Correction TD 3 : Dimensionality Reduction

Exercise n°5 : Computing the largest eigenvalue.

1. Remark : In real applications, $P(\ker(A) \neq \emptyset) = 0$

So $u_{t+1} = \frac{Au_t}{\|Au_t\|}$ is well defined a.s.

Proof by recurrence :

- $t = 0$ $\quad \|u_0\|^2 = \sum_i c_i^2 = 1$

Furthermore, $v_1, \dots, v_n$ is orthonormal so $u_0 = \sum_{i=1}^{n} \langle u_0, v_i \rangle v_i$

and furthermore, $\|u_0\|^2 = \sum_{i=1}^{n} \langle u_0, v_i \rangle^2$

So the property is true for $t = 0$

- $t \Rightarrow t+1$ Let's suppose the property true for $t \geq 0$

$$\|u_{t+1}\| = \left\| \frac{Au_t}{\|Au_t\|} \right\| = 1$$

Furthermore, $u_{t+1} = \frac{Au_t}{\|Au_t\|} = \frac{AA^t u_0}{\|AA^t u_0\|} = \frac{A^{t+1} u_0}{\|A^{t+1} u_0\|}$

$$A^{t+1} u_0 = A^{t+1} \sum_{i=1}^{n} \langle u_0, v_i \rangle v_i = \sum_{i=1}^{n} \lambda_i^{t+1} \langle u_0, v_i \rangle v_i$$

$$\text{So} \quad u_{t+1} = \frac{\sum_{i=1}^{n} \lambda_i^{t+1} \langle u_0, v_i \rangle v_i}{\sqrt{\sum_{i=1}^{n} (\lambda_i^{t+1} \langle u_0, v_i \rangle v_i)^2}}$$

By recurrence, the property is true $\forall t \geq 1$

2. Let's note $v = v^{(1)} e_1 + \cdots + v^{(n)} e_n$ in the canonical $\mathbb{R}^n$ basis.

Then $$\langle u_0, v\rangle = \frac{\varepsilon_1 v^{(1)}}{\sqrt{n}} + \cdots + \frac{\varepsilon_n v^{(n)}}{\sqrt{n}}$$

So $E\langle u_0, v\rangle = 0$

and $$\operatorname{Var}\langle u_0, v\rangle = \frac{(v^{(1)})^2}{n} \operatorname{Var} \varepsilon_1 + \cdots + \frac{(v^{(n)})^2}{n} \operatorname{Var} \varepsilon_n$$

$$= \frac{(v^{(1)})^2}{n} \quad 1 \quad + \cdots$$

$$= \frac{1}{n} \|v\|^2 = \frac{1}{n}$$

3. $$E(Z) = \operatorname{Var}(\langle u_0, v\rangle) + E(\langle u_0, v\rangle)^2$$

$$= \frac{1}{n} + 0 \qquad E(Z) = \frac{1}{n}$$

$$E(Z^2) = E\left(\left(\frac{\varepsilon_1 v^{(1)}}{\sqrt{n}} + \cdots + \frac{\varepsilon_n v^{(n)}}{\sqrt{n}}\right)^4\right)$$

$$= \frac{1}{n^2}\left[6 \sum_{i \neq j} v^{(i)^2} v^{(j)^2} E(\varepsilon_i^2) E(\varepsilon_j^2) + \sum_i v^{(i)^4} E(\varepsilon_i^4)\right]$$

$$= \frac{1}{n^2}\left[3\left(\sum_i v_i^2\right)^2 - 3\left(\sum_i v_i^4\right) + \sum_i v_i^4\right]$$

$$\leq 3/n^2$$

4. Let X be non-negative with finite variance,

$$E(X) = E(X\mathbb{1}_{X<\delta E(X)}) + E(X\mathbb{1}_{X\geq \delta E(X)})$$

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$$\leq \delta E(X) + \sqrt{E(X^2)}\sqrt{E(\mathbb{1}_{X\geq\delta E(X)}^2)}$$

$$\leq \delta E(X) + \sqrt{E(X^2)}\sqrt{P(X\geq \delta E(X))}$$

$$\text{So, } \underline{P(X\geq \delta E(X)) \geq (1-\delta)^2 \frac{E(X)^2}{E(X^2)}}$$

5. For our problem, it gives with Z which is non-negative:

$$P\left(X \geq \frac{\delta}{n}\right) \geq (1-\delta)^2\, \frac{1}{3}$$

Let's use it with $\delta = \frac{1}{4}$:

$$\underline{P\left(X\geq \frac{1}{4n}\right) \geq \frac{3}{16}}$$

6. We suppose that $\langle u_0, v\rangle^2 \geq \frac{1}{4n}$,

$$|\langle u_t, v\rangle| = \left|\frac{\lambda_1^t \langle u_0, v\rangle}{\sqrt{\sum(\lambda_i^t\langle u_0, v_i\rangle)^2}}\right|$$

$$= \left|\frac{1}{1+\frac{1}{\langle u_0, v\rangle^2}\sum_{i=2}^{n}\langle u_0, v_i\rangle^2\left(\frac{\lambda_i}{\lambda_1}\right)^{2t}}\right| \geq \left|\frac{1}{1+4n\sum_{i=2}^{n}\langle u_0, v_i\rangle^2\left(\frac{\lambda_2}{\lambda_1}\right)^{2t}}\right|$$

$$\geq \frac{1}{\sqrt{1+4n\|v_0\|^2\left(\frac{\lambda_2}{\lambda_1}\right)^{2t}}} \geq \frac{1}{\sqrt{1+4n\left(\frac{\lambda_2}{\lambda_1}\right)^{2t}}} \geq 1-2n\left(\frac{\lambda_2}{\lambda_1}\right)^{2t}$$

convexity, $\forall x \geq 0$, $\frac{1}{\sqrt{1+x}} \geq 1-\frac{1}{2}x$

7. With a positive probability, the iterates can be come as close as we want to $\{v, -v\}$

8. In the case where $\langle v_0, v\rangle^2 > \frac{1}{4n}$,

$$1-2n\left(\frac{\lambda_2}{\lambda_1}\right)^{2t} \geq 1-\varepsilon \implies |\langle v_t, v\rangle| \geq 1-\varepsilon$$

So we need $t \geq \frac{1}{2}\left\lceil \frac{\log\left(\frac{2n}{\varepsilon}\right)}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} \right\rceil$ to have $|\langle v_t, v\rangle| \geq 1-\varepsilon$

Now we still have to decide how many runs we have to do.

Let's suppose we run the algorithm n times, The probability that we never meet the condition $\langle v_0, v\rangle^2 \geq \frac{1}{4n}$ is $\left(1-3/_{16}\right)^n$

So $\left(1-\frac{3}{16}\right)^n \leq 5/100$

iff $n \geq \frac{\log(5/100)}{\log\left(1-\frac{3}{16}\right)}$

So we have to perform $\left\lceil\frac{\log(5/100)}{\log\left(1-3/_{16}\right)}\right\rceil \frac{1}{2}\left\lceil\frac{\log\left(\frac{2n}{\varepsilon}\right)}{\log\left(\frac{\lambda_1}{\lambda_2}\right)}\right\rceil$ iterations to obtain a vector that satisfies $|\langle\cdot, v\rangle| \geq 1-\varepsilon$ with probability 95%.