ContFrac[riccati_to_cfrac] - Compute a Corresponding Continued Fraction expansion for a power series solution of a differential equation.

Calling Sequence

riccati_to_cfrac(eq, y(z), S, [names, prooftable, minnbused=m, maxnbused=M])

Parameters

`eq` – Riccati differential equation {y' = p y^2 + q y + r, y(0)=v}, with rational coefficients p,q,r and initial condition v at zero.
`y(z)` – name; function and variable name
`S` – procedure, mapping (n,z) to the truncation ot order n of the power series solution y(z) of the Riccati equation.

`names` (optional) – list of names used in notations and during the formal convergence proof:
`n, a, alpha`: index, coefficients and exponents names : the main coefficient is a(n) * x^(alpha(n)).
`P, Q` : name of the numerator and denominator sequences
`H` : remainder, s.t. H_n tends to zero (as forml power series) iff. convergence is achieved.
HH, k (both optional) : intermediate sequence name. Intuitvely, HH(k) = H(2k).

`prooftable` (optional) - table used for storing proof elements along the computation. These elements certify the formula.

`minnbused` – (optional) minimal number of terms in the C-fraction to be used for computing the formula
`maxnbused` - (optional) idem.

Description

- This procedure computes and proves a corresponding continued fraction formula, for the solution of a given Riccati equation with initial conditions at zero. It fails if no such formula could be found, or if the automatic proof process did not succeed.

- If the argument 'prooftable' is given, it is added the following entries:
  - `rem_def`: the definition of a remainder H(n,z) such that H(n,z)=O(z^n) implies convergence.
    It is an expression using a(n), the coeficients of the C-fraction, P(n,x) and Q(n,x) the canonical numerators and denominators of the C-fraction convergent of order n.
  - `rem_rec`: a recurrence satisfied by this remainder, with coefficients polynomial in x, a(n), a(n+1), and other shifts of a(n).

  - in the case where the period of the coefficients is 1 (rational coefficients):
    + `small_rec`: a recurrence equivalent to rem_rec which proves convergence.
  - in the case where the period is 2,
    + `rem_seq2`: an equation of shape HH(k)=H(2*n) defining an auxiliary sequence
    + `rec2`: a recurrence for HH(k)
    + `smallrec2`: a recurrence equivalent to rec2 which proves convergence.

- It returns either:
  - a `CFrac` structure, giving a valid expansion for the function,
  - or FAIL if no formulas could be found, or if the correspondence could not be proved automatically. In the first case, you can increase the number of first terms used for guessing the formula, using minnbused and maxnbused accordingly.

Examples

```
> restart;
> with(gfun): with(ContFrac):
> eq := {diff(y(z),z)-1-y(z)^2, y(0)=0};
```

$$eq := \left\{ \frac{d}{dz}\, y(z) - 1 - y(z)^2, y(0) = 0 \right\} \qquad \textbf{(1.4.1)}$$

```
> riccati_to_cfrac( eq, y(z), proc(n,x) series(tan(x),x,n) end
  );
```

$$\cfrac{z}{1 + \cfrac{a_1 z^2}{1 + \cfrac{\dots}{1 + \cfrac{a_n z^2}{1 + \dots}}}}, a_n = -\frac{1}{(2n-1)(2n+1)} \qquad \textbf{(1.4.2)}$$

use the information level 'gfuncontfrac' to get more information, from 1 to 5.

```
> infolevel[gfuncontfrac] := 2:
> cf := riccati_to_cfrac( eq, y(z), proc(n,x) series(tan(x),x,
  n) end );
Guessing a formula
conjecture formula (on 8 coefficients)
```

$$\cfrac{z}{1 + \cfrac{a_1 z^2}{1 + \cfrac{\dots}{1 + \cfrac{a_n z^2}{1 + \dots}}}}, a_n = -\frac{1}{(2n-1)(2n+1)}$$

```
defining a sequence H(n,z), which relates to convergence.
```

$$H(n,z) = \left(\frac{\partial}{\partial z} P(n,z)\right) Q(n,z) - Q(n,z)^2 - P(n,z)^2 - P(n,z)\left(\frac{\partial}{\partial z} Q(n,z)\right)$$

```
lemma: the conjecture holds iff there exists an unbounded i(n) s.t.,
H(i(n),z) tends to 0.
  (i.e., their valuations tend to infinity)
proving Limit(val(H(n,z)),n = infinity) = infinity:
- computing a recurrence for H(n,z).
  (which does not conclude)
```

$$\dots z^8 H(n) + \dots z^4 H(n+1) + \left(\dots z^4 + \dots z^2\right) H(n+2) + \dots H(n+3) + \dots H(n+4) = 0$$

```
- reducing the recurrence order for H(n,z)...
- ... done.
```

$$\left\{ -z^2 H(n) + (2n+3)^2 H(n+1), H(0) = -z^2 \right\}$$

```
QED.
```

(1.4.3)

$$cf := \cfrac{z}{1+\cfrac{a_1 z^2}{1+\cfrac{\ldots}{1+\cfrac{a_n z^2}{1+\ldots}}}}, a_n = -\frac{1}{(2n-1)(2n+1)} \quad (1.4.3)$$

```
> riccati_to_cfrac( {diff(y(z),z)-y(z), y(0)=1}, y(z), proc(n,
  x) series(exp(x),x,n) end );
Guessing a formula
conjecture formula (on 8 coefficients)
```

$$1+\cfrac{z}{1+\cfrac{a_1 z}{1+\cfrac{\ldots}{1+\cfrac{a_n z}{1+\ldots}}}}, a_n = \begin{cases} \frac{1}{2(n+1)} & n::even \\ -\frac{1}{2n} & n::odd \end{cases}$$

```
defining a sequence H(n,z), which relates to convergence.
```

$$H(n,z) = \left(\frac{\partial}{\partial z} P(n,z)\right) Q(n,z) - Q(n,z)^2 - P(n,z)\, Q(n,z) - P(n,z) \left(\frac{\partial}{\partial z} Q(n,z)\right)$$

```
lemma: the conjecture holds iff there exists an unbounded i(n) s.t.,
H(i(n),z) tends to 0.
  (i.e., their valuations tend to infinity)
proving Limit(val(H(n,z)),n = infinity) = infinity:
- computing a recurrence for H(n,z).
  (which does not conclude)
```

$$\ldots z^4 H(n) + \ldots z^2 H(n+1) + (\ldots z^2 + \ldots z) H(n+2) + \ldots H(n+3) + \ldots H(n+4) = 0$$

```
- computing a P-recurrence for H(2*n,z) using the rational formulas.
  (does not conclude either)
- reducing equation for H(2*n,z)...
- ... done.
```

$$\{(n+2) z^2 H(2n) + 4(n+1)(2n+3)^2 H(2n+2), H(0) = -z\}$$

```
QED.
```

$$1+\cfrac{z}{1+\cfrac{a_1 z}{1+\cfrac{\ldots}{1+\cfrac{a_n z}{1+\ldots}}}}, a_n = \begin{cases} \frac{1}{2(n+1)} & n::even \\ -\frac{1}{2n} & n::odd \end{cases} \quad (1.4.4)$$

See Also

gfun, ContFrac, expr_to_cfrac