THERMODYNAMICS OF FOFONOFF FLOWS

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MOTIVATION

A class of solutions of idealized ocean circulation model

Fofonoff modes are stable steady solutions of an idealized model of oceanic circulation at midlatitude. They are presented in each textbook of oceanography and have played a historical rôle in this field.

Organization of oceanic flows into large-scale coherent structures

This is a generic property of 2D turbulent flows. An explanation of this organization in terms of equilibrium statistical mechanics was first proposed by Onsager (1949) for point vortices and then by Robert-Miller-Sommeria (1990) for the continuum Euler equation, which can be extented to geophysical flows. Fofonoff modes are examples of large scale structures predicted by this theory.

A system with long range interactions, at equilibrium

2D turbulence is a promising field of application for statistical mechanics of systems with long range interactions. One striking feature that share those sytems is the existence of negative specific heat : temperature decreases when the energy increases. This is a consequence of the lack of additivity of the energy.

We propose for the first time a theoretical description of such Fofonoff flows in the context of statistical theories

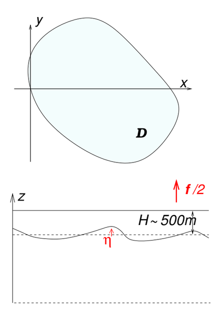
MODEL

1-1/2 layer quasi-geostrophic (QG) equations

$$\partial_t q+\mathbf{v}\cdot\nabla q=0 \quad \text{with} \quad q=\Delta\psi-\frac{\psi}{R^2}+h\,, \quad \mathbf{v}=\mathbf{e}_z\times\nabla\psi, \quad \psi=\psi_{fr} \text{ on } \partial\mathcal{D}\,,\ <\psi>=0$$

Stationnary states: $\partial_t q=0 \Leftrightarrow \mathbf{v}\cdot\nabla\, q=0 \Leftrightarrow \boldsymbol{q=f(\psi)}$

Fofonoff solutions ($h=by$) are obtained assuming a **linear** $\boldsymbol{q-\psi}$ relation.



METHOD

Equilibrium states : variational problem

Stable (thermodynamically) Fofonoff flows are solutions of the following variational problem :

$$\max_{q}\{\mathcal{S}[q]\mid\mathcal{E}[q]=E\ \&\ \mathcal{C}[q]=\Gamma\} \quad \textbf{(microcanonical ensemble)}$$

- $\mathcal{E}[q]=-\frac{1}{2}\langle\mathcal{O}[q-h]|q-h\rangle$: energy (in QG 1-1/2 layer, $\mathcal{E}=\frac{1}{2}\left\langle(\nabla\psi)^2+\frac{\psi^2}{R^2}\right\rangle$ and thus $\mathcal{O}=(\Delta-R^{-2})^{-1}$)
- $\mathcal{C}[q]=<q>$: circulation
- $\mathcal{S}[q]=-\frac{1}{2}<q|q>$ is a generalized entropy, here the opposite of the enstrophy.

Critical points : $\delta\mathcal{S}-\beta\delta\mathcal{E}-\gamma\delta\mathcal{C}=0\Leftrightarrow f(\psi)=\beta\psi-\gamma$ (Original Fofonoff solution: $\beta\gg1$)

The solutions of a variational problem are necessary the solutions of a more constrained dual problem. We will see that is it sufficient to consider two easier dual subproblems in order to solve the first one:

$$J(\beta,\gamma)=\min_{q}\{-\mathcal{S}[q]\ +\beta\,\mathcal{E}[q]+\gamma\mathcal{C}\,[q]\} \quad \textbf{(grand canonical ensemble)}$$

$$F(\beta,\Gamma)=\min_{q}\{-\mathcal{S}[q]\ +\beta\,\mathcal{E}[q]\mid\mathcal{C}[q]=\Gamma\} \quad \textbf{(canonical ensemble)}$$

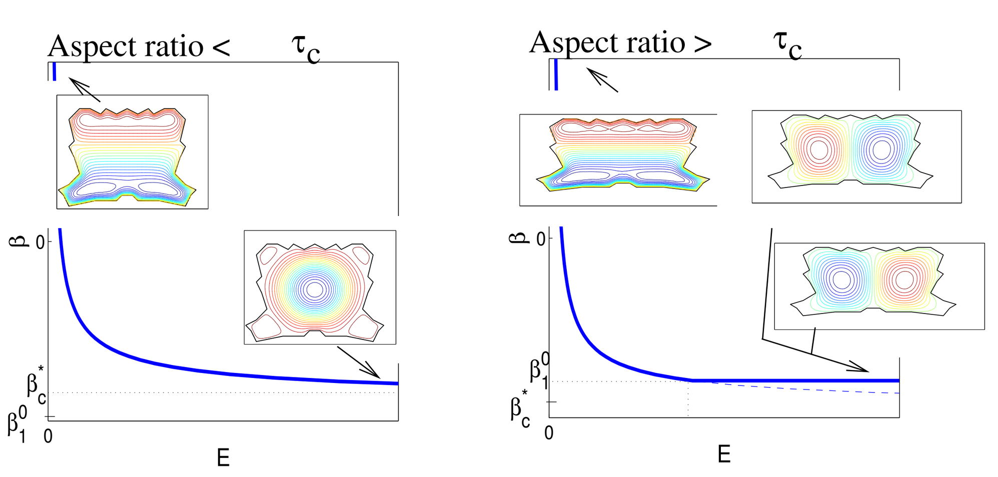
Ensemble inequivalence

By computing the energy and circulation of grand canonical solutions, we prove that the constraints values are not achieved above a parabola $\mathcal{P}$ *(see phase diagrams for 2D Euler)* in this ensemble : it is an ensemble inequivalence area. We look then for solutions of the canonical variational problem in this area. We find that it exists a solution for each value of the constraints : this ensemble is equivalent to the microcanonical one.

Geometry governed second order phase transitions

- **i) Domain with symmetry:** let us call h_0 the projection of the topography h on the laplacian eigenvector e^0 associated to the smallest eigenvalue while satisfying $\langle e^0\rangle=0$
  - **i-a)** $h_0=0$ and the domain is sufficiently stretched in a direction perpendicular to its symmetry axis (aspect ratio greater than a critical value : $\tau>\tau_c$): there is a microcanonical second order transition line $\mathcal{P}_0$ *(see right panel of the 2D Euler phase diagram and of the figure below).*
  - **i-b)** $h_0=0$ and $\tau<\tau_c$: there is a microcanonical first order transition line *(see left panel of the 2D Euler phase diagram and of the figure below)*
  - **i-c)** $h_0\neq0$ and $\tau>\tau_c$: there is no microcanonical transitions.
  - **i-d)** $h_0\neq0$ and $\tau<\tau_c$: see case **i-b)**.
- **ii) Domain without symmetry:** see case **i-b)**.

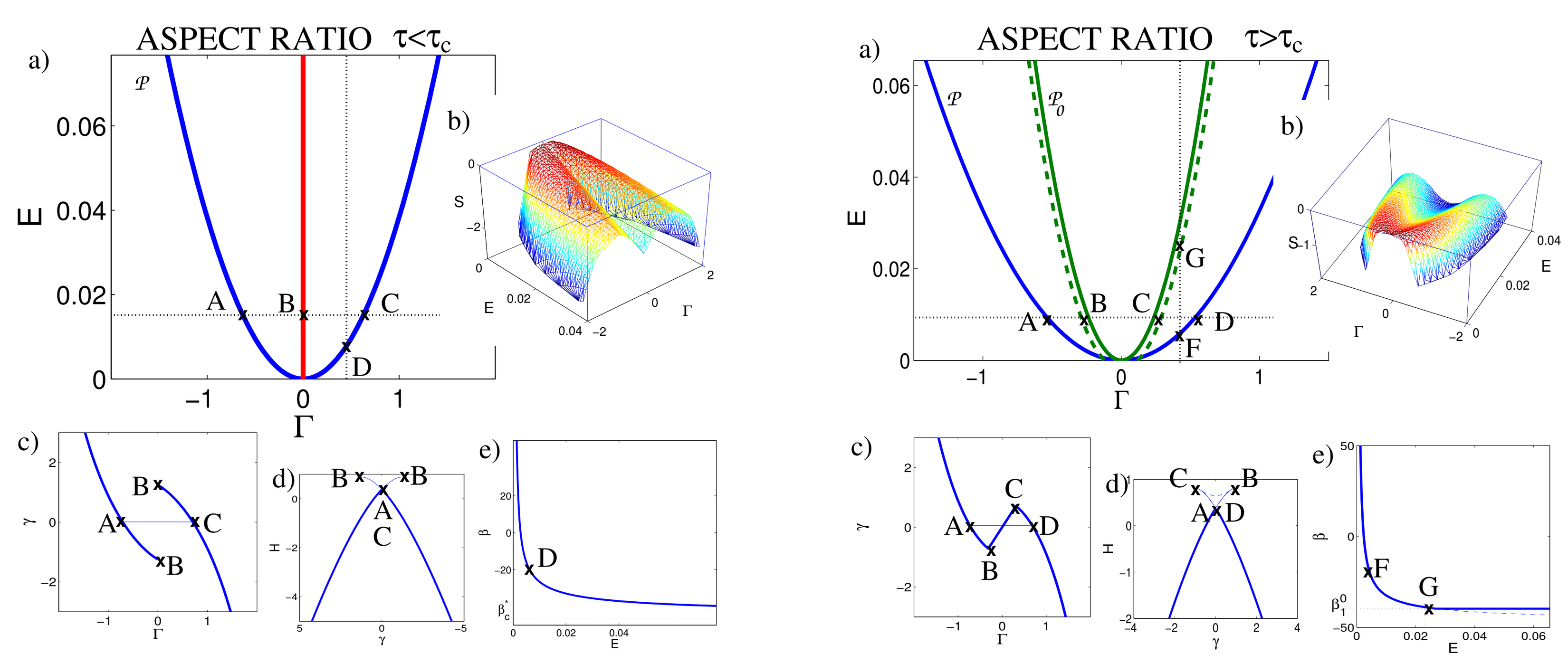
We obtain an explicit criteria to compute the sign of $\tau-\tau_c$



Domain with a symmetry axis in the y direction and with $h=by$. Γ fixed. When the domain is sufficiently stretched, there is a 2nd order phase transition from Fofonoff modes at low energy to a dipole at high energy which breaks the symmetry. *Left panel : case i-b) . Right panel : case i-a).*

PHASE DIAGRAMS

Phase diagram for the 2D Euler equation ($R=+\infty$, $h=0$)



a) Green line : microcanonical 2nd order phase transition. Red line : microcanonical first order phase transition. Blue line : boundary of ensemble inequivalence area. b) Equilibrium entropy S vs the parameters E and Γ. c) E fixed, $\gamma=\partial S/\partial\Gamma$ vs Γ e) Γ fixed. $\gamma=\partial S/\partial E$ vs E. The ensemble inequivalence area is associated with a first order transition in the ensemble where there is only one constraint on the energy (see the maxwell construction figure c and the corresponding free energy $H=-S+\gamma\Gamma$ figure d).

Addition of R, of a topography, generalisation to n-layers

For the 1-1/2 layer model, the criteria for the phase transition does not depend on R.

The addition of a topography $h\neq0$ only changes the location of the minimum of the red, blue and green lines in the phase diagram of the 2D Euler equation.

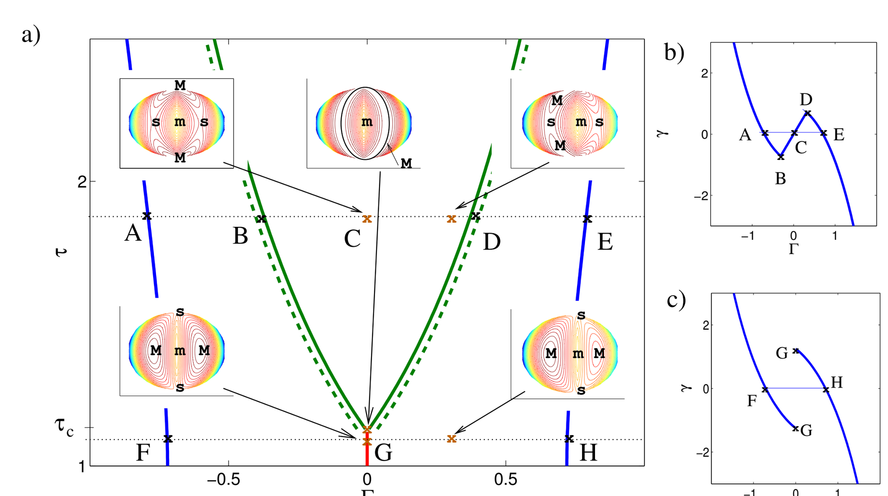
The criteria and the associated phase transitions presented here still hold for multi layers QG equations. In this case q is a vector whose each component is advected by a non divergent velocity field.

NEW PHASE TRANSITIONS

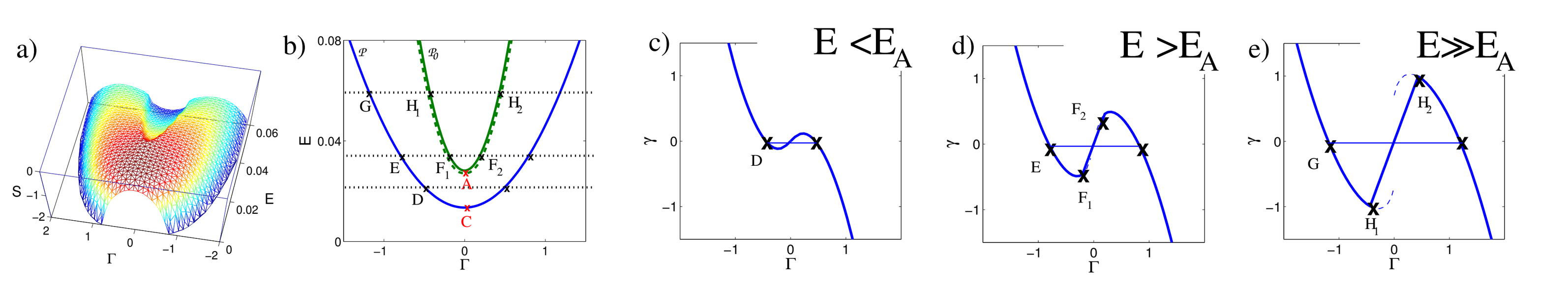
We present phase transitions whose possible existence was predicted in the context of systems with long range interactions but yet never observed in any physical system.

Bicritical point

We consider a fixed energy for 2D Euler equation in a rectangular domain of aspect ratio τ, taken as an external parameter. In the microcanonical ensemble, there is a bifurcation from a first order transition line to two second order transition lines at a critical value $\tau=\tau_c$. Such a bifurcation is referred as a bicritical point. Insets are projection of the Entropy $\mathcal{S}[q]$ in a plane (q_1^0,q_1) for fixed energy and circulation (**s**addle, **M**aximum,**m**inimum) .



Second order azeotropy



In this example we considered a domain with a symmetry axis in the y direction such that $\tau>\tau_c$ and a a topography $h(y)=by$ in a 1-1/2 layer QG model. If we take the energy as an external parameter, there is the simultaneous appearance of two second order phase transitions in the microcanonical ensemble, at point A, figure b. This is the signature of second order azeotropy. The corresponding flow is presented in the left column of the poster.

CONCLUSION

We obtained analytically a general, explicit criteria on the domain shape to determine the existence of a 2nd order phase transition and to compute the critical energy. Strikingly this is valid whatever the topography, for a large class of models including Euler and multilayers QG equations.

This shows that many new thermodynamical phenomena as negative specific heat, bicritical points and second order azeotropy are generic for many idealized 2D turbulent flows.Those flows may be studied experimentally using magnetized electron columns or three dimensional tanks with small height compared to the horizontal scale, with a further ordering (strong rotation or a transverse magnetic field).