Perfect Simulation of Processes with Long Memory `[arXiv:1106.5971]`

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Outline

Stationary Markov Chains

Markov Chain $(X_t)_{t\in\mathbb{Z}}$ on the finite set $G = \{1, \dots, K\}$

Dynamical System $X_{t+1} = \phi(U_t, X_t)$

Kernel $P(i, \cdot) \in \mathcal{M}_1(G)$, such that

$$\forall i, j \in G, \quad \mathbb{P}(X_{t+1} = j | X_t = i) = P(i, j)$$

Stationary distribution π such that $\pi P = \pi$

Simulating the chain

Problem given a kernel P, simulate a sample path $X_0, X_1, \ldots, X_n$ from the stationary Markov Chain with kernel P

Update rule $\phi : [0,1[\times \{1,\ldots,K\} \to \{1,\ldots,K\}$ such that

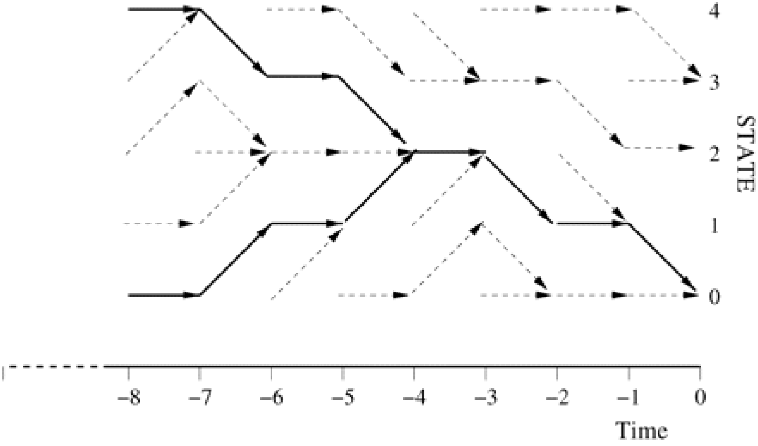
$$\forall i,j \in G : \quad \lambda\Big(\{u : \phi(u,i) = j\}\Big) = P(i,j)$$

Recursion Given X_t, taking $X_{t+1} = \phi(U_t, X_t)$ works

$\Longrightarrow$ it is sufficient to sample X_0 from π.

Coupling from the Past: the idea

Idea: given the sequence $(U_t)_{t \leq 0}$, I may know X_0 even if I do not know the value of X_{-8}!



Coupling from the Past: more formally

Local transition for each $t < 0$ let $f_t : G \to G$ be defined by

$$f_t(g) = \phi(U_t, g)$$

Iterated transition $F_t = f_{-1} \circ \cdots \circ f_t$

Propp-Wilson: if you know U_t for all $t \geq \tau(n)$, where

$$\tau(n) = \sup\{t < 0 : F_t \text{ is constant}\} ,$$

then you know X_0.

Prop: $\tau(n)$ is of the same order of magnitude as the *mixing time* of the chain!

The Nummelin update rule

Nummelin coefficient:

$$A_1 = \sum_{j=1}^{K} \min_{1 \le i \le K} P(i,j)$$

Update rule $\phi : [0,1[\times G \to G$ such that

$$u \le A_1 \implies \forall i, i' \in G,\ \phi(u,i) = \phi(u,i')$$

Regeneration if $U_t \le A_1$, then $X_{t+1}, X_{t+2} \dots,$ is independent from $X_t, X_{t-1}, \dots$.

$\Longrightarrow$ alternative coupling from the past: wait for a regeneration!

Outline

Context Tree Sources

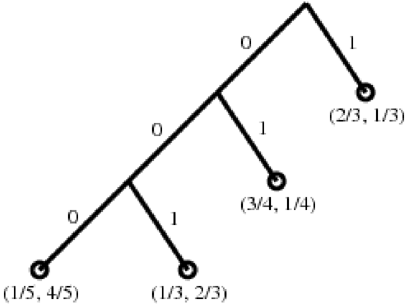
Variable Length Markov Chains: the order of the chain is allowed to depend on the past according to some *tree structure*

t r y i n g _ v a n i l l a _ q u i e t

Example : $T = \{1, 10, 100, 000\}$

$$\mathbb{P}(X_1^4 = 00110 | X_{-1}^0 = 10)$$

$$\begin{array}{rl} = & \mathbb{P}(X_1 = 0 | X_{-1}^0 = 10) \\ \times & \mathbb{P}(X_2 = 0 | X_{-1}^1 = 100) \\ \times & \mathbb{P}(X_3 = 1 | X_{-1}^2 = 1000) \\ \times & \mathbb{P}(X_4 = 1 | X_{-1}^3 = 10001) \\ \times & \mathbb{P}(X_5 = 0 | X_{-1}^4 = 100011) \end{array}$$



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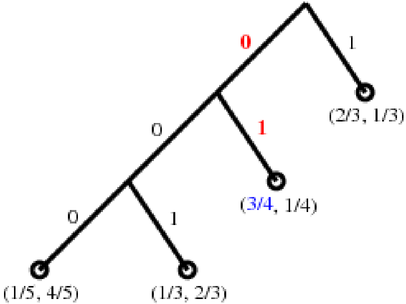

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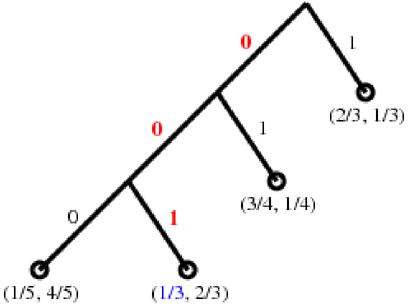

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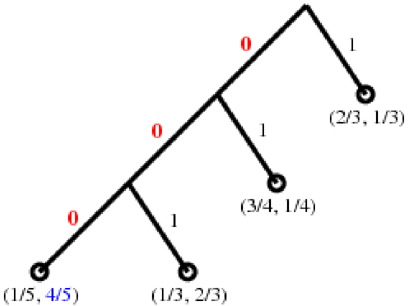

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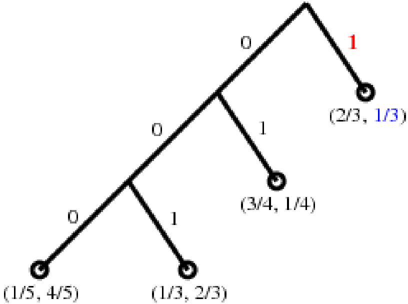

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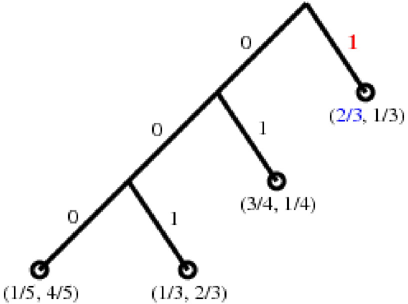

Histories

History $\underline{w} = w_{-\infty:-1} \in G^{-\mathbb{N}^*}$

Ultrametric distance $\delta(\underline{w}, \underline{z}) = 2^{\sup\{k<0: w_k \neq z_k\}}$

$\Longrightarrow$ $(G^{-\mathbb{N}^*}, \delta)$ is a complete and compact set.

Ball $B \subset G^{-\mathbb{N}^*}$ is a (closed or open) ball if

$$B = \left\{\underline{z}s : \underline{z} \in G^{-\mathbb{N}^*}\right\} \text{ for some } s \in G^*$$

Trees and roots $B = \mathcal{T}(s)$, $s = \mathcal{R}(B)$

Ex: $\mathcal{T}(\varepsilon) = G^{-\mathbb{N}^*}$, the radius of $\mathcal{T}(s)$ is $2^{-|s|-1}$

Piecewise constant A mapping f defined on $G^{-\mathbb{N}^*}$ is *piecewise constant* if the exists a family $\{s_j\}_{j\in\mathbb{N}}$ of elements of $G^{-\mathbb{N}^*}$ such that f is constant on each ball $\mathcal{T}(s_j)$.

Projection $\Pi^n : G^{-\mathbb{N}^*} \to G^n$ be defined by $\Pi^n(\underline{w}) = w_{n:-1}$.

Kernels

Kernel $P : G^{-\mathbb{N}^*} \to \mathcal{M}_1(G)$

Total Variation distance: for $p, q \in \mathcal{M}_1(G)$,

$$|p - q|_{TV} = \frac{1}{2} \sum_{a \in G} |p(a) - q(a)| = 1 - \sum_{a \in G} p(a) \wedge q(a)$$

Process $(X_t)_{t \in \mathbb{Z}}$ with distribution ν on $G^{\mathbb{Z}}$ is *compatible* with kernel P if the latter is a version of the one-sided conditional probabilities of the former:

$$\nu \left(X_i = g | X_{i+j} = w_j, j \in -\mathbb{N}^* \right) = P(g | \underline{w})$$

for all $i \in \mathbb{Z}, g \in G$ and ν-almost every $\underline{w}$.

Kernel continuity

continuity $P : \left(G^{-\mathbb{N}^*}, \delta\right) \to (\mathcal{M}_1(G), |\cdot|_{TV})$

oscillation of P on the ball $\mathcal{T}(s)$

$$\eta(s) = \sup\left\{\left|P(\cdot|\underline{w}) - P(\cdot|\underline{z})\right|_{TV} : \underline{w}, \underline{z} \in \mathcal{T}(s)\right\}.$$

P1: P is continuous if and only if $\forall \underline{w} \in G^{-\mathbb{N}^*}, \eta(w_{-k:-1}) \to 0$ as k goes to infinity.

P2: P is continuous if and only if $\sup\{\eta(s) : s \in G^{-k}\} \to 0$ as k goes to infinity.

P3: P is uniformly continuous if and only it is continuous.

Outline

Existing CFP algorithms

Comets, Fernandez, Ferrari 2002 simulation algortithm using a Kalikow-type decomposition of the kernel as a mixture of Markov Chains of all orders. Require strong continuity conditions.

De Santis, Piccioni mix the ideas of CFF and the algorithm of PW: they propose an hybrid simulation scheme working with a Markov regime and a long-memory regime.

Gallo 2010 Relaxes the continuity condition, replaced by conditions on the *shape* of the memory tree.

Our goal: describe a single procedure that generalizes the sampling schemes of CFF and PW in an unified framework.

Update rules

Def: $\phi : [0,1[\times G^{-\mathbb{N}^*} \to G$ is called an *update rule* of P if

$$U \sim \mathcal{U}([0,1[) \implies \phi(U,\underline{w}) \sim P(\cdot|\underline{w})$$

for all $\underline{w} \in G^{-\mathbb{N}^*}$.

Prop: There exists an update rule ϕ of P such that:

$$\forall s \in G^*, 0 \leq u < 1 - |G|\eta(s) \implies \phi(u,\cdot) \text{ cst on } \mathcal{T}(s) .$$

Prop: If P is continuous, then for all $u \in [0,1[$ the mapping $\underline{w} \to \phi(u,\underline{w})$ is continuous, i.e, piecewise constant.

Perfect Simulation Scheme

Goal: draw $(X_n, \ldots, X_{-1})$ from a stationary distribution compatible with P

Tool: semi-infinite sequence of i.i.d. random variables $U_t \sim \mathcal{U}([0,1[)$

Idea: $S_t = (\ldots, X_{t-1}, X_t), t \in \mathbb{Z}$ is a Markov Chain on $G^{-\mathbb{N}^*}$, with kernel Q given by:

$$\forall \underline{w}, \underline{z} \in G^{-\mathbb{N}^*}, \quad Q(\underline{w}|\underline{z}) = P(w_{-1}|\underline{z}) \mathbb{1}_{w_{i-1}=z_i : i<0} \; .$$

A Propp-Wilson Scheme

Local transition $f_t : G^{-\mathbb{N}^*} \to G^{-\mathbb{N}^*}$ be defined by

$$f_t(\underline{w}) = \underline{w}\phi(U_t, \underline{w});$$

Iterated transition $F_t = f_{-1} \circ \cdots \circ f_t$

Projection $H_t^n = \Pi^n \circ F_t$

Continuity: H_t^n is a piecewise constant mapping

Propp-Wilson: if you wait for

$$\tau(n) = \sup\{t < n : H_t^n \text{ is constant}\} ,$$

you will know $(X_n, \ldots, X_{-1})$

Local Continuity Coefficients

For every $\underline{w} \in G^{-\mathbb{N}^*}$ the continuity of kernel P is locally characterized by the coefficients

$$\begin{aligned}
a_k(g|w_{-k:-1}) &= \inf\{P(g|\underline{z}) : \underline{z} \in \mathcal{T}(w_{-k:-1})\} \\
A_k(w_{-k:-1}) &= \sum_{g\in G} a_k(g|w_{-k:-1}) \\
A_k^- &= \inf_{s\in G^{-k}} A_k(s) \\
\alpha_k(g|w_{-k:-1}) &= A_{k-1}(w_{-k+1:-1}) + \sum_{h<g} \{a_k(h|w_{-k:-1}) - a_{k-1}(h|w_{-k+1:-1})\} \\
\beta_k(g|w_{-k:-1}) &= A_{k-1}(w_{-k+1:-1}) + \sum_{h\le g} \{a_k(h|w_{-k:-1}) - a_{k-1}(h|w_{-k+1:-1})\}
\end{aligned}$$

Local characterization of the kernel continuity

Let P be a fixed kernel on G.

Prop: For all $s \in G^*$,

$$1 - |G|\eta(s) \leq A_{|s|}(s) \leq 1 - \eta(s) \ .$$

Prop: The three assertions are equivalent:

(i) the kernel P is continuous;

(ii) $\forall \underline{w} \in G^{-\mathbb{N}^*}$, $A_k(w_{-k:-1}) \to 1$ as $k \to \infty$;

(iii) $A_k^- \to 1$ as k goes to infinity.

Construction of the update rule

Prop: For every $\underline{w} \in G^{-\mathbb{N}^*}$,

$$[0,1[= \bigsqcup_{g \in G, k \in \mathbb{N}} [\alpha_k(g|w_{-k:-1}), \beta_k(g|w_{-k:-1})[\ .$$

Def: The mapping $\phi : [0,1[\times G^{-\mathbb{N}^*} \to G$ is defined as follows:

$$\phi(u, \underline{w}) = \sum_{g \in G, k \in \mathbb{N}} g \mathbb{1}_{[\alpha_k(g), \beta_k(g)[}(u) \ .$$

Prop: ϕ is an update rule such that $\forall s \in G^*, \forall u \in [0,1]$:

$$\forall \underline{w}, \underline{z} \in \mathcal{T}(s), \quad u < A_{|s|}(s) \implies \phi(u, \underline{w}) = \phi(u, \underline{z}) \ .$$

Illustration

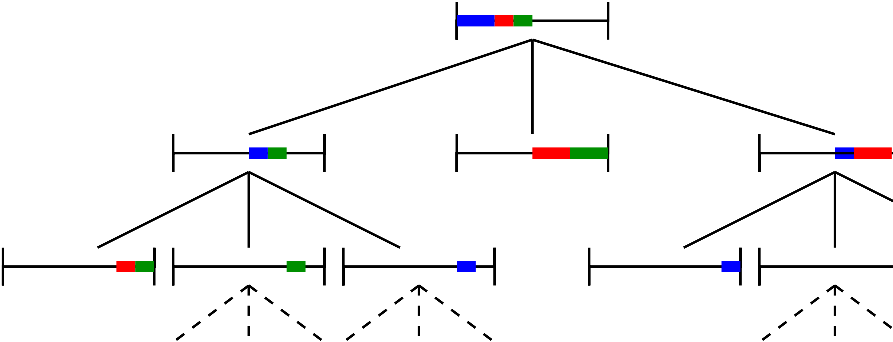

Figure: Graphical representation of an update rule ϕ on alphabet $\{0,1,2\}$: for each $w_{-k:-1}$, the intervals $[\alpha_k(g|w_{-k:-1}), \beta_k(g|w_{-k:-1})[$ are represented in blue ($g=0$), red ($g=1$) and green ($g=2$). For example, $P(1|1) = \alpha_0(1|\varepsilon) + \alpha_1(1|1) = 1/8 + 1/4$, and $P(0|00) = \alpha_0(0|\varepsilon) + \alpha_1(0|0) + \alpha_2(0|00) = 1/4 + 1/8 + 0$.

Outline

Complete suffix Dictionaries

Def: a (finite or infinite) set of words $D \subset \mathcal{P}(G^*)$ is a CSD if one of the following equivalent properties is satisfied:

- every $\underline{w} \in G^{-\mathbb{N}^*}$ has a unique suffix in D:

$$\forall \underline{w} \in G^{-\mathbb{N}^*}, \exists! s \in D : \underline{w} \succeq s \ ;$$

- $\{\mathcal{T}(s) : s \in D\}$ is a partition of $G^{-\mathbb{N}^*}$:

$$G^{-\mathbb{N}^*} = \sqcup_{s \in D} \mathcal{T}(s) \ .$$

The *depth* of D is

$$\mathrm{d}(D) = \sup\{|s| : s \in D\}$$

The smallest possible CSD is $\{\epsilon\}$: it has depth 0 and size 1.
The second smallest is G, it has depth 1.

Representation as a trie

A CSD D can be represented by a *trie*, that is, a tree with edges labelled by elements of G such that the path from the root to any leaf is labelled by an element of D.

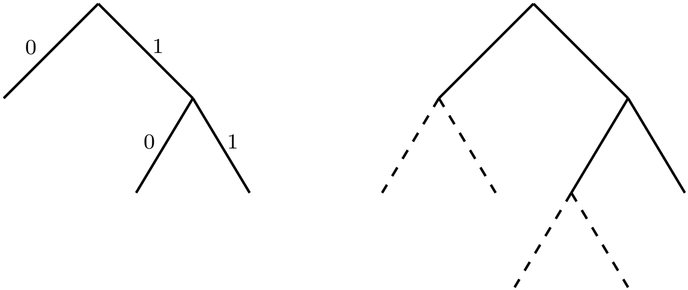

Figure: Left: the trie representing the Complete Suffix Dictionary $D = \{0, 01, 11\}$. Right: $\{00, 10, 001, 101, 11\} \succeq \{0, 01, 11\}$. Both examples concern the binary alphabet.

If D and D' are such that $\forall s \in D', s \succeq D$, then we note $D' \succeq D$.

Piecewise constant functions

Def: For a CSD D, we say that a function f defined on $G^{-\mathbb{N}^*}$ is *D-constant* if

$$\forall s \in D, \forall w \in \mathcal{T}(s), f(\underline{w}) = f(\underline{0}s) \; .$$

Def: For every $h \in G^{-\mathbb{N}^*} \cup G^*$ we define $f(h) = f(\mathcal{T}(h)) = f\left(\vec{D}(h)\right)$ and note that if $h \succeq D, f(h)$ is a singleton.

Minimal CSD D^f = CSD with minimal cardinality such that f is constant on each of its elements.

Pruning if f is D-constant, then D^f can be obtained by recursive pruning of D.

Recursive construction of H_t^n

The mapping H_t^n being piecewise constant, we define $D_t^n = D^{H_t^n}$.

- Initialization: $D_{-1}^{-1} = G, \quad \forall g \in G, \forall \underline{w} \in \mathcal{T}(s), H_{-1}^{-1}(\underline{w}) = g$.
- For $t < -1$, $s \in D(U_t)$ denote $\{g_t(s)\} = \phi(U_t, s)$ and define $E_t^n(s)$ as follows:
  - if $sg_t(s) \succeq D_{t+1}^n$, let $E_t^n(s) = \{s\}$;
  - otherwise, let

$$E_t^n(s) = \bigcup_{hg_t(s) \in D_{t+1}^n(sg_t(s))} \{h\} .$$

- Let

$$E_t^n = \bigcup_{s \in D(U_t)} E_t^n(s) .$$

  E_t^n is a CSD, and H_t^n is E_t^n-constant.
- D_t^n is obtained by pruning E_t^n
- for $t = n$, D_t^t is equal to D_t^{t+1} unless $D_t^{t+1} = \{\epsilon\}$, in which case $D_t^t = G$.

How it works

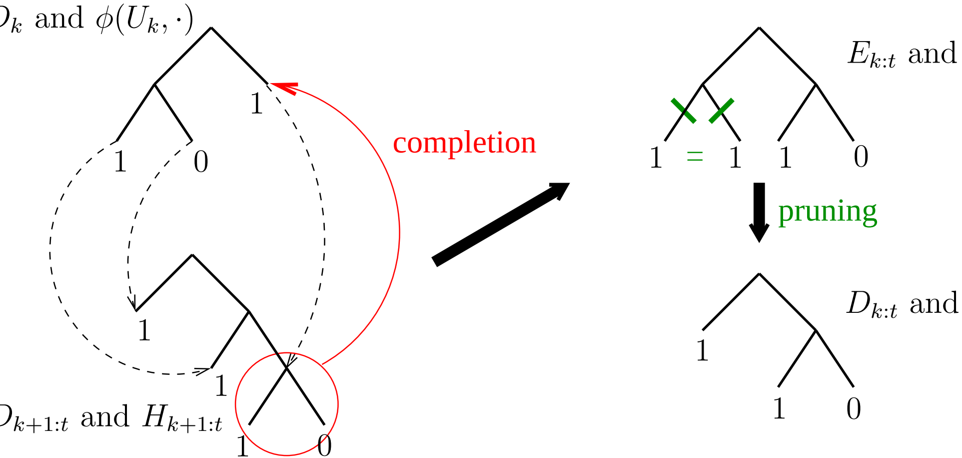

Figure: Obtaining D_t^n from D_t and D_{t+1}^n. For each function $\phi(U_t, \cdot), D_{t+1}^n$ and D_t^n, we represent a CSD on which it is constant, and the values taken in each leaf; here, $G = \{0, 1\}$.

Example

Renewal process For all $k \geq 0$, let

$$P(0|01^k) = 1 - 1/\sqrt{k}$$

Not Harris Observe that $P(1|0) = \lim_{k\to\infty} P(0|01^k) = 1$, so that $a_0 = 0$.

Slow continuity for $k \geq 0$, $A_{k+1} = A_k(01^k) = 1 - 1/\sqrt{k}$, so that

$$\sum_n \prod_{k=2}^{n} A_k^- < \infty$$

$\Longrightarrow$ the continuity conditions of [Comets, Fernandez, Ferrari] and [De Santis, Piccioni] do not apply.

yet the algorithm works well

Example: the coupling illustrated

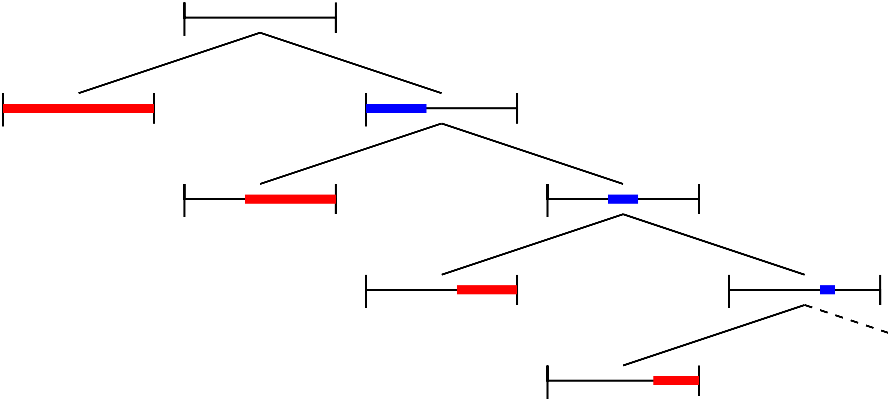

Figure: Graphical representation of the of P - blue stands for 0, red stands for 1

Conclusion

The perfect simulation scheme described in this presentation is

- **Versatile:** works as well for Markov Chains and for (mixing) infinite memory processes
- **Powerful:** needs weak continuity assumptions to converge
- **Fast:** for (large order) Markov chains, much faster than Propp-Wilson's algorithm on the extended chain: all the tries encountered in the algorithm are of size at most $|D| \times d(D) \ll 2^{|D|}$.
- **but** a little hard to implement...