The *dwtrans2d* package

Managing 2-dimensional dyadic wavelet transforms
Computation of the corresponding extrema
Reconstruction of the original image from the extrema

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Documentation

Chapter 1

Using the 2d dyadic wavelet transform and extrema (`dwtrans2d`) package

1.1 Introduction

This package allows to perform dyadic wavelet transforms of images and computation of the corresponding extrema. Moreover, it allows the recontruction of the original image from these extrema and lets you manipulate these extrema so that you can perform high level denoising. This denoising procedure and the theory corresponding to the dyadic wavelet transform extrema can be found in the article *Characterization of signals from multiscale edges* by Stephane Mallat and Sifen Zhong, which appeared in IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 14, No. 7, p. 710-732 in July 1992.

1.2 loading the `&dwtrans2d` package

As for any package, you must load the `dwtrans2d` package for using 1d-wavelet transforms. This is done by the original startup file or directly by typing `package load dwtrans2d` in the terminal.

1.3 The `&dwtrans2` structure

1.3.1 Introduction

In LastWave, a dyadic wavelet transform is stored in a structure called a *dwtrans2* structure and corresponds to the type `&dwtrans2`. (In the present version of this package, there is no type defined for extrema related structure). The resulting images as well as the original images are stored in a 2d array of images. The first index corresponds to the octave number `oct`. Only $oct \geq 1$ are used by a wavelet transforms. Consequently the images with `oct=0` are working images available for the user. The second index, that we will referred to as i could be one of

- $i = 0$: the image corresponding to the smoothed image at the corresponding octave number `oct` (i.e., the projection on the corresponding V space)
- $i = 1$: the image corresponding to the x-wavelet image at the corresponding octave number `oct` (i.e., the projection on the corresponding W_x space)
- $i = 2$: the image corresponding to the y-wavelet image at the corresponding octave number `oct` (i.e., the projection on the corresponding W_y space)
- $i = 3$: the image corresponding to the modulus wavelet image at the octave number `oct` (i.e., the modulus of the complex image which real part is the x-wavelet image and which imaginary part is the y-wavelet image)
- $i = 4$: the image corresponding to the phase wavelet image at the octave number `oct` (i.e., the phase in $[0, 2\pi]$ of the complex image which real part is the x-wavelet image and which imaginary part is the y-wavelet image)

1.3.2 The fields of `&dwtrans2`

The main fields of the `dwtrans2` structure are

- `noct` : the number of octaves used for the decomposition. An octave value of 1 means that the image is decomposed on a single level corresponding to scale 2. The dyadic wavelet transform at this scale corresponds to 2 images of the same size as the original image. One image corresponds to the variation along the x-coordinate (the x-wavelet image) and the other one to the y-coordinate (the y-wavelet image). The number of octaves *noct* can take any value between 1 and $\log_2 N + 1$ where N is the number of row of the original **square** image
- `[oct,i]` : an 2d array of images as explained in the previous section.

1.4 A simple example of wavelet decomposition/reconstruction

In this section, we basically describe the demo file `scripts/dwtrans2d/DemoDWtrans2d` (the `DemoDWtrans2dWT` command). We want to perform the dyadic wavelet transform of the *lenna* image. Loading the `dwtrans2d` package creates 2 variables `a2d` and `b2d` of type `&dwtrans2`. Moreover, it also defines two commands `a2d` (resp. `b2d`) which sets the current object to be the variable `a2d` (resp. `b2d`). Indeed, in order to access the image `[<oct>,<i>]` of `a2d`,one can either use the regular syntax

```
a2d[<oct>,<i>]
```

or the abreviated syntax

```
<oct><i>a2d
```

or, in the case `a2d` is the current object, the syntax

```
a2d> <oct><i>
```

We are going to use the variable `a2d` for all our computations. Thus it is convenient to set the current object to `a2d` :

```
(&wtrans) a> a2d
(&dwtrans2) a2d>
```

Since we are dealing with black and white images, we should set the current colormap to be the `grey` colormap (which should have been defined in your `startup` file) :

```
(&dwtrans2) a2d> colormap current 'grey'
```

Then we read the "lenna" image as the original image to be decomposed (i.e., in the image `[0,0]` of the `a2d` variable). The file is in the directory `image/` of the original scripts directory. The original script directory is stored in the global variable `_scriptDir`. Thus

```
(&dwtrans2) a2d> iread 0 '$_scriptDir/image/lenna.char' -c
```

Then we perform the dyadic wavelet decomposition on 5 octaves and display it :

```
(&dwtrans2) a2d> dwt2d 5
(&dwtrans2) a2d> dw2disp
```

The `dw2disp` is a script command that is defined in the `scripts/dwtrans2d/dwtrans2d.pkg` file. It displays the x-wavelet transform image at the different octaves in the first row. The y-wavelet transform images are displayed in the second row. The modulus in the third and the phases in the fourth. The current colormap (i.e., the `grey` one) is used and all the images are displayed using the normalization method `max` (see Section on images in the main LastWave manual) except the modulus images which are coded using the default normalization method (`+max`) (see Section on images in the main LastWave manual) using the inversed default colormap (i.e., if the current colormap is the `grey` colormap, black pixels will correspond to high modulus values).

Since the reconstruction command reconstructs the image in the image `[0,0]`, in order not to loose the original image we are going to make a copy into the image `[0,1]` :

```
(&dwtrans2) a2d> 1 = 0a2d
```

then perform the reconstruction (by default it is set in in `[0,0]`)

```
(&dwtrans2) a2d> dwt2r
```

Then we can display in the same window the original image, the reconstructed image and the error image :

```
(&dwtrans2) a2d> disp {0 1 0a2d-1a2d} -title "Original/Reconstruction/Error" -pos 10 40
```

Let us note that the `dwt2f` command lets you choose the filter you want to use for decomposition/reconstruction. However, for now, only one filter (named `p3`) can be used.

1.5 A simple example of extrema decomposition and reconstruction

In this section, we basically describe the demo file `scripts/dwtrans2d/DemoDWtrans2d` (the `DemoDWtrans2dExtrema` command). We want to perform the dyadic wavelet transform of the "lenna" image, compute the corresponding extrema and reconstruct the image from the extrema or from the thresholded extrema.

In order to perform the decomposition, we just do as in the previous section :

```
(&wtrans) a> a2d
(&dwtrans2) a2d> colormap current 'grey'
(&dwtrans2) a2d> iread 0 '$_scriptDir/image/lenna.char' -c
(&dwtrans2) a2d> dwt2d 5
```

Then in order to compute the extrema, one just needs to call the `extrema2` command :

```
(&dwtrans2) a2d> extrema2
```

There is no way to display the extrema as a specific graphic object. In order to dispaly them, you need to use an image graphic object. The `e2image` command fills up 2 images. The first one (the modulus image) is filled up with zeros where there are no extrema and with the extrema modulus values where there are extrema. The second one is filled up with zeros where there are no extrema and with the extrema phase values where there are extrema. Thus, for instance, if you want to set the images `[0,1]` and `[0,2]` with the extrema modulus and phases of the extrema at octave 3, you should type

```
(&dwtrans2) a2d> e2image 1 1 2 -..view.* -cm '_'
(&dwtrans2) a2d> disp {1 2}
```

Let us note that the `-..view.* -cm '_'` option allows to set the colormap of the two images to be the inversed current colormap. In our case the current colormap if the `grey` one. Thus the high modulus values will be displayed using black whereas white will correspond to pixels where there are no extrema.

If you just want to get a binary image (indicating just the position of the extrema and not their values), you should use the `-p` option

```
(&dwtrans2) a2d> e2image 1 1 2 -p
(&dwtrans2) a2d> disp 1 -..1 -cm '_'
```

Before performing the extrema reconstruction (using 10 iterations) we must save the original image (in `[0,2]` for instance)

```
(&dwtrans2) a2d> 2=0a2d
(&dwtrans2) a2d> e2recons 10
```

Let us copy the result in the image `[0,1]`

```
(&dwtrans2) a2d> 1=0a2d
```

Actually before performing the extrema reconstruction, we could erase the extrema at all octaves whose modulus are small (e.g., for compression or for denoising). This is done using the `e2thresh` command

```
(&dwtrans2) a2d> e2thresh 0 20
```

The first `0` argument means that extrema at all octaves should be thresholded (otherwise you should have specified an octave number instead of 0) and `20` sets the threshold (it assumes a power law scaling across scales, read the corresponding help). Then we can perform the reconstruction

```
(&dwtrans2) a2d> e2recons 10
```

and display in the same window the reconstructed image using the thresholded extrema, the reconstructed image using all the extrema and the original image :

```
(&dwtrans2) a2d> disp {0 1 2} -title "Threshold/reconstruct/original"
```

Let us note that there is a another algorithm for extrema reconstruction using the conjugate gradient method and which corresponds to the command `e2reconsgrad`.

1.6 Other use of the `dwtrans2d` package (including denoising)

In the `scripts/dwtrans2d/WaveletTour` directory, there are 4 files which (when sourced) display figures that where use as examples in the book *A wavelet tour of signal processing* by S.Mallat (Academic Press, 1998). The figures are the following

- Figure 6.9 (file `fig69`) : It corresponds to a dyadic wavelet decomposition and extrema of a simple "circle" image,
- Figure 6.10 (file `fig610`) : It corresponds to a dyadic wavelet decomposition and extrema of the "lenna" image and reconstruction using all the extrema or just the strongest extrema,
- Figure 6.11 (file `fig611`) : Same as figure 6.11 except that the extrema reconstruction method is the conjugate gradient method.

- Figure 10.6 (file `fig106`) : Denoising of the "pepper" image using a very sophisticated algorithm described in the paper *Characterization of signals from multiscale edges* by Stephane Mallat and Sifen Zhong, which appeared in IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 14, No. 7, p. 710-732 in July 1992. It chains the extrema along scales and then denoise the noisy image using the `denoise2` interactive command which deals directly with these chains (not with individual extrema). The `denoise2` command will ask you questions about which chains you want to keep. In our example, all the answered to those questions have been stored in a file called `param` (in the same directory) and we just redirected the standard input of the command. In a future version, we will give more details about this procedure.

Part II

Reference

Chapter 2

Package `dwtrans2d` 2.0

Package allowing to perform 2d dyadic wavelet transform decomposition, reconstruction and extrema reconstruction
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2.1 Defined types

2.1.1 Type `&dwtrans2`

This type is the basic type for 2d dyadic wavelet transforms. It contains a 2d array of images. The first index 'oct' corresponds to the octave number and the second index is called 'orient'. When a wavelet transform is performed, the analyzed image must be in [0,0]. The projection on the V_oct spaces are stored in [oct,0]. [oct,1] corresponds to the vertical details at octave oct and [oct,2] corresponds to the horizontal details at octave oct. [oct,3] corresponds to the norm of ([oct,1],[oct,2]) and [oct,4] to its phase. All the other images are not used by the wavelet transform and can be used as working images.

- `&dwtrans2 [oct,orient]`
  it returns the image which corresponds to octave <oct> and orientation <orient>. The image [0,0] corresponds to the analyzed image. The projection on the V_oct spaces are stored in [oct,0]. [oct,1] corresponds to the vertical details at octave oct and [oct,2] corresponds to the horizontal details at octave oct. [oct,3] corresponds to the norm of ([oct,1],[oct,2]) and [oct,4] to its phase. All the other images are not used by the wavelet transform and can be used as working images.

- `&dwtrans2.name [= <name>]`
  Sets/Gets the name of a dwtrans2

- `&dwtrans2.noct [= <noct>]`
  Gets the number of octave of a 2d dyadic wavelet transform.

- `&dwtrans2.wavelet`
  Gets the analyzing wavelet used for the dyadic wavelet transform.

2.2 Commands related to wavelet 2D transform

• **dw2read** `[<dwtrans2>=objCur] (<filename> | <stream>)`
Reads a wavelet transform from a file named <filename> or a <stream>.

• **dw2write** `[<dwtrans2>=objCur] (<filename> | <stream>)`
Writes a wavelet transform in a file named <filename> or a <stream>.

• **dwt2d** `[<dwtrans2>=objCur] <noct> [-N] [-m]`
Performs a dyadic wavelet decomposition on the current <dwtrans2> on <noct> octaves. The image to be analyzed is supposed to be in the image 0 of the <dwtrans2>. If the number of columns and the number of rows are N = 2ˆ p, then <noct> must range from 1 to p+1. When <oct> is equal to p+1, the coarsest is a constant. Options are :

• **dwt2f** `<filterFileName>`
Specifies the filename of the filters that must be used in case a new dyadic wavelet decomposition is performed (using the 'dwt2d' command). The filter directory (defined in the file 'scripts/dwtrans2d/dwtrans2d.pkg') contains all the filter available files. The filenames ending by the '.1' suffixes correspond to decomposition filters and the ones ending with the '.2' suffixes correspond to reconstruction filters. For this command, you must specify the name without the suffix (it will load both the reconstruction and decomposition filters at the same time). For now, only one pair of decomposition/reconstruction filters are available? Its name is 'p3'. More details about filters can be found in the Appendix A of the paper :
Characterization of signals from multiscale edges
by Stephane Mallat and Sifen Zhong,
IEEE Transactions on Pattern Analysis and Machine Intelligence,
Vol. 14, No. 7, p. 710-732, July 1992.

• **dwt2r** `[<dwtrans2>=objCur] [-N]`
Performs a dyadic wavelet reconstruction from a dyadic wavelet decomposition (made with the 'dwt2d' command). It reconstructs the image which corresponds to the dyadic wavelet transform stored in <dwtrans2>. It uses the same number of octaves as the decomposition has been made with. The reconstructed image is put in image 0 of <dwtrans2>.

2.3 Commands related to ext used in wavelet 2D transform

• **e2image** `[<dwtrans2>=objCur] <octaveNumber> <modImage> <phaseImage> [-p]`
Copies the modulus and phase of extrema at octave number <octaveNumber> to the images <modImage> for the modulus and <phaseImage> for the phase. If '-p' is set then the <modImage> is just made of 1 (to indicate an extrema) or 0.

• **e2read** `[<dwtrans2>=objCur] (<file> | <stream>)`
Reads an extrema representation in a <file> or a <stream>

• **e2recons** `[<dwtrans2>=objCur] <iteration> [-N][-i][-n]`
Reconstructs the image from the extrema of the wavelet transform (computing using the 'extrema2' command). The algorithm reconstructs an approximation of the original image,

by alternatively projecting between two affine spaces. The reconstructed image is in image 0 of <dwtrans2>. The parameter <iteration> gives the number of alternative projection desired for the reconstruction. 20 iterations are sufficient to reconstruct an approximation with an SNR larger than 25 db. Options are :
-i : Does not initialize the reconstruction algorithm. It begins the iterations from an image (in 0) obtained after <n> iterations previously performed and from the associated extrema. The resultant image is stored in 0 and corresponds to an approximation of the original image after <n>+<iteration> iterations.

• **e2reconsgrad** `[<dwtrans2>=objCur] <iterations> [<threshold>=0]`
Reconstruction from the wavelet maxima with the conjugate gradient algorithm. Before the reconstruction is performed, the extrema are thresholded using <threshold>.

• **e2thresh** `[<dwtrans2>=objCur] <octaveNumber> <threshold>`
Thresholds the extrema at scale <octaveNumber> and whose modulus are smaller than <threshold>2ˆ (<octaveNumber>/2)/3. If <octaveNumber> is 0 then all the scales are thresholded.

• **e2write** `[<dwtrans2>=objCur] (<file> | <stream>)`
Writes an extrema representation in a <file> or a <stream>

• **extrema2** `[<dwtrans2>=objCur] [-o] [-n] [-N]`
Computes the extrema representation of the current wavelet transform. The extrema are defined as the modulus local maxima along some gradient direction. The command returns the number of extrema found. Options are :
-n : Due to the discretization of the filter values, the wavelet modulus maxima of a step edge do not have the same amplitude at all scales. In the extrema computation, we have correlated the extremas modulus such that the extremas of a step edge in each scale will have the same values. We called this process the 'normalization process'. With -n, we eliminate the normalization process in the computation of the extrema.
-o : The extrema are defined as the modulus local maxima in any gradient direction. It is possible that, for example, the modulus at location (x,y) is bigger than the modulus at (x-1,y) but less than that at (x+1,y), however the gradient at location (x+1,y) is opposite to the gradient at (x,y). We would expect in this case that the intensity at the location (x,y) is a local maxima and include the (x,y) as a extrema location. With -o, we will eliminate this inclusion.Properties of the extrema are described in the paper :

2.4 Commands related to the noise in pictures

• **denoise2** `[<dwtrans2>=objCur]`
Removes the white noise in images by thresholding extrema chains corresponding to particular values of Lipschitz exponents. Lipschitz exponents are calculated from the decay of the wavelet transform modulus local extrema along chains. The denoising process proceeds as follows :
1. thresholds all the extrema in a chain whose Lipschitz exponent is smaller than a given threshold.

2. thresholds all the chains at any level according to their lengths or average amplitudes.
3. removes all the extrema not propagating to a given scale.
4. rechains the chains such that all the chains in a fine scale corresponding to the same coarser chain are grouped together. We know that a contour has chains at many scales. Moreover, the white noise's energy distributes mostly in fine scales. Thus, the chains in fine scales are distorted more severely than those in coarse scales. This is the reason why we need to rechain the chains (in fine scales) by making use of the chains in a coarse scales.
5. smoothes the abscissa and amplitude along the chains at some scales by a Gaussian kernel with a given standard deviation.
6. predicts a chain's phases in fine scales from the abscissa of the chain and smoothes the resultant phases.

2.5 Script Commands

• **dw2disp** (in file `scripts/dwtrans2d/dwtrans2d.pkg`) `[<owtrans2>=objCur]`
This function displays a dyadic 2d wavelet transform (i.e., a variable of type '&dwtrans2') in a window. If no <dwtrans2> is specified then the current object is used. The display is organized as follows : each row corresponds to a different scale from small scales (top) to large scales (bottom). The first column corresponds to the horizontal wavelet component. The second one corresponds to the vertical component. The third one to the modulus and the last one to the phase. Except the modulus (which uses a '+max' normalization and the inverted current colormap), all the other images use a 'max' normalization and the current colormap. Let us note that you can us the mouse on each image of the display in order to perform zooms.

2.6 Demos

Here is a list of all the Demo files and for each of them all the corresponding Demo commands. To try a Demo command, you should first source the corresponding Demo file then run the command. (When sourcing the Demo file, LastWave tells you about all the commands included in this file).
The Demo files corresponding to this package are :

Demo file **DemoDWtrans2d**

• **DemoDWtrans2dExtrema** (in file `scripts/dwtrans2d/DemoDWtrans2d`)

Demo command that computes the extrema of the dyadic wavelet transform of the 'lenna' image on 5 octaves. It displays the original image, the reconstruction image (reconstructed from the extrema) as well as the error image.

• **DemoDWtrans2dWT** (in file `scripts/dwtrans2d/DemoDWtrans2d`)

Demo command that computes the dyadic wavelet transform of the 'lenna' image on 5 octaves and displays it. It also displays the reconstruction image as well as the error image.

Demo file **DemoDWtrans2dWT**

• **DemoDWtrans2dWT_10_6** (in file `scripts/dwtrans2d/DemoDWtrans2dWT`)

Demo command that reproduces the figure 10.6 of the book 'A Wavelet Tour in Signal Processing' by S. Mallat.

• **DemoDWtrans2dWT_6_10** (in file `scripts/dwtrans2d/DemoDWtrans2dWT`)

Demo command that reproduces the figure 6.10 of the book 'A Wavelet Tour in Signal Processing' by S. Mallat.

• **DemoDWtrans2dWT_6_11** (in file `scripts/dwtrans2d/DemoDWtrans2dWT`)

Demo command that reproduces the figure 6.11 of the book 'A Wavelet Tour in Signal Processing' by S. Mallat.

• **DemoDWtrans2dWT_6_9** (in file `scripts/dwtrans2d/DemoDWtrans2dWT`)

Demo command that reproduces the figure 6.9 of the book 'A Wavelet Tour in Signal Processing' by S. Mallat.

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