Topics in Advanced Machine Learning: Reinforcement Learning

Master 2 Machine Learning and Data Mining - Saint-Etienne

Aurélien Garivier

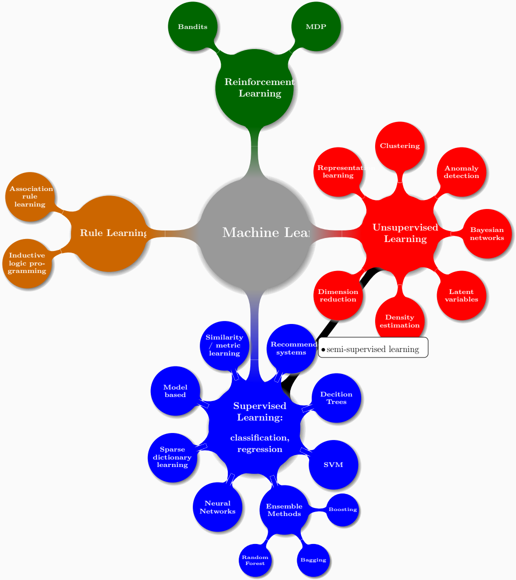
2019-2020

Table of contents

What is Reinforcement Learning?

Outline

Different Types of Learning

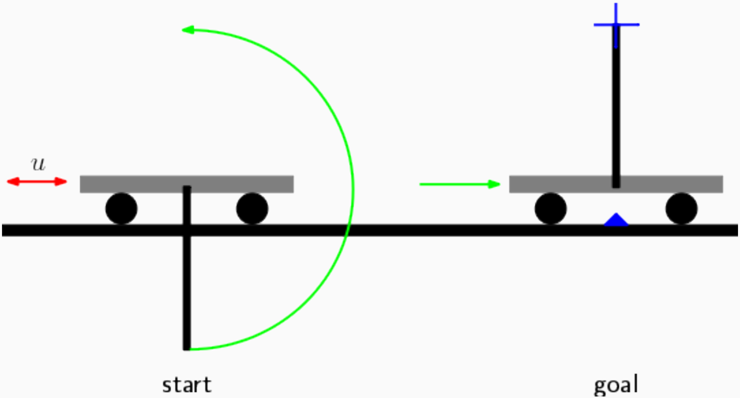


Reinforcement Learning

- Dates back to 1950's (Bellman)
- Stochastic Optimal Control
- Dynamic Programming
- Strong revival with the work of

DeepMind

Example: Inverted Pendulum



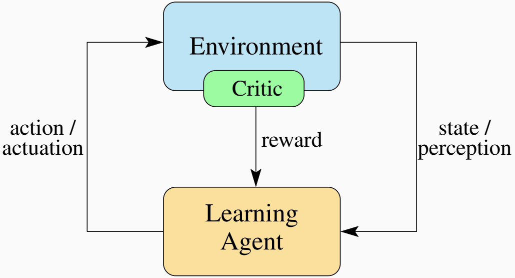
The Learning algorithm used by Martin is *Neural Fitted Q iteration*, a version of Q-iteration where neural networks are used as function approximators

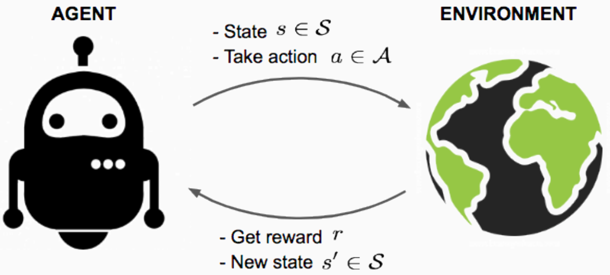
Some Applications

- TD-Gammon. [Tesauro '92-'95]: backgammon world champion
- KnightCap [Baxter et al. '98]: chess (2500 ELO)
- Computer poker [Alberta, '08...]
- Computer go [Mogo '06], [AlphaGo '15, Alphazero '18]
- Atari, Starcraft, etc. [Deepmind '10 sqq]
- Robotics: jugglers, acrobots, ... [Schaal et Atkeson '94 sqq]
- Navigation: robot guide in Smithonian Museum [Thrun et al. '99]
- Lift command [Crites et Barto, 1996]
- Internet Packet Routing [Boyan et Littman, 1993]
- Task Scheduling [Zhang et Dietterich, 1995]
- Maintenance [Mahadevan et al., 1997]
- Social Networks [Acemoglu et Ozdaglar, 2010]
- Yield Management, pricing [Gosavi 2010]
- Load forecasting [S. Meynn, 2010]
- ...

Outline

A Model for RL: MDP





exploration
vs
exploitation
dilemma

src: https://lilianweng.github.io

Model: Markov Decision Process

Markov Decision Process $=$ 4-uple $(\mathcal{S}, \mathcal{A}, k, r)$:

- State space $\mathcal{S} = \{1, \dots, p\}$
- Action space $\mathcal{A} = \{1, \dots, K\}$
- Transition kernel $k \in \mathfrak{M}_1(\mathcal{S})^{\mathcal{S}\times\mathcal{A}}$
- Random reward function $r \in \mathfrak{M}_1(\mathbb{R})^{\mathcal{S}\times\mathcal{A}}$

Dynamic $=$ controlled Markov Process:

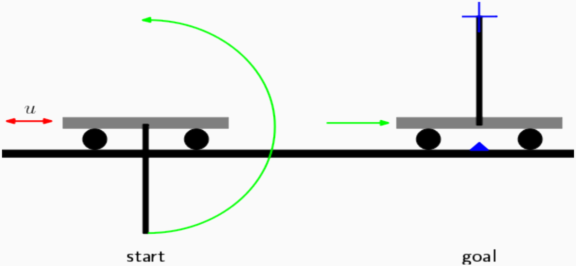
- Initial state S_0
- At each time $t \in \mathbb{N}$:
    - choose action A_t
    - get reward $X_t \sim r(\cdot|S_t, A_t)$
    - switch to new state $S_{t+1} \sim k(\cdot|S_t, A_t)$

Cumulated reward: $W = \sum_{t=0}^{\infty} \gamma^t X_t$ where $\gamma \in (0, 1)$ is a *discount parameter*

Goal

choose the actions so as to **maximize the cumulated reward** in expectation.

Example: inverted pendulum



- State: horizontal position, angular position and velocity
  State space: $\mathcal{S} = [0, 1] \times [-\pi, \pi] \times \mathbb{R}$
- Action: move left or right
  Action space: $\mathcal{A} = \{-1, +1\}$
- Reward = proportional to height of the stick end: if $S_t = (x_t, \theta_t, \dot{\theta}_t)$,

$$X_t = \sin(\theta_t)$$

- Transition: given by the laws of physics

Example: Retail Store Management 1/2

You owe a bike store. During week t, the (random) demand is D_t units. On Monday morning you may choose to command A_t additional units: they are delivered immediately before the shop opens. For each week:

- Maintenance cost: $h(s)$ for s units in stock left from the previous week
- Command cost: $C(a)$ for a units
- Sales profit: $f(q)$ for q units sold
- Constraint:
  - your warehouse has a maximal capacity of M unit (any additionnal bike gets stolen)
  - you cannot sell bikes that you don't have in stock

Example: Retail Store Management 2/2

- State: number of bikes in stock on Sunday
  State space: $\mathcal{S} = \{0, \ldots, M\}$
- Action: number of bikes commanded at the beginning of the week
  Action space: $\mathcal{A} = \{0, \ldots, M\}$
- Reward = balance of the week: if you command A_t bikes,

$$X_t = -C(A_t) - h(S_t) + f\big(\min(D_t, S_t + A_t, M)\big)$$

- Transition: you end the week with

$$S_{t+1} = \max\big(0, \min(M, S_t + A_t) - D_t\big) \qquad \text{bikes}$$

We may assume for example that $h(s) = h \cdot s$, $f(q) = p \cdot q$ and $C(a) = c_0 \mathbb{1}\{a > 0\} + c \cdot a$

Policies: Controlled Markov Chain

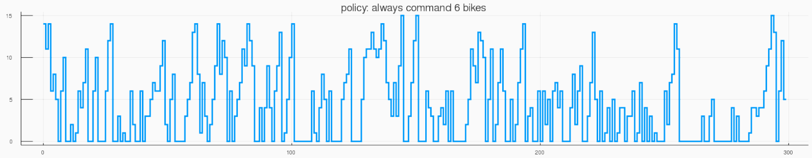
Policy $\pi : \mathcal{S} \to \mathcal{A}$
$\pi(s)$ = action chosen every time the agent is in state s

- can be randomized $\pi : \mathcal{S} \to \mathfrak{M}_1(\mathcal{A})$
  $\pi(s)_a$ = probability to choose action a in state s
- can be non-stationary $\pi : \mathcal{S} \times \mathbb{N} \to \mathfrak{M}_1(\mathcal{A})$
  $\pi(s, t)_a$ = probability to choose action a in state s at time t
- ...but it is useless: stationary, deterministic policies can do as well

For a given policy π, the sequence of states $(S_t)_{t\geq 0}$ is a Markov chain of kernel K_π:

$$K_\pi(s, s') = k\big(s'|s, \pi(s)\big)$$

and the sequence of rewards $(X_t)_{t\geq 0}$ is a hidden Markov chain



Policy Evaluation

Outline

Policy Value Function

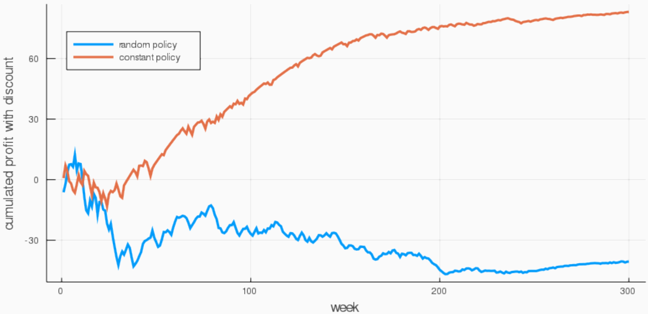
Avg reward function $\bar{r}(s,a) = \mathbb{E}\big[X_t \big| S_t = s, A_t = a\big] =$ mean of $r(\cdot|s,a)$

The **value function** of π is $V_\pi : \mathcal{S} \to \mathbb{R}$ defined by

$$V_\pi(s) = \mathbb{E}_\pi\left[\sum_{t\geq 0} \gamma^t X_t \middle| S_0 = s\right]$$

$$= \bar{r}\big(s,\pi(s)\big) + \gamma \sum_{s_1} k\big(s_1|s,\pi(s)\big)\,\bar{r}\big(s_1,\pi(s_1)\big) + \gamma^2 \sum_{s1,s2} k\big(s_1|s,\pi(s)\big)\,k\big(s_2|s_1,\pi(s_1)\big)\,\bar{r}\big(s_2,\pi(s_2)\big) + \ldots$$

One can simulate runs of the policy and estimate V_π by Monte-Carlo



Bellman's Equation for a Policy

Average reward function for policy π: $\bar{R}_\pi = \left[s \mapsto \bar{r}(s, \pi(s)\right]$

Matrix notation: identify functions $\mathcal{S} \to \mathbb{R}$ with $\mathbb{R}$-valued vectors
Coordinatewise partial order: $\forall U, V \in \mathbb{R}^{\mathcal{S}}, U \leq V \iff \forall s \in \mathcal{S}, U_s \leq V_s$

Bellman's Equation for a policy

The values $V_\pi(s)$ of a policy π at states $s \in \mathcal{S}$ satisfy the linear system:

$$\forall s \in \mathcal{S}, V_\pi(s) = \bar{r}\big(s, \pi(s)\big) + \gamma \sum_{s' \in \mathcal{S}} k\big(s' | s, \pi(s)\big)\, V_\pi(s')$$

In matrix form:

$$V_\pi = \bar{R}_\pi + \gamma K_\pi V_\pi$$

Theorem

Bellman's equation for a policy admits a unique solution given by

$$V_\pi = (I_{\mathcal{S}} - \gamma K_\pi)^{-1} \bar{R}_\pi$$

Operator View

Bellman's Transition Operator

Bellman's Transition Operator $T_\pi : \mathbb{R}^{\mathcal{S}} \to \mathbb{R}^{\mathcal{S}}$ is defined by

$$T_\pi(V) = \bar{R}_\pi + \gamma K_\pi V$$

It is **affine**, **isotonic** ($U \leq V \implies T_\pi U \leq T_\pi V$) and γ-**contractant**:
$\forall U, V \in \mathbb{R}^{\mathcal{S}}, \|T_\pi U - T_\pi V\|_\infty \leq \gamma \|U - V\|_\infty$

Proof: As a Markov kernel, K_π is 1-contractant:

$$|||K_\pi|||_\infty = \max_{\|x\|_\infty \leq 1} \|K_\pi x\|_\infty = \max_{\|x\|_\infty \leq 1} \max_{s \in \mathcal{S}} \left| \sum_{s' \in \mathcal{S}} K_\pi(s, s') x_{s'} \right| \leq \max_{\|x\|_\infty \leq 1} \max_{s \in \mathcal{S}} \sum_{s' \in \mathcal{S}} |K_\pi(s, s')| \, |x_{s'}| \leq 1$$

and thus

$$\|T_\pi U - T_\pi V\|_\infty = \|\bar{R}_\pi + \gamma K_\pi U - \bar{R}_\pi - \gamma K_\pi V\|_\infty = \gamma \|K_\pi (U - V)\|_\infty \leq \gamma |||K_\pi|||_\infty \|U - V\|_\infty \leq \gamma \|U - V\|_\infty$$

Thus, T_π **has a unique fixed point equal to** V_π
Moreover, for all $V_0 \in \mathbb{R}^{\mathcal{S}}$, $T_\pi^n V_0 \underset{n \to \infty}{\to} V_\pi$: denoting $V_n = T_\pi^n V_0$,

$$\|V_\pi - V_n\|_\infty = \|T_\pi V_\pi - T_\pi V_{n-1}\|_\infty \leq \gamma \|V_\pi - V_{n-1}\|_\infty \leq \gamma^n \|V_\pi - V_0\|_\infty$$

Also note that $T_\pi^n V_0 = \bar{R}_\pi + \gamma K_\pi R_\pi + \dots + \gamma^n K_\pi^n R_\pi + \gamma^n K_\pi^n V_0$

$$\to \left(I_{\mathcal{S}} + \gamma K_\pi + \gamma^2 K_\pi^2 + \dots \right) \bar{R}_\pi = (I_{\mathcal{S}} - \gamma K_\pi)^{-1} \bar{R}_\pi = V_\pi$$

Sample-based Policy Evaluation: $TD(0)$

As an alternative to plain Monte-Carlo evaluation, the **Temporal Difference** method is based on the idea of *stochastic approximation*

Algorithm 1: $TD(0)$

Input : $V_0 =$ any function (e.g. $V_0 \leftarrow 0_{\mathcal{S}}$)
$T =$ number of iterations

$V \leftarrow V_0$
for $t \leftarrow 0$ *to* T **do**
  $r' \leftarrow \mathrm{reward}(s, \pi(s))$
  $s' \leftarrow \mathrm{next\_state}(s, \pi(s))$
  $V(s) \leftarrow (1 - \alpha_t)V(s) + \alpha_t\big(r' + \gamma V(s')\big)$
end

Return: V

Stochastic Approximation

Let $(X_n)_{n\geq 1}$ be a sequence of iid variables with expectation μ. A *sequential estimator* of μ is: $\hat{\mu}_1 = X_1$ and for all $n \geq 2$,

$$\hat{\mu}_n = (1 - \alpha_n)\hat{\mu}_{n-1} + \alpha_n X_n$$

Proposition

When $(\alpha_n)_n$ is a decreasing sequence such that $\sum_n \alpha_n = \infty$ and $\sum_n \alpha_n^2 < \infty$, if the $(X_n)_n$ have a finite variance, $\hat{\mu}_n$ converges almost-surely to μ.

Case $\alpha_n = \frac{1}{n}$: $\hat{\mu}_n = \frac{X_1 + \cdots + X_n}{n}$ and $\mathbb{E}\left[(\hat{\mu}_n - \mu)^2\right] = \frac{\mathbb{V}ar[X_1]}{n}$

In $TD(0)$: $V(s) \leftarrow (1 - \alpha_t)V(s) + \alpha_t\big(r' + \gamma V(s')\big)$

At every step, if $V = V_\pi$ then the expectation of the rhs is equal to $V(s)$

Outline

What are Optimal Policies – and How to Find them?

Goal

Among all possible policies $\pi : \mathcal{S} \to \mathcal{A}$, find an *optimal* one π^* maximizing the expected value *on all states at the same time*:

$$\forall \pi : \mathcal{S} \to \mathcal{A}, \forall s \in \mathcal{S} : V_{\pi^*}(s) \geq V_\pi(s)$$

Questions:

- Is there always an optimal policy π^*?
- How to find π^*...
    - ... when the model (k, r) is known?
      $\to$ *planning*
    - ... when the model is unknown, but only sample trajectories can be observed?
      $\to$ *learning*

Bellman's Optimality Operator

Bellman's Optimality Operator

Bellman's Optimality Operator $T_* : \mathbb{R}^{\mathcal{S}} \to \mathbb{R}^{\mathcal{S}}$ defined by

$$\big(T_*(V)\big)_s = \max_{a\in\mathcal{A}} \left\{ \bar{r}(s,a) + \gamma \sum_{s'\in\mathcal{S}} k(s'|s,a) V_{s'} \right\}$$

is **isotonic** and γ-**contractant**. Besides, for every policy π, $T_\pi \leq T_*$ in the sense that $\forall U \in \mathbb{R}^S$, $T_\pi U \leq T_* U$

Note that T_* is not affine, due to the presence of the max

Proof: Since for all functions f and g we have $|\max f - \max g| \leq \max |f - g|$,

$$\begin{aligned}
\|T_* U - T_* V\|_\infty &= \max_{s\in\mathcal{S}} \left| \max_{a\in\mathcal{A}} \left\{ \bar{r}(s,a) + \gamma \sum_{s'\in\mathcal{S}} k(s'|s,a) U_{s'} \right\} - \max_{a'\in\mathcal{A}} \left\{ \bar{r}(s,a') - \gamma \sum_{s'\in\mathcal{S}} k(s'|s,a') V_{s'} \right\} \right| \\
&\leq \max_{s\in\mathcal{S}} \max_{a\in\mathcal{A}} \left| \bar{r}(s,a) + \gamma \sum_{s'\in\mathcal{S}} k(s'|s,a) U_{s'} - \bar{r}(s,a) - \gamma \sum_{s'\in\mathcal{S}} k(s'|s,a) V_{s'} \right| \\
&= \gamma \max_{s\in\mathcal{S}} \left| \max_{a\in\mathcal{A}} \left\{ \sum_{s'\in\mathcal{S}} k(s'|s,a)(U_{s'} - V_{s'}) \right\} \right| \leq \gamma \max_{s\in\mathcal{S}} \max_{a\in\mathcal{A}} \sum_{s'\in\mathcal{S}} |k(s'|s,a)| \|U - V\|_\infty \leq \gamma \|U - V\|_\infty
\end{aligned}$$

Policy Improvement

Greedy Policy

For every $V \in \mathbb{R}^{\mathcal{S}}$, there exist at least one policy π such that $T_\pi V = T_* V$. It is called **greedy w.r.t.** V, and is characterized as:

- $\forall s \in \mathcal{S}, \pi(s) \in \arg\max_{a \in \mathcal{A}} \left\{ \bar{r}(s,a) + \gamma \sum_{s' \in \mathcal{S}} k(s'|s,a) V_{s'} \right\}$
- $\pi \in \arg\max_{\pi'} \bar{R}_\pi + \gamma K_\pi V$

Policy Improvement Lemma

For any policy π, the greedy policy π' wrt V_π improves on π: $V_{\pi'} \geq V_\pi$

Proof Using successively $T_* \geq T_\pi$ and the isotonicity of $T_{\pi'}$:

$$T_{\pi'} V_\pi = T_* V_\pi \geq T_\pi V_\pi = V_\pi \implies T_{\pi'}^2 V_\pi \geq T_{\pi'} V_\pi \geq V_\pi \implies \cdots \implies T_{\pi'}^n V_\pi \geq V_\pi$$

for all $n \geq 1$, and since $T_{\pi'}^n V_\pi \underset{n \to \infty}{\to} V_{\pi'}$ we get $V_{\pi'} \geq V_\pi$

Optimal Value Function

Since T_* is γ-contractant, it **has a unique fixed point** V_* and $\forall V \in \mathbb{R}^{\mathcal{S}}, T_*^n V \underset{n\to\infty}{\to} V_*$

Bellman's Optimality Theorem

V_* is the optimal value function:

$$\forall s \in \mathcal{S}, V_*(s) = \max_\pi V_\pi(s)$$

and any policy π such that $T_\pi V_* = V_*$ is optimal

Proof: For any policy π, since $T_\pi \leq T_*$, $V_\pi = T_\pi^n V_\pi \leq T_*^n V_\pi \underset{n\to\infty}{\to} V_*$, and $V_* \geq V_\pi$.
Now, let π_* be the *greedy policy* w.r.t. V^*: then $T_{\pi_*} V_* = T_* V_* = V_*$, and hence its value V_{π_*} is V_*, the only fixed point of T_{π_*}. It is simultaneously optimal for all states $s \in \mathcal{S}$.

Corollary

Any finite MDP admits an optimal (deterministic and stationary) policy

This optimal policy is not necessarily unique

Planning

Value Iteration

If you know V_*, computing the greedy policy w.r.t V_* gives an optimal policy. And V_* is the fixed point of Bellman's optimality operator T_*, hence can be computed by a simple iteration process:

Algorithm 2: Value Iteration

Input : $\epsilon =$ required precision, $V_0 =$ any function (e.g. $V_0 \leftarrow 0_{\mathcal{S}}$)

1 $V \leftarrow V_0$

2 **while** $\|V - T_*(V)\| \geq \frac{(1-\gamma)\epsilon}{\gamma}$ **do**

3 $\quad V \leftarrow T_* V$

4 **end**

Return: $T_* V$

Theorem

The Value Iteration algorithm returns a value vector V such that $\|V - V_*\|_\infty \leq \epsilon$ using at most $\frac{\log \frac{M}{(1-\gamma)\epsilon}}{1-\gamma}$ iterations where $M = \|T_* V_0 - V_0\|_\infty$

Remark: if V_0 is the value function of some policy π_0 and if π_t is the sequence of policies obtained on line 3 (i.e. π_t is the greedy policy w.r.t. V_{t-1}), then the returned function obtained after T iterations is the value of the (non-stationary) policy $(\pi'_t)_t$, where $\pi'_t = \pi_{(T-t)_+}$.

Proof

Denoting $V_n = T_*^n V_0$,

$$\|V_* - V_n\|_\infty \leq \|V_* - T_* V_n\|_\infty + \|T_* V_n - V_n\|_\infty \leq \gamma\|V_* - V_n\|_\infty + \gamma\|V_n - V_{n-1}\|_\infty$$

gives $\|V_* - V_n\|_\infty \leq \frac{\gamma}{1-\gamma}\|V_n - V_{n-1}\|_\infty$. Hence, if $\|V_n - V_{n-1}\|_\infty \leq \frac{(1-\gamma)\epsilon}{\gamma}$, then $\|V_* - V_n\|_\infty \leq \epsilon$.

Now,

$$\|V_{n+1} - V_n\|_\infty = \|T_* V_n - T_* V_{n-1}\|_\infty \leq \gamma\|V_n - V_{n-1}\|_\infty \leq \gamma^n\|T_* V_0 - V_0\|_\infty$$

Hence, if $n \geq \dfrac{\log \frac{M}{(1-\gamma)\epsilon}}{1-\gamma} \geq \dfrac{\log \frac{M\gamma}{(1-\gamma)\epsilon}}{-\log(\gamma)}$, then $\gamma^n \leq \frac{(1-\gamma)\epsilon}{M\gamma}$ and $\|V_{n+1} - V_n\|_\infty \leq \frac{(1-\gamma)\epsilon}{\gamma}$.

Policy Iteration

The Policy Improvement lemma directly suggests Policy Iteration: starting from any policy, evaluate it (by solving the linear system $T_\pi V_\pi = V_\pi$) and improve π greedily:

Algorithm 3: Policy Iteration

Input : $\pi_0 =$ any policy (e.g. chosen at random)

$\pi \leftarrow \pi_0$

$\pi' \leftarrow \mathrm{NULL}$

while $\pi \neq \pi'$ **do**

  compute V_π

  $\pi' \leftarrow \pi$

  $\pi \leftarrow$ greedy policy w.r.t. V_π

end

Return: π

NB: the iterations of PI are much more costly than those of VI

Convergence of Policy Iteration

Theorem

The Policy Iteration algorithm always returns an optimal policy in at most $|\mathcal{A}|^{|\mathcal{S}|}$ iterations.

Proof: the Policy Improvement lemma shows that the value of π raises strictly at each iteration before convergence, and there are only $|\mathcal{A}|^{|\mathcal{S}|}$ different policies. Remark: better upper-bounds in $O\left(\frac{|\mathcal{A}|^{|\mathcal{S}|}}{|\mathcal{S}|}\right)$ are known.

Lemma

Let (U_n) be the sequence of value functions generated by the Value Iteration algorithm, and (V_n) be the one for the Policy Iteration algorithm. If $U_0 = V_0$ (i.e. if U_0 is the value function of π_0), then

$$\forall n \geq 0, U_n \leq V_n$$

Proof: Assume by induction that $U_n \leq V_n$. Since T_* and $T_{\pi_{n+1}}$ are isotonic, and since $V_n \leq V_{n+1}$ by the policy improvement lemma:

$$U_{n+1} = T_* U_n \leq T_* V_n = T_{\pi_{n+1}} V_n \leq T_{\pi_{n+1}} V_{n+1} = V_{n+1}$$

Linear Programming

Proposition

Let $\alpha : \mathcal{S} \to (0, +\infty)$. V_* is the only solution of the linear program

$$\min_V \sum_{s \in \mathcal{S}} \alpha(s) V(s)$$

$$\text{subject to } \forall s \in \mathcal{S}, \forall a \in \mathcal{A}, V(s) \geq \bar{r}(s,a) + \gamma \sum_{s' \in \mathcal{S}} k(s'|s,a) V(s')$$

Proof: By Bellman's optimality equation $T_* V_* = V_*$, V_* satisfies the constraint with equality. If V satisfies the condition, then $W = V - V_*$ is such that $\forall s, a$, $W(s) \geq \gamma \sum_{s' \in \mathcal{S}} k(s'|s,a) W(s')$; thus if $s_- \in \arg\min_{s \in \mathcal{S}} W(s)$ one gets $W(s_-) \geq \gamma \sum_{s' \in \mathcal{S}} k(s'|s,a) W(s') \geq -\gamma \big|W(s_-)\big|$, hence $W(s_-) \geq 0$ and $W \geq 0$, and thus $\sum_{s \in \mathcal{S}} \alpha(s) V(s) \geq \sum_{s \in \mathcal{S}} \alpha(s) V_*(s)$ with equality iff $V = V_*$.

This linear program has $|\mathcal{S}| \cdot |\mathcal{A}|$ rows (constraints) and $|\mathcal{S}|$ columns (variables). Solvers have a complexity typically larger in the number of rows than columns. Hence, it may be more efficient to consider the dual problem.

Learning

Outline

State-Action Value Function

Definition

The state-action value function $Q_\pi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ for policy π is the expected return for first taking action a in state s, and then following policy π:

$$\begin{aligned} Q_\pi(s,a) &= \text{``}\mathbb{E}_{a,\pi}\text{``} \left[\sum_{t=0}^{\infty} \gamma^t r(S_t, A_t) \,\middle|\, S_0 = s, A_0 = a \right] \\ &= \bar{r}(s,a) + \gamma \sum_{s'} k(s'|s,a) V_\pi(s') \end{aligned}$$

The state-action value function is a key-tool in the study of MDPs

Observe that $Q_\pi\big(s, \pi(s)\big) = V_\pi(s)$.

Policy Improvement Lemma

Lemma

For any two policies π and π',

$$\left[\forall s \in \mathcal{S}, Q_\pi\left(s, \pi'(s)\right) \geq Q_\pi\left(s, \pi(s)\right)\right] \implies \left[\forall s \in \mathcal{S}, V_{\pi'}(s) \geq V_\pi(s)\right]$$

Furthermore, if one of the inequalities in the LHS is strict, then at least one of the inequalities in the RHS is strict

Proof: for any $s \in \mathcal{S}$,

$$\begin{aligned}
V_\pi(s) &= Q_\pi\left(s, \pi(s)\right) \leq Q_\pi\left(s, \pi'(s)\right) \\
&= \bar{r}\left(s, \pi'(s)\right) + \gamma \sum_{s_1 \in \mathcal{S}} k(s_1 | s, \pi'(s)) \underbrace{V_\pi(s_1)}_{= Q_\pi\left(s_1, \pi(s_1)\right)} \\
&\leq \bar{r}\left(s, \pi'(s)\right) + \gamma \sum_{s_1 \in \mathcal{S}} k\left(s_1 | s, \pi'(s)\right) Q_\pi\left(s_1, \pi'(s_1)\right) \\
&= \bar{r}\left(s, \pi'(s)\right) + \gamma \sum_{s_1 \in \mathcal{S}} k\left(s_1 | s, \pi'(s)\right) \bar{r}\left(s_1, \pi'(s_1)\right) + \gamma^2 \sum_{s_1, s_2 \in \mathcal{S}} k\left(s_1 | s, \pi'(s)\right) k\left(s_2 | s_1, \pi'(s_1)\right) V_\pi(s_2) \\
&\cdots = V_{\pi'}(s)
\end{aligned}$$

Furthermore, we see that $Q_\pi\left(s, \pi(s)\right) < Q_\pi\left(s, \pi'(s)\right)$ implies $V_\pi(s) < V_{\pi'}(s)$

Bellman's Optimality Condition: Q-table formulation

Theorem

A policy π is optimal if and only if

$$\forall s \in \mathcal{S},\ \pi(s) \in \arg\max_{a \in \mathcal{A}} Q_\pi(s, a)$$

Proof:

- A policy π such that

$$\pi(s) \in \arg\max_{a \in \mathcal{A}} Q_\pi(s, a) = \arg\max_{a \in \mathcal{A}} \left\{ \bar{r}(s, a) + \gamma \sum_{s' \in \mathcal{S}} k(s'|s, a) V_\pi(s') \right\}$$

  is greedy w.r.t. V_π and thus $T_* V_\pi = T_\pi V_\pi = V_\pi$: V_π is the unique fixed point V_* of T_*
- If $\exists s_0 \in \mathcal{S}, a \in A$ such that $\pi(s_0) < Q_\pi(s_0, a)$, then by the policy improvement lemma the policy π' defined by $\pi'(s) = \pi(s)$ for $s \neq s_0$ and $\pi'(s_0) = a$ is better: $V_{\pi'}(s_0) > V_\pi(s_0)$

Outline

Q-Learning

Algorithm 4: Q-learning

Input : $Q_0 =$ any state-value function (e.g. chosen at random)
$s_0 =$ initial state (possibly chosen at random)
$\pi =$ learning policy (may be ϵ-greedy w.r.t. current Q)
$T =$ number of iterations

1 $Q \leftarrow Q_0$
2 $s \leftarrow s_0$
3 **for** $t \leftarrow 0$ *to* T **do**
4 $\quad a \leftarrow \mathrm{select\_action}(\pi(Q), s)$
5 $\quad r' \leftarrow \mathrm{random\_reward}(s, a)$
6 $\quad s' \leftarrow \mathrm{next\_state}(\mathrm{s}, \mathrm{a})$
7 $\quad Q(s, a) \leftarrow Q(s, a) + \alpha_t \big[r' + \gamma \max_{a' \in \mathcal{A}} Q(s', a') - Q(s, a)\big]$
8 $\quad s \leftarrow s'$
9 **end**

Return: Q

Off-policy learning: update rule $\neq$ learning policy (on l.7, a' may be different from played action a)

Convergence of Q-learning

Denote by $(S_t)_t$ (resp. $(A_t)_t$) the sequence of states (resp. actions) visited by the Q-learning algorithm. For all $(s, a) \in \mathcal{S} \times \mathcal{A}$, let $\alpha_t(s, a) = \alpha_t \mathbb{1}\{S_t = s, A_t = a\}$

Theorem

If for all $s \in \mathcal{S}$ and $a \in \mathcal{A}$ it holds that $\sum_{t \geq 0} \alpha_t(s, a) = +\infty$ and $\sum_{t \geq 0} \alpha_t^2(s, a) < +\infty$, then with probability 1 the Q-learning algorithm converges to the optimal state-value function Q_*

This condition implies in particular that the policy *select_action* guarantees an infinite number of visits to all state-action pairs (s, a)

The **proof** is more involved, and based on the idea of **stochastic approximation**

SARSA

Algorithm 5: SARSA

Input : Q_0 = any state-value function (e.g. chosen at random)
s_0 = initial state (possibly chosen at random)
π = learning policy (may be ϵ-greedy w.r.t. current Q)
T = number of iterations

$Q \leftarrow Q_0$
$s \leftarrow s_0$
$a \leftarrow \mathrm{select\_action}(\pi(Q), s)$
for $t \leftarrow 0$ *to* T **do**
  $r' \leftarrow \mathrm{random\_reward}(s, a)$
  $s' \leftarrow \mathrm{next\_state}(\mathrm{s}, \mathrm{a})$
  $a' \leftarrow \mathrm{select\_action}(\pi(Q), s')$
  $Q(s, a) \leftarrow Q(s, a) + \alpha_t \left[r' + \gamma Q(s', a') - Q(s, a) \right]$
  $s \leftarrow s'$
end
Return: Q

On-policy learning: update rule = learning policy

Q-learning with function approximation

If $\mathcal{S} \times \mathcal{A}$ is large, it is necessary

- to do **state aggregation**
- or to assume a model $Q_\theta(s, a)$ for $Q(s, a)$, where θ is a (finite-dimensional) parameter to be fitted. The obvious extension of Q-learning is:

$$\theta_{t+1} = \theta_t + \alpha_t \big[r' + \gamma \max_{a' \in \mathcal{A}} Q(s', a') - Q(s, a) \big] \nabla_\theta Q_{\theta_t}(S_t, A_t)$$

For example, with a linear approximation method with $Q_\theta = \theta^T \phi$ with features map $\phi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$, line 8 of Q-learning is replaced by:

$$\theta \leftarrow \theta + \alpha \big[r' + \gamma \max_{a' \in \mathcal{A}} \theta^T \phi(s', a') - \theta^T \phi(s, a) \big] \phi(s, a)$$

- possibility to use any function approximator, typically *splines* or *neural networks*
- ...but very unstable and few guarantees of convergence!
- possiblity to update θ in *batch* and not at each step

Conclusion: What more?

- a lot !
- $TD(\lambda)$ and eligibility traces
- Model-based learning: KL-UCRL

  Build optimistic estimates of Q-table, and play greedily w.r.t. these estimates
- POMDP: Partially Observed Markov Decision Process
- Bandit models

  $=$ MDPs with only 1 state, but already a dilemma exploration vs exploitation
- MCTS: AlphaGo / AlphaZero

References

- **C. Szepesvári** Algorithms for Reinforcement Learning. Morgan & Claypool, 2010
- **M. Mohri, A. Rostamizadeh and A. Talwalkar** *Foundations of Machine Learning, 2nd Ed.*, MIT Press, 2018
- **T. Lattimore and C. Szepesvári** *Bandit Algorithms*, Cambridge Univeristy Press, 2019
- **D. Bertsekas and J. Tsitsiklis** *Neuro-Dynamic Programming*, Athena Scientific, 1996
- **M. L. Puterman.** *Markov Decision Processes, Discrete Stochastic Dynamic Programming.* Wiley-Interscience, 1994.
- **R. S. Sutton & A. G. Barto.** *Reinforcement Learning, an Introduction (2nd Ed.)* MIT Press, 2018.