A General Solution to Eigenvalue Distributions of Hermitian Random Matrix Models

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joint work with

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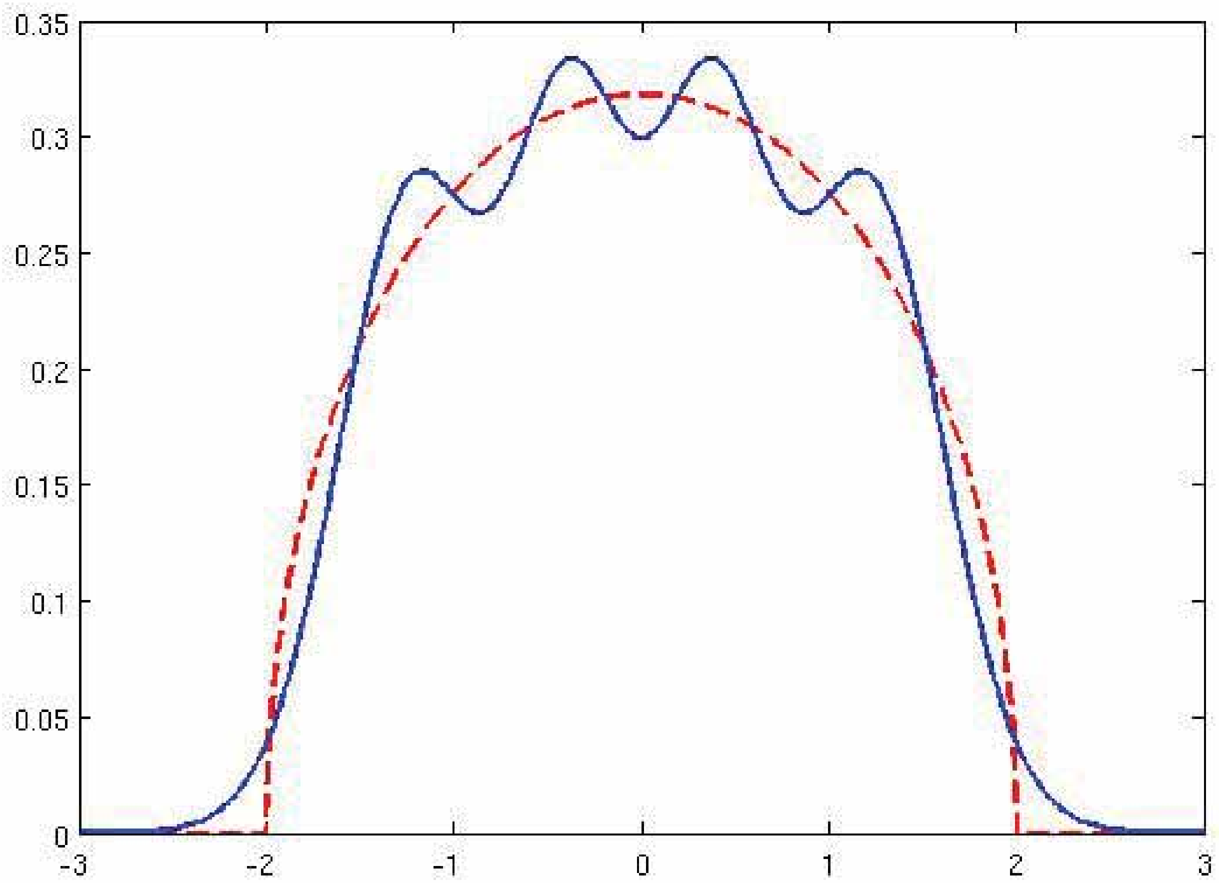
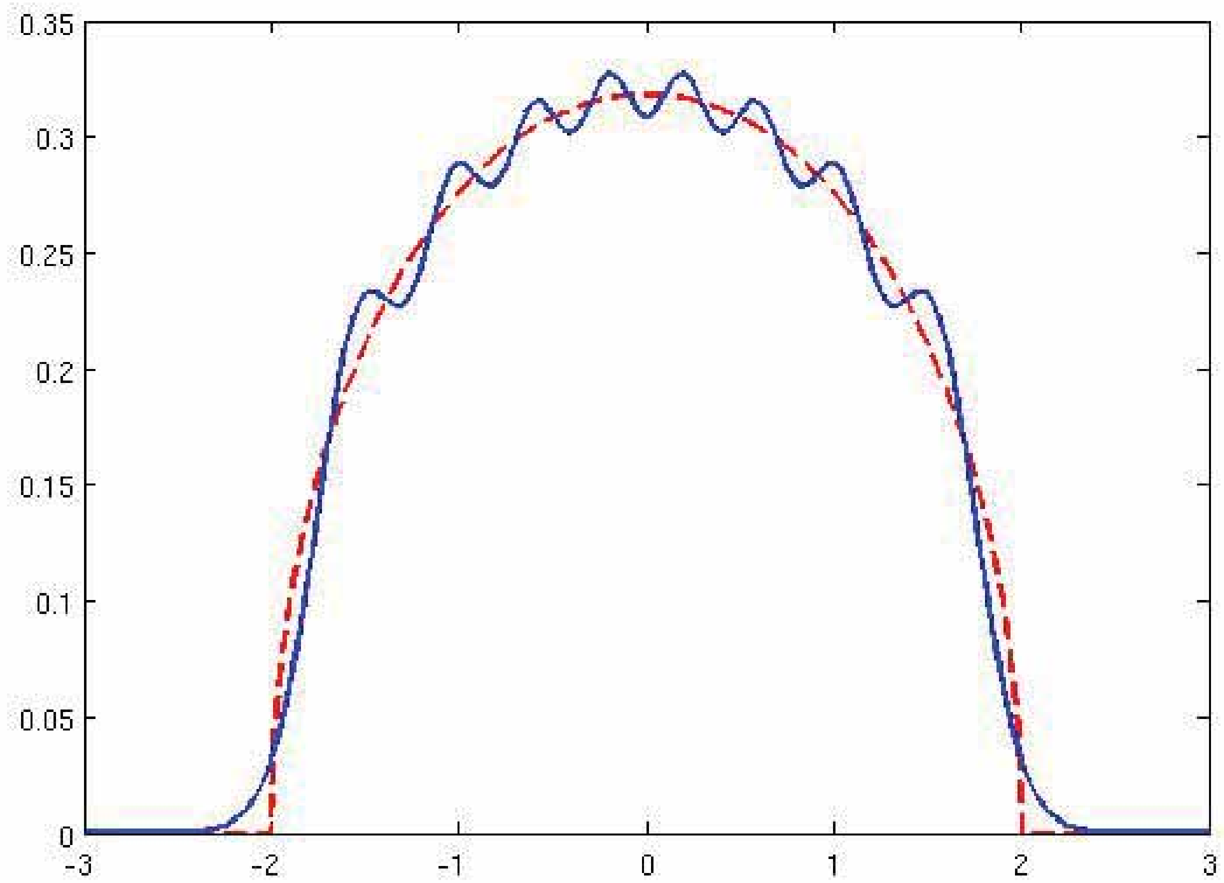
We only ask that the monomials of $P(X_1, \ldots, X_n)$ are square and have the same size.

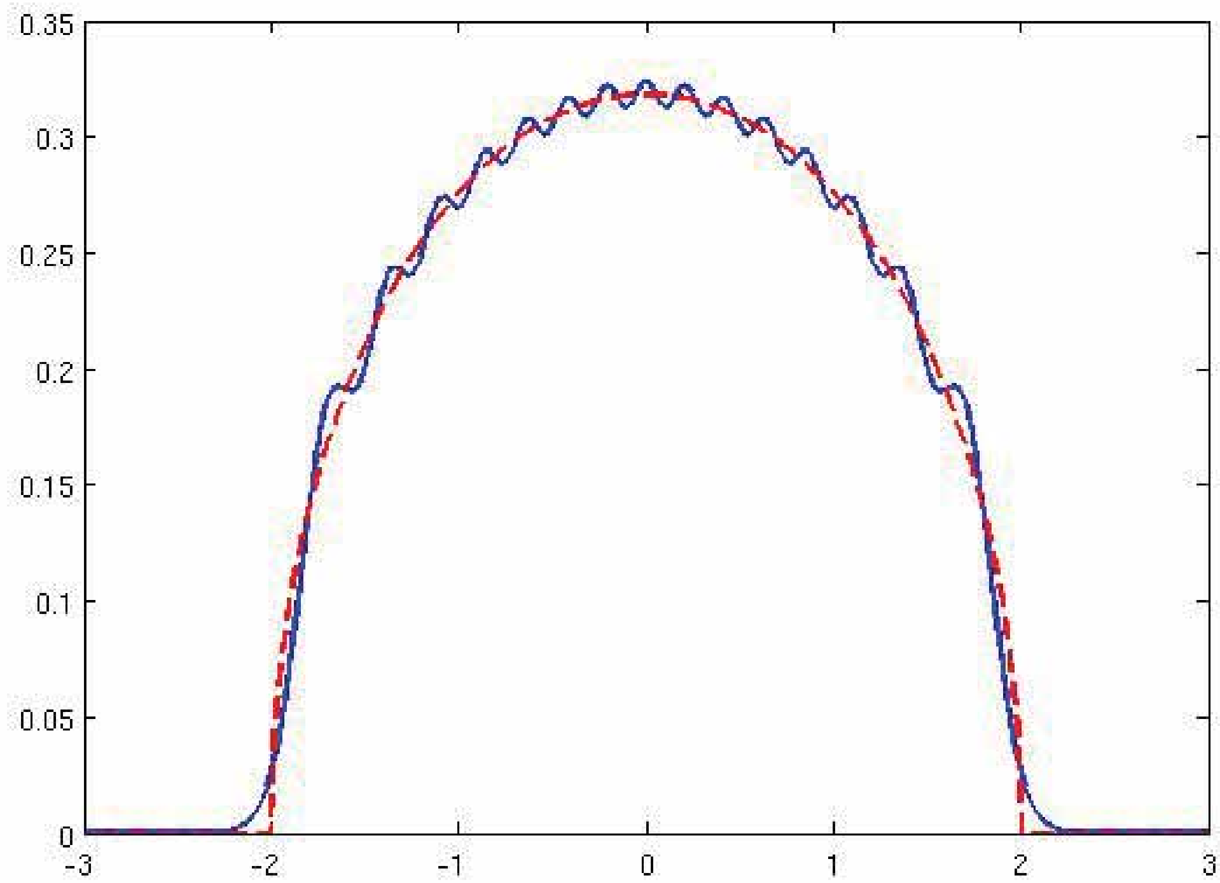
GUE (N=4):



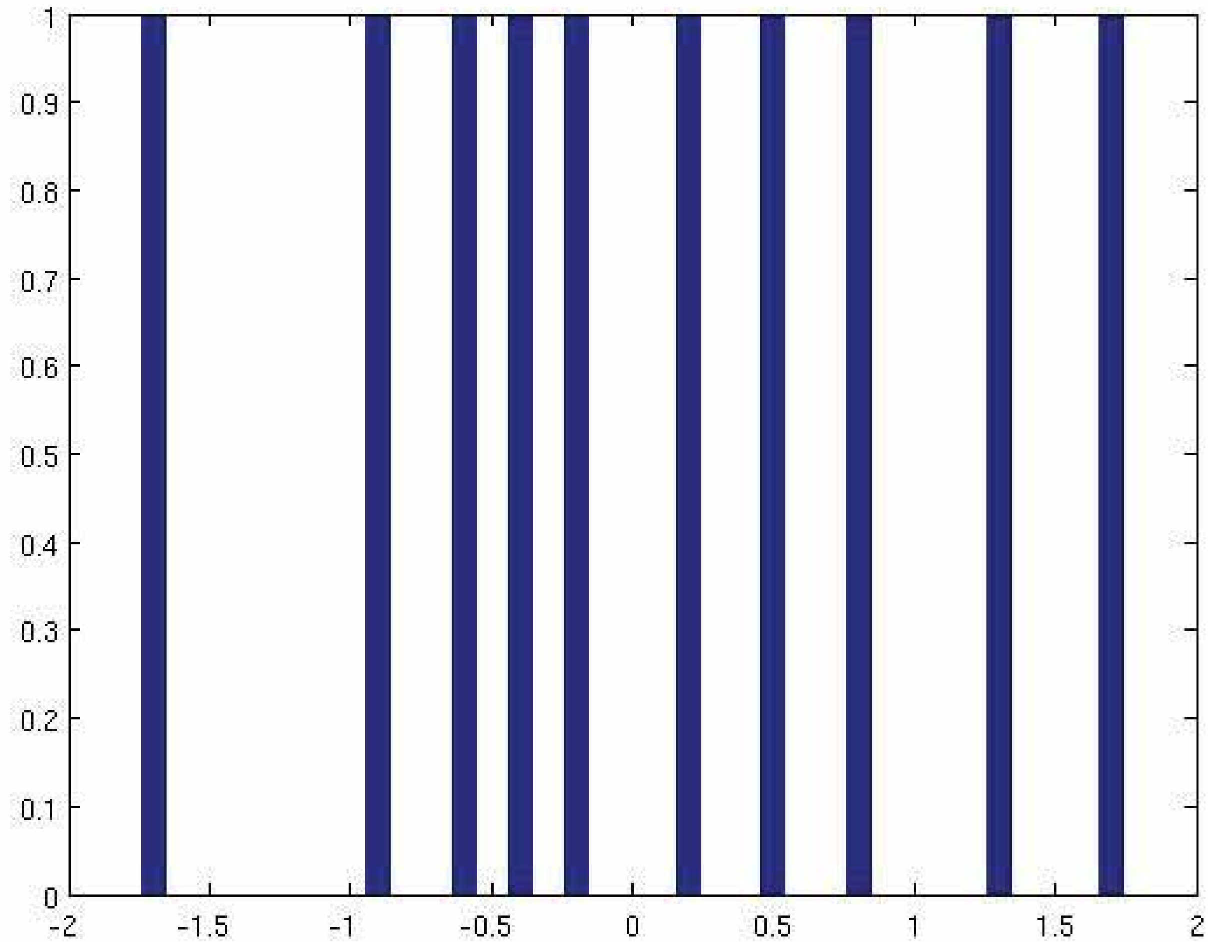
GUE (N=8):



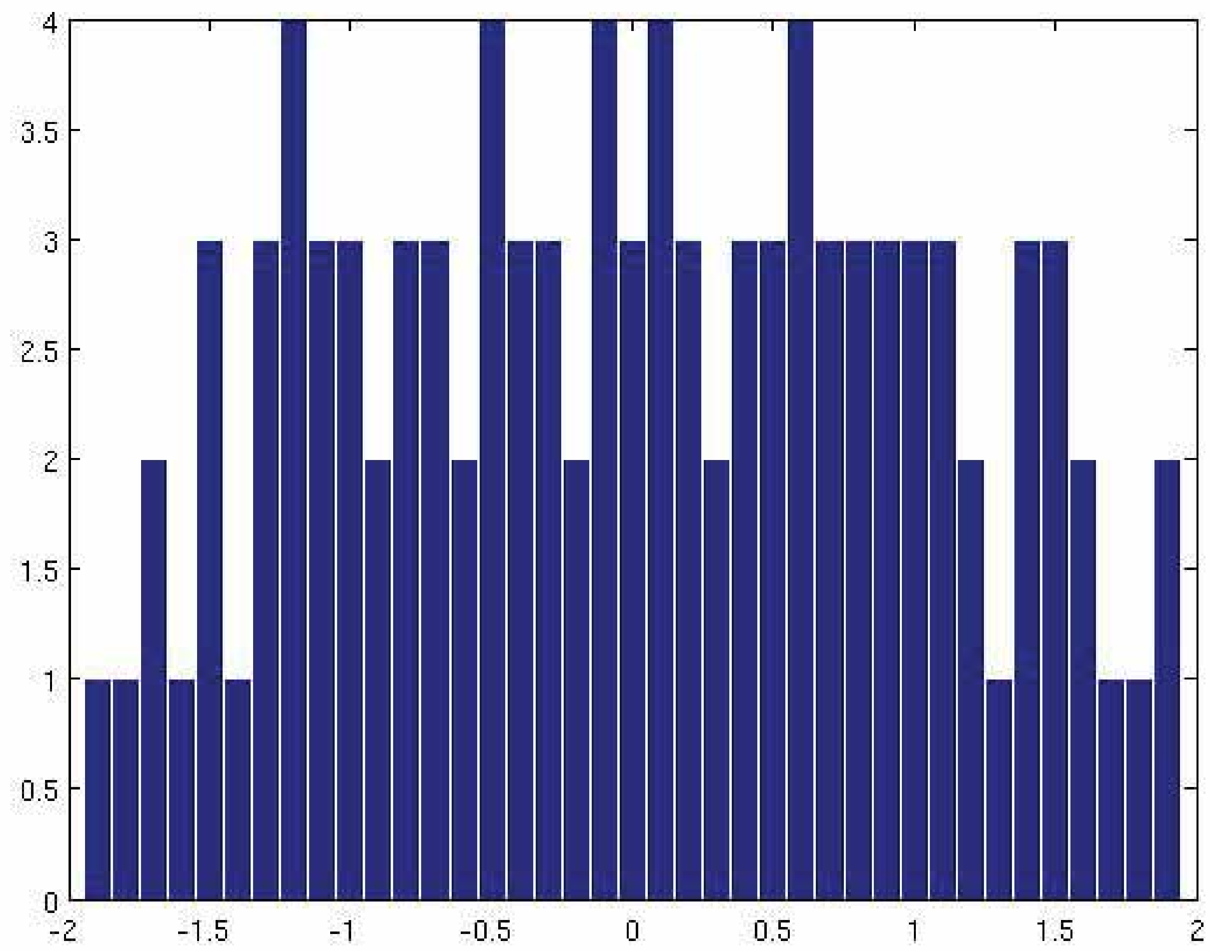
GUE (N=15):



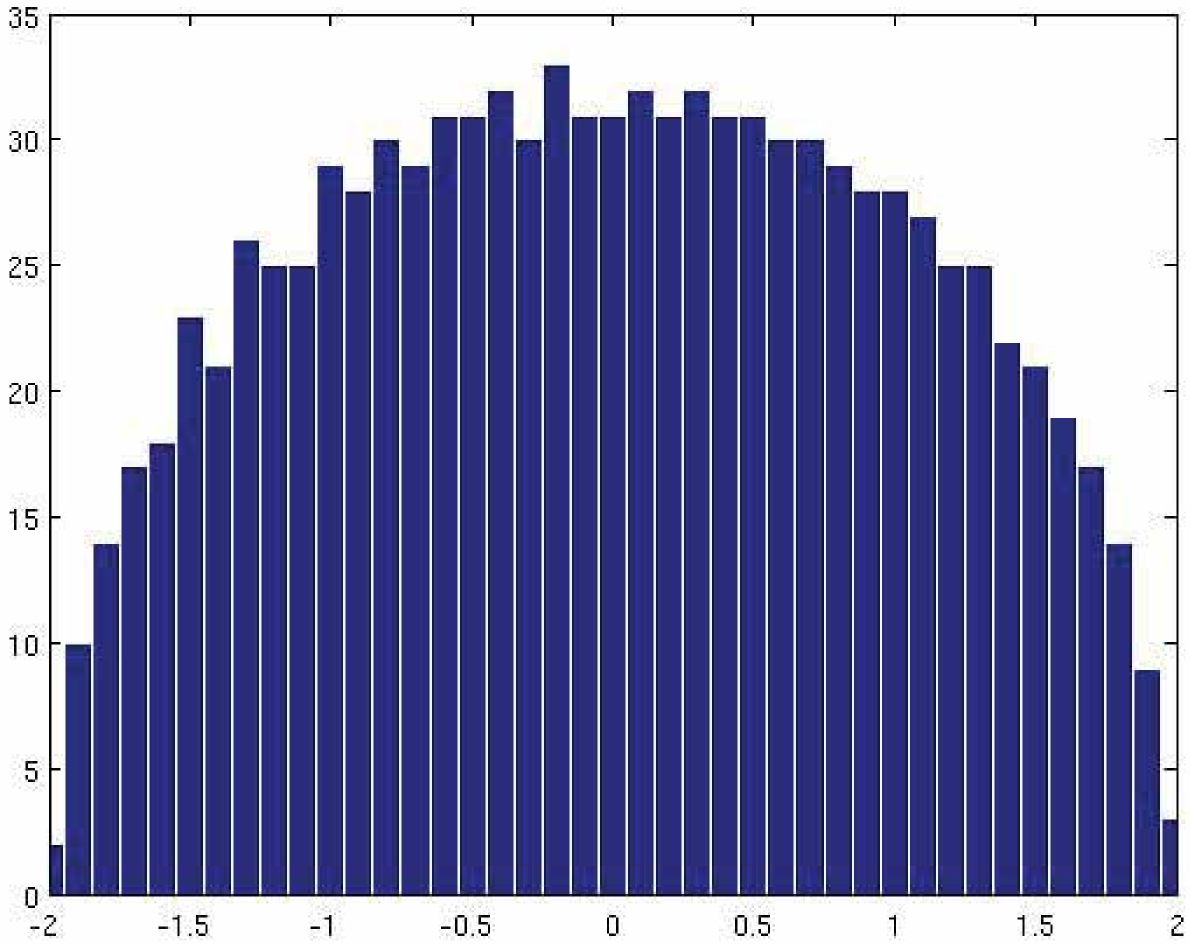
EED of Bernoulli Wigner Matrix (N=10):



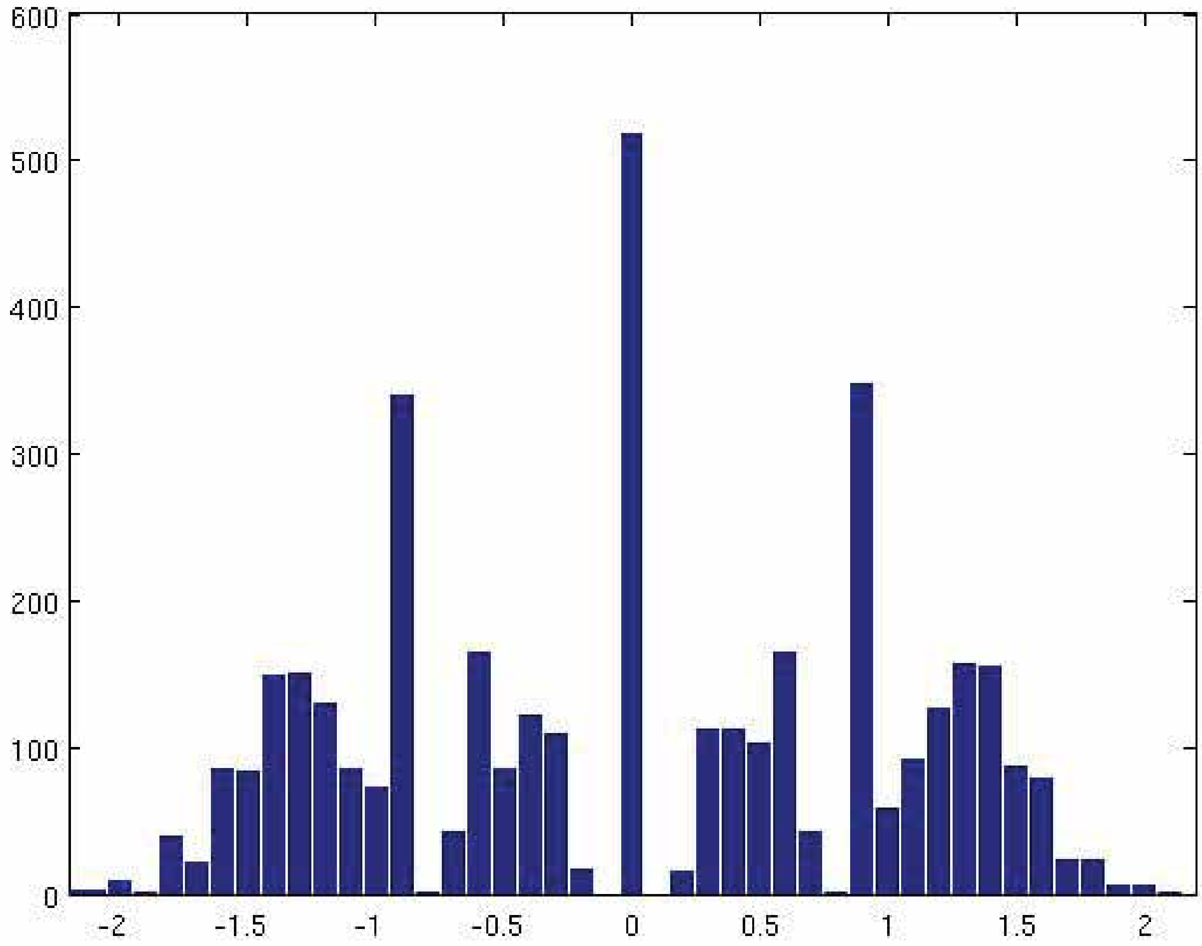
EED of Bernoulli Wigner Matrix (N=100):



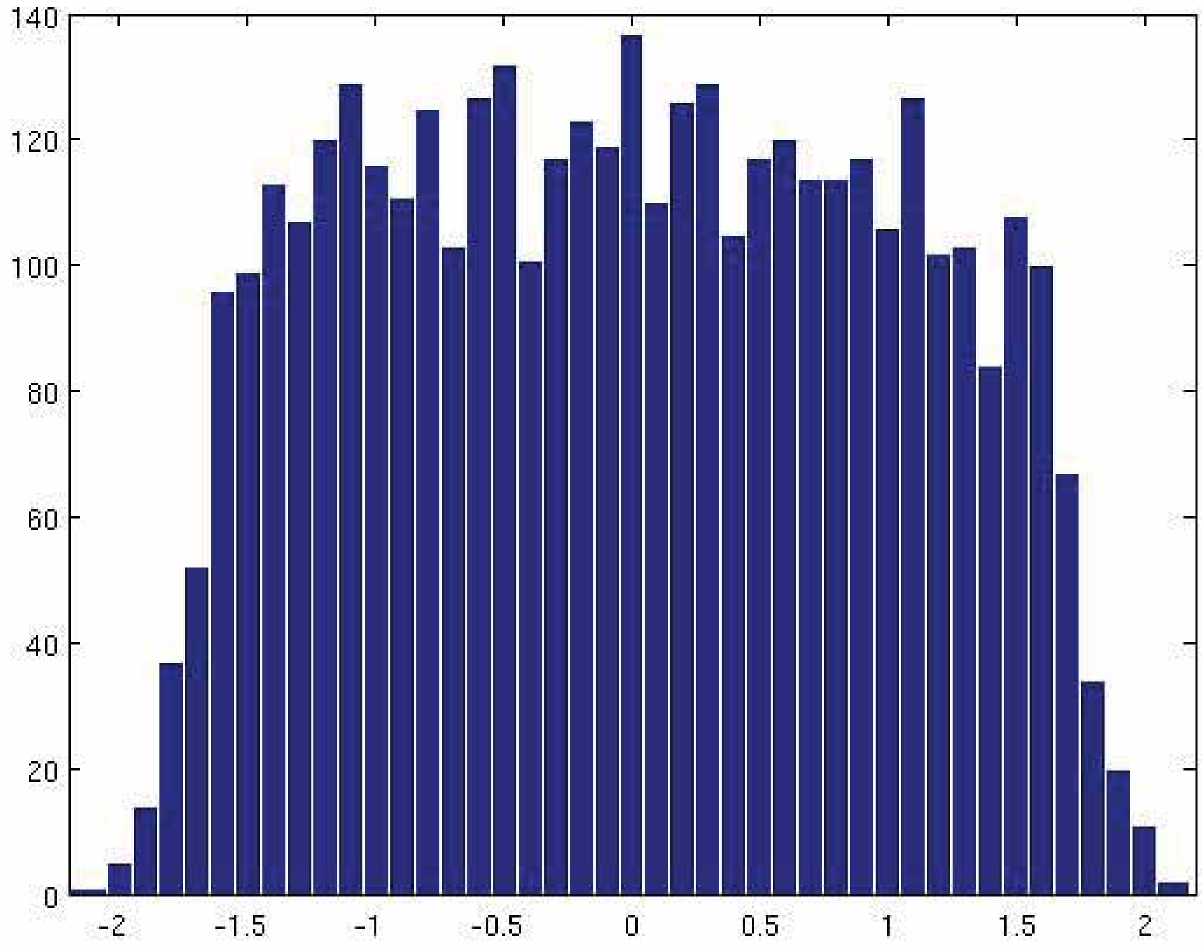
EED of Bernoulli Wigner Matrix (N=1000):



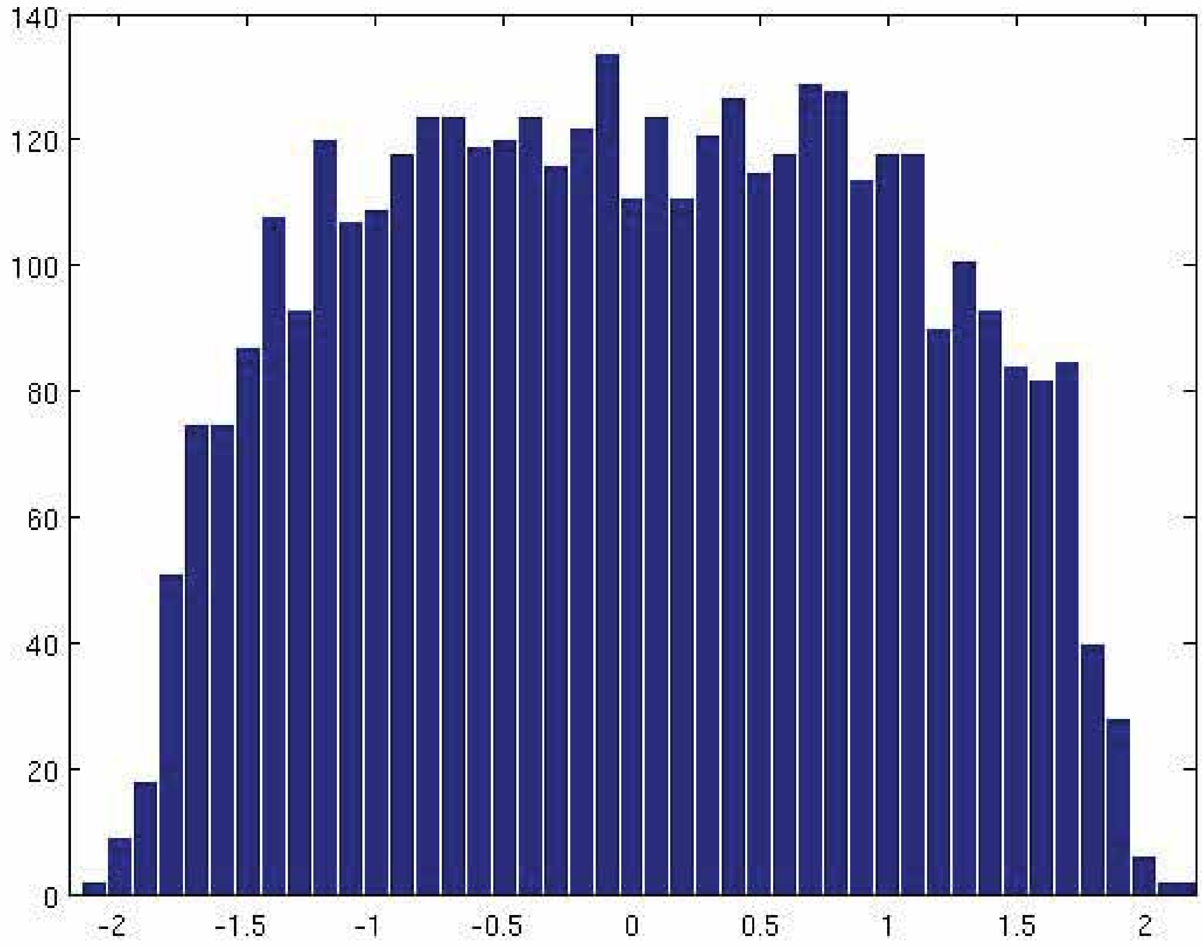
4000 Eigenvalues of Bernoulli Matrices (N=5):



4000 Eigenvalues of Bernoulli Matrices (N=10):



4000 Eigenvalues of Bernoulli Matrices (N=20):



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- Keeping joint distributions of deterministic matrices.
- Matrices of different sizes/ rectangular matrices.

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We can blow-up the model by considering
$P_N = Q_N U_N R_N U_N^* Q_N^* + S_N V_N T_N V_N^* S_N^*$, where $A_N := A \otimes I_N$ for $A \in \{Q, R, S, T\}$ and letting $U_N \in \mathcal{U}(8N)$ $V_N \in \mathcal{U}(4N)$ be independent, with uniform distribution.

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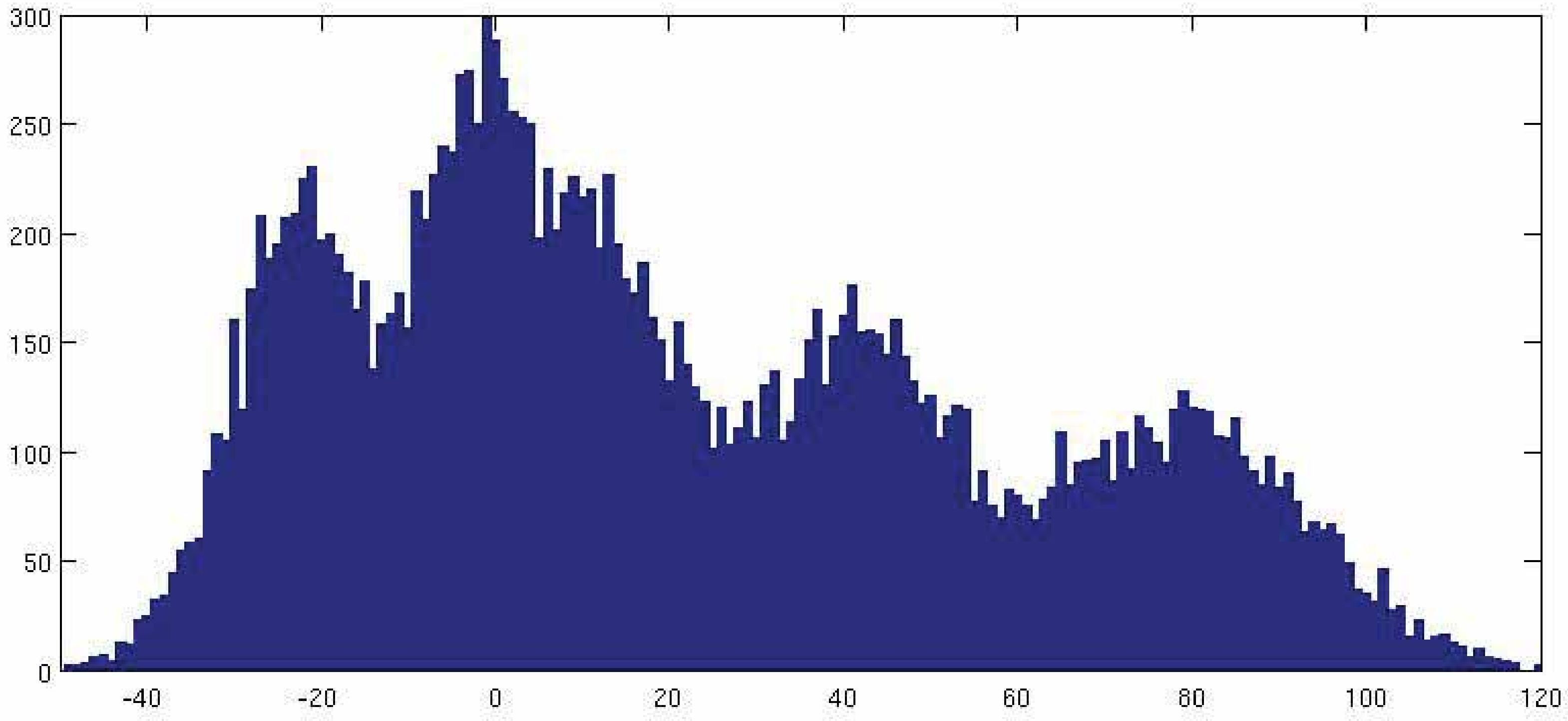
While the question of obtaining the AED μ_{P_N} seems now completely out of reach, one observes once more that the measures μ_{P_N} converge towards a deterministic shape.

A Rectangular Example

20000 eigenvalues of P_N (N=1):



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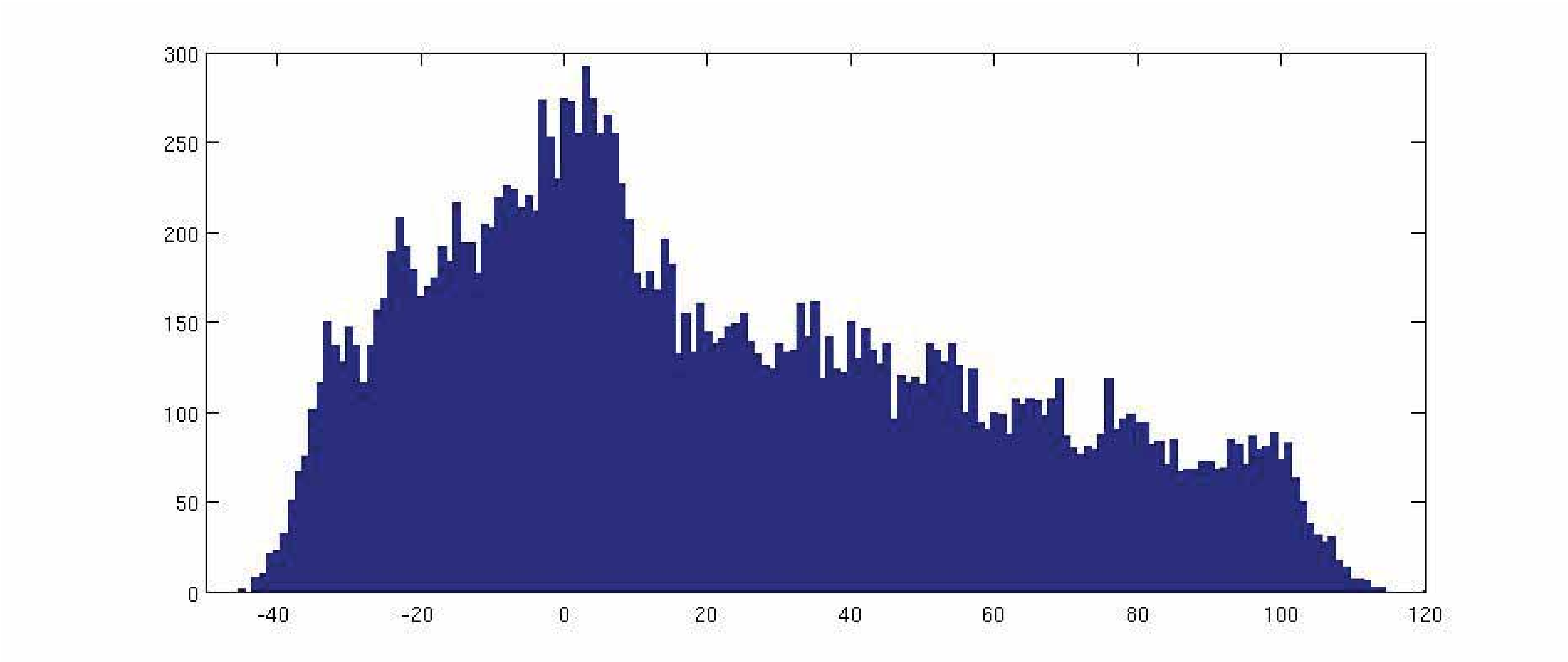
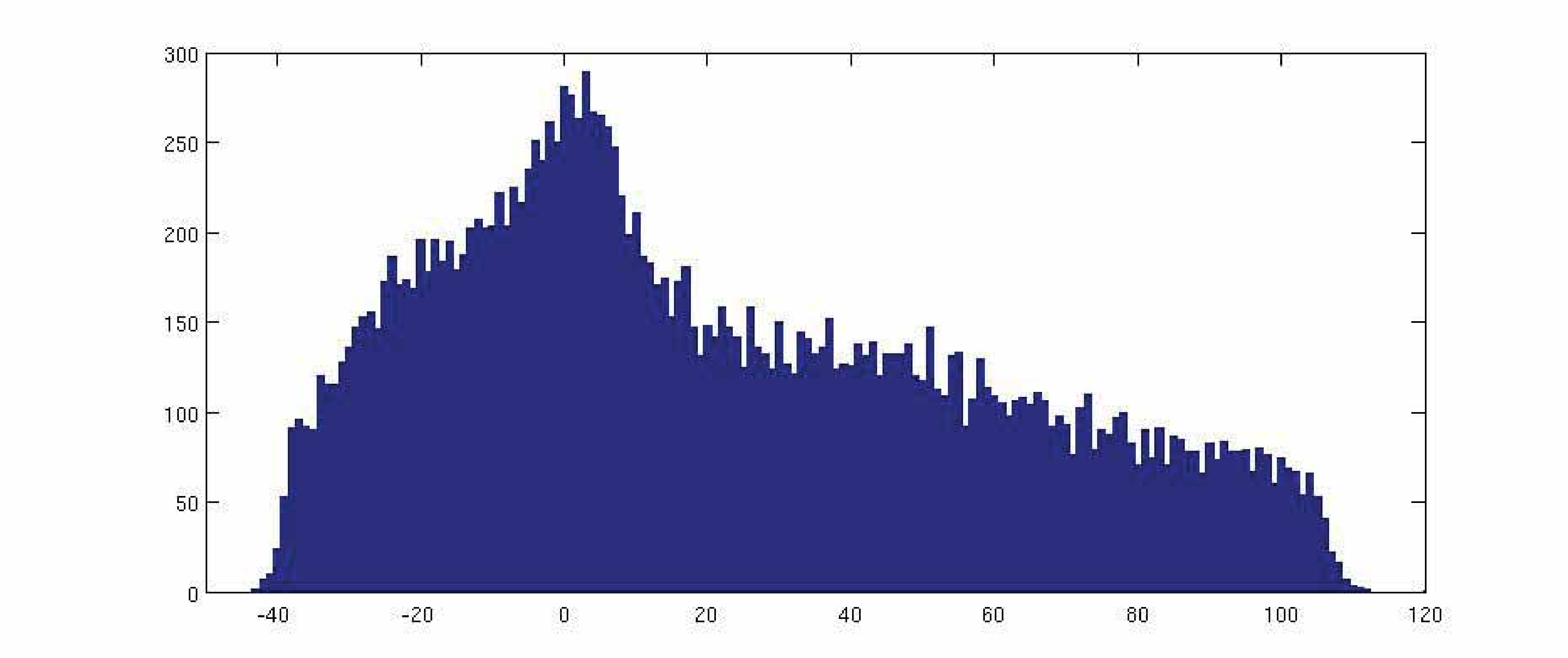
20000 eigenvalues of P_N (N=3):



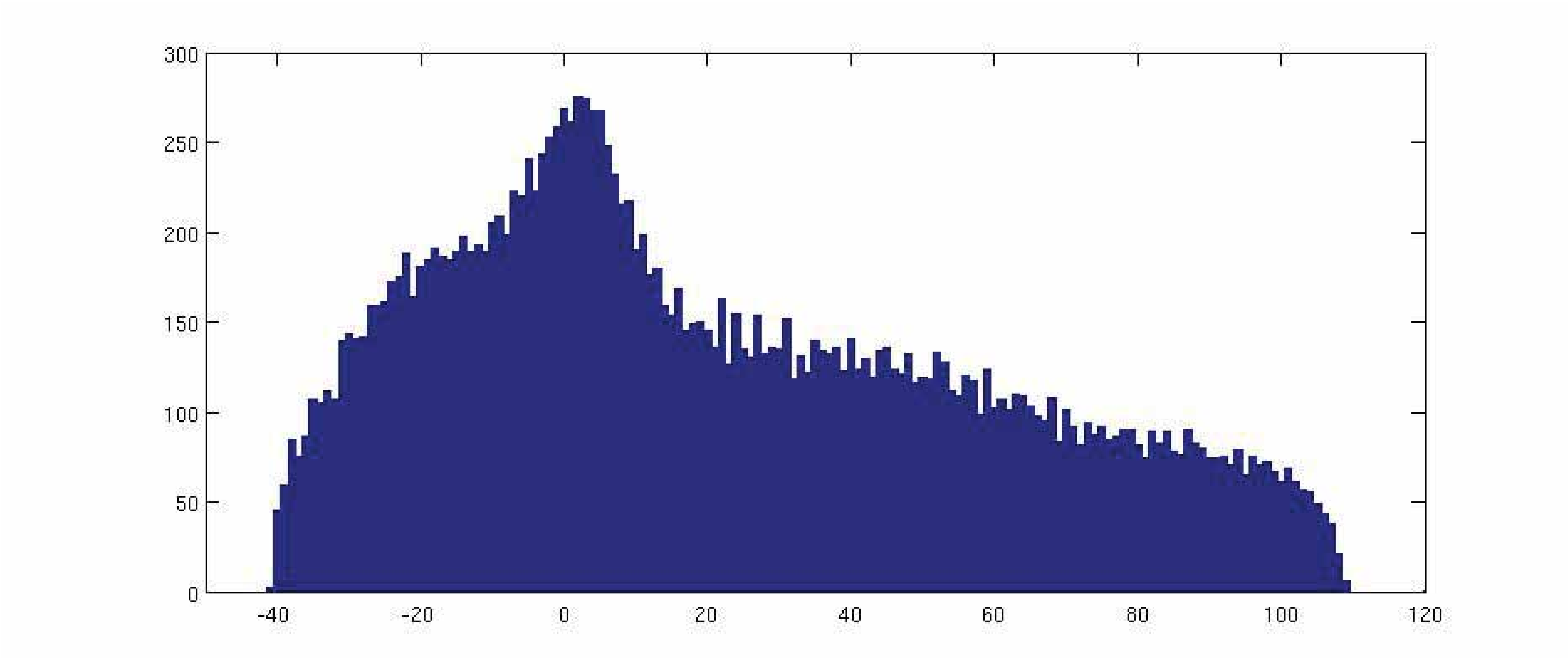
A Rectangular Example

20000 eigenvalues of P_N (N=10):

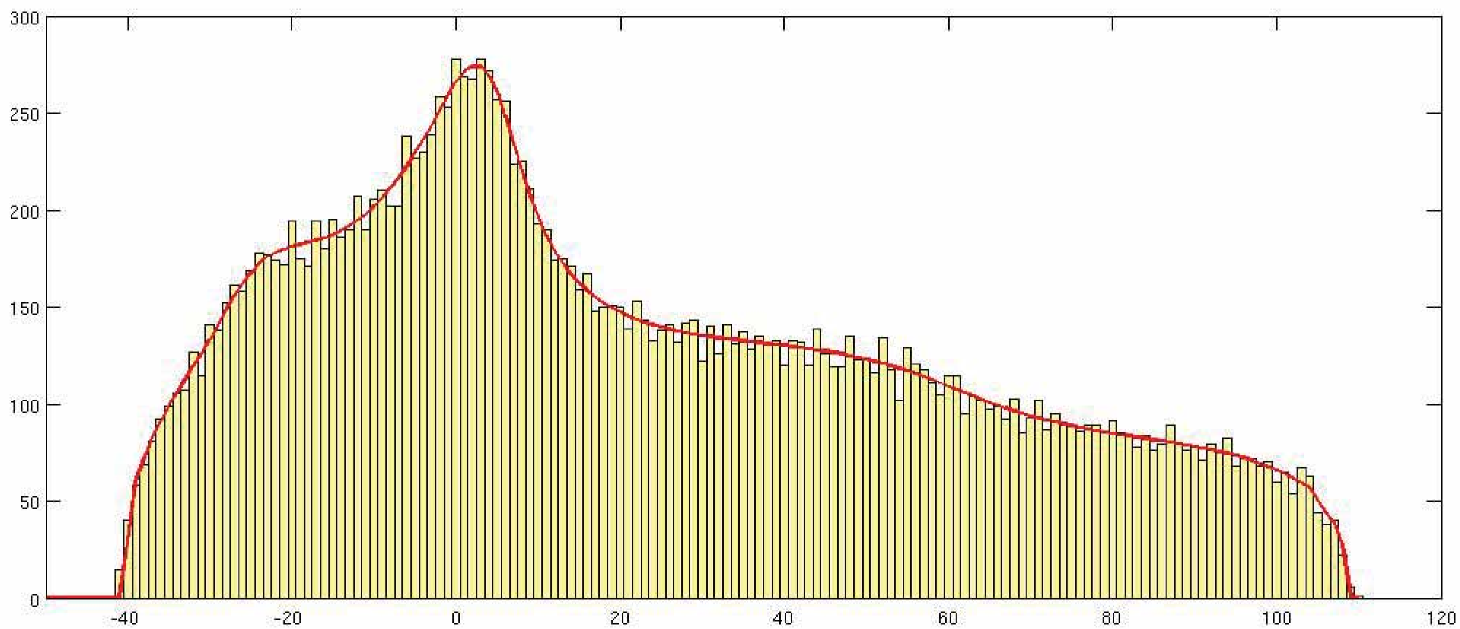


A Rectangular Example

20000 eigenvalues of P_N (N=40):



Main Goal



Main Result

Let $P = P(X_1, X_1^*, \ldots, X_n, X_n^*) = P(X_1, \ldots, X_n)$ be a self-adjoint polynomial on non-commutative indeterminates $X_1, \ldots, X_n$ (and its adjoints $X_1^*, \ldots X_n^*$).

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- There exists a probability measure μ such that $\mu_m \to \mu$.
- μ is the spectral distribution of the free deterministic equivalent $P^\square = P(y_1, \ldots, y_n)$ of P_1.
- μ can be nummerically computed.

Definitions

(1). Let $\mathcal{A}$ be a unital *-algebra. A *$\mathcal{B}$-probability space* is a pair $(\mathcal{A}, \mathbf{F})$ and a linear map $\mathbf{F} : \mathcal{A} \to \mathcal{B} \subseteq \mathcal{A}$ satisfying

$$\begin{aligned} \mathbf{F}\left(bab'\right) &= b\mathbf{F}(a)b', \qquad \forall b, b' \in \mathcal{B}, a \in \mathcal{A} \\ \mathbf{F}(1) &= 1. \end{aligned}$$

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(2). Let $(\mathcal{A}, \mathbf{F})$ be a $\mathcal{B}$-probability space and let $\bar{a} := a - \mathbf{F}(a)1_{\mathcal{A}}$ for any $a \in \mathcal{A}$. The *-subalgebras $\mathcal{B} \subseteq A_1, \dots, A_k \subseteq \mathcal{A}$ are $\mathcal{B}$-free (or free over $\mathcal{B}$, or free with amalgamation over $\mathcal{B}$) (with respect to $\mathbf{F}$) iff

$$\mathbf{F}(\bar{a_1}\bar{a_2}\cdots\bar{a_m}) = 0, \tag{1}$$

where for all $m \geq 1$ and all tuples $a_1, \dots, a_m \in \mathcal{A}$ such that $a_i \in A_{j(i)}$ with $j(1) \neq j(2) \neq \cdots \neq j(m)$.

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Then there exists a unique conditional expectation $\mathbf{F} : \mathcal{A} \to \mathcal{D}$ such that $\tau \circ \mathbf{F} = \tau$, which is given by

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With this, $(\mathcal{A}, \mathbf{F})$ becomes a $\mathcal{D}$-valued probability space.

Rectangular-Probability Spaces

Consider the model $\Phi = \sum_{i=1}^{K} R_i U_i T_i U_i^* R_i^*$. We embed our matrices in a rectangular probability space:

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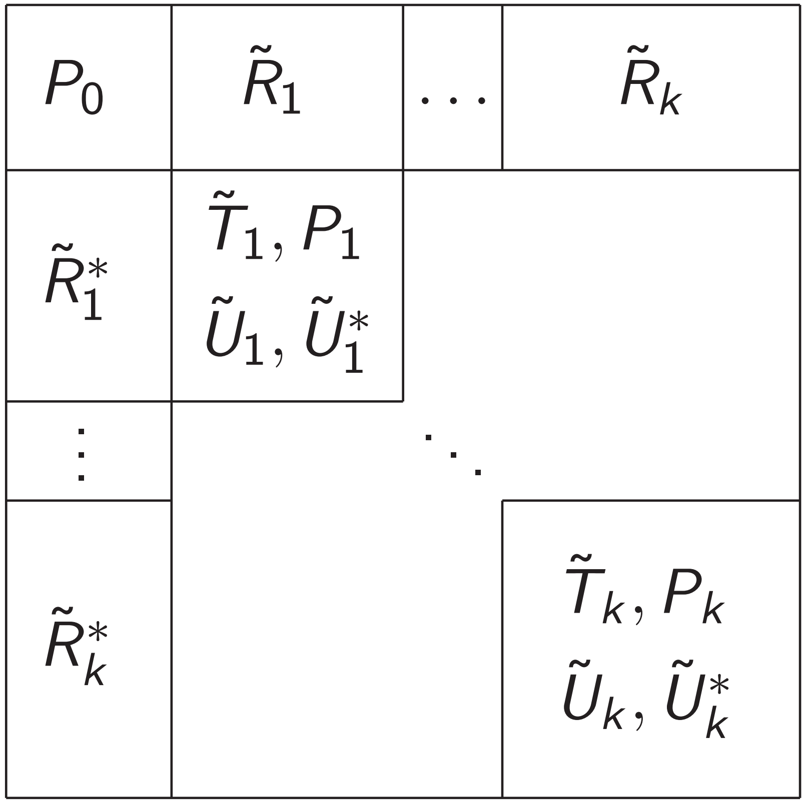

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P_0	$\tilde{R}_1$	$\dots$	$\tilde{R}_k$
$\tilde{R}_1^*$	$\tilde{T}_1, P_1$ $\tilde{U}_1, \tilde{U}_1^*$		
$\vdots$		$\ddots$	
$\tilde{R}_k^*$			$\tilde{T}_k, P_k$ $\tilde{U}_k, \tilde{U}_k^*$

Theorem (Benaych-Georges)

$\{\tilde{R}_1 \otimes I_m, \dots, \tilde{R}_K \otimes I_m, \tilde{T}_1 \otimes I_m, \dots, \tilde{T}_k \otimes I_m\}$ *and* $\{\tilde{U}_1^m, \dots \tilde{U}_K^m\}$ *are asymptotically free over* $\langle P_0 \otimes I_m, \dots, P_k \otimes I_m \rangle$.

Example (Matrix-valued probability spaces)

Let $(\mathcal{A}, \tau)$ be a $*$-probability space and consider the algebra $M_n(\mathcal{A}) \cong M_n(\mathbb{C}) \otimes \mathcal{A}$ of $n \times n$ matrices with entries in $\mathcal{A}$. The maps

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are respectively, conditional expectations onto the algebras $M_n(\mathbb{C}) \supset D_n(\mathbb{C}) \supset I_n(\mathbb{C})$ of constant matrices, diagonal matrices and multiples of the identity.

Matrices on free elements are matrix-valued free!

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Proposition

Let $(\mathcal{A}, \mathbf{F})$ be a $\mathcal{B}$-probability space, and consider the $M_n(\mathcal{B})$-valued probability space $(M_n(\mathbb{C}) \otimes \mathcal{A}, id \otimes \mathbf{F})$. If $A_1, \ldots, A_k \subseteq \mathcal{A}$ are $\mathcal{B}$-free, then $(M_n(\mathbb{C}) \otimes A_1), \ldots, (M_n(\mathbb{C}) \otimes A_k) \subseteq (M_n(\mathbb{C}) \otimes \mathcal{A})$ are $(M_n(\mathcal{B}))$-free.

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Very often $(\mathcal{A})$ may have several operator-valued structures $\mathbf{F}_i : \mathcal{A} \to \mathcal{B}_i$ simmultaneusly, with $\mathbb{C} = \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_k$, and $\mathbf{F}_i \circ \mathbf{F}_{i+1} = \mathbf{F}_i$.

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$$G_{c\otimes x}^{M_n(\mathbb{C})}(b) = \int_{\mathbb{R}} (b - c \otimes t)^{-1} d\mu_x(t)$$

Theorem (Belinschi, Mai, Speicher 2013)

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Additive Free Convolution via Analytic Subordination

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$$\omega(b) = \lim_{n\to\infty} f_b^{\circ n}(w),$$

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where, for any $b, w \in \mathbb{H}^+(\mathcal{B})$*,* $f_b(w) = h_y(h_x(w) + b) + b$ *and* h *is the auxiliary analytic self-map* $h_x(b) = (E((b - x)^{-1}))^{-1} - b$ *on* $\mathbb{H}^+(\mathcal{B})$*.*

Theorem

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Let $P = P(x_1, \dots, x_{n_1}, d_1, \dots, d_{n_2})$ *be a self-adjoint polynomial evaluated on* $x_1, \dots, x_{n_1}, d_1, \dots, d_{n_2}$ *and their adjoints.*

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where $\hat{d} = diag(d, 0, 0, \dots, 0) \in M_m(\mathcal{D})$.

Linearization: example

If $P(X_1, X_2, \ldots, X_n) = X_1 X_2 \cdots X_n X_n^* \cdots X_2^* X_1^*$ then

$$L_P(X_1, \ldots, X_n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & X_1 \\ 0 & 0 & 0 & \cdots & 0 & X_2 & -1 \\ 0 & 0 & 0 & \cdots & X_3 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & X_3^* & \cdots & 0 & 0 & 0 \\ 0 & X_2^* & -1 & \cdots & 0 & 0 & 0 \\ X_1^* & -1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

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5. Since $(c_1 \otimes y_1 + c_1^* \otimes y_1^*), \ldots (c_n \otimes y_n + c_n^* \otimes y_n^*), c_0$ are $M_m(\mathbb{C}) \otimes \langle p_1, \ldots, p_k \rangle$-free and self-adjoint, compute $G_{L_{P^{\square}}}$ by the fixed point method of Belinschi et al.

Thanks for your attention!

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