Limit theorems for nearly unstable Hawkes processes

Thibault Jaisson and Mathieu Rosenbaum

Journées MAS 2014

Thursday 28^{th} August 2014

Table of contents

Table of contents

Definition (Hawkes 1971)

A Hawkes process N is a point process on $\mathbb{R}_{(+)}$ of intensity:

$$\begin{aligned} \lambda_t &= \mu + \int_{-\infty}^{t} \phi(t-s)dN_s & (1) \\ &= \mu + \sum_{J_i<t} \phi(t-J_i) & (2) \end{aligned}$$

where $\mu \in \mathbb{R}_+^*$ is the exogenous intensity and ϕ is a positive kernel supported in $\mathbb{R}_+$ which satisfies $\int \phi < 1$ and the J_i are the points of N.

Basic properties

Proposition (Hawkes 1971)

The process is well defined and admits a version with stationary increments under the stability condition:

$$|\phi| := \int \phi < 1.$$

Proposition

The average intensity of a stationary Hawkes process is

$$E[\lambda_t] = \frac{\mu}{1 - |\phi|}.$$

Endogeneity of a Hawkes process.

- μ can be seen as the exogenous part of the intensity.
- $\lambda_t - \mu = \int_0^t \phi(t-s)dN_s$ as the endogenous part of the intensity.
- $\frac{\mathbb{E}[\lambda]-\mu}{\mathbb{E}[\lambda]} = |\phi|$ is thus a measure of the endogeneity of the process.
- $|\phi|$ close to one means that the process is very endogenous.

Proposition (Dayri et al. 2012)

The correlation of the h-increments of stationary Hawkes processes

$$C_\tau^h = Cov(N_{t+\tau+h} - N_{t+\tau}, N_{t+h} - N_t)$$

can be computed:

$$C_\tau^h = h\Lambda(g_\tau^h + (g^h * \psi)_{-\tau} + (g^h * \psi)_\tau + (g^h * \tilde{\psi} * \psi)_\tau)$$

where $\Lambda = \mu/(1-|\phi|)$, $\psi = \sum_{k=1}^{+\infty} \phi^{*k}$, $\tilde{\psi}(x) = \psi(-x)$ *and* $g_\tau^h = (1 - |\tau|/h)^+$.

Proposition (Dayri et al. 2012)

Reversely, given an empirical correlation function, it is possible to numerically find a ϕ *which fits it.*

Table of contents

Various applications

- Ecology (Hawkes, Oakes 1974).
- Seismology (Ogata 1998).
- Genomic analysis (Reynaud-Bouret, Schbath 2010).
- Sociology (Mohler et al. 2011).

Applications to finance

- Midquotes and transaction prices: Bowsher (07), Bauwens and Hautsch (04), Hewlett (06), Bacry, Delattre, Hoffmann, Muzy (13).
- Order books: Large (07).
- Financial contagion: Aït-Sahalia, Cacho-Diaz, Laeven (10).
- Credit Risk: Errais, Giesecke, Goldberg (10).
- Market activity.

Financial modelling of market activity

Definition

The order flow process is the cumulated number of market orders which arrived during the day.

- Hawkes processes are a natural way to reproduce the clusterization of this process.
- Nice branching interpretation (endogenous vs. exogenous orders).
- Tractability.

Clustering at low time scales

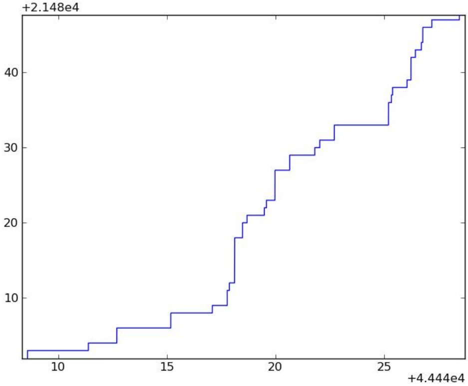

Figure : Cumulated number of trades as a function of time over 20 seconds (DAX 01/07/2013).

Intermediate scale

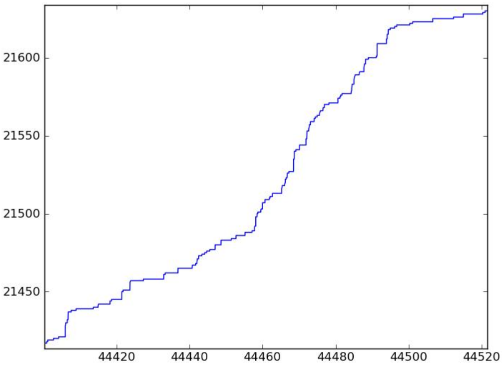

Figure : Cumulated number of trades as a function of time over 3 minutes (DAX 01/07/2013).

Persistence at high time scales

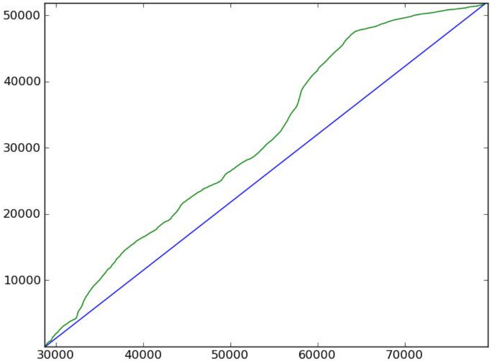

Figure : Cumulated number of trades as a function of time (green) over a trading day (DAX 01/07/2013).

A first result

As Poisson processes, at large time scales, Hawkes processes behave as deterministic processes. They thus cannot fit the data.

Theorem (Bacry et al. 2013)

The sequence of renormalized Hawkes processes

$$X_v^T = \frac{N_{vT}}{T}$$

is asymptotically deterministic, in the sense that the following convergence in L^2 holds:

$$\sup_{v \in [0,1]} |X_v^T - \frac{\mu}{1 - |\phi|} v| \to 0.$$

Table of contents

Formal framework

- Most estimation procedures applied to the financial order flow yield a parameter $|\phi|$ close to one. This is due to the persistence in the order flow.
- We want to study the long term behaviour of Hawkes processes close to criticality (whose kernel's norm is close to one).
- Formally, we consider a sequence of Hawkes processes $(A_T N^T_{Tt})_{t\geq 0}$ indexed by the observation scale T of intensity μ and of kernel
$$\phi^T = a_T \phi$$
with $\int \phi = 1$ and $a_T \to 1$ but $a_T < 1$.

Our asymptotic

Assumption

$$T(1-a_T) \underset{T\to+\infty}{\to} \lambda. \tag{3}$$

$$\int_0^{+\infty} s\phi(s)ds = m < \infty. \tag{4}$$

ϕ is differentiable with derivative ϕ' such that

$$\|\phi'\|_\infty < +\infty \text{ and } \|\phi'\|_1 < +\infty.$$

Finally, $\|\psi^T\|_\infty$ is bounded.

The theorem

Let us denote $C_t^T = (1 - a_T)\lambda_{Tt}^T$.

Theorem (Jaisson, Rosenbaum 2013)

The sequence of renormalized Hawkes intensities (C_t^T) *converges in law, for the Skorohod topology, towards the law of the unique strong solution of the following Cox Ingersoll Ross stochastic differential equation on* $[0,1]$*:*

$$C_t = \int_0^t (\mu - C_s)\frac{\lambda}{m}ds + \frac{\sqrt{\lambda}}{m}\int_0^t \sqrt{C_s}dB_s.$$

The theorem

Theorem

Furthermore, the sequence of renormalized Hawkes processes

$$V_t^T = \frac{1-a_T}{T} N_{tT}^T$$

converges in law, for the Skorohod topology, towards the process

$$\int_0^t C_s ds, \;\; t \in [0,1].$$

Conclusion

We have essentially shown that if one looks at a Hawkes process of kernel's norm close to one at a time scale of the order $1/(1-|\phi|)$ then one sees an integrated CIR process.

- At macroscopic time scales, the cumulated order flow is empirically proportional to the integrated variance:

$$V_t = \kappa \int_0^t \sigma_s^2 ds.$$

- In many usual frameworks, the macroscopic squared volatility is modelled as a CIR process.

The model

Bidimensional Hawkes process

- We consider the model for the mid price of Bacry *et al* (11):

$$P_t^T = N_t^{T+} - N_t^{T-},$$

with (N^{T+}, N^{T-}) a bidimensional Hawkes process with intensity

$$\begin{pmatrix} \lambda_t^{T+} \\ \lambda_t^{T-} \end{pmatrix} = \begin{pmatrix} \mu \\ \mu \end{pmatrix} + \int_0^t \begin{pmatrix} \phi_1^T(t-s) & \phi_2^T(t-s) \\ \phi_2^T(t-s) & \phi_1^T(t-s) \end{pmatrix} \begin{pmatrix} dN_s^{T+} \\ dN_s^{T-} \end{pmatrix}.$$

The model

Assumption

- We assume

$$\phi_i^T(t) = a_T \phi_i(t),$$

where $(a_T)_{T \geq 0}$ is a sequence of positive numbers converging to one such that for all T, $a_T < 1$ and ϕ_1 and ϕ_2 such that

$$\int_0^{+\infty} \phi_1(s) + \phi_2(s) ds = 1 \text{ and } \int_0^{+\infty} s\big(\phi_1(s) + \phi_2(s)\big) ds = m.$$

The model

Properties of the model

The preceding model takes into account the discreteness and the negative autocorrelation of the prices at the microstructure level.

Figure : Traded price as a function of time(Bund 01/07/2013).

Convergence to a Heston model

Theorem

Let $\phi = \phi_1 - \phi_2$. *The renormalized process*

$$P_t^T = \frac{1}{T}(N_{Tt}^{T+} - N_{Tt}^{T-})$$

converges in law, for the Skorohod topology, towards a Heston type process P *on* $[0,1]$ *defined by:*

$$\begin{cases} dC_t = (\frac{2\mu}{\lambda} - C_t)\frac{\lambda}{m}dt + \frac{1}{m}\sqrt{C_t}dB_t^1 & C_0 = 0 \\ dP_t = \frac{1}{1-\|\phi\|_1}\sqrt{C_t}dB_t^2 & P_0 = 0, \end{cases}$$

with (B^1, B^2) *a bidimensional Brownian motion.*

Thank you for your attention!