TD n° 1 - Standard laws

**Exercice 1.**  
**Bernoulli law**

Let \( p \in [0, 1] \). A random variable \( X \) follows a Bernoulli law \( \mathcal{B}(p) \) if \( X \) takes its values \( \{0, 1\} \) with \( \mathbb{P}(X = 1) = p \) and \( \mathbb{P}(X = 0) = 1 - p \).

Compute the mean, the variance and the moment-generating series of \( X \).

**Exercice 2.**  
**Binomial law**

Let \( p \in [0, 1] \) and \( n \in \mathbb{N}^* \). A random variable \( X \) follows a binomial law \( \mathcal{B}(n, p) \) if \( X \) takes its values in \( \{0, \ldots, n\} \) and \( \mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \).

1. Give an example of a phenomenon following a binomial distribution \( \mathcal{B}(n, p) \).

2. Compute the mean, the variance and the moment-generating series of \( X \).

**Exercice 3.**  
**Poisson law**

Let \( \lambda > 0 \). A random variable \( X \) follows a Poisson law \( \mathcal{P}(\lambda) \) if \( X \) takes its values in \( \mathbb{N} \) and \( \mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \).

1. Give an example of a phenomenon following a Poisson distribution \( \mathcal{P}(\lambda) \).

2. Compute the mean, the variance and the moment-generating series of \( X \).

3. Show that if \( X \) and \( Y \) are two independent random variables following Poisson laws of parameters \( \lambda \) and \( \mu \), then \( X + Y \) follows the Poisson law of parameter \( \lambda + \mu \).

The Poisson approximation principle states that the sum \( S_n \) of many independent Bernoulli random variables with small parameters almost follows a Poisson law of parameter \( \mathbb{E}(S_n) \).

4. Illustrate this principle by proving that for \( \lambda > 0 \), any sequence of random variables \( (X_n)_{n \in \mathbb{N}} \) respectively following a binomial law \( \mathcal{B}(n, \lambda/n) \) converges in distribution (Fr : convergence en loi) to a Poisson law \( \mathcal{P}(\lambda) \).

5. 1000 competitors attend a fishing competition, each one having independently a probability 0.001 to hook a fish and win a victory medal. The organizing committee thinks that with two medals, they have a probability larger than 80% to award all the winners. Are they right?

**Exercice 4.**  
**Geometric law**

Let \( p \in [0, 1] \). A random variable \( X \) follows a geometric law \( \mathcal{G}(p) \) if \( X \) takes its values in \( \mathbb{N}^* \) and \( \mathbb{P}(X = k) = (1 - p)^k - 1 p \).

1. Give an example of a phenomenon following a geometric law \( \mathcal{G}(p) \).

2. Compute the mean, the variance and the moment formal series for \( X \).

3. Show that geometric laws are memoryless : for all \( n, k \in \mathbb{N} \), \( \mathbb{P}(X = k + n \mid X > n) = \mathbb{P}(X = k) \).

**Exercice 5.**  
**Exponential law**

Let \( \lambda > 0 \), a random variable \( X \) follows an exponential law \( \text{Exp}(\lambda) \) if \( X \) takes its values in \( \mathbb{R}_+ \) with density \( f(x) = \lambda e^{-\lambda x} \).

1. Give an example of a phenomenon following an exponential law \( \text{Exp}(\lambda) \).

2. Compute the mean and the variance of \( X \).

3. Show that exponential laws are memoryless : for all \( a, b \in \mathbb{R}_+ \), \( \mathbb{P}(X \geq a + b \mid X \geq b) = \mathbb{P}(X \geq a) \).
4. Show the converse: any memoryless continuous law is exponential.

5. Let $X_1, \ldots, X_n$ be independent exponential random variables of parameters $\lambda_1, \ldots, \lambda_n$. Show that the random variable $\min(X_1, \ldots, X_n)$ also follows an exponential law and find its parameter. Then, compute $P(\min(X_1, \ldots, X_n) = X_i)$ for $1 \leq i \leq n$. Suggestion: start with $n = 2$.

**Exercice 6.**

Let $m, \sigma^2 \in \mathbb{R}_+$, a random variable $X$ follows a normal law $\mathcal{N}(m, \sigma^2)$ if $X$ takes its values in $\mathbb{R}$ and has density $f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-m)^2}{2\sigma^2}}$.

1. Give an example of a phenomenon following a normal law.

2. Compute the mean and variance of $X$.

3. Let $X, Y$ be independent normal random variables of laws $\mathcal{N}(m_1, \sigma_1^2)$ and $\mathcal{N}(m_2, \sigma_2^2)$, show that $X + Y$ follows the law $\mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

**Exercice 7.**

Homework question (from P. Brémaud)

A long time ago, a number $p$ was chosen at random uniformly between 0 and 1, but this value was never revealed to mankind. Since this time, the sun rises every day with probability $p$ (still unknown). What happened during the preceding days is independent of what happens today. You know that the sun has risen every day from the beginning, that is $n$ times (and you know this number), what is the probability that it will rise tomorrow?