Proofs and Programs

Semaine 4, TD 4 - Curry-Howard Expension

Philippe Audebaud, Aurore Alcolei

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Lemma 1 (Generation lemma)**.** *Let $\Delta \vdash t : T$,*

- *if t is a variable x then $x : T$ in Δ*
- *if $t \equiv a\ b$ then there exists S such that $\Delta \vdash a : S \to T$ and $\Delta \vdash b : S$*
- *if $t \equiv \lambda x.a$ with $x \notin dom(\Delta)$ then $T = S \to U$ such that $\Delta, x : S \vdash a : U$*

Proof. See *Lectures on the Curry Howard Isomorphism* for technical details. □

Exercice 1 (Subject reduction)**.** Prove that $\lambda_\to$ has the subject reduction property:

$$\text{If } \Delta \vdash m : T \text{ then for every } m \longrightarrow_\beta^* m', \text{ the typing judgement } \Delta \vdash m' : T \text{ holds}$$

1. (term substitution) Prove that if $\Delta, y : S \vdash t : T$ and $\Delta \vdash s : S$, then $\Delta \vdash t\langle s/y\rangle : T$;

   Let us show the above property by induction on t, using the generation lemma:

   - (base case) if t is a variable s then $x : T \in \Delta, y : S$ and so either $x \neq y$ hence $t\langle s/y\rangle = x$ and $\Delta \vdash x : T$ is a valid type derivation or $x = y$, $S = T$ hence $t\langle s/y\rangle = s$ and by hypothesis $\Delta \vdash s : T$ is a valid typing judgement.
   - if $t \equiv a\ b$ then $t\langle s/y\rangle = a\langle s/y\rangle\ b\langle s/y\rangle$ and there exists R such that $\Delta, y : S \vdash a : R \to T$ and $\Delta, y : S \vdash b : R$. Applying the induction hypothesis we get $\Delta \vdash a\langle s/y\rangle : R \to T$ and $\Delta \vdash b\langle s/y\rangle : R$ and so (applying the $\to_E$ rule) $\Delta \vdash t\langle s/y\rangle : S$.
   - if $t \equiv \lambda x.a$, then by α-conversion we can assume that $x \notin \mathrm{dom}(\Delta) \cup y$ and so there exist R, U, such that $T = R \to U$ and $\Delta, y : S, x : R \vdash a : U$. By induction hypothesis $\Delta, x : R \vdash a\langle s/y\rangle : U$ and so, applying the $\to_I$ rule, $\Delta \vdash t\langle s/y\rangle : S$ since $t\langle s/y\rangle = (\lambda x.a)\langle s/y\rangle = \lambda x.a\langle s/y\rangle$.

2. Now, show the subject reduction property.

   By induction over β, let us consider the basic reduction step $m \to_\beta m_1$:

   - if $m = (\lambda x.a)b$ (wlog we assume that $x \notin \mathrm{dom}(\Delta)$) and $m_1 = a\langle b/x\rangle$ then by generation lemma (applied twice) there exists S such that $\Delta, x : S \vdash a : T$ and $\Delta \vdash b : S$. Hence, by the above substitution lemma $\Delta \vdash a\langle b/x\rangle : T$, as desired.
   - if $m = a\ b$ and $m_1 = a'\ b$ then by generation lemma there exists S such that $\Delta \vdash a : S \to T$ and $\Delta \vdash b : S$, and by induction hypothesis $\Delta \vdash a' : S \to T$, so, applying the $\to_E$ rule, $\Delta \vdash a'\ b : T$ is a valid type judgement.
   - cases with $m = a\ b$, $m_1 = a\ b'$ and $m = \lambda x.a$, $m' = \lambda x.a'$ are similar.

3. Show that the contraposite does not hold. (Hint: $\boldsymbol{\Omega}$ can help you to build a counter example) $(\lambda x.y.y)\Omega$ reduces to $\lambda x.x$ which is typable, but Ω is not.