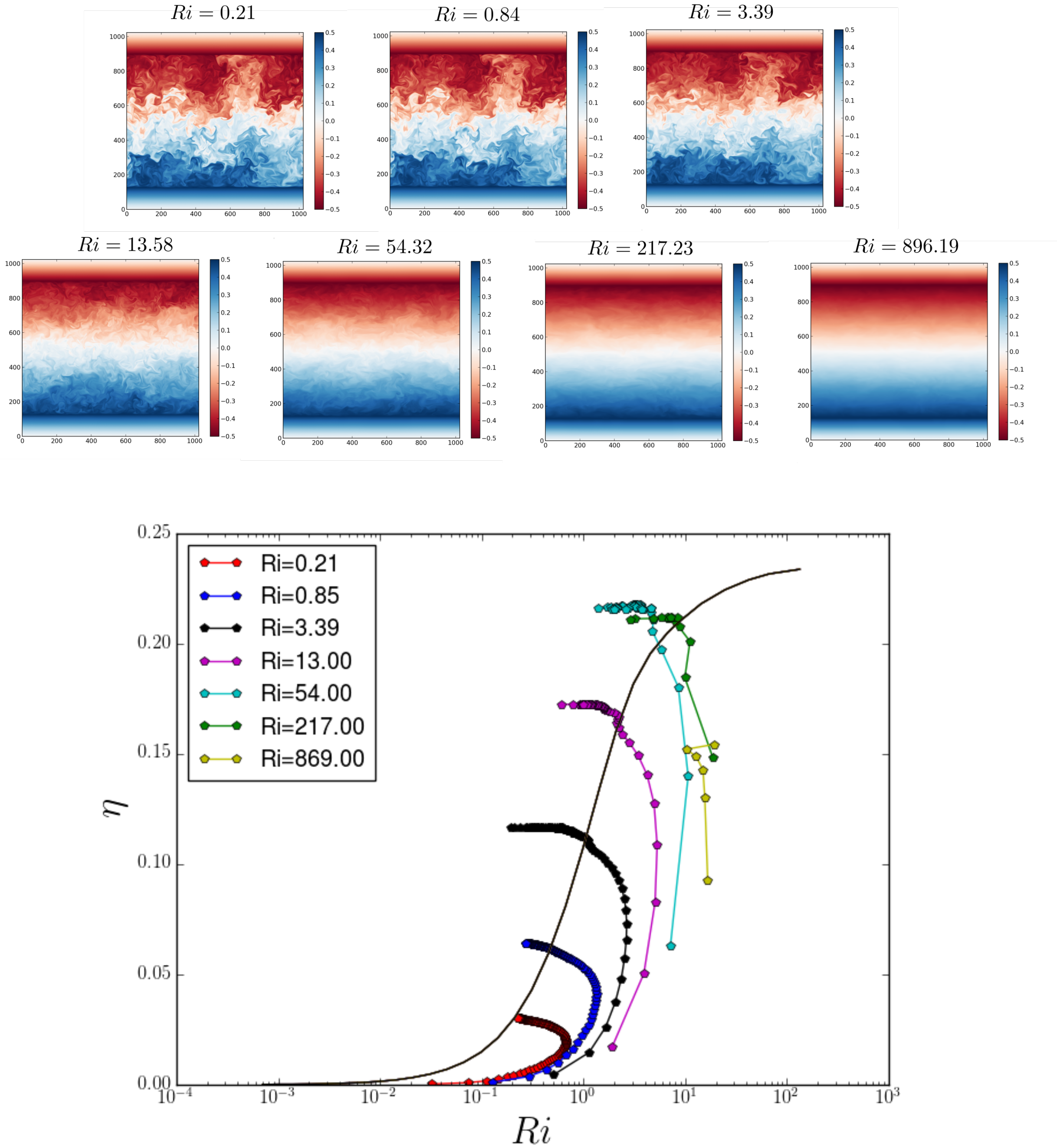

Figure 5.4: Preliminary results on mixing efficiency in decaying numerical experiments when energy is injected on the form of an initial turbulent velocity field. **Top:** Snapshot of the buoyancy field (denoted $-\theta$) in the case of a linear stratification, at time $t \simeq \tau_0$, with $\tau_0 = L_t/U_t$ the eddy turnover time in the initial condition, for different buoyancy profiles characterized by a global Richardson number $Ri_g = N^2 L_t^2/U_t^2$ (the index g is missing on the figure). For a given value of Ri_g we compute at each time the cumulative mixing efficiency ($\eta = \Delta E_p/E_{inj}$ with $\Delta E_p(t)$ the irreversible increase of background potential energy and E_{inj} the energy initially injected) and the Richardson number ($Ri = N^2 L_{th}^2/e_c$, with $e_c(t)$ the cumulative amount of specific kinetic energy dissipated at small scale, and $L_{th}(t)$ the Thorpe's scale quantifying the typical length scale of overturning events. **Botttom:** Cumulative mixing efficiency as a function of the cumulative Richardson number. Each color corresponds to a different numerical experiment. The black line is the statistical mechanics prediction. Figure provided by A. Delache.

One striking prediction of the equilibrium theory is the qualitative difference between the variations of mixing efficiency η with the Richardson number. While the shape in monotonic in the linearly stratified case, we expect a bell shape in the two-layer case. Preliminary analysis for numerical simulations by A. Delache and E. Horne are also confirming this prediction.

To conclude, these comparisons between statistical mechanics theory and actual decaying stratified fluids show that the theory is useful at a phenomenological level to rationalize observations, but that it must be complemented with supplementary information from the dynamics. I hope that these ideas will provide a useful guide for new subgrid-scale parameterizations of small scale turbulent mixing.

Chapter 6

Project: topological waves in geophysical fluids

I present in this chapter a research project on the topological properties of geophysical waves and their physical manifestations. This builds upon promising first results obtained in collaboration with Pierre Delplace and Brad Marston, showing that Kelvin and Yanai equatorial waves, two important components of our climate system, have an origin in topology. We have shown that shallow water waves present topological singularities as a consequence of Earth rotation that breaks time-reversal symmetry. Northern and Southern hemispheres can then be interpreted as two different topological phases, by analogy with topological insulators in condensed matter; Kelvin and Yanai waves, that propagate energy eastward along the equator, can be understood in this framework as topological edge states. Their existence and their robustness can be related to a topological invariant, the Chern number. More details are available in the following preprint (with link to the pdf file):

Delplace, P., Marston, J. B., and Venaille, A. (2017). Topological Origin of Equatorial Waves. arXiv preprint arXiv:1702.07583, in review for Science

Summary of the project. Owing to rotation, stratification or compressibility, geophysical flows support the existence of propagating waves. Those waves can be surprisingly robust to perturbations. Is this robustness a purely dynamical feature, or is there a deeper principle at work? We hypothesize that concepts from topology can bring a new understanding to the peculiar physical properties of those fluid waves. In condensed matter, topology explains the precise quantization of the Hall effect and the protection of surface states in topological insulators against scattering from disorder. Those ideas have recently been applied to optical and mechanical systems, but have so far played little role in our understanding of ocean and atmosphere dynamics. The aim of this project is to compute topological invariants in a hierarchy of flow models, to relate their existence to the discrete symmetries of these models, and to show the physical manifestation of these topological features. This will provide a new classification of geophysical waves based on topology. We have shown in a preliminary work the existence of non-trivial topological properties in the rotating shallow-water model, manifesting as singularities in the set of eigenmodes for this model. We found that celebrated equatorial Kelvin and Yanai waves are topological edge states as a consequence of Earth rotation breaking time-reversal symmetry. We will build on this result to look for physical manifestations of topology in geophysical context. We will consider the linear response to an external forcing, the effect of dissipation, and wave-mean flow interactions, as different ways of probing topological properties. We will use a combination of theory and numerical simulations, with an emphasis on the shallow-water system. We argue that this model is also a promising and natural platform to address the role of nonlinearities (or strong interactions) in topological phases of matter, and, at longer term, to propose a fluid analogue of quantized hall effect. Rotating shallow-water flows are two-dimensional and break time reversal symmetry. We will address the effect of changing flow dimensionality and breaking other discrete symmetries by considering inertia-gravity waves in Boussinesq equations, or acoustic-gravity waves in compressible stratified fluids. We argue that these models may exhibit other topological features such as the fluid analogous of Weyl/Dirac-like points. We will pay particular attention to three dimensional inertia-gravity waves, that are currently thoroughly studied in laboratory experiments at ENS de Lyon.

6.1 Objectives and scientific hypotheses

Owing to rotation, stratification or compressibility, geophysical flows support the existence of propagating waves [184]. Some of those waves can be surprisingly robust to perturbations. This is for instance the case of equatorial Kelvin waves that travel across the Pacific ocean during an El Nino event [124], or of Lamb waves that transmit over great distances the pressure pulses from large explosions in the atmosphere [28]. Is this robustness a purely dynamical feature, or is there a deeper principle at work? We hypothesise that concepts from topology will bring a new understanding to this problem.

Topology describes global properties that are invariant under continuous deformations. It explains the precise quantization of the Hall effect [179, 10] and the robustness of edge states against scattering from disorder at the surface of certain insulators [70, 128]. The interest in the physics of topological insulators has been reinvigorated over the last decade, after the discovery of new phases of matter with exotic topological properties [84, 14]. In this context, the topological invariants are integers such as the first Chern number, that reflect the existence of singularities for a family of eigenstates. The existence of these topological invariants can be related to the discrete symmetries of the physical model, and topological insulators have been classified according to these symmetries [161, 89]. The interplay between **discrete symmetries and topology has so far played little role in thinking about the fluid dynamics of oceans and atmospheres. The aim of this project is to fill this gap**.

It has actually been realized over the last few years that properties initially discovered in topological insulators can be found in other domains of physics, such as cold atoms [17], optical systems [103], metamaterials in acoustics and solid mechanics systems [77]. These artificial lattices support the existence of edge states that propagate in only one direction with no backscattering, among other peculiarities that can be related to discrete symmetries and topology. We ask in this project if such topological waves can occur at geophysical scale.

In a preliminary work, we have shown that, as a consequence of the rotation of the Earth that breaks time reversal symmetry, two equatorially trapped waves known as Kelvin and Yanai waves have a topological origin, manifesting as edge modes in the rotating shallow-water model [44]. As those waves are an important component of the El Niño phenomenon and the Madden-Julian Oscillation, our results have demonstrated that topology plays a surprising role in Earth's climate system, and that atmosphere or oceans share basic physics with topological insulators. A number of fundamental questions need then to be addressed: i) are those results generic to more complex flow models and configurations? ii) Can we find other intrinsic topological properties in those systems, and iii) what would be their physical manifestations?

The first objective is to propose a new classification of geophysical waves based on topology and on the discrete symmetries of the underlying flow models. In practice, this will require the analytical computations of global topological properties for the set of eigenmodes in a given geophysical flow model. Just as condensed matter physicists have discovered long after band theory that eigenstates in unbounded geometry carry information on a physical system that is not encoded in the dispersion relation, we argue that computing the global topological properties of fluid waves will bring new physical insights into geophysical flow models, even if the waves and their dispersion relation are already known and well documented. This additional information correspond to topological invariants describing singularities in families of eigenmodes for a given system. Our aim will be to compute these singularities and to relate their existence to peculiar discrete symmetries in the system. In addition to a better understanding of the topological origin of some of the geophysical waves, this classification may also provide physical realization of systems whose existence has been predicted in 'periodic tables of topological isulators', without known examples.

The second objective is then to decipher the physical consequences of fluid waves having non-trivial global topological singularities. We will address this question within the rotating shallow-water framework, as we know already that non-trivial topological properties appear in this model. There exists several ways of probing global topological properties of a physical system. The most common one use bulk-edge correspondence, that relates the topological invariants of the waves to the number of unidirectional (chiral) edge states in the presence of boundaries. A first step will be to clarify this correspondence in this fluid case. In the absence of boundaries, other physical observables can

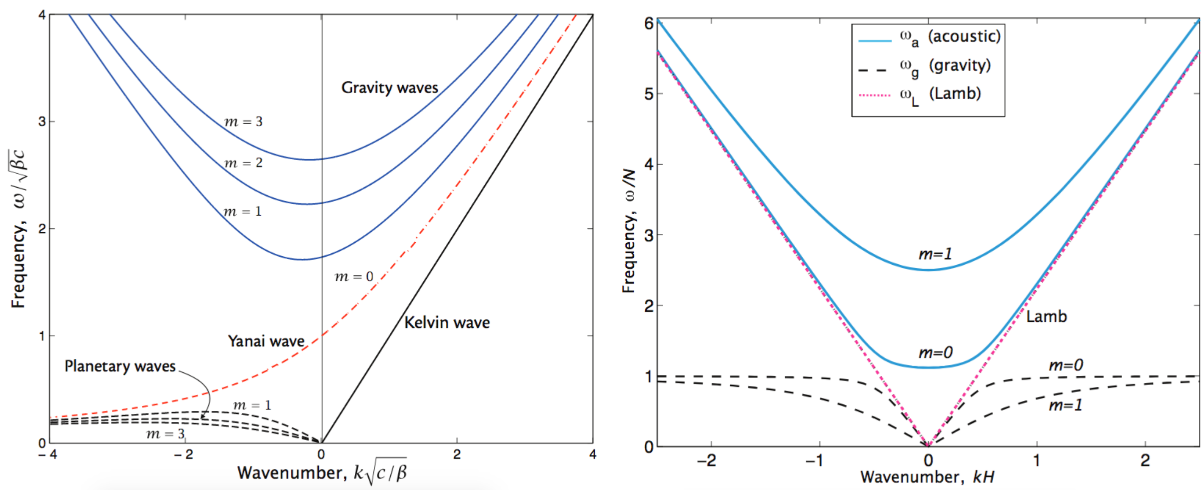

Figure 6.1: **Left panel:** equatorial shallow water waves on an equatorial beta plane (the Coriolis parameter is $f = \beta y$). There is a gap between the gravity (Poincaré) wave band and the planetary (Rossby) wave band. The presence of this gap is due to the Earth rotation, that breaks time-reversal symmetry. This gap is filled by two equatorially trapped modes: the mixed gravity - Rossby (Yanai) wave mode, and the equatorial Kelvin mode. Using bulk-edge correspondence, we have shown that these two edge waves have a topological origin [44]. **Right panel:** acoustic gravity waves in an isothermal atmosphere. In that case, the mirror symmetry in z-direction is broken in the presence of gravity. There is a frequency gap separating acoustic waves from internal gravity waves. The Lamb wave is an edge mode filling this frequency gap. Wether or not this mode has a topological origin remain to be addressed. Both figures are reproduced from Vallis's Book *Atmospheric and Oceanic Fluid Dynamics*, second edition 2017.

be related to non-trivial global topological properties of eigenmodes. One of them is to consider the response of a system fo an external forcing. This is actually how topology first came into play in the context of quantized Hall effect through the celebrated Thouless-Kohmoto-Nightingale-den Nijs (TKNN) formula. A second step will be to see wether similar features could be recovered in fluids. Finally, it was recently suggested to use losses (dissipation) as a way to describe global topological properties of a system. Since boundaries, external forcing, dissipation, nonlinearities and disorder are essential ingredients of geophysical flows, we hypothetize that those three approaches will be useful to probe the topological nature of flow models.

The third objective is to address with tools from topology and within the shallow-water framework the longstanding problem of wave-mean flow interactions in geophysics [32]: what are the waves that may appear in the presence of a mean flow, and what is their feedback on the mean flow? A first question concerns the existence of transitions in the global topological properties of linearized eigenmodes around a prescribed mean flow, when the mean flow is varied adiabatically. This goes beyond the more familiar framework of topological insulators described by Hermitian Hamiltonians, as the operator of the linearized dynamics is in general not self-adjoint in the presence of a mean flow, and can for instance admit unstable eigenmodes. A second question it to study waves propagating on a spatially periodic mean flow. In that case the mean flow brings a macroscopic lattice into the problem, and one may expect the emergence of robust edge waves in this macroscopic lattice. At longer term, those fluids will be a natural framework to address the combined effect of topology and nonlinearities (for instance through the feedback on waves on the mean flow), which may be thought as the analogous of strong interactions in condensed matter system. This could be a step towards a fluid analogue of quantized Hall effect.

6.2 Originality and relevance in relation to the state of the art

Waves, symmetries and topology in fluids

Geophysical waves. Oceanic and atmospheric phenomena are studied in a hierarchy of flow models of increasing complexities, from incompressible two-dimensional Euler flows to comprehensive three-dimensional primitive equations [184]. Simple models are usually very useful to interpret observations or output from more complicated models [73]. The waves associated with a given model are obtained by computing eigenmodes of the linearized dynamics around a prescribed state of the system. These eigenmodes carry useful physical information on the model, and are therefore presented in most classical textbooks [99, 63]. The models may include rotation, that support the propagation of inertial plane waves, stratification, that supports the propagation of gravity waves, and compressibility, that supports the propagation of acoustic waves. Most textbooks usually present the structure of a given wave in physical space, and the dispersion relation for these waves. We show in figure 6.1 two classical examples of such dispersion relations. The first one is the dispersion relation of equatorial waves in the rotating shallow-water model. The second one is the dispersion relation of acoustic-gravity waves in compressible stratified turbulence. There are three striking common points: the existence of gaps separating different classes of waves, the existence of edge modes (Kelvin, Yanai, Lamb waves) that fill those gaps in the presence of a boundary, and the existence of crossing points between waves. **The originality of our project is to compute information on the linearized dynamics in unbounded geometry that is not encoded in the dispersion relation**, and to relate those properties to the appearance of unidirectional edge modes in the presence of boundaries. This supplementary information will be provided by global properties of the set of eigenmodes corresponding to a given frequency band (say for instance, the Planetary wave band or the Gravity wave band of figure 1), and will be related to discrete symmetries in the flow models. This new information will in turn offer new insights into the nature of the gaps, edge modes and crossing points encountered in the dispersion relation of geophysical waves.

Discrete symmetries Continuous symmetries have long played an important role in geophysical fluids. According to Noether theorem, these symmetries are associated with conservation laws. For instance invariance through time translation implies energy conservation, and invariance through particle relabelling implies conservation of an infinite number of Casimir functionals in two-dimensional Euler flows, among which the enstrophy, and the conservation of Helicity in three-dimensional Euler flows. A number of studies have addressed the physical consequences of these conservation laws, such as the existence of multiple stationary states, the stability of some of those states, and the direction of the turbulent cascade. Other studies have used (both discrete and continuous) symmetries to derive amplitude equations describing the robust emergence of patterns in fluid flows [61]. By contrast, the effect of breaking discrete symmetries have so far play little role in the context of geophysical waves. This is surprising, as propagating waves in geophysical fluids often owe their existence to a term that breaks some of the discrete symmetries in the flow model: the Coriolis force breaks time reversal symmetry, and buoyancy forces due to stratification (and gravity) breaks inversion symmetry in the vertical direction. In fact, properties of fluid waves in the presence of discrete symmetries such as time reversal symmetry are still an active subject of research [11, 56]. More generally, Berry showed how the structure of a given eigenstate in both quantum and classical system can be drastically changed when breaking time reversal symmetry [16], and explained with those ideas the emergence of amphidromic points in ocean tides (points where the tidal amplitude vanishes). This work emphasized the role of symmetries in the topology of a given wave. **The novelty of our work will be to compute global topological invariant in a set of waves, that emerge as a consequence of enforcing or breaking discrete symmetries** of the operator $\mathcal{H}$ of the linear dynamics.

Topological fluids dynamics. Ideas from topology appeared in fluid dynamics after the seminal work of Arnold [6], who interpreted the Euler equation as the geodesic equation on the group of volume-preserving diffeomorphisms, and after the introduction by Moffatt [125] (and others) of the helicity invariant in fluids and magnetohydrodynamics. Helicity is a dynamical invariant sometimes referred to as Hopf invariant [7], that measures the linkage and knottedness of vortex lines. It plays a fundamental role in various areas of fluids mechanics, including dynamo theory, and vortex entanglements in superfluids [127]. Thanks to recent observations in superfluids [57], and thanks to recent technical methods to control the generation of vortices in classical fluids [90], and numerical simulations of vortices reconnection and excitation of Kelvin waves in Gross-Pitaevskii equation [46, 201], the physics of knots in turbulent and quantum fluid is currently a very active subject (note that the Kelvin waves appearing in this context are perturbation of vortex lines, and thus of a very different nature that coastal and equatorial Kelvin waves encountered in large scale geophysical flows). **The novelty of our project is to apply ideas from topology to families of geophysical waves in parameter space**.

Chern invariants Topological invariants are integers characterising global properties such as the winding number of a closed curve in a plane around a hole, the genus of a connected orientable surface (the number of holes), or singularities (vortices) in a vector field. Remarkably, these global topological invariants can sometimes be related to local geometrical properties. For instance, Gauss-Bonnet theorem states that the total Gauss curvature of a closed surface S is related to the genus g of the surface through

$$\frac{1}{2\pi}\int_S K da = I \tag{6.1}$$

with $K = 1/(r_1 r2)$ the local Gauss curvature, and $I = 2\,(1-g)$ an even integer called the Euler characteristics. This theorem has been generalized in the 40ties by Shiing-shen Chern to fibre bundles. In our project, the base space of the fibre bundle will be a close surface in parameter space, and the fibres will be given by the (complex) eigenmodes Ψ of the linearized dynamics in this parameter space. Importantly, the eigenmodes are defined up to a phase. Local geometrical properties of the fibre bundle are then characterised by the Berry curvature $\mathbf{B}$, that can be expressed in term of a vector potential:

$$\mathbf{B} = \nabla_p \times \mathbf{A}, \quad \mathbf{A} = -i\,\langle \Psi, \nabla_p \Psi\rangle \tag{6.2}$$

with $< .,. >$ the usual inner product, ∇_p the gradient operator in parameter space. According to the generalized Gauss-Bonnet theorem, the global topological properties of the fibre bundle are characterised by an integer I called the Chern number, related to the Berry curvature through formula (6.1), with $K = \mathbf{B}\cdot\mathbf{n}$, where $\mathbf{n}$ is a unit vector normal to the surface. A non-zero Chern number reflects the impossibility of continuously defining the phase of the eigenmodes everywhere on the surface S, with I quantifying the 'vorticity' of the singularity (different than the fluid vorticity). Although the importance of geometric phase in classical systems in the presence of broken time-reversal symmetry has been emphasised by Berry [15, 69], **there has been to our knowledge no study computing a Chern number (or other topological invariant) for fluid waves**, excepted in our preliminary work [44] where we computed the integrated Berry curvature for the shallow-water model. In that case, parameter space is given by the wavenumbers (k_x, k_y) and the Coriolis parameter f, and the eigenmodes are described by $\Psi = (u, v, h)$, with (u, v) the velocity and h the height.

Topology of waves in condensed matter and metamaterials

Chern invariants and bulk-edge correspondence in condensed matter. Physical reasoning based on the topology of waves can be traced back to Dirac argument for the quantification in the presence of magnetic monopole [48], but such monopoles have yet not been observed. In condensed matter, Thouless and collaborators (TKNN) showed that the observed plateaus of Hall conductance in integer quantized Hall effect can be related to Chern invariants, that emerge naturally as a bulk property of ground eigenstates in linear response theory, when computing the Kubo formula. Complementary to this argument in unbounded geometry, Halperin explained the crucial role of edge states at the boundaries of a finite size system in integer quantized Hall effect [95, 68], and Buttiker showed the robustness of these edge states with respect to backscattering [33]. The correspondence between Chern invariants computed in the bulk by TKNN and the number of unidirectional edge states filling the energy gaps was established by Hatsugai [72, 71]. **Our project is to search for similar physical manifestations of topology in fluid waves**: can we find topological edge states robust to backscattering, and can be find signature of topology as a bulk property in response to an external forcing?

Classification of topological phases. Interest in the topological properties of physical systems has been renewed after the discovery of the quantum spin Hall effect [83, 84, 14, 91], that involves different topological invariants ($\mathbb{Z}_2$ invariants). Following the discovery of states of matter with peculiar topological properties, a periodic table classifying the topological insulators and superconductors has been created[161, 89, 151]. The table organizes the possible topological states according to their space-time dimension and the symmetries that must remain protected: time-reversal, particle-hole, and chiral symmetries. This work in the theory of topological insulators showed that an important consideration is not only which symmetries the state breaks, but which symmetries must be preserved to ensure the stability of the state. This classification has been extented to systems admitting additional reflection or inversion symmetries [60, 78, 39, 27]. **We will use such classifications to search for peculiar topological features in fluids**, and to find analogies or differences with other physical systems encountered in this context.

Topology of waves in metamaterials Recently topologically protected edge excitations have been found in artificial lattice structures that support waves of various types. Experimental simulations of topological properties encountered in condensed matter systems were first realized with cold atoms, and this continues to be an active field of research [17]. Those last few years, a number of studies have shown that topological phases are actually ubiquitous in physics. They can be found in optical systems [103], for instance by providing analogues of graphene lattices [13]. They can even be found in classical systems such as isostatic lattices in mechanics [82, 77], phonons supported by a lattices of coupled pendulum [173], acoustic waves propagating through an array of rotating cylinder, or in lattices composed of annular channels filled with a spontaneously flowing active liquid [171]. In this context, the effect of rotation has been addressed in an array of masses and springs [203], or in arrays of gyroscopes [130]. Those previous studies have the common point that they reproduce the lattice structure of condensed matter systems at a macroscopic level. As shown in our preliminary work on the shallow-water model, the originality of fluid models with respect to those systems is threefold: first, **topological properties emerge in fluids without having to build a lattice structure**, as in plasma [214]; second, it is remarkable that these topological properties exist in nature at geophysical scale, and perhaps even at astrophysical scale. This contrasts with previous artificial realizations of classical systems presenting topological features. Finally, we stress that many physical systems can share similar topological properties, but that experimental access to the manifestation of these topological properties differ from one system to another, in a complementary manner. Any advance in a given field may be useful for the others.

Analogies with exotic states of matter

Hydrodynamics of electron flows. Fluid models of strongly interacting electrons in condensed matter are currently developed in conjunction with new experimental measurements in graphene [105, 98, 217]. These approaches aim at describing the electron movements at a macroscopic level in a limit where their mutual interactions are sufficiently strong, to ensure a good scale separations with other typical lengths of the system. So far these models have been developed and applied in situations where the flow model is highly viscous (Stokes flows). A major question concerning those approaches concerns the addition of a magnetic field (playing the role of rotation), which just begins to be addressed [158], and which could lead to useful flow models describing fractional Hall quantum effect. The fractional Hall effect corresponds to a Hall conductance that presents plateaus where electrons are strongly interacting, contrary to the integer Hall effect case. **In our project, we propose to attack the problem from a different angle, by considering straight from the beginning a flow model including rotation (or other external fields breaking a discrete symmetry), and nonlinearities**, but excluding viscous effects.We will ask wether we can recover in our flow models some effects that are analogous to those encountered in condensed matter.

Fluid analogues of quantum Hall effects Cold atoms systems [17] and synthetic photonic systemscite [160] have already be proven useful to explore quantum many-body physics and novel states of matter under controlled conditions. It has also been argued that Fractional quantum Hall effect can also be phenomenologically described as a special flow of a quantum incompressible Euler liquid, that the Laughlin wave function naturally emerges as a stationary flow of the system of vortices in quantum fluid dynamics, and that Benjamin-Ono solitons carrying fractional charges of vortices can propagate at the edge of the system [209, 210]. There has also been attempts to simulate some aspects of fractional quantum Hall with actual fluids by Leo Maas [108], but at this stage there is no direct evidence for a quantized Hall effect in fluids. We our new understanding on topological properties of some flow models may allow us to go one step further, building on those previous results, and may help us **to propose a fluid analogue of the quantized Hall effect**.

Weyl points and other exotic fermions Weyl fermions are hypothetical two-component massless relativistic particles in three-dimensional space. Their band-crossing points, called 'Weyl points', carry a topological charge and are therefore highly robust. Such points have been observed in semimetals [202], in optical systems [132], and in mechanical materials [149]. Other exotic band crossing points can occur, depending on the symmetries of the systems, and the number of bands in it, e.g. [101, 27]. We will search for such peculiar points in flow models, and at the physical manifestations associated with them.

Non-Hermitian effects Real fluid are dissipative. Dissipation effects such as harmonic viscosity or linear friction correspond to non-Hermitian effects in quantum mechanical systems. Taking into account non-hermician effects is interesting for three reasons. First, it is natural to add those effects to test the robustness of topological edge states in real systems [52]. Second, non-hermicity break time-reversal symmetry, and can bring new topological properties to a systems [12]. Third, it can be a way of probing bulk topological properties of a system, instead of computing or observing edge modes at the boundaries [218, 9].

The effects of dissipation has been addressed in the context of topological mechanical systems [213], but remained to be understood for fluids. Tthe issue of time-reversal symmetry breaking and of anomalous dissipation in turbulent flows is actually a long standing problem in fluid mechanics, see e.g. [81, 58] and references therein. In three-dimensional incompressible turbulence, the rate of energy dissipation is nonvanishing in the limit of zero-viscosity, which results in a fundamental anomaly of time irreversibility. Understanding topological properties of the linearized flow dynamics in the presence of dissipative effects is a natural first step before addressing the full nonlinear problem with those tools.

6.3 Research program

1) A classification of geophysical waves based on topology

Objective and rationale The common aim of the three sub-tasks below will be to understand the topological consequences of discrete symmetries (time reversal and/or inversion symmetry), of dimensionality (2D vs 3D), and of the number of fields involved in the flow model. Just as in our previous work in the shallow-water case, the topological properties will be obtained by computing bulk topological invariants such as a Chern number, as well as corresponding topological edge modes.

Task 1a: Breaking the inversion symmetry: are Lamb waves and inertia-gravity waves topologically protected? Inversion symmetry refers to the invariance of the dynamical system by reversing one spatial direction. The arguably simplest and yet non-trivial configuration of geophysical interest where inversion symmetry is broken is an isothermal, stationary ideal gas in the presence of gravity. This system supports both acoustic waves as the fluid is incompressible and internal gravity waves as the fluid at rest is stably stratified. The dispersion relation is presented in many textbooks, see e.g. [99, 63, 184]. The existence of a frequency gap between acoustic waves and internal gravity waves is pointed out in such books, but the reasons for the existence remains mysterious. The aim of will be to compute Chern invariants for these two different wave bands, and to relate their properties to the existence of the celebrated Lamb wave, and edge mode localized at the bottom boundary of the system, with constant group velocity, and filling the gap between acoustic and gravity waves. We will then attempt to generalize those results on the effect of breaking inversion symmetry to a broader class of models in fluids and elastic system. We will also search for a general explanation for the emergence of non-dispersive edge modes in different dynamics systems (Lamb waves for compressible-stratified fluids, Kelvin waves for rotating flow).

Once the effect of breaking the inversion symmetry will be understood, we will address the effect of breaking simultaneously time-reversal symmetry and inversion symmetry. The natural framework to do this is three-dimensional rotating stratified fluids, that support the existence of inertia-gravity waves. It will be simpler to start with the case of an incompressible Boussinesq fluid, but the compressible case including acoustic waves could be studied later if needed. These inertia-gravity waves currently attract a lot of attention in fluid dynamics, as they have peculiar properties, as for instance an angle of propagation set by the frequency of oscillations. These waves are thoroughly studied at ENS Lyon in the groups of Thierry Dauxois and Philippe Odier, using theory and experiments. Any interesting results obtained within the framework of topology could then lead to experimental observations, and we intend to search for such observables. For instance, inertia-gravity waves are known to present a gap in their dispersion relation, provided that both rotation and stratification are present, and we expect non-trivial topological properties to emerge in that case, which should correspond to the existence of topological edge states filling the frequency gaps. Furthermore, the case of inertia gravity waves may be seen as the generalization in three dimensions of the two-dimensional shallow-water waves that we have studied in our preliminary work. It is therefore crucial to consider this case in order to address the robustness of our results obtained in two dimensions to three-dimensional perturbations.

Both the acoustic-gravity wave case and the rotating stratified case contain band crossing points. We will ask wether these points can be interpreted as Weyl/Dirac-like point. We note that Weyl and Dirac points correspond to two cases with two and four bands, respectively. We will search for such cases. However, we know that the shallow-water case and incompressible Boussinesq case involve crossing points with three bands. Such points are different than Weyl/Dirac points, and attract currently a lot of attention in the context of semi-metals [27]. It will be useful to understand their fluid analogue, both in terms of edge states, and in terms of linear response in unbounded geometries (see task 2).

Task 1b: From continuous fluid models to lattice models. In many condensed matter, bands are defined on a Brillouin zone, which provides a natural closed manifold to compute a Chern invariant associated with a given band. Fluid are continuum media, and therefore frequency bands are not defined on a closed manifold. For instance, shallow-water waves are defined on the whole plane (k_x, k_y). In our preliminary work on those waves, we bypassed this issue by defining the Chern invariant for the transition from one topological phase to another, by considering the symmetry breaking parameter as an external parameter of the problem that can be varied adiabatically, and by constructing a closed manifold in parameter space. For numerical simulations, we used another trick, by considering a discrete version of the continuous flow model (known as C-grid finite elements model in geophysical), and by noticing that this model can be interpreted as an Haldane-like model on a Lieb-lattice [44]. It remains to understand the relation between the continuum model and the discrete model, and this will be the aim of the second task. We stress that the discretization of a fluid problem usually leads to non-trivial issues: for instance, it breaks important continuous symmetries (such as particle relabelling symmetry). The main goal and difficulty for this subtask will be to find a way to assign a topological invariant to a given frequency band in a fluid model without considering a Brillouin zone. A clue may be given by recent studies performed in the context of electromagnetic waves [166], but the relevance of these ideas remains to be addressed in our case.

The bulk-edge correspondence relates the Chern invariants of the different frequency bands in unbounded geometry to the number of edge states filling the frequency gaps in the presence of a boundary. This correspondence has be proven in a number of cases, and is conjectured to be valid generically. In our preliminary work, we obtained results consistent with this correspondence. However, it remains to explain how these edge states are related to the bulk properties, just as Hatsugai explained in 1993 how the edge states in the Quantum Hall effects could be deduced geometrically from the bulk properties of Landau eigenmodes [72, 71]. We propose to adapt his approach to the fluid case.

At the end of this task, we will be able to assign a given topological number to a class of geophysical waves and a given value of the symmetry breaking parameter, and to prove the correspondence between these bulk properties and the number of topological edge states filling the frequency gaps.

Task 1c: a general classification of geophysical waves. The final sub-task will be to propose the general classification of geophysical waves according to their symmetries and topology. This will not only summarize the results of task 1a and 1b, but also explain how different models of the hierarchy of geophysical fluids may break or recover discrete symmetries. Those models all stems from primitive equations, using multiple scale analysis. This procedure amounts to filtering out some waves, and the concomitant effect of this filtering is that simpler flow model can recover a symmetry that is broken in more complex model. For instance, rotating shallow-water equations on on f-plane break time reversal symmetry, but the quasi-geostrophic dynamics, derived in the large rotation limit (small Rossby number) by filtering out the inertia-gravity waves, recover time reversal symmetry. Additional terms (such as the spatial variation of the rotation, as a beta plane) must be added to break this symmetry in the quasi-geostrophic models. We will then explain the interplay between of discrete symmetries, asymptotic derivations of reduced flow model, and topological properties. We will also include in the classification the case of magnetohydrodynamics (with a magnetic field advected by a conducting flow, and acting on it through a Lorenz force).

2) Physical manifestation of topological properties in geophysical waves

Objective and rationale The aim of the second task will be to describe manifestations of topological properties in geophysical waves. We will test different ideas within the framework of the shallow-water equations, arguably the simplest flow model with non-trivial topological properties. For that purpose, it will be necessary to develop ourself a numerical shallow-water model to perform intensive simulations in simple geometries (in a doubly-periodic and in a channel geometry).

Task 2a. Thermalization of low frequency eigenmodes in a narrow frequency band within gaps. In condensed matter systems, the presence of chiral edge modes in the energy gap implies a net transport of electrons when the fermi energy is chosen in the gap. In that case, this net transport can be related to topological invariants, that set the algebraic number of chiral edge modes within the gap, according to bulk-edge correspondence. The analogy with flow models is not straightforward, and the aim of this task will be to establish this by studying the thermalization between Rossby waves, Yanai waves and Kelvin waves in the presence of a large frequency gap within the framework of the shallow-water model. By thermalization, we mean here equipartition of energy among the different modes in a narrow frequency range.

The first step will be to develop the shallow-water model on a doubly-periodic geometry and a channel geometry (see above). The second thing will be to choose relevant numerical setup for the scattering experiments. First, there is no such thing as a Fermi energy level set by a substrate in geophysical flows. We will consider a both a freely decaying case to study the energy partition between waves within the frequency gaps, and the case of an external forcing either stochastic or with a prescribed frequency. Second, while in condensed matter a given point of the energy band corresponds to a single electronic state (maybe more in the presence of degeneracies), a given point in frequency-wavenumber space in fluids corresponds is associated to an eigenmode that can carry a priori an arbitrary amplitude (controlled by the total energy). Consequently, nothing guarantees in fluids that the total energy at a given frequency will be transported in the direction of the group velocity of the topological edge modes, contrary to the condensed matter case. We will ask when and how the presence of disorder (topography, weak random eddies) or weak interactions (nonlinearities) can lead to the thermalization of modes within a frequency range, and thus to a net eastward energy transport.

If the idea of a thermalization of modes within a frequency gap does not work, this will at least provide a new perspective on backscattering in equatorial waves [117, 111, 146, 41]. In addition, just as in many wave turbulence problems, a limitation of direct numerical simulations may be the very long time needed to reach an effective thermalization. In that case, and at longer term, this will be a strong incentive to start laboratory experiments on the interactions between a coastal Kelvin waves (at the boundary of a rotating tank), and Rossby waves (by considering a varying topography that plays the role of a beta plane in the rotating tank).

Task 2b. Response to a forcing within the system bulk. The aim of the second task will be to search for signatures of topological properties in fluids by considering the response to an external forcing within the shallow-water framework. This is actually by considering linear response theory that topology first came into play in condensed matter, when Thouless and collaborators explained that the quantized Hall conductivity is related to a Chern invariant, using Kubo formula [179]. Applying those ideas to fluids is not straightforward for several reasons. First, we know already that this model does not belong the same symmetry class as the quantized Hall effect. Second, we will have to choose a relevant forcing term, either for its geophysical significance (i.e. a wind stress in the momentum equation), or for its direct interpretation in term of Hall effect (an imposed gradient of height as a difference of electric potential). We will also have to choose relevant observables: the analogue of a current could be given by the velocity field itself, and the analogue of an electric potential could be given by the gradients of the height field, but other choices may be possible and will be investigated. We will also look for the simplest possible choice that make possible a shallow-water analogue of Laughlin argument in a Corbino disk geometry in the context of Hall effect [95]. The most difficult part will be to find the equivalent of generalized boundary conditions for the fluid case. Indeed, fluids involve fields are not defined up to a phase (only the eigenmodes of the system are defined up to a phase). Even if some of the analogies are not successful, this first part will be a useful preparation of task 3b on a possible quantized Hall effect in fluids. We will for instance consider linear response for flow models having a Weyl-like band crossing point, and search for the equivalent of a negative magnetoresistance observed recently in Weyl semi-metals. For this we will consider an external forcing in the same direction as the rotation vector, and compute the system response.

Task 2c. Dissipation as a way to probe topological edge states. The last sub-task will be to address the possibility to use dissipation as a way of probing bulk topological properties, building on recent works concerning topological properties in non-Hermitian systems. We identified a promising analogy between these recent works and two-layer quasi-geostrophic dynamics with bottom friction. In the case of a bi-partite one-dimensional lattice, Ref. [150] predicted the emergence of non-decaying 'dark states' corresponding to states where particles occupy only sites without losses, emerging at a topological transitions. This dark state bears strong similarities with solutions of the two-layer quasi-geostrophic model with all the kinetic energy located in the upper layer that I studied previously [194]. Additional work in relation with task 3 concerning the effect of a mean flow will be necessary to establish this analogy on firmer ground. We will also work on the role of dissipative effects within the shallow-water model by looking for non-trivial bulk topological properties in the presence of linear friction and viscosity, using the same methodology as in task 1. We will address in this framework the robustness of results obtained in an inviscid case in our preliminary study.

3) Mean flows and nonlinearities: towards a quantized Hall effect in fluids.

Objective and rationale The aim of the third task will be to address wave-mean flow interactions in the framework of topology. The first question will be to determine if some large scale geophysical flows are topologically protected. The second question, probably a longer term, will be to find fluid analogues of quantum Hall effect. This third task is more ambitious as the other ones, and we are less sure about the results. However, a large par of the work will involve numerical studies with a geophysical interest on their own, and we have identified initial steps that are feasible before looking at more complicated problems.

Task 3a. Topological properties in the presence of a prescribed mean flow. The aim of this task will be to extend the classification of task1 obtained for a flow at rest to situations where the base flow is non-zero. A prescribed mean-flow will generically break time-reversal symmetry and inversion symmetry; the operator of the linearized dynamics will in general be non-Hermitian, even in the absence of dissipative effects; and this operator may admit unstable modes. In addition, imposing a mean flow can be a way to reintroduce a macroscopic lattice structure into the flow model, for instance by considering an array of vortices. For those reasons, there is a strong potential for the appearance of new topological peculiarities in this context. We will begin by addressing the robustness of (topological) Kelvin and Yanai waves in the presence of a mean flow within the shallow-water framework. Even if analytical computations may be possible in limiting cases, they will be mostly addressed numerically, using the analogy between discrete representation of this flow model on a C-grid and Haldane-like model on a Lieb lattice. We will use those results to interpret direct numerical simulations of the shallow-water model in the presence of a mean flow.

We will then build on this work to address the scattering of Poincaré (inertia-gravity) waves of the shallow-water model in a lattice of geostrophic vortices that may be seen as a crude model of the field of oceanic mesoscale eddies. It will be interesting to see wether the lattice structure of large scale vortices allow for the propagation of topological edge modes (as Coastal Kelvin waves) along continental margins, and to compare these predictions with direct numerical simulations.

Using similar methods, we will then consider the case of multilayer quasi-geostrophic model in the presence of a mean-flow that is homogeneous on the two horizontal directions, but that varies along the vertical direction. This model contains as many frequency bands as layers, and this is the simplest possible framework to explain baroclinic instability, arguably the backbone of geophysical fluid dynamics. A. Venaille routinely perform numerical simulations of this class of models. As far as topology is concerned, this model will be interesting because it may allow us to describe bifurcations in topological properties emerging simultaneously with the appearance of unstable modes. It will also be possible to address in the framework the effect of varying the mean flow on the horizontal, and on adding dissipation.

Task 3b. Are potential vorticity staircases topologically protected? The aim of this task will be to see wether topology plays a role in the observed robustness of potential vorticity staircases commonly encountered in geophysical flows [49]. Statistical mechanics arguments, cascade phenomenology and kinetic theories have been invoked to describe such staircases, but none of them is fully satisfactory, and the reason for their spontaneous emergence remains poorly understood. We will use the results of the first task on the properties of the operator of the linearized quasi-geostrophic dynamics around a potential vorticity staircase to see wether topological mode exists, and we will then study the feedback of these modes on the mean flow. We will first consider the equivalent barotropic quasi-geostrophic equations for this problem. This model can be derived from the shallow-water model in a strong rotation limit, and by filtering out inertia gravity waves. In order to prepare the next task, it will be interesting in a second step to see wether these inertia-gravity waves can also play a role in this problem.

Task 3c. Wind driven circulation in a Channel as a quantized Hall effect? The aim of the last task will be to propose a fluid analogue to the quantized Hall effect in the presence of nonlinearities at the end of the project. We deliberately neither use the term integer nor quantum Hall effect as the fluid analogue of quantized Hall effect may be of a different nature. Our starting point will be to revisit the problem of wind-driven shallow-water dynamics in a reentrant channel (periodic in the x-direction), that bears strong similarities with the Hall effect: an external field (the wind) is applied in the zonal direction, which induces a flow in this zonal direction. In the presence of rotation, and if the flow is at geostrophic equilibrium, i.e. if horizontal pressure gradients are balanced by the Coriolis force, the currents is associated with a gradient of height in the meridional y-direction. The zonal current is the analogue of an electric current, and the meridional gradient of height is the analogue of an electric field. In this context, geostrophic balance can be interpreted as the classical Hall effect, as a different of stream-function (electric potential) in one direction is proportional to a current in a perpendicular direction. We will look for a regime parameter where this balance can be broken due to nonlinearities and lead to the quantization of the zonal current for a given transverse height gradient. This part is more exploratory than the others, but results from the previous subtasks will bring useful insights for this problem.

Chapter 7

Conclusion: main results and prospects

Over the last 9 year, I have addressed several aspects of rotating-stratified fluids, paying particular attention to the redistribution of energy by waves and turbulence in geophysical flows, and to phenomena whose macroscopic or global behavior can be understood without describing the details of the dynamics.

I first used equilibrium statistical mechanics as a poor's man approach to quantify the combined effect of turbulence and dynamical invariant on the energy partition in a variety of flow models. This allowed me for instance to explain why weak planetary vorticity gradients favor a energy transfers of surface intensified mesoscale eddies from the top to the bottom of the ocean [198], or how much of the energy released by the breaking of an internal wave can be used to irreversibly raise the potential energy of the system [192]. I then used quasi-linear approaches to describe interactions between internal gravity waves and large scale vortical flows in the presence of dissipative effects [42], motivated initially by experimental observations [19], and more recently by applications to quasi-biennial oscillation of equatorial stratospheric winds [144]. Since last year, I have been interested in topological properties of geophysical waves, to explain the robust emergence of unidirectional edge states trapped at boundaries, such as equatorial Kelvin waves. This work allowed me (and co-workers) to relate these modes with time-reversal symmetry breaking in two-dimensional fluids and the existence of a topological invariant, the Chern number, that had so far not been used in hydrodynamics [44]. My research interests are thus drifting from midlatitudes to the tropics, and oscillate between turbulence and waves.

My contributions concerning the equilibrium theory has been to generalize the Miller-Robert-Sommeria approach to different flow models, to compute explicitly phase diagrams for these models, and to use those diagrams to interpret idealized numerical simulations related to various geophysical phenomena: continuously stratified quasi-geostrophic equations for the vertical structure of geostrophic turbulence [198, 186], shallow-water model for the energy partition between waves and vortices [145], non-rotating stratified Boussinesq fluids for the partition between small scale kinetic energy and potential energy [192].

Because of the strong assumptions underlying the equilibrium statistical mechanics, I have confronted the predictions to numerical experiments to test the validity of the underlying hypothesis. I showed in particular that the range of interactions between fluid particles plays an important role in selecting the large-scale flow structure in decaying configurations [190]. Because oceans and atmospheres are forced dissipative systems, the equilibrium theory can only be a first step before more comprehensive out-of equilibrium studies addressing the role of forcing and dissipation. This first step allowed us to explain the spontaneous emergence of meandering sharp eastward jets (ribbons) in regimes with large bottom friction [194], the existence of an abrupt transition between two stationary states during sudden stratospheric warming events [216], or the spontaneous formation of bottom-trapped recirculations such as the Zapiola anticyclone in the Argentine basin [186].

Independently from the statistical mechanics approach, I addressed the problem of energy injection for geostrophic turbulence in the ocean, by questioning the locality hypothesis for energy injection through baroclinic instability [197]. I also proposed a simple model for the observed huge mass fluctuations of the Zapiola anticyclone, interpreted as the response to a stochastic lateral eddy flux [193].

So far I studied one mechanism isolated fro the others: the role of bottom friction, planetary vorticity gradients, stratification, potential vortical flows and inertia-gravity waves. My current aim is to put some of these pieces together to address what sets the mass transport of the antarctic circumpolar current in response to surface winds, which is a longstanding question in physical oceanography. At longer term, I would like to address the interplay between baroclinic instability, geostrophic turbulence and the global stratification of oceans and atmosphere. I also would like to address the interplay between geostrophic currents and bottom topography, to understand for instance the emergence of the sedimentary Zapiola rise; this project would concern much longer time scales related to the formation and the erosion of the bathymetry.

I think that the most promising perspective of my previous work is to further explore topological properties of geophysical waves, in relation with discrete symmetries and dimensionality of the flow models. My main objective in the next few years is to classify such topological properties of geophysical waves and to decipher their physical manifestations. This subject could bring new understanding of rotating-stratified fluids, from laboratory to planetary scales, and foster new analogies with other physical systems. I will propose a PhD project on this subject next year, in collaboration with P. Delplace.

Finally, I would like to mention my activities concerning teaching, supervizion of students and organization of scientific events. Since 2013, it has been my pleasure to teach tutorial classes on continuum mechanics at undergraduate level (about 30h each year) and to participate to several events for popularizing geophysical fluid dynamics and the physics of our climate system. I am currently preparing lectures notes on statistical mechanics and nonlinear physics for geophysical flows (lectures scheduled for spring 2018, M2 physique ENS de Lyon). Over the last few years, I had the good fortune to supervize the one-year Master internship and the PhD thesis of A. Renaud (defense scheduled in 2018 in co-supervizion with F. Bouchet).

I have organized several scientific events on different subjects and formats: one workshop gathering European researchers on geostrophic turbulence with Freddy Bouchet, one workshop gathering European researchers on internal waves with Louis Gostiaux and Caroline Muller, and I am co-organizing one workshop gathering French researchers on topological phenomena beyond electronic systems with Pierre Delplace, Alberto Amo and Matthieu Bellec. I also co-organized with Freddy Bouchet and Tapio Schneider a summer school on turbulence in Climate. This summer school took place in les Houches, a beautiful place to finish this habilitation thesis.

Antoine VENAILLE
Born March 16th, 1983
Pacsé, two children (2012, 2015)
Chargé de Recherche, CNRS
Laboratoire de Physique, ENS de Lyon
perso.ens-lyon.fr/antoine.venaille/

Previous positions, education and diploma

2017	**Habilitation à Diriger des Recherches** ENS de Lyon
2016–	**CR1 CNRS** ENS de Lyon *Section 2*
2012– 2016	**CR2 CNRS** ENS de Lyon *Section 2*
2011– 2012	**Post-doc ENS de Lyon** ANR LORIS *with S. Ruffo and T. Dauxois*
2009– 2011	**Post-doc Princeton University** GFDL *AOS program, hosted by G. Vallis*
2005–2008	**Thèse LEGI, Grenoble** and INLN, Nice *with J. Sommeria and F. Bouchet*
2002–2006	**Elève Normalien** ENS-Lyon *Physics*
2000–2002	**Classes préparatoires** Lycée Clemenceau, Nantes

Supervision of students

2018	M. Perrot (ENS Paris, mathématiques), Stage M1 de physique (with P. Delplace)
2016-2017	K. Gianchandani (NISER India) L3 summer intern and M1 scientific project.
2015-2018	A. Renaud, PhD, (with F. Bouchet)
2015	Y. Yasuda, PhD in Tokyo, 4 month visite in France (with F. Bouchet)
2013-2014	A. Renaud, stage long M1 Cachan (with F. Bouchet)

Participation to Scientific Projects

2018–2019	Partner of AO LEFE INSU held by B. Deremble at LMD ENS
2014–2017	Partner of ERC Transition held by F. Bouchet at ENS de Lyon
2013–2017	Partner of ANR STRATIMIX held by L. Gostiaux at LMFA Centrale Lyon.
2015	PEPS Défi Inphyniti, project NewWave (principal investigator)

Reviews for journals, proposals and participation to committees

2017	Participation to the PhD committee of Qu Bo, LMFA Ecole Centrale Lyon
2009–	Regular reviews for Phys. Rev., JFM, JPO, JAS, GRL, JSTAT and others
2012–	Ponctual reviews for NSF, Cambridge U. Press, U. Grenoble-Alpes (AGIR)

Organization of scientific events

2017 (3 days)	**Topological phenomena**, with A. Amo, M. Bellec & P. Delplace
2017 (1 month)	**Les Houches Summer School**, with F. Bouchet & T. Schneider
2013 (3 days)	**NewWave workshop**, with L. Gostiaux & C. Muller
2012 (3 days)	**GeoTurb workshop**, with F. Bouchet
2012-2017	**Colloquium** Physics Lab of ENS de Lyon, each Monday.

Teaching

2018–	Lectures on Geophysical Fluid Dynamics, M2 ENS de Lyon (18 hours per year)
2013–2016	Tutorials on continuum mechanics, L3 ENS de Lyon (28 hours per year).
2006–2009	Tutorial in Physics and Mechanics L3/M1 at UJF Grenoble (64 hours per year)

Outreach

2012-2016	Regular participation to fête de la sciences, including interventions on climate or demonstration of physics experiments.
2017	Ordre et chaos dans l'atmophère, présentation dans le cadre de *Pint of Science*

Invited presentations or lectures in conferences and workshops

[8] March 2017	*2D flows : From Graphene to Planet Atmosphere*, Simons Center for geometry, Stony Brook Workshop
[7] Sept 2016	*SIAM Mathematics of the Planet Earth*, Philadelphie PA (US), mini-symposium
[6] Mars 2015	*Theoretical advances in planetary flows and climate dynamics*, Les Houches, conference
[5] Mai 2014	*EUROMECH Colloquium : dimensionality of Turbulence*, Coventry (UK), lecture
[4] Mai 2014	*Fundamentals of climate, atmosphere and ocean dynamics*, Hambourg, workshop
[3] Mars 2013	*Geostrophic turbulence and active tracer transport in two dimensions* PCTS Princeton University, workshop
[2] Avril 2012	*European Geophysical Union*, Vienne (Autriche)
[1] Sept 2011	*50 years after the Turbulence Colloquium Marseille 1961*, Marseille, invited junior participant, conference

Seminars

2018	Toronto (colloquium of the Physics Department)
2018	IsTerre Grenoble
2016	IRPHE Marseille, GFDL Princeton
2015	LPO Brest
2014	BAS Cambridge U.
2013	LMFA ECL Lyon, CPT Marseille
2011	PAOC MIT, CAOS NYU, LEGI, Scripps UCSD
2010	LPT Toulouse, LPTMC Orsay, LPS ENS, McGill, Ladhyx
2009	ENS Lyon, GFDL Princeton

Publications soumises

[21] 2018 **A. Renaud, L.-P. Nadeau, A. Venaille** Periodicity Disruption in a Model Quasi-Biennial Oscillation *Submitted to Nature Physics*

[20] 2018 **A. Renaud, A. Venaille** Boundary streaming by internal waves *In review for Journal of Fluid Mechanics, arXiv :1708.00068*

Publications (see perso.ens-lyon.fr/antoine.venaille/ for preprints)

[19] 2018 **T. Dauxois, S. Joubaud, P. Odier, A. Venaille,** Instabilities of internal wave beams *2018 : Annu. Rev. Fluid Mech. 2018. 50 :128, arXiv :1702.07762*

[18] 2017 **P. Deplace, B. Marston, A. Venaille** Topological origin of equatorial waves *Science, arXiv :1702.07583*

[17] 2017 **Y. Yasuda, F. Bouchet, A. Venaille** A New Interpretation of Vortex-Split Stratospheric Sudden Warmings in Terms of Equilibrium Statistical Mechanics *Journal of Atmospheric Science, arXiv :1702.03716*

[16] 2016 **A. Venaille, L. Gostiaux, J. Sommeria,** A statistical mechanics approach to mixing in stratified fluids *J. Fluid Mechanics, 810, 554-583*

[15] 2016 **A. Renaud, A. Venaille, F. Bouchet,** Equilibrium statistical mechanics and energy partition for the shallow water model *J. Stat. Phys., 163(4), 784-843*

[14] 2015 **A. Venaille, T. Dauxois, S. Ruffo,** Violent relaxation in system with long range interactions *Phys. Rev. E-Rapid Communication, 92, 011001(R).*

[13] 2014 **A. Venaille, L.P. Nadeau, G. Vallis,** Ribbon Turbulence *Phys. of Fluids, 26(12), 126605.*

[12] 2012 **G. Bordes, A. Venaille, S. Joubaud, P. Odier, T. Dauxois,** Experimental observation of a strong mean flow induced by internal gravity waves *Physics of Fluids, 24(8), 086602.*

[11] 2012 **A. Venaille** , Bottom-trapped currents as statistical equilibrium states above topographic anomalies *J. Fluid Mechanics, 699, 500-510.*

[10] 2012 **A. Venaille, G. Vallis, S. Griffies,** The catalytic role of the beta effect in barotropization processes *J. Fluid Mechanics, 709, 490-515.*

[9] 2012 **F. Bouchet, A. Venaille** , Statistical mechanics of two-dimensional and geophysical turbulence *Physics Reports, 515(5), 227-295.*

[8] 2011 **A. Venaille, J. Le Sommer, J.M. Molines, B. Barnier** Stochastic variability of eddy driven oceanic flows above topography *Geo. Res. Lett., 38 : 16611*

[7] 2011 **A. Venaille, F. Bouchet** Oceanics rings and jets as statistical equilibria *Journal of Physical Oceanography : 41(10), 1860-1873*

[6] 2011 **A. Venaille, K.S. Smith, G. Vallis** Geography of baroclinic turbulence in the ocean : analysis with primitive equation and quasi-geostrophic simulations *Journal of Physical Oceanography : 41(9), 1605-1623*

[5] 2011 **A. Venaille, F. Bouchet** Solvable phase diagrams and ensemble inequivalence for two-dimensional and geophysical turbulent flows *J. Stat. Phys., 143 :2356*

[4] 2009 **A. Venaille, F. Bouchet** Ensemble inequivalence, bicritical points and azeotropy for generalized Fofonoff flows *Phys. Rev. Lett. 102 : 104501*

[3] 2008 **A. Venaille, J. Sommeria** Is mixing a self convolution process ? *Phys. Rev. Lett. 100 : 234506*

[2] 2007 **A. Venaille, J. Sommeria.** A dynamical equation for the distribution of a scalar advected by turbulence *Phys. of Fluids 19 : 028101*

Publications (cont.)

[1] 2005 **A. Venaille, P. Varona, M. Rabinovich** Synchronization and coordination of sequences in two neural ensemble *Phys Rev E 71 : 061909*

Book Chapters

[2] 2017 **F. Bouchet, A. Venaille** , Zonal flows as statistical equilibria *in Zonal Jets, Editor B. Galperin, P. Read, in press for Cambridge University Press arXiv :1602.06714*

[1] 2012 F. Bouchet, **A. Venaille**, Application of equilibrium statistical mechanics to atmospheres and oceans *Peyresq Lectures on Nonlinear Phenomena, vol. 3*

Conference Proceedings

[6] 2016 **A. Venaille**, L. Gostiaux, J. Sommeria, A statistical theory for mixing in stratified fluids *ISSF San Diego, Editor Kraig Winters.*

[5] 2013 **L. Gostiaux, A. Venaille, J. Sommeria,** Diffusion turbulente Ã travers une interface de densité : observations expérimentales et approche statistique . *Congrès Francais de Mécanique (pp. 4-12). Bordeaux, France.*

[4] 2012 **A. Venaille**, F. Bouchet, Are rings and jets statistical equilibrium states ? *ETC13, Journal of Physics : Conference Series.*

[3] 2011 **A. Venaille, J. Sommeria,** Modeling mixing in two-dimensional turbulence and stratified fluids *Turbulence in the atmosphere and oceans, IUTAM proceedings, Springer, Ed. D. Dritschel*

[2] 2008 **F. Bouchet, J. Barré, A. Venaille** , Equilibrium and out of equilibrium phase transitions in systems with long range interactions and in 2D flows , *AIP conference proceedings vol 970, editors : A. Campa et al*

[1] 2007 **A. Venaille, J. Sommeria** Turbulent mixing of a stably stratified fluid *comptes rendus du congrès francais de mécanique CFM2007-0505*

Bibliography

[1] F. Ait-Chaalal, T. Schneider, B. Meyer, and J. B. Marston. Cumulant expansions for atmospheric flows. *New Journal of Physics*, 18(2):025019, 2016.

[2] M. H. Alford. Redistribution of energy available for ocean mixing by long-range propagation of internal waves. *Nature*, 423(6936):159–162, 2003.

[3] D. G. Andrews and M. E. McIntyre. An exact theory of nonlinear waves on a lagrangian-mean flow. *Journal of Fluid Mechanics*, 89(04):609–646, 1978.

[4] B. K. Arbic and G. R. Flierl. Coherent vortices and kinetic energy ribbons in asymptotic, quasi two-dimensional f-plane turbulence. *Physics of Fluids*, 15:2177–2189, August 2003.

[5] B. K. Arbic and G. R. Flierl. Baroclinically Unstable Geostrophic Turbulence in the Limits of Strong and Weak Bottom Ekman Friction: Application to Midocean Eddies. *Journal of Physical Oceanography*, 34:2257, 2004.

[6] V. I. Arnold. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. In *Annales de l'institut Fourier*, volume 16, pages 319–361, 1966.

[7] V. I. Arnold. The asymptotic hopf invariant and its applications. In *Vladimir I. Arnold-Collected Works*, pages 357–375. Springer, 1974.

[8] V. I. Arnold and B. A. Khesin. *Topological methods in hydrodynamics*, volume 125. Springer Science & Business Media, 1999.

[9] J. K. Asboth, T. Rakovszky, and A. Alberti. Detecting topological invariants via losses in chiral symmetric floquet insulators. *arXiv preprint arXiv:1611.09670*, 2016.

[10] J. E. Avron, D. Osadchy, and R. Seiler. A topological look at the quantum hall effect. *Physics Today*, 56(8):38–42, 2003.

[11] V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort. Time reversal and holography with spacetime transformations. *Nature Physics*, 2016.

[12] C. E. Bardyn, M. A. Baranov, C. V. Kraus, E. Rico, A. İmamoğlu, P. Zoller, and S. Diehl. Topology by dissipation. *New Journal of Physics*, 15(8):085001, 2013.

[13] M. Bellec, U. Kuhl, G. Montambaux, and F. Mortessagne. Topological transition of dirac points in a microwave experiment. *Physical review letters*, 110(3):033902, 2013.

[14] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang. Quantum spin Hall effect and topological phase transition in HgTe quantum wells. *Science*, 314(5806):1757–1761, 2006.

[15] M. V. Berry. Classical adiabatic angles and quantal adiabatic phase. *Journal of Physics A: Mathematical and General*, 18(1):15, 1985.

[16] M. V. Berry. The bakerian lecture, 1987: quantum chaology. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 413, pages 183–198. The Royal Society, 1987.

[17] I. Bloch, J. Dalibard, and S. Nascimbene. Quantum simulations with ultracold quantum gases. *Nature Physics*, 8(4):267–276, 2012.

[18] S.J. Bolton et al. Jupiter's interior and deep atmosphere: The initial pole-to-pole passes with the juno spacecraft. *Science*, 356(6340):821–825, 2017.

[19] G. Bordes, A. Venaille, S. Joubaud, P. Odier, and T. Dauxois. Experimental observation of a strong mean flow induced by internal gravity waves. *Physics of Fluids*, 24(8):086602, 2012.

[20] F. Bouchet and M. Corvellec. Invariant measures of the 2D Euler and Vlasov equations. *Journal of Statistical Mechanics: Theory and Experiment*, 8:21, August 2010.

[21] F. Bouchet, C. Nardini, and T. Tangarife. Non-equilibrium statistical mechanics of the stochastic Navier–Stokes equations and geostrophic turbulence. In Warsaw University Press, editor, *5th Warsaw School of Statistical Physics*. 2014.

[22] F. Bouchet, F. Ragone, and J. Wouters. Sampling extreme heat waves using a large deviation algorithm. In *EGU General Assembly Conference Abstracts*, volume 19, page 12681, 2017.

[23] F. Bouchet and J. Sommeria. Emergence of intense jets and jupiter's great red spot as maximum-entropy structures. *Journal of Fluid Mechanics*, 464:165–207, 2002.

[24] F. Bouchet and A. Venaille. Statistical mechanics of two-dimensional and geophysical flows. *Physics reports*, 515(5):227–295, 2012.

[25] F. Bouchet and A Venaille. Zonal flows as statistical equilibria. *arXiv preprint arXiv:1602.06714*, 2016.

[26] D. Bouffard and L. Boegman. A diapycnal diffusivity model for stratified environmental flows. *Dynamics of Atmospheres and Oceans*, 61:14–34, 2013.

[27] B. Bradlyn, J. Cano, Z. Wang, M. G. Vergniory, C. Felser, R. J. Cava, and B. A. Bernevig. Beyond dirac and weyl fermions: Unconventional quasiparticles in conventional crystals. *Science*, 353(6299):aaf5037, 2016.

[28] F. P. Bretherton. Lamb waves in a nearly isothermal atmosphere. *Quarterly Journal of the Royal Meteorological Society*, 95(406):754–757, 1969.

[29] F. P. Bretherton and D. B. Haidvogel. Two-dimensional turbulence above topography. *Journal of Fluid Mechanics*, 78:129–154, 1976.

[30] F.P. Bretherton. Critical layer instability in baroclinic flows. *Quart. J. Roy. Meteor. Soc.*, 92:325–334, 1966.

[31] C. Brouzet, E. V. Ermanyuk, S. Joubaud, I. Sibgatullin, and T. Dauxois. Energy cascade in internal-wave attractors. *EPL (Europhysics Letters)*, 113(4):44001, 2016.

[32] O. Bühler. *Waves and mean flows*. Cambridge University Press, 2014.

[33] M. Büttiker. Absence of backscattering in the quantum hall effect in multiprobe conductors. *Physical Review B*, 38(14):9375, 1988.

[34] A. Campa, T. Dauxois, and S. Ruffo. Statistical mechanics and dynamics of solvable models with long-range interactions. *Physics Reports*, 480(3):57–159, 2009.

[35] J. G. Charney. *On the scale of atmospheric motions*. Cammermeyer in Komm., 1948.

[36] J. G. Charney. Geostrophic Turbulence. *Journal of Atmospheric Sciences*, 28:1087–1094, September 1971.

[37] P. H. Chavanis and J. Sommeria. Statistical mechanics of the shallow water system. *Phys. Rev. E*, 65(2):026302, February 2002.

[38] R. Chemke and Y. Kaspi. The latitudinal dependence of the oceanic barotropic eddy kinetic energy and macroturbulence energy transport. *Geophysical Research Letters*, 43(6):2723–2731, 2016.

[39] C.-K. Chiu, H. Yao, and S. Ryu. Classification of topological insulators and superconductors in the presence of reflection symmetry. *Physical Review B*, 88(7):075142, 2013.

[40] A. Colin de Verdiere. Keeping the freedom to build idealized climate models. *EOS*, 90, 2009.

[41] B. Cushman-Roisin and J.-M. Beckers. *Introduction to Geophysical Fluid Dynamics – Physical and Numerical Aspects*. Elsevier, 2nd edition, 2011.

[42] T. Dauxois, S. Joubaud, P. Odier, and A. Venaille. Instabilities of internal gravity wave beams. *arXiv preprint arXiv:1702.07762, accepted for Annual Review of Fluid Mechanics*, 2018.

[43] A. P. de Miranda, B. Barnier, and W. K. Dewar. On the dynamics of the Zapiola Anticyclone. *J. Geophys. Res.*, 104:21137 – 21150, September 1999.

[44] P. Delplace, J. B. Marston, and A. Venaille. Topological origin of geophysical waves. *arXiv preprint arXiv:1702.07583*, 2017.

[45] W.K. Dewar. Topography and barotropic transport control by bottom friction. *J. Mar. Res.*, 56:295–328, 1998.

[46] P. C. di Leoni, P. D. Mininni, and M. E. Brachet. Helicity, topology, and kelvin waves in reconnecting quantum knots. *Physical Review A*, 94(4):043605, 2016.

[47] H. A. Dijkstra and M. Ghil. Low-frequency variability of the large-scale ocean circulation: A dynamical systems approach. *Reviews of Geophysics*, 43(3), 2005.

[48] P. A. M. Dirac. Quantised singularities in the electromagnetic field. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 133(821):60–72, 1931.

[49] D. G. Dritschel and M. E. McIntyre. Multiple jets as pv staircases: the phillips effect and the resilience of eddy-transport barriers. *Journal of the Atmospheric Sciences*, 65(3):855–874, 2008.

[50] D. G. Dritschel, W. Qi, and J. B. Marston. On the late-time behaviour of a bounded, inviscid two-dimensional flow. *Journal of Fluid Mechanics*, 783:1–22, 2015.

[51] E. T. Eady. Long waves and cyclone waves. *Tellus*, 1(3):33–52, 1949.

[52] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto. Edge states and topological phases in non-hermitian systems. *Physical Review B*, 84(20):205128, 2011.

[53] M. Farge and R. Sadourny. Wave-vortex dynamics in rotating shallow water. *Journal of Fluid Mechanics*, 206:433–462, September 1989.

[54] S. Fauve. 1991 summer study program in geophysical fluid dynamics: patterns in fluid flow. Technical report, Woods Hole Oceanographic Institution, 1991.

[55] R. Ferrari and C. Wunsch. Ocean Circulation Kinetic Energy: Reservoirs, Sources, and Sinks. *Annual Review of Fluid Mechanics*, 41:253–282, January 2009.

[56] M. Fink. From loschmidt daemons to time-reversed waves. *Phil. Trans. R. Soc. A*, 374(2069):20150156, 2016.

[57] E. Fonda, D. P. Meichle, N. T. Ouellette, S. Hormoz, and D. P. Lathrop. Direct observation of kelvin waves excited by quantized vortex reconnection. *Proceedings of the National Academy of Sciences*, 111(Supplement 1):4707–4710, 2014.

[58] A. Frishman and G. Falkovich. New type of anomaly in turbulence. *Physical review letters*, 113(2):024501, 2014.

[59] D. C. Fritts and M. J. Alexander. Gravity wave dynamics and effects in the middle atmosphere. *Reviews of geophysics*, 41(1), 2003.

[60] L. Fu and C. L. Kane. Topological insulators with inversion symmetry. *Physical Review B*, 76(4):045302, 2007.

[61] B. Gallet, J. Herault, C. Laroche, F. Pétrélis, and S. Fauve. Reversals of a large-scale field generated over a turbulent background. *Geophysical & Astrophysical Fluid Dynamics*, 106(4-5):468–492, 2012.

[62] M. Ghil, M. D. Chekroun, and E. Simonnet. Climate dynamics and fluid mechanics: Natural variability and related uncertainties. *Physica D: Nonlinear Phenomena*, 237(14):2111–2126, 2008.

[63] A. E. Gill. *Atmosphere-ocean dynamics*, volume 30. Academic press, 1982.

[64] N. Grisouard. *Réflexions et réfractions non linéaires d'ondes internes de gravité*. PhD thesis, PhD thesis (in French), University of Grenoble, France, 2010.

[65] N. Grisouard and O. Bühler. Forcing of oceanic mean flows by dissipating internal tides. *Journal of Fluid Mechanics*, 708:250–278, 2012.

[66] N. Grisouard, M. Leclair, L. Gostiaux, and C. Staquet. Large scale energy transfer from an internal gravity wave reflecting on a simple slope. *Procedia IUTAM*, 8:119–128, 2013.

[67] I. Grooms, K. S. Smith, and A. J. Majda. Multiscale models for synoptic–mesoscale interactions in the ocean. *Dynamics of Atmospheres and Oceans*, 58:95–107, 2012.

[68] B. I. Halperin. Quantized Hall Conductance, Current-Carrying Edge States, and the Existence of Extended States in a Two-Dimensional Disordered Potential. *Physical Review B (Condensed Matter)*, 25(4):2185–2190, 1982.

[69] J. H. Hannay. Angle variable holonomy in adiabatic excursion of an integrable hamiltonian. *Journal of Physics A: Mathematical and General*, 18(2):221, 1985.

[70] M. Z. Hasan and C. L. Kane. Colloquium: Topological insulators. *Reviews of Modern Physics*, 82(4):3045–3067, November 2010.

[71] Y. Hatsugai. Chern number and edge states in the integer quantum Hall effect. *Physical review letters*, 71(22):3697, 1993.

[72] Y. Hatsugai. Edge states in the integer quantum hall effect and the riemann surface of the bloch function. *Physical Review B*, 48(16):11851, 1993.

[73] I. M. Held. The gap between simulation and understanding in climate modeling. *Bulletin of the American Meteorological Society*, 86(11):1609–1614, 2005.

[74] I. M. Held and V. D. Larichev. A Scaling Theory for Horizontally Homogeneous, Baroclinically Unstable Flow on a Beta Plane. *Journal of Atmospheric Sciences*, 53:946–952, April 1996.

[75] G. Holloway. Representing topographic stress for large-scale ocean models. *Journal of Physical Oceanography*, 22(9):1033–1046, 1992.

[76] E. Horne, A. Delache, and L. Gostiaux. Energetics aspects in direct numerical simulations of a turbulent stratified flow: irreversible mixing. In *VIIIth International Symposium on Stratified Flows*, volume 1, 2016.

[77] S. D. Huber. Topological mechanics. *Nature Physics*, 12(7):621–623, 2016.

[78] T. L. Hughes, E. Prodan, and B. A. Bernevig. Inversion-symmetric topological insulators. *Physical Review B*, 83(24):245132, 2011.

[79] M. Ilicak. Energetics and mixing efficiency of lock-exchange flow. *Ocean Modelling*, 83:1–10, 2014.

[80] G.N. Ivey, K.B. Winters, and J.R. Koseff. Density stratification, turbulence, but how much mixing? *Annual Review of Fluid Mechanics*, 40(1):169, 2008.

[81] J. Jucha, H. Xu, A. Pumir, and E. Bodenschatz. Time-reversal-symmetry breaking in turbulence. *Physical review letters*, 113(5):054501, 2014.

[82] C. L. Kane and T. C. Lubensky. Topological boundary modes in isostatic lattices. *Nature Physics*, 10(1):39–45, 2014.

[83] C. L. Kane and E. J. Mele. Quantum spin Hall effect in graphene. *Physical review letters*, 95(22):226801, 2005.

[84] C. L. Kane and E. J. Mele. Z2 topological order and the quantum spin Hall effect. *Physical review letters*, 95(14):146802, 2005.

[85] F. Karimpour and S.K. Venayagamoorthy. A simple turbulence model for stably stratified wall-bounded flows. *Journal of Geophysical Research: Oceans*, 119(2):870–880, 2014.

[86] T. Kataoka and T. R. Akylas. On three-dimensional internal gravity wave beams and induced large-scale mean flows. *Journal of Fluid Mechanics*, 769:621–634, 2015.

[87] E. Kazantsev, J. Sommeria, and J. Verron. Subgrid-scale eddy parameterization by statistical mechanics in a barotropic ocean model. *Journal of Physical Oceanography*, 28(6):1017–1042, 1998.

[88] G. N. Kiladis, M. C. Wheeler, P. T. Haertel, K. H. Straub, and P. E. Roundy. Convectively coupled equatorial waves. *Reviews of Geophysics*, 47(2), 2009.

[89] A. Kitaev. Periodic table for topological insulators and superconductors. *arXiv preprint arXiv:0901.2686*, 2009.

[90] D. Kleckner and W. T. M. Irvine. Creation and dynamics of knotted vortices. *Nature physics*, 9(4):253–258, 2013.

[91] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang. Quantum spin Hall insulator state in HgTe quantum wells. *Science*, 318(5851):766–770, 2007.

[92] R. H. Kraichnan. Inertial Ranges in Two-Dimensional Turbulence. *Physics of Fluids*, 10:1417–1423, July 1967.

[93] G. Lapeyre. Surface quasi-geostrophy. *Fluids*, 2(1):7, 2017.

[94] G. Lapeyre and P. Klein. Dynamics of the upper oceanic layers in terms of surface quasigeostrophy theory. *Journal of Physical Oceanography*, 36:165–176, 2006.

[95] R. B. Laughlin. Quantized Hall conductivity in two-dimensions. *Phys. Rev.*, B23(10):5632–5733, 1981.

[96] M. Leclair, N. Grisouard, L. Gostiaux, C. Staquet, and F. Auclair. Reflexion of a plane wave onto a slope and wave-induced mean flow. In *Stratified Flows Conference, Roma*, 2011.

[97] Y. Levin, R. Pakter, F. B. Rizzato, T. N. Teles, and F. P. C. Benetti. Nonequilibrium statistical mechanics of systems with long-range interactions. *Physics Reports*, 535(1):1–60, 2014.

[98] L. Levitov and G. Falkovich. Electron viscosity, current vortices and negative nonlocal resistance in graphene. *Nature Physics*, 2016.

[99] J. Lighthill. *Waves in fluids*. Cambridge university press, 2001.

[100] P.F. Linden. Mixing across a density interface produced by grid turbulence. *Journal of Fluid Mechanics*, 100(04):691–703, 1980.

[101] T. Louvet, P. Delplace, A. A. Fedorenko, and D. Carpentier. On the origin of minimal conductivity at a band crossing. *Physical Review B*, 92(15):155116, 2015.

[102] I.D. Lozovatsky and H.J.S. Fernando. Mixing efficiency in natural flows. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 371(1982):20120213, 2013.

[103] L. Lu, J. D. Joannopoulos, and M. Soljačić. Topological photonics. *Nature Photonics*, 8(11):821–829, 2014.

[104] V. Lucarini, R. Blender, C. Herbert, F. Ragone, S. Pascale, and J. Wouters. Mathematical and physical ideas for climate science. *Reviews of Geophysics*, 52(4):809–859, 2014.

[105] A. Lucas, J. Crossno, K. C. Fong, P. Kim, and S. Sachdev. Transport in inhomogeneous quantum critical fluids and in the Dirac fluid in graphene. *Physical Review B*, 93(7):075426, 2016.

[106] D. Lynden-Bell. Statistical mechanics of violent relaxation in stellar systems. *Monthly Notices of the Royal Astronomical Society*, 136:101, 1967.

[107] L. Maas. Wave focusing and ensuing mean flow due to symmetry breaking in rotating fluids. *Journal of Fluid Mechanics*, 437:13–28, 2001.

[108] L. Maas. Experiments on rotating flows: impact of rotation on flow through tilted rectangular ducts. *unpuplished*, 2008.

[109] A. Maffioli, G. Brethouwer, and E. Lindborg. Mixing efficiency in stratified turbulence. *Journal of Fluid Mechanics*, 794:R3, 2016.

[110] A. Majda and X. Wang. *Nonlinear dynamics and statistical theories for basic geophysical flows*. Cambridge University Press, 2006.

[111] A. J. Majda, R. R. Rosales, E. G. Tabak, and C. V. Turner. Interaction of large-scale equatorial waves and dispersion of Kelvin waves through topographic resonances. *Journal of Atmospheric Sciences*, 56(24):4118–4133, 1999.

[112] J. B. Marston. Trend: Looking for new problems to solve? consider the climate. *Physics*, 4:20, 2011.

[113] A. Mashayek, R. Ferrari, S. Merrifield, J. R. Ledwell, L. St Laurent, and A. N. Garabato. Topographic enhancement of vertical turbulent mixing in the southern ocean. *Nature Communications*, 8:14197, 2017.

[114] A. Mashayek, H. Salehipour, D. Bouffard, C. P. Caulfield, R. Ferrari, M. Nikurashin, W. R. Peltier, and W. D. Smyth. Efficiency of turbulent mixing in the abyssal ocean circulation. *Geophysical Research Letters*, 2017.

[115] A.D. McEwan. Internal mixing in stratified fluids. *Journal of Fluid Mechanics*, 128:59–80, 1983.

[116] A.D. McEwan. The kinematics of stratified mixing through internal wavebreaking. *Journal of Fluid Mechanics*, 128:47–57, 1983.

[117] M. J. McPhaden and A. E. Gill. Topographic scattering of equatorial Kelvin waves. *Journal of Physical Oceanography*, 17(1):82–96, 1987.

[118] J. C. McWilliams. Submesoscale currents in the ocean. In *Proc. R. Soc. A*, volume 472, page 20160117. The Royal Society, 2016.

[119] J. C. McWilliams, J. B. Weiss, and I. Yavneh. Anisotropy and Coherent Vortex Structures in Planetary Turbulence. *Science*, 264:410–413, April 1994.

[120] G. L. Mellor and T. Yamada. Development of a turbulence closure model for geophysical fluid problems. *Reviews of Geophysics*, 20(4):851–875, 1982.

[121] M. J. Mercier, D. Martinand, M. Mathur, L. Gostiaux, T. Peacock, and T. Dauxois. New wave generation. *Journal of Fluid Mechanics*, 657:308–334, 2010.

[122] W. J. Merryfield. Effects of stratification on quasi-geostrophic inviscid equilibria. *Journal of Fluid Mechanics*, 354:345–356, January 1998.

[123] D. Micard, Y. Dossmann, L. Gostiaux, and A. Venaille. Mixing efficiency in a lock exchange experiment. In *VIIIth International Symposium on Stratified Flows*, volume 1, 2016.

[124] L. Miller, R. E. Cheney, and B. C. Douglas. GEOSAT altimeter observations of Kelvin waves and the 1986-87 El Nino. *Science*, 239(4835):52, 1988.

[125] H. K. Moffatt. The degree of knottedness of tangled vortex lines. *Journal of Fluid Mechanics*, 35(01):117–129, 1969.

[126] H. K. Moffatt. The topology of turbulence. Lesieur, M (ed.) et al., Ecole de Physique des Houches UJF INPG Grenoble. *Les Houches session LXXIV*, 31, 2001.

[127] H. K. Moffatt. Helicity and singular structures in fluid dynamics. *Proceedings of the National Academy of Sciences*, 111(10):3663–3670, 2014.

[128] J. E. Moore. The birth of topological insulators. *Nature*, 464(7286):194–198, 2010.

[129] B. Moudjed, V. Botton, D. Henry, H. Ben Hadid, and J.-P. Garandet. Scaling and dimensional analysis of acoustic streaming jets. *Physics of Fluids*, 26(9):093602, 2014.

[130] L. M. Nash, D. Kleckner, A. Read, V. Vitelli, A. M. Turner, and W. T. M. Irvine. Topological mechanics of gyroscopic metamaterials. *PNAS*, 112(47):14495–14500, November 2015.

[131] A. Naso, P. H. Chavanis, and B. Dubrulle. Statistical mechanics of two-dimensional Euler flows and minimum enstrophy states. *European Physical Journal B*, 77:187–212, September 2010.

[132] J. Noh, S. Huang, D. Leykam, Y. D. Chong, K. Chen, and M. C. Rechtsman. Experimental observation of optical weyl points. *arXiv preprint arXiv:1610.01033*, 2016.

[133] W. L. Nyborg. Acoustic streaming. *Physical acoustics*, 2(Pt B):265, 1965.

[134] L. Onsager. Statistical hydrodynamics. *Il Nuovo Cimento (1943-1954)*, 6:279–287, 1949.

[135] T.R. Osborn. Estimates of the local rate of vertical diffusion from dissipation measurements. *Journal of Physical Oceanography*, 10(1):83–89, 1980.

[136] W.R. Peltier and C.P. Caulfield. Mixing efficiency in stratified shear flows. *Annual review of fluid mechanics*, 35(1):135–167, 2003.

[137] R. T. Pierrehumbert, I. M. Held, and K. L. Swanson. Spectra of local and nonlocal two-dimensional turbulence. *Chaos, Solitons & Fractals*, 4(6):1111–1116, 1994.

[138] R. A. Plumb. The interaction of two internal waves with the mean flow: Implications for the theory of the quasi-biennial oscillation. *Journal of the Atmospheric Sciences*, 34(12):1847–1858, 1977.

[139] R. A. Plumb and A. D. McEwan. The instability of a forced standing wave in a viscous stratified fluid: A laboratory analogue of. the quasi-biennial oscillation. *Journal of the atmospheric sciences*, 35(10):1827–1839, 1978.

[140] T Prastowo, R W Griffiths, G O Hughes, and A M Hogg. Mixing efficiency in controlled exchange flows. *Journal of Fluid Mechanics*, 600:235–244, 2008.

[141] H. Punzmann, N. Francois, H. Xia, G. Falkovich, and M. Shats. Generation and reversal of surface flows by propagating waves. *Nature Physics*, 10(9):658–663, 2014.

[142] E. M. Purcell. Life at low reynolds number. *American journal of physics*, 45(1):3–11, 1977.

[143] J. Rayleigh. *The theory of sound*, volume 2. Macmillan, 1896.

[144] A. Renaud and A. Venaille. Boundary streaming by internal waves. *arXiv preprint 0000*, 2017.

[145] A. Renaud, A. Venaille, and F. Bouchet. Equilibrium statistical mechanics and energy partition for the shallow water model. *Journal of Statistical Physics*, 163(4):784–843, 2016.

[146] G. M. Reznik and V. Zeitlin. The interaction of free rossby waves with semi-transparent equatorial waveguide–wave-mean flow interaction. *Nonlinear Processes in Geophysics*, 16(3):381, 2009.

[147] P. Rhines, editor. *The dynamics of unsteady currents*, volume 6. Wiley and Sons, 1977.

[148] N. Riley. Steady streaming. *Annual Review of Fluid Mechanics*, 33(1):43–65, 2001.

[149] D. Z. Rocklin, B. G. Chen, M. Falk, V. Vitelli, and T. C. Lubensky. Mechanical Weyl Modes in Topological Maxwell Lattices. *Physical Review Letters*, 116(13):135503–5, April 2016.

[150] M. S. Rudner and L. S. Levitov. Topological transition in a non-hermitian quantum walk. *Physical review letters*, 102(6):065703, 2009.

[151] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W.W. Ludwig. Topological insulators and superconductors: tenfold way and dimensional hierarchy. *New Journal of Physics*, 12(6):065010, 2010.

[152] H. Salehipour and W.R. Peltier. Diapycnal diffusivity, turbulent prandtl number and mixing efficiency in boussinesq stratified turbulence. *Journal of Fluid Mechanics*, 775:464–500, 2015.

[153] H. Salehipour, W.R. Peltier, and A. Mashayek. Turbulent diapycnal mixing in stratified shear flows: the influence of prandtl number on mixing efficiency and transition at high reynolds number. *Journal of Fluid Mechanics*, 773:178–223, 2015.

[154] R. Salmon. Hamiltonian fluid mechanics. *Annual review of fluid mechanics*, 20(1):225–256, 1988.

[155] R. Salmon. *Lectures on geophysical fluid dynamics*. Oxford University Press, 1998.

[156] R. Salmon, G. Holloway, and M. C. Hendershott. The equilibrium statistical mechanics of simple quasi-geostrophic models. *Journal of Fluid Mechanics*, 75:691–703, 1976.

[157] P. M. Saunders and B. A. King. Bottom Currents Derived from a Shipborne ADCP on WOCE Cruise A11 in the South Atlantic. *Journal of Physical Oceanography*, 25:329–347, March 1995.

[158] T. Scaffidi, N. Nandi, B. Schmidt, A. P. Mackenzie, and J E Moore. Hydrodynamic Electron Flow and Hall Viscosity. *ArXiv e-prints*, 2017.

[159] D. A. Schecter. Maximum entropy theory and the rapid relaxation of three-dimensional quasi-geostrophic turbulence. *Physical Review E*, 68(6):066309, December 2003.

[160] N. Schine, A. Ryou, A. Gromov, A. Sommer, and J. Simon. Synthetic landau levels for photons. *Nature*, 2016.

[161] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig. Classification of topological insulators and superconductors in three spatial dimensions. *Physical Review B*, 78(19):195125, 2008.

[162] R. B. Scott and F. Wang. Direct Evidence of an Oceanic Inverse Kinetic Energy Cascade from Satellite Altimetry. *Journal of Physical Oceanography*, 35:1650, 2005.

[163] B. Semin, G. Facchini, F. Pétrélis, and S. Fauve. Generation of a mean flow by an internal wave. *Physics of Fluids*, 28(9):096601, 2016.

[164] A. Shapere and F. Wilczek. Geometry of self-propulsion at low reynolds number. *Journal of Fluid Mechanics*, 198:557–585, 1989.

[165] T. G. Shepherd. Symmetries, conservation laws, and hamiltonian structure in geophysical fluid dynamics. *Advances in Geophysics*, 32:287–338, 1990.

[166] M. G. Silveirinha. Chern invariants for continuous media. *Physical Review B*, 92(12):125153, 2015.

[167] K. S. Smith. The geography of linear baroclinic instability in earth's oceans. *Journal of Marine Research*, 65(5):655–683, 2007.

[168] K. S. Smith and G. K. Vallis. The Scales and Equilibration of Midocean Eddies: Freely Evolving Flow. *Journal of Physical Oceanography*, 31:554–571, 2001.

[169] K. S. Smith and G. K. Vallis. The scales and equilibration of midocean eddies: Forced–dissipative flow. *Journal of Physical Oceanography*, 32(6):1699–1720, 2002.

[170] J. Sommeria. Two-dimensional turbulence. In *New trends in turbulence Turbulence: nouveaux aspects*, pages 385–447. Springer, 2001.

[171] A. Souslov, B. C. van Zuiden, D. Bartolo, and V. Vitelli. Topological sound in active-liquid metamaterials. *arXiv preprint arXiv:1610.06873*, 2016.

[172] P. H. Stone. Baroclinic adjustment. *Journal of the Atmospheric Sciences*, 35(4):561–571, 1978.

[173] R. Süsstrunk and S. D. Huber. Observation of phononic helical edge states in a mechanical topological insulator. *Science*, 349(6243):47–50, 2015.

[174] B. R. Sutherland. *Internal gravity waves*. Cambridge University Press, 2010.

[175] E.G. Tabak and F.A. Tal. Mixing in simple models for turbulent diffusion. *Communications on pure and applied mathematics*, 57(5):563–589, 2004.

[176] P. Tabeling. Two-dimensional turbulence: a physicist approach. *Physics Reports*, 362(1):1–62, 2002.

[177] S. Thalabard, B. Dubrulle, and F. Bouchet. Statistical mechanics of the 3d axisymmetric euler equations in a taylor–couette geometry. *Journal of Statistical Mechanics: Theory and Experiment*, 2014:P01005, 2014.

[178] A. F. Thompson and W. R. Young. Scaling Baroclinic Eddy Fluxes: Vortices and Energy Balance. *Journal of Physical Oceanography*, 36:720, 2006.

[179] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. Dennijs. Quantized Hall Conductance in a Two-Dimensional Periodic Potential. *Physical Review Letters*, 49(6):405–408, 1982.

[180] A. M. Treguier. Topographically generated steady currents in barotropic turbulence. *Geophysical and Astrophysical Fluid Dynamics*, 47(1-4):43–68, 1989.

[181] A. M. Treguier and B. Lien Hua. Oceanic quasi-geostrophic turbulence forced by stochastic wind fluctuations. *Journal of physical oceanography*, 17(3):397–411, 1987.

[182] A.M. Treguier and B.L. Hua. Influence of bottom topography on stratified quasi-geostrophic turbulence in the ocean. *Geophysical and Astrophysical Fluid Dynamics*, 43(3-4):265–305, 1988.

[183] G. K. Vallis. Geophysical fluid dynamics: whence, whither and why? In *Proc. R. Soc. A*, volume 472, page 20160140. The Royal Society, 2016.

[184] G. K. Vallis. *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation*. Cambridge University Press, Cambridge, U.K., 2nd edition, 2017.

[185] J. Vanneste. Balance and Spontaneous Wave Generation in Geophysical Flows. *Annual Review of Fluid Mechanics*, 45:147–172, 2013.

[186] A. Venaille. Bottom-trapped currents as statistical equilibrium states above topographic anomalies. *Journal of Fluid Mechanics*, 699:500–510, 2012.

[187] A. Venaille and F. Bouchet. Statistical Ensemble Inequivalence and Bicritical Points for Two-Dimensional Flows and Geophysical Flows. *Physical Review Letters*, 102:104501, March 2009.

[188] A. Venaille and F. Bouchet. Oceanic rings and jets as statistical equilibrium states. *Journal of Physical Oceanography*, 41(10):1860–1873, October 2011.

[189] A. Venaille and F. Bouchet. Solvable phase diagrams and ensemble inequivalence for two-dimensional and geophysical flows. *Journal of Statistical Physics*, 143(2):346–380, 2011.

[190] A. Venaille, T. Dauxois, and S. Ruffo. Violent relaxation in two-dimensional flows with varying interaction range. *Physical Review E*, 92(1):011001, 2015.

[191] A. Venaille, L. Gostiaux, and J. Sommeria. A. statistical theory of mixing in stratified fluids. In *VIIIth International Symposium on Stratified Flows*, volume 1, 2016.

[192] A. Venaille, L. Gostiaux, and J. Sommeria. A. statistical mechanics approach to mixing in stratified fluids. *Journal of Fluid Mechanics*, 810:554–583, 2017.

[193] A. Venaille, J. Le Sommer, J. M. Molines, and B. Barnier. Stochastic variability of oceanic flows above topography anomalies. *Geophysical Research Letters*, 38(16611), 2011.

[194] A. Venaille, L.-P. Nadeau, and G. K. Vallis. Ribbon turbulence. *Physics of Fluids*, 26(12):126605, 2014.

[195] A. Venaille and J. Sommeria. A. dynamical equation for the distribution of a scalar advected by turbulence. *Physics of Fluids*, 19:028101, February 2007.

[196] A. Venaille and J. Sommeria. Is Turbulent Mixing a Self-Convolution Process? *Physical Review Letters*, 100:234506, June 2008.

[197] A. Venaille, G. K. Vallis, and K. S. Smith. Baroclinic turbulence in the ocean: analysis with primitive equation and quasi-geostrophic simulations. *Journal of Physical Oceanography*, 41(9):1605–1623, 2011.

[198] A. Venaille, G.K. Vallis, and S.M. Griffies. The catalytic role of the beta effect in barotropization processes. *Journal of Fluid Mechanics*, 709:490–515, 2012.

[199] A. Venaille, P. Varona, and M. I. Rabinovich. Synchronization and coordination of sequences in two neural ensembles. *Phys. Rev. E*, 71(6):061909, June 2005.

[200] S. K. Venayagamoorthy and J. R. Koseff. On the flux richardson number in stably stratified turbulence. *Journal of Fluid Mechanics*, 798:R1 (10 pages), 7 2016.

[201] A. Villois, G. Krstulovic, and D. Proment. (non)-universality of vortex reconnections in superfluids. *arXiv preprint arXiv:1612.00386*, 2016.

[202] A. Vishwanath. Viewpoint: Where the weyl things are. *Physics*, 8:84, 2015.

[203] Y.-T. Wang, P.-G. Luan, and S. Zhang. Coriolis force induced topological order for classical mechanical vibrations. *New Journal of Physics*, 17(7):073031, 2015.

[204] T. Warn. Statistical mechanical equilibria of the shallow water equations. *Tellus Series A*, 38:1–11, January 1986.

[205] P. B. Weichman. Competing turbulent cascades and eddy-wave interactions in shallow water equilibria. *Physical Review Fluids*, 2(3):034701, 2017.

[206] P. B. Weichman and D. M. Petrich. Statistical Equilibrium Solutions of the Shallow Water Equations. *Physical Review Letters*, 86:1761–1764, February 2001.

[207] W. Weijer, M. E. Maltrud, W. B. Homoky, K. L. Polzin, and L. R. M. Maas. Eddy-driven sediment transport in the argentine basin: Is the height of the zapiola rise hydrodynamically controlled? *Journal of Geophysical Research: Oceans*, 120(3):2096–2111, 2015.

[208] J. S. Wettlaufer. Editorial: Climate science: An invitation for physicists. *Physical review letters*, 116(15):150002, 2016.

[209] P. B. Wiegmann. Nonlinear hydrodynamics and fractionally quantized solitons at the fractional quantum hall edge. *Physical review letters*, 108(20):206810, 2012.

[210] P. B. Wiegmann. Hydrodynamics of euler incompressible fluid and the fractional quantum hall effect. *Physical Review B*, 88(24):241305, 2013.

[211] C. Wunsch and R. Ferrari. Vertical mixing, energy, and the general circulation of the oceans. *Annu. Rev. Fluid Mech.*, 36:281–314, 2004.

[212] J.-H. Xie and J. Vanneste. Boundary streaming with navier boundary condition. *Physical Review E*, 89(6):063010, 2014.

[213] Y. Xiong, T. Wang, and P. Tong. The effects of dissipation on topological mechanical systems. *Scientific Reports*, 6, 2016.

[214] B. Yang, M. Lawrence, W. Gao, Q. Guo, and S. Zhang. One-way helical electromagnetic wave propagation supported by magnetized plasma. *Scientific reports*, 6, 2016.

[215] E. Yarom and E. Sharon. Experimental observation of steady inertial wave turbulence in deep rotating flows. *Nature Physics*, 10(7):510–514, 2014.

[216] Y. Yasuda, F. Bouchet, and A. Venaille. A. new interpretation of vortex-split stratospheric sudden warmings in terms of equilibrium statistical mechanics. *arXiv preprint arXiv:1702.03716, accepted for Journal of Atmospheric Science*, 2017.

[217] J. Zaanen. Electrons go with the flow in exotic material systems. *Science*, 351(6277):1026–1027, 2016.

[218] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit. Observation of a topological transition in the bulk of a non-hermitian system. *Physical review letters*, 115(4):040402, 2015.

[219] H. P. Zhang, B. King, and H. L. Swinney. Experimental study of internal gravity waves generated by supercritical topography. *Physics of Fluids*, 19(9):096602, 2007.