ContFrac[riccati_to_cfrac] - Compute a Corresponding Continued Fraction expansion for a power series solutions of a differential equation.

Calling Sequence

riccati_to_cfrac(eq, y(z), S, [names=Names, minnbused=m, maxnbused=M])

Parameters

`eq` – Riccati differential equation {y' = p y^2 + q y + r, y(0)=v}, with rational coefficients p,q,r and initial condition v at zero.
`y` – name; function name
`z` – name; variable of the function y
`S` – procedure, such that S(n,x) is the of the Riccati equation, truncated at order n.
`Names` – (optional) list of names used in the continued fraction and in the formal convergence proof:
  n, a, alpha: index, coefficients and exponents names : the main coefficient is a(n) * x^(alpha(n)).
  P, Q : name of the numerator and denominator sequences
  H : remainder, s.t. H_n tends to zero (as forml power series) iff. convergence is achieved.
   it is polynomial in P,Q, their shifts and their derivatives.

`minnbused` – (optional) minimal number of terms in the C-fraction to be used for computing the formula
maxnbused - (optional) idem.

Description

- This procedure computes and proves a Corresponding Fraction formula if it exists.
- It returns either:
  - a `cfrac` structure (use pretty_print_cfrac to visualize it)
  - FAIL if none was found, in the limit of the number of initial terms used for "guessing". If a formula exists, using more and more terms ends up finding it.

Examples

```
> restart;
> with(gfun): with(ContFrac):
> eq := {diff(y(z),z)-1-y(z)^2, y(0)=0};
```

$$eq := \left\{ \frac{d}{dz} y(z) - 1 - y(z)^2, y(0) = 0 \right\} \tag{1.4.1}$$

```
> riccati_to_cfrac( eq, y(z), proc(n,x) series(tan(x),x,n) end )
;
```

$$\cfrac{z}{1 + \cfrac{a_1 z^2}{1 + \cfrac{\dots}{1 + \cfrac{a_n z^2}{1 + \dots}}}}, a_n = - \frac{1}{(2n-1)(2n+1)} \tag{1.4.2}$$

use the information level 'gfuncontfrac' to get more information, from 1 to 5.

```
> infolevel[gfuncontfrac] := 2:
```

```
> cf := riccati_to_cfrac( eq, y(z), proc(n,x) series(tan(x),x,n)
  end );
```
Guessing a formula
conjecture formula (on 8 coefficients)

$$\cfrac{z}{1+\cfrac{a_1 z^2}{1+\cfrac{\dots}{1+\cfrac{a_n z^2}{1+\dots}}}}, a_n = -\frac{1}{(2n-1)(2n+1)}$$

defining a sequence H(n,z), which relates to convergence.

$$H(n,z) = \left(\frac{\partial}{\partial z} P(n,z)\right) Q(n,z) - Q(n,z)^2 - P(n,z)^2 - P(n,z) \left(\frac{\partial}{\partial z} Q(n,z)\right)$$

lemma: the conjecture holds iff there exists an unbounded i(n) s.t., H(i(n),z) tends to 0.
(i.e., their valuations tend to infinity)
proving Limit(val(H(n,z)),n = infinity) = infinity:
- computing a recurrence for H(n,z).
(which does not conclude)

$$\dots z^8 H(n) + \dots z^4 H(n+1) + \left(\dots z^4 + \dots z^2\right) H(n+2) + \dots H(n+3) + \dots H(n+4) = 0$$

- reducing the recurrence order for H(n,z)...
- ... done.

$$\left\{-z^2 H(n) + (2n+3)^2 H(n+1), H(0) = -z^2\right\}$$

QED.

$$cf := \cfrac{z}{1+\cfrac{a_1 z^2}{1+\cfrac{\dots}{1+\cfrac{a_n z^2}{1+\dots}}}}, a_n = -\frac{1}{(2n-1)(2n+1)} \qquad \textbf{(1.4.3)}$$

```
> riccati_to_cfrac( {diff(y(z),z)-y(z), y(0)=1}, y(z), proc(n,x)
  series(exp(x),x,n) end );
```
Guessing a formula
conjecture formula (on 8 coefficients)

$$1+\cfrac{z}{1+\cfrac{a_1 z}{1+\cfrac{\dots}{1+\cfrac{a_n z}{1+\dots}}}}, a_n = \begin{cases} \frac{1}{2(n+1)} & n::even \\ -\frac{1}{2n} & n::odd \end{cases}$$

defining a sequence H(n,z), which relates to convergence.

$$H(n,z) = \left(\frac{\partial}{\partial z} P(n,z)\right) Q(n,z) - Q(n,z)^2 - P(n,z)\, Q(n,z) - P(n,z) \left(\frac{\partial}{\partial z} Q(n,z)\right)$$

lemma: the conjecture holds iff there exists an unbounded i(n) s.t., H(i(n),z) tends to 0.
(i.e., their valuations tend to infinity)
proving Limit(val(H(n,z)),n = infinity) = infinity:
- computing a recurrence for H(n,z).
(which does not conclude)

$$\dots z^4 H(n) + \dots z^2 H(n+1) + \left(\dots z^2 + \dots z\right) H(n+2) + \dots H(n+3) + \dots H(n+4)$$

$= 0$

```
- computing a P-recurrence for H(2*n,z) using the rational formulas.
  (does not conclude either)
- reducing equation for H(2*n,z)...
- ... done.
```

$$\left\{(n+2)\,z^2\,H(2\,n) + 4\,(n+1)\,(2\,n+3)^2\,H(2\,n+2), H(0) = -z\right\}$$

```
QED.
```

$$1 + \cfrac{z}{1 + \cfrac{a_1\,z}{1 + \cfrac{\dots}{1 + \cfrac{a_n\,z}{1 + \dots}}}}, a_n = \begin{cases} \dfrac{1}{2\,(n+1)} & n::even \\ -\dfrac{1}{2\,n} & n::odd \end{cases} \tag{1.4.4}$$

See Also

gfun, ContFrac, expr_to_cfrac