Proofs and Programs

TD 3 - The heart of Curry-Howard

Philippe Audebaud, Aurore Alcolei

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HW is due on 27 February 8am

Highlights

- Working on basics of type theory (typability - ex 1, inhabitance - ex 2).
- Understanding the C.-H. Correspondence between NJ($\Rightarrow$) derivation trees, and $\lambda_\rightarrow$ terms (ex 3,5);
- Taking advantage of that understanding, by dealing with other logical connectors (ex-4).

Let $X, Y, \ldots.$ be *type variables* and $x, y, \ldots$ be *term variables*. Types (capital letter) and pure terms (small letters) are inductively defined by:

$$\begin{aligned} S, T, \ldots \quad &::= \quad X \mid S \rightarrow T \\ a, b, \ldots \quad &::= \quad x \mid \lambda x.a \mid ab \end{aligned}$$

The *simply typed* λ*-calculus* $\lambda_\rightarrow$ is defined as the set of *typable* λ-terms in the following *type system*:

$$\frac{}{\Delta, x : A \vdash x : A}\ (\text{Hyp}) \qquad \frac{\Delta, x : A \vdash m : B}{\Delta \vdash \lambda x.m : A \rightarrow B}\ (\rightarrow_I) \qquad \frac{\Delta \vdash f : A \rightarrow B \quad \Delta \vdash n : A}{\Delta \vdash fn : B}\ (\rightarrow_E)$$

where every variable appears at most once in the *context* (hence viewed as a set of typed variables). Simply typed λ-terms correspond to annotations of proof derivations in the implicative fragment of NJ, NJ($\Rightarrow$). Then, for every proposition P:

$$P \text{ is provable in NJ}(\Rightarrow) \quad \text{iff} \quad P \text{ (viewed as a type) is inhabited in } \lambda_\rightarrow$$

Exercice 1 ($\vdash t : ?$)**.** Provide the "most general types" that solve the following *type inference problems* or explain why it is not possible:

1. $t \equiv \lambda f.\lambda g.g\,(f\ \mathsf{F})\ (f\ \mathsf{T})$; $\qquad t \equiv \lambda f.\lambda g.g\,(f\ \mathsf{I})\ (f\ \mathsf{T})$;
2. $t \equiv \lambda f.\lambda x.f\ (f\ x)$; $\qquad t \equiv \lambda f.\lambda x.f\ (x\ x)$;
3. **HW** $t \equiv \lambda x.x(\mathsf{I\,I})$; $\qquad t \equiv \mathbf{\Upsilon}$.

Exercice 2 ($\vdash ? : A$)**.** Recall that λ-terms in β-normal form are described by the following grammar:

$$n := a \mid \lambda x.n \qquad\qquad a := x \mid a\ n$$

Solve the following *inhabitance problems* using terms in normal form or explain why it is not possible:

1. $A \equiv (S \rightarrow T) \rightarrow (T \rightarrow U) \rightarrow S \rightarrow U$;
2. $A \equiv (S \rightarrow T) \rightarrow ((S \rightarrow U) \rightarrow U) \rightarrow (T \rightarrow U) \rightarrow U$.
3. $A \equiv ((S \rightarrow T) \rightarrow S) \rightarrow S$ (Pierce's law)

Exercice 3 (Avoiding detours)**.** 1. Give a term associated to the following NJ($\Rightarrow$) proof

$$\dfrac{\dfrac{\dfrac{}{P \vdash P}\ (Hyp)}{\vdash P \Rightarrow P}\ (\Rightarrow_I) \qquad \dfrac{\dfrac{\dfrac{}{P \Rightarrow P, Q \vdash P \Rightarrow P}\ (Hyp)}{P \Rightarrow P \vdash Q \Rightarrow P \Rightarrow P}\ (\Rightarrow_I)}{\vdash (P \Rightarrow P) \Rightarrow Q \Rightarrow P \Rightarrow P}\ (\Rightarrow_I)}{Q \Rightarrow P \Rightarrow P}\ (\Rightarrow_E)$$

2. Reduce the term from question 1. and draw its type derivation tree. What is changed compare to the tree above?

3. Assuming the *subject reduction* and *strong normalization* properties of $\lambda_\to$ (cf. Lecture notes: Lemma 4 and Theorem 6). Prove the following equivalences:

$$\begin{array}{lll} \text{P is provable in NJ}(\Rightarrow) & \text{iff} & P \text{ (viewed as a type) is inhabited by a } \beta\text{-normal form of } \lambda_\to \\ & \text{iff} & P \text{ is provable without (I/E)-detours} \end{array}$$

4. Deduce that Pierce's law is not derivable in NJ.

Exercice 4 (Product Type)**.** In this exercise we are interested in extending the CH correspondence to NJ($\Rightarrow, \wedge$), the NJ fragment with implication and conjunction. Following the BHK interpretation, a witness for the *conjunction* $A \wedge B$ will correspond to a *pair* of witnesses for A and B.

1. Recall the encoding of **pair**, π_1 and π_2 *combinators* seen in TD1. Can we use them to encode *general* pairs and projection in $\lambda_\to$?

2. Instead, we extend types with *products* ($\times$) and pure λ-terms with three new *constants* **pair**, π_1, π_2 (hence making the calculus unpure...):

$$\begin{array}{rcl} S, T, \ldots & ::= & X \mid S \to T \mid S \times T \\ a, b, \ldots & ::= & x \mid \lambda x.a \mid ab \mid \textbf{pair}(a, b) \mid \pi_i(a) \end{array}$$

   Give typing rules for the new constants so that they annotate the intro-elim rules of $\wedge$ in NJ.

3. Similarly to the I/E-detour of $\Rightarrow$ in NJ($\Rightarrow$), explain what detours can be created by $\wedge$ in proof derivations and how to eliminate them.

4. Deduce new reductions (β-rules) for the extended λ-calculus. We will denote $\lambda_{\to,\times}$ this new calculus.

5. Inhabit the following types of $\lambda_{\to,\times}$:

   a) $(A \to B \to C) \to A \times B \to C$; b) $(A \times B \to C) \to A \to B \to C$.

6. What is the grammar for β-normal terms in $\lambda_{\to,\times}$?

7. Let $\Delta \vdash p : A \times B$, what proof simplification in NJ($\Rightarrow, \wedge$) corresponds to the following reduction?

$$\textbf{pair}(\pi_1\ p)(\pi_2\ p) \longrightarrow_\eta p$$

Exercice 5 (Making a choice)**. HW** Let A be a propositional variable and $P \equiv (A \Rightarrow A) \Rightarrow (A \Rightarrow A)$

1. Give a term corresponding to the following proof π_1:

$$\frac{}{\vdash P}(\pi_1) \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A, A \vdash A \Rightarrow A}(Hyp) \quad \dfrac{}{A \Rightarrow A, A \vdash A}(Hyp)}{A \Rightarrow A, A \vdash A}(\Rightarrow_E)}{A \Rightarrow A \vdash A \Rightarrow A}(\Rightarrow_I)}{\vdash P}(\Rightarrow_I)$$

2. Give a term corresponding to the following proof π_2:

$$\frac{}{\vdash P}(\pi_2) \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, P \vdash P}(Hyp)}{P \vdash P \Rightarrow P}(\Rightarrow_I)}{\vdash P \Rightarrow P \Rightarrow P}(\Rightarrow_I) \quad \dfrac{\dfrac{}{A \Rightarrow A \vdash A \Rightarrow A}(Hyp)}{\vdash P}(\Rightarrow_I)}{\vdash P \Rightarrow P}(\Rightarrow_E) \quad \dfrac{}{\vdash P}(\pi_1)}{\vdash P}(\Rightarrow_E)$$

3. Reduce the previous term to its β-normal form and give at each step of β reduction the corresponding proof simplification.

4. What would have happened if you had choose a different annotation/witness for the first hypothesis $\dfrac{}{P, P \vdash P}(Hyp)$?