An introduction to econophysics and the Multifractal Random walk

Vincent Vargas

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Why put probability in financial markets ?

- Typical daily return $r_t = \ln S_t/S_{t-1} \approx (S_t - S_{t-1})/S_{t-1}$:

$$r_t \approx 10^{-4}, \;\; |r_t| \approx 10^{-2}.$$

Why put probability in financial markets ?

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Why put probability in financial markets ?

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- Huge amounts of data, interacting agents : probabilistic approach is natural (statistical physics)
- Don't let finance to economists because they really have a great sense of humour :
  - Eugene Fama (Nobel prize 2013) : markets are efficient.
  - Robert Shiller (Nobel prize 2013) : markets are unefficient !

Modelling financial markets

A model of asset price $(S_t)_{t\geq 0}$ must take into account :

- Week-end, holidays
- Overnight ($\sim$ 2h)
- open/close
- News Macro (14h30)
- Discretization effects : tick size at high frequency

Modelling financial markets

In the sequel, we will denote $X_t = \ln(S_t)$. All time scales τ are interesting ; we set :

$$r_t = r_t^{(\tau)} = X_{t+\tau} - X_t.$$

There are discrete models with fixed τ (GARCH, etc...) and continuous models which therefore give a rule to relate the distribution of returns at different scales τ (Local or Stochastic volatility models, Multifractal models, etc...).

Modelling financial markets

One must distinguish two scales (we denote τ_c the time which corresponds to 100 trades : typically, $\tau_c \sim 1 - 10$ mins.) :

- $\tau \leq \tau_c$: High frequency trading (HF). The price process is not well defined : tick, bid-ask spread effect. Study of the order book, limit orders, market orders, etc... Returns can be correlated. See talk of T. Jaisson and I. Mastromatteo
- $\tau > \tau_c$: Returns are decorrelated.

There is no benchmark continuous model which models all time scales. In the sequel of this talk, we consider $\tau > \tau_c$.

Modelling financial markets

In the discrete case, we write $(r_t = \ln S_{(t+1)\tau}/S_{t\tau})$:

- $r_t = \sigma_t \epsilon_t$
- $(\sigma_t)_{t\in\mathbb{Z}}$ is the volatility (highly correlated).
- $(\epsilon_t)_{t\in\mathbb{Z}}$ is an i.i.d. sequence of variance 1 (typically with normal or student distribution).

In the continuous case, we write :

- $dS_t/S_t = \sigma_t dW_t + (\text{Jumps})$
- $(\sigma_t)_{t\in\mathbb{R}}$ is the volatility (highly correlated).
- $(W_t)_{t\geq 0}$ is Brownian motion (BM)

Popular models in mathematical finance or econophysics

- Asymmetric GARCH(1,1) $(\alpha + \beta_-/2 + \beta_+/2 < 1)$:

$$\sigma_t^2 = \sigma^2 + \alpha(\sigma_{t-1}^2 - \sigma^2) + \beta_-(r_{t-1}^2 - \sigma^2)1_{r_{t-1}<0} + \beta_+(r_{t-1}^2 - \sigma^2)1_{r_{t-1}>0}.$$

- Levy process : dS_t/S_t is a Lévy process (Black Scholes : BM+drift)
- Local volatility : $dS_t/S_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$
- Stochastic volatility : $dS_t/S_t = \sigma_t dW_t$ and $(\sigma_t)_{t\geq 0}$ solution of an SDE.

Multifractal random walk (MRW)

Discrete time (at scale τ) :

- $r_t = \sigma_t \epsilon_t, \quad (\sigma_t)_t \perp (\epsilon_t)_t$
- $(\epsilon_t)_{t \in \mathbb{Z}}$ i.i.d. standard Gaussian
- $\sigma_t = \sigma e^{\lambda \omega_t^\tau - \lambda^2 E[(\omega_t^\tau)^2]}$, $(\omega_t^\tau)_t$ is a centered Gaussian sequence :

$$E[\omega_s^\tau \omega_t^\tau] = \ln^+ \frac{T}{|t-s|\tau + \tau}$$

  - σ : average volatility
  - λ^2 : intermittency parameter
  - T : integral scale (cut-off)

Multifractal random walk (MRW)

Continuous time : $dS_t/S_t = \sigma_t dW_t$ where the volatility process is

$$\sigma_t = \sigma e^{\lambda\omega_t - \lambda^2 E[(\omega_t)^2]}$$

where ω is a centered Gaussian "process" (independent of W) with covariance

$$E[\omega_s \omega_t] = \ln^+ \frac{T}{|t-s|}$$

Problem : it makes no sense !

Multifractal random walk (MRW)

Write :

$$\int_{[0,t]} \sigma_s dW_s = B_{\int_{[0,t]} \sigma_s^2 ds}.$$

where B is a BM. Then, one can define the integrated volatility process M (Gaussian multiplicative chaos) :

$$M[0,t] = \sigma^2 \int_{[0,t]} e^{2\lambda\omega_s - 2\lambda^2 E[\omega_s^2]} ds, \quad t \geq 0$$

Definition (Bacry, Delour, Muzy, 2002)

The Multifractal Random walk (MRW) is simply $B_{M[0,t]}$.

Remark

The model is a time changed BM. This idea appears already in Mandelbrot, Taylor (1967).

Historics

- 1962 : Kolmogorov-Obukhov : lognormal model (*Journal of Fluid Mechanics*).
- 1972 : Mandelbrot defines the limit lognormal model.
- 1985 : Kahane defines the theory of Gaussian multiplicative chaos (Sur le chaos multiplicatif, *Annales Scientifiques et Mathematiques Quebec*).

Gaussian multiplicative chaos (volatility)

M is defined by a limit procedure $M = \lim_{\tau \to 0} M_\tau$:

$$M_\tau[0,t] = \sigma \int_{[0,t]} e^{2\lambda \omega^\tau_{s/\tau} - 2\lambda^2 E[(\omega^\tau_{s/\tau})^2]} ds, \quad t \geq 0$$

where $(\omega^\tau_s)_s$ is the discrete Gaussian process :

$$E[\omega^\tau_{s/\tau} \omega^\tau_{t/\tau}] = \ln^+ \frac{T}{|t-s| + \tau}$$

Local volatility : $dS_t/S_t = \sigma(S_t)dW_t$ good model ?

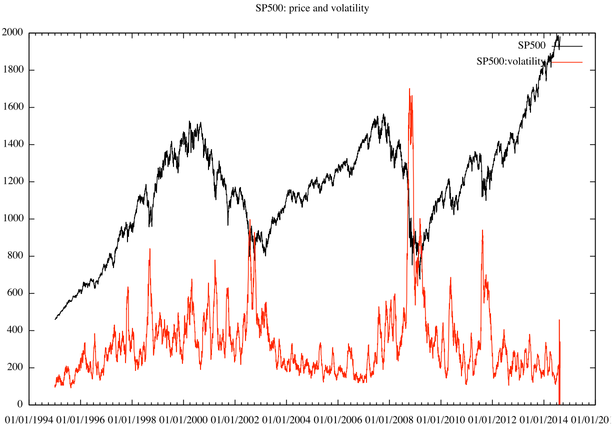

FIGURE: SP500 :1995-2014. Notice that volatility is NOT a function of price

Lévy process : good model ?

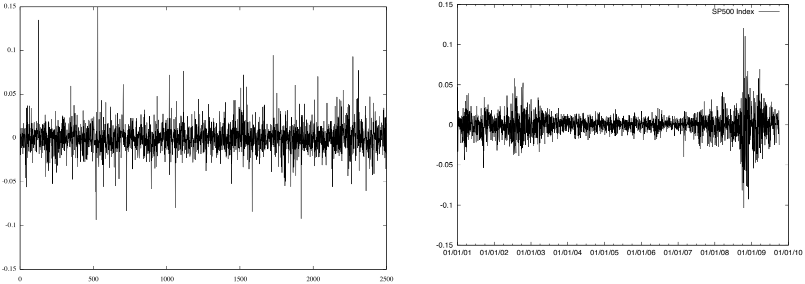

FIGURE: Simulation of independent Student(3) and SP500 (2001-2009). Notice that i.i.d. variables do not exhibit clustering.

Intermittency of MRW as a function of λ^2

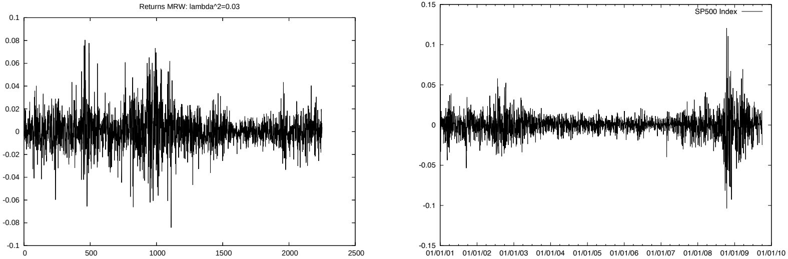

FIGURE: Returns of MRW : $\lambda^2 = 0.03$ and SP500 (2001-2009).

Intermittency of MRW as a function of λ^2

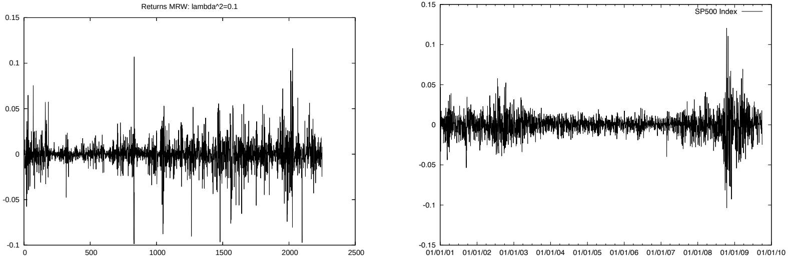

FIGURE: Returns of MRW : $\lambda^2 = 0.1$ and SP500 (2001-2009).

Intermittency of MRW as a function of λ^2

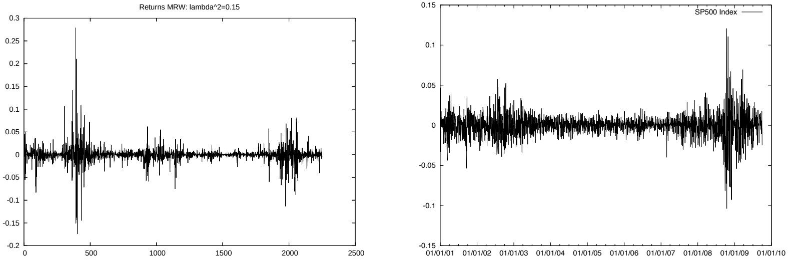

FIGURE: Returns of MRW : $\lambda^2 = 0.15$ and SP500 (2001-2009).

Intermittency of the SP500

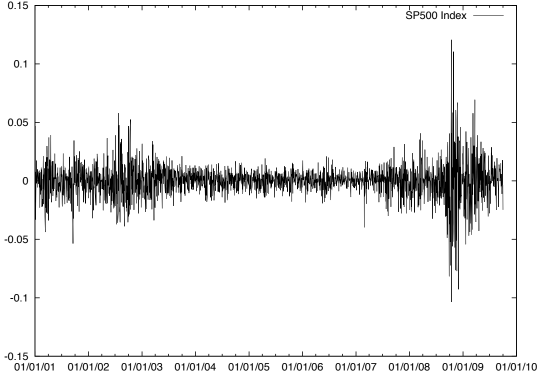

FIGURE: Returns of the SP500 on the period 2001-2009.

Intermittency of the Nasdaq 100 index

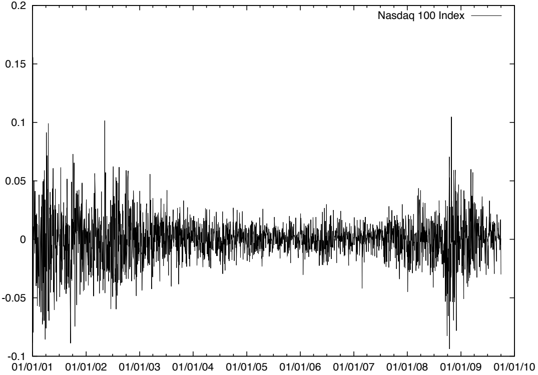

FIGURE: Returns of the Nasdaq index on the period 2001-2009.

Empirical stylized facts

Stylized fact : "universal" (statistical) signature of all assets : stocks, indices, currencies, bonds, etc... But some are specific like the leverage effect for stocks or indices.

Next slides : empirical study of the indices SP500 and the Nasdaq 100 Index.

Returns are approximately decorrelated

The log price $X_t = \ln S_t$ of a good model in finance must satisfy :

$$E[X_s(X_t - X_s)] = 0.$$

Correlation of returns of the Nasdaq 100

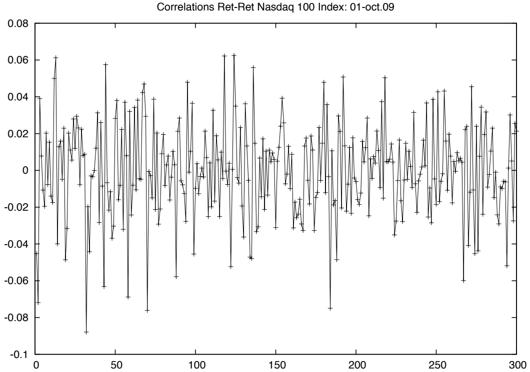

FIGURE: Empirical correlation of the daily returns of the Nasdaq index on the period 2001-2009.

Correlation of returns of the SP500

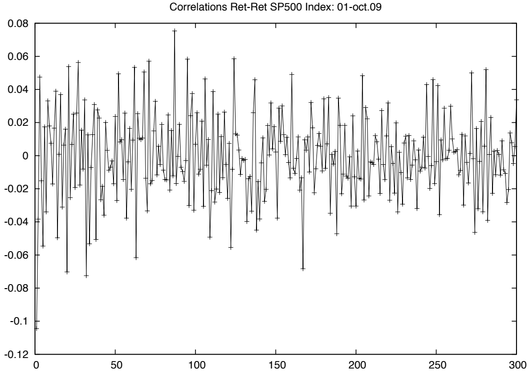

FIGURE: Empirical correlation of the daily returns of the SP500 index on the period 2001-2009.

Volatility correlations

The definition of volatility is ambiguous ; if $X_t = \ln S_t$ is the log price, we define

- the theoretical volatility (hard to observe : filtering, etc...) is the square root of the quadratic variation $< X >_{s,t}$ of X :

$$< X >_{s,t} = \lim_{n\to\infty} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2,$$

  where $s = t_0 < \ldots < t_n = t$ is a subdivision of $[s, t]$ with mesh going to 0.

- in practice, volatility between s and t can be (one speaks of proxy) :

$$|X_t - X_s|, \ \sup_{u\in[s,t]} X_u - \inf_{u\in[s,t]} X_u, \ etc...$$

Long range volatility correlations

One observes the following volatility correlations on markets (Corr denotes correlation) :

$$\text{Corr}(< X >_{0,1}, < X >_{t,t+1}) = A/(1+t)^{\mu},$$

where $\mu \in [0, 0.5]$.

The MRW model corresponds approximately to taking $\mu \to 0$:

$$\text{Corr}(< X >_{0,1}, < X >_{t,t+1}) = A - B \ln(t+1).$$

In the next graphics, we will take $\sigma_{HL}(t) = \sup_{u \in [t,t+1]} X_u - \inf_{u \in [t,t+1]} X_u$ as proxy for $\sqrt{< X >_{t,t+1}}$ (t is in days).

Volatility correlations of the SP500

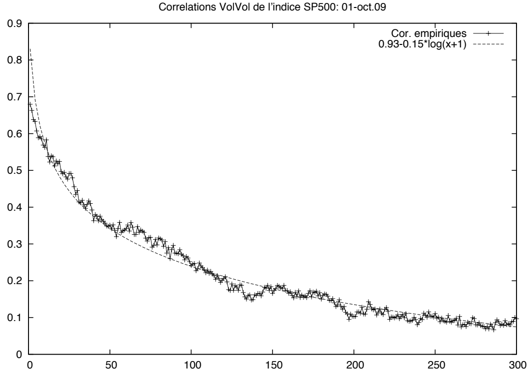

FIGURE: Empirical volatility correlations of the SP500 on the period 2001-2009.

Volatility correlations of the Nasdaq 100

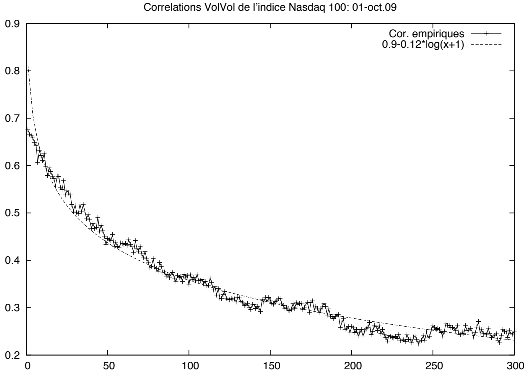

FIGURE: Empirical volatility correlations of the Nasdaq 100 on the period 2001-2009.

Distribution of volatility

The distribution of $< X >_{s,t}$ for $s < t$ is approximately lognormal. In the next graphics, we will consider the empirical distribution of the following daily renormalized proxy of $\sqrt{< X >_{t,t+1}}$:

$$\sigma_{HL}(t) = \sup_{u \in [t,t+1]} X_u - \inf_{u \in [t,t+1]} X_u$$

We will fit the empirical distribution of $\sigma_{HL}(t)$ with :

- a lognormal distribution of density : $f(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$.
- an inverse gamma distribution with density : $f(x) = \frac{A^\nu}{\Gamma(\nu)x^{1+\nu}} e^{-\frac{A}{x}}$.

Distribution of the SP500 volatility

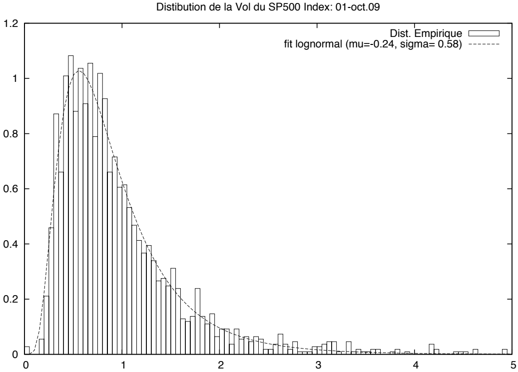

FIGURE: Empirical volatility distribution of the SP500 on the period 2001-2009.

Distribution of the Nasdaq 100 volatility

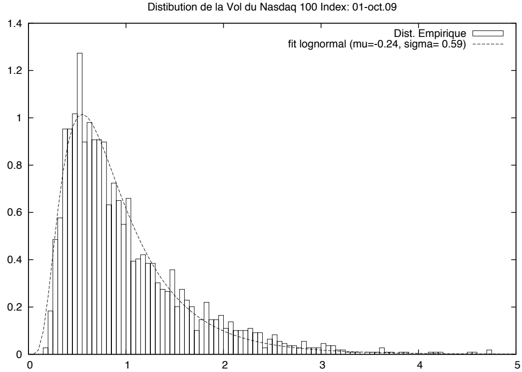

FIGURE: Empirical volatility distribution of the Nasdaq 100 on the period 2001-2009.

Distribution of the volatility of the SP500 stocks

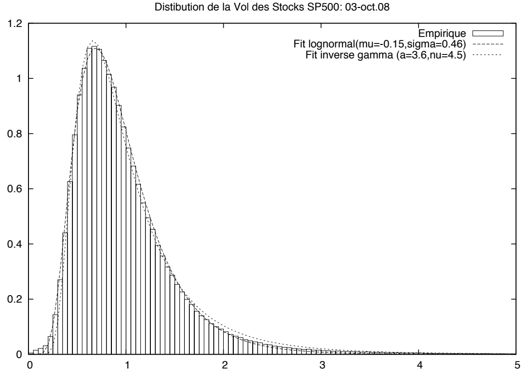

FIGURE: Empirical volatility distribution of the SP500 stocks on the period 2001-2008.

Distribution of the volatility of the SP500 stocks

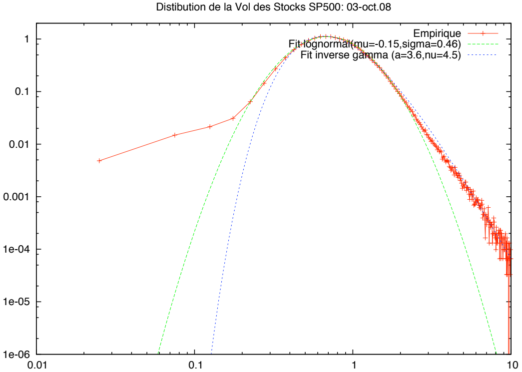

FIGURE: Empirical volatility distribution (log-log) of the SP500 stocks on the period 2001-2008.

Effet Levier : corrélations Ret-Vol

One observes the following Return-Volatility correlations on the market

$$\frac{E[X_1 < X >_{t,t+1}]}{E[< X >_{t,t+1}]^2} = -Ae^{-t/L}, \quad \text{(Corrélations Ret-Vol)}$$

where $A > 0$ and L is the decorrelation scale.

In the next graphics, we consider the correlations Ret-Vol with $\sigma_{HL} = \sup_{u \in [t,t+1]} X_u - \inf_{u \in [t,t+1]} X_u$ as proxy for $\sqrt{< X >_{t,t+1}}$ (t is in days).

Leverage effect of the SP500

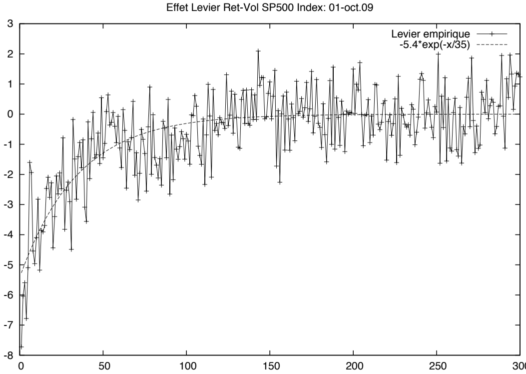

FIGURE: Leverage effect of the SP500 on the period 2001-2009.

Leverage effect of the Nasdaq 100 index

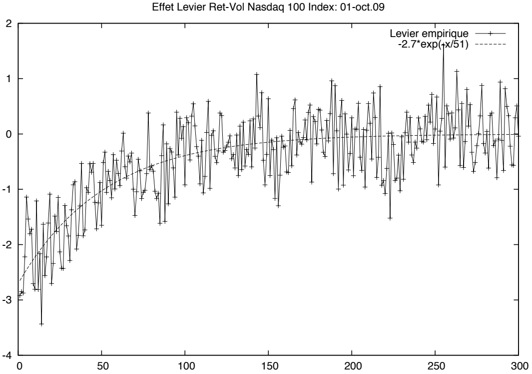

FIGURE: Leverage effect of the Nasdaq 100 index on the period 2001-2009.

Intraday stylized facts

Most intrady daily stylized effects are similar to daily stylized effects : we will see a few empirical curves for the SP500 index. It is important though to take properly into account the intrady seasonality, i.e. the U-effect of volatility.

U-effect of volatility as a function of time in the day

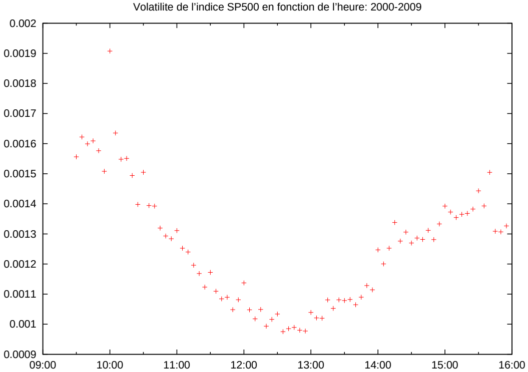

FIGURE: 5 mins. volatility log(H/L) of the SP500 index on the period 2000-2009.

Distribution of 5 mins. volatility of the SP500

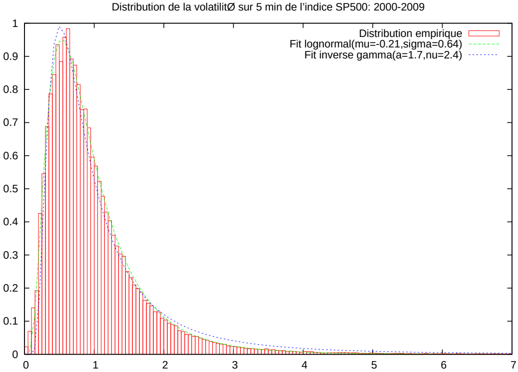

FIGURE: Empirical distribution of the 5 mins. log(H/L) volatility of the SP500 on the period 2000-2009.

Distribution of 5 mins. volatility of the SP500

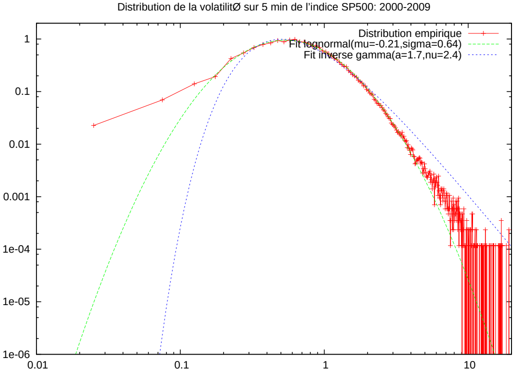

FIGURE: Empirical distribution of the 5 mins. log(H/L) volatility (log-log) of the SP500 on the period 2000-2009.

Forecasting volatility with the MRW and the 1/f noise

The integrated volatility process $M := M^T$ of MRW :

$$M^T[0,t] = \sigma \int_{[0,t]} e^{2\lambda\omega_s - 2\lambda^2 E[\omega_s^2]} ds, \quad t \geq 0,$$

where ω is a centered Gaussian "process" (independent of W) with covariance

$$E[\omega_s \omega_t] = \ln^+ \frac{T}{|t-s|}$$

Forecasting the volatility with the MRW and the 1/f noise

Definition (Invariance by integral scale change)

For all $T < T'$:

$$(M^{T'}[0,t])_{t\in[0,T]} \underset{(distribution)}{=} e^{\Omega_{T'/T} - 2\lambda^2 \ln \frac{T'}{T}} (M^T[0,t])_{t\in[0,T]}$$

where $\Omega_{T'/T}$ is a centered Gaussian variable of variance $2\lambda^2 \ln \frac{T'}{T}$ that is independent of $(M^T([0,t]))_{t\in[0,T]}$.

Consequence : if the observation window of M^T is of length less than or equal to T, it is impossible de determine σ and T (ill-posed problem).
Idea : let $T \to \infty$ to get rid of σ and T !

Exact definition of the log-volatility ω

ω is a gaussian measure on the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ (in the sense of Schwartz) :

$$\forall \phi \in \mathcal{S}(\mathbb{R}), \quad E(e^{i \int_{\mathbb{R}} \phi(t)\omega_t dt}) = e^{-1/2 E((\int_{\mathbb{R}} \phi(t)\omega_t dt)^2)}$$
$$= e^{-1/2 \int \int_{\mathbb{R}^2} \phi(t)\phi(s) \ln^+(T/|t-s|) ds dt}.$$

We want to let $T \to \infty$ in the above formula.

Exact definition of the log-volatility ω for $T \to \infty$

ω is a Gaussian measure on the quotient space $\mathcal{S}'(\mathbb{R})/\mathbb{R}$ (1/f noise)
If one considers $\mathcal{S}_0(\mathbb{R}) = \{\phi \in \mathcal{S}(\mathbb{R})\ ;\ \int_{\mathbb{R}} \phi(t)dt = 0\}$ then :

$$
\begin{aligned}
\forall \phi \in \mathcal{S}_0(\mathbb{R}), \quad E(e^{i\int_{\mathbb{R}} \phi(t)\omega_t^{\infty} dt}) &= e^{-1/2 E((\int_{\mathbb{R}} \phi(t)\omega_t^{\infty} dt)^2)} \\
&= e^{-1/2 \int\int_{\mathbb{R}^2} \phi(t)\phi(s) \ln(1/|t-s|) ds dt} \\
&= e^{-1/4 \int_{\mathbb{R}} \frac{|\hat{\phi}(\xi)|^2}{|\xi|} d\xi}.
\end{aligned}
$$

See also Duplantier, Rhodes, Sheffield, V. : Log-correlated Gaussian fields : an overview.

Exact prediction formulas for the 1/f noise

We consider the reproducing kernel Hilbert space of ω :

$$\mathcal{H}^{1/2}(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R})/\mathbb{R} \; ; \; \int_{\mathbb{R}} |\xi||\hat{f}(\xi)|^2 d\xi < \infty\}.$$

Theorem (Duchon, Robert, V., 2007)

For all $f \in \mathcal{H}^{1/2}(\mathbb{R})$,

$$\forall t > 0 \quad E[\omega_t^\infty | (\omega_s^\infty)_{s<0} = f] = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\sqrt{t}}{(t-s)\sqrt{-s}} f(s) ds.$$

Exact prediction formulas for the log-volatility ω

Let L be some observation window and T the integral scale. There is an explicit kernel $K_{L,T}(t,s)$ such that :

Corollary (Duchon, Robert, V., 2007)

For all $f \in H^{1/2}(\mathbb{R})$,

$$\forall t \in]0, T-2L[, \; E[\omega_t | (\omega_s)_{-2L<s<0} = f] = \int_{-2L}^{0} K_{L,T}(t,s) f(s) ds.$$

Remark

On can discretize the above formulas to get approximate formulas for the discrete model. In that case, one can also transfer the prediction formulas to the volatility itself.

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