Coefficient Asymptotics of Multivariate Rational Functions

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Asymptotics of Multiple Binomial Sums

Input:
Multiple
Binomial Sum

$$S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

easy

Generating function
is a diagonal

$$S(z) = \sum_{n\geq 0} S_n z^n = \operatorname{Diag}\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

Griffiths-Dwork
reduction

$S(z)$ satisfies
a LDE

analytic
combinatorics
in several
variables

$$z^2(4z+1)\left(16z-1\right)S^{(3)}(z)+\cdots+2\,(30z+1)\,S(z)=0$$

analytic combinatorics

Aim:
compare
these
approaches

Output:
Asymptotic
behaviour

$$S_n = 16^n\, n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1-\frac{9}{16n}+O\left(\frac{1}{n^2}\right)\right),\quad n\to\infty$$

I. Multiple Binomial Sums, Diagonals and Multiple Integrals

$$S_n = \sum_{r\ge 0}\sum_{s\ge 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

$$S(z) = \sum_{n\ge 0} S_n z^n = \operatorname{Diag}\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

$$z^2(4z+1)\left(16z-1\right)S^{(3)}(z) + \cdots + 2\,(30z+1)\,S(z) = 0$$

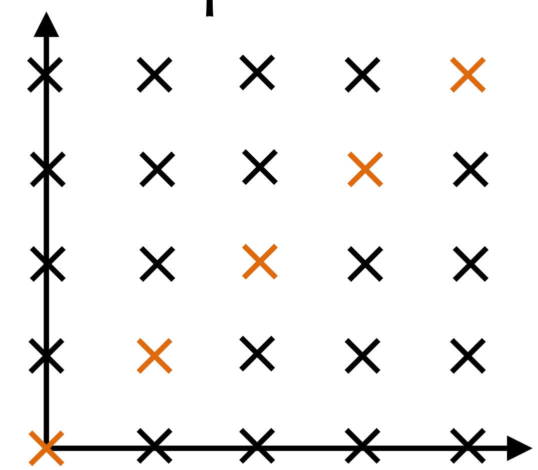
$$S_n = 16^n\, n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1 - \frac{9}{16n} + O\left(\frac{1}{n^2}\right)\right), \quad n\to\infty$$

Definition

in this talk

If $F(\underline{z}) = \dfrac{G(\underline{z})}{H(\underline{z})}$ is a multivariate rational function with Taylor expansion

$$F(\underline{z}) = \sum_{\underline{i} \in \mathbb{N}^n} c_{\underline{i}} \underline{z}^{\underline{i}},$$



its **diagonal** is $\operatorname{Diag} F = \sum_{k \geq 0} c_{k,\dots,k} z^k$.

$$\binom{2k}{k}: \qquad \frac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$$

$$\frac{1}{k+1}\binom{2k}{k}: \qquad \frac{1-2x}{(1-x-y)(1-x)} = 1 + y + 1xy - x^2 + y^2 + \cdots + 2x^2y^2 + \cdots$$

$$\text{Apéry's } a_k: \qquad \frac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$$

Multiple Binomial Sums

over a field $\mathbb{K}$

Sequences constructed from

the binomial sequence $(n, k) \mapsto \binom{n}{k}$;

geometric sequences $n \mapsto C^n, C \in \mathbb{K}$;

Kronecker's $\delta : n \mapsto \delta_n$

using algebra operations and

affine changes of indices $(u_{\underline{n}}) \mapsto (u_{\lambda(\underline{n})})$;

indefinite summation $(u_{\underline{n},k}) \mapsto \left(\sum_{k=0^m} u_{\underline{n},k} \right)$.

From Sum to Residue to Diagonal

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \left(= \binom{2n}{n} \right)$$

$$\binom{n}{k} := [x^k](1+x)^n = \frac{1}{2\pi i} \oint (1+x)^n \frac{dx}{x^{k+1}}$$

$$\binom{n}{k}^2 = \frac{1}{(2\pi i)^2} \oint (1+x_1)^n (1+x_2)^n \frac{dx_1 dx_2}{x_1^{k+1} x_2^{k+1}}$$

Geometric sums

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{1}{(2\pi i)^2} \oint (1+x_1)^n (1+x_2)^n \frac{1 - 1/(x_1 x_2)^{n+1}}{x_1 x_2 - 1} dx_1 dx_2$$

$$\sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 z^n = \frac{1}{(2\pi i)^2} \oint \left(\frac{1}{x_1 x_2 - z(1+x_1)(1+x_2)} + \frac{1}{1 - z(1+x_1)(1+x_2)} \right) \frac{dx_1 dx_2}{1 - x_1 x_2}$$

$\times\, x_1 x_2$ and
$z \mapsto z x_1 x_2$

$$= \mathrm{Diag} \left(\left(\frac{1}{1 - z(1+x_1)(1+x_2)} + \frac{1}{1 - z x_1 x_2 (1+x_1)(1+x_2)} \right) \frac{1}{1 - x_1 x_2} \right)$$

Can be turned into a general algorithm

Diagonals & Multiple Binomial Sums

$$S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

Thm. Diagonals ≡ binomial sums with 1 free index.

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> BinomSums[sumtores](S,u): (…)
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$$\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

has for diagonal the generating function of S_n

[Bostan-Lairez-S.17]

II. Aside: Griffiths-Dwork Reduction

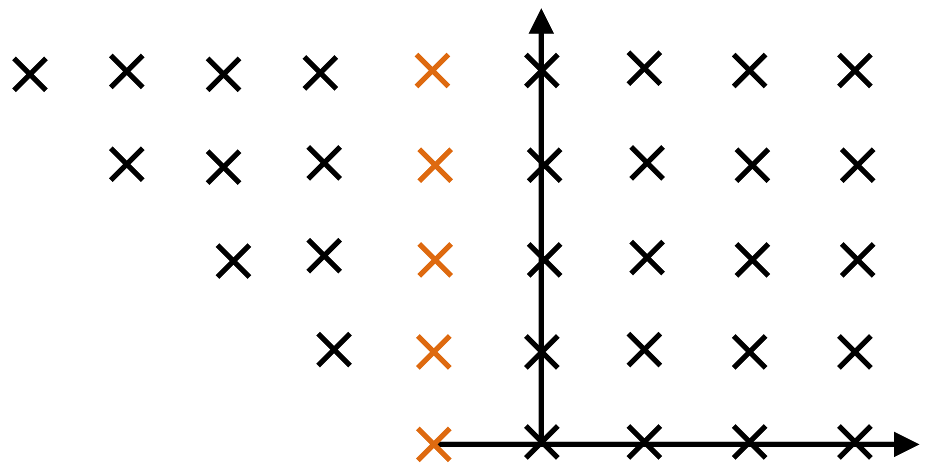
$$S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

$$S(z) = \sum_{n\geq 0} S_n z^n = \mathrm{Diag}\,\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

$$z^2(4z+1)\left(16z-1\right)S^{(3)}(z) + \cdots + 2(30z+1)S(z) = 0$$

$$S_n = 16^n\, n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1 - \frac{9}{16n} + O\left(\frac{1}{n^2}\right)\right),\quad n\to\infty$$

Diagonals as Integrals



$$F(\underline{z}) = \frac{G(\underline{z})}{H(\underline{z})} \Rightarrow \operatorname{Diag} F = \left(\frac{1}{2\pi i}\right)^{n-1} \oint F\left(z_1, \ldots, z_{n-1}, \frac{t}{z_1 \cdots z_{n-1}}\right) \frac{dz_1 \cdots dz_{n-1}}{z_1 \cdots z_{n-1}}$$

LDE for Integrals: Griffiths-Dwork Method

$$I(t) = \oint \frac{P(t,\underline{x})}{Q^m(t,\underline{x})}\,\underline{dx}$$

Q square-free
Int. over a cycle
where $Q \neq 0$.

Basic idea

1. While $m > 1$, reduce P modulo $J := \langle \partial_1 Q, \dots, \partial_n Q \rangle$ and integrate by parts

$$\frac{P}{Q^m} = \frac{r + v_1\partial_1 Q + \dots + v_n \partial_n Q}{Q^m} = \frac{r}{Q^m} + \frac{\tilde{P}}{Q^{m-1}} + \text{derivatives}$$

2. Apply to $I, I', I'', \dots$ until a linear dependency is found.

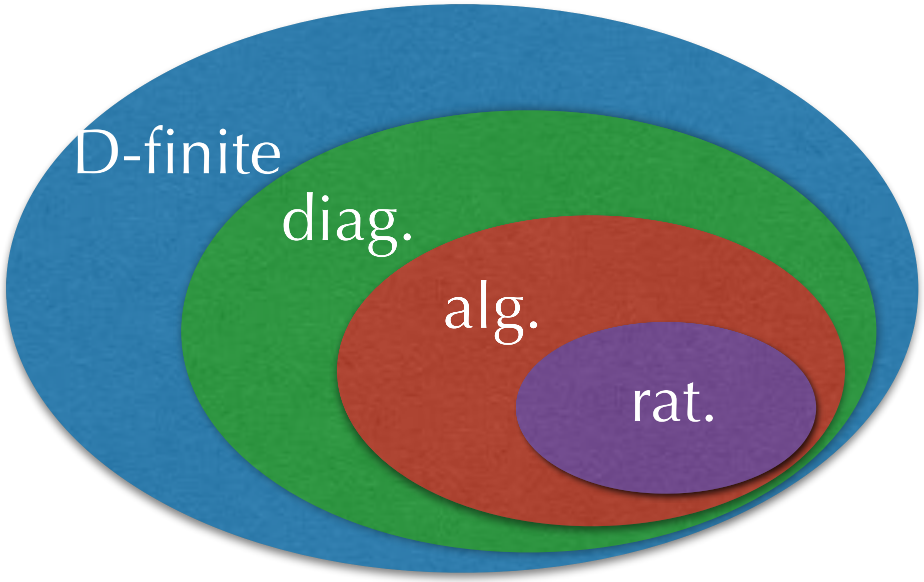
If P/Q is the GF of $\mathrm{Diag}(G/H)$,
J becomes $\langle z_1\partial_1 H - z_n\partial_n H, \dots, z_{n-1}\partial_{n-1} H - z_n \partial_n H\rangle$

Ideal of critical points later

[Griffiths70;Christol84;Bostan-Lairez-S.13;Lairez16]

Diagonals are Differentially Finite
[Christol84,Lipshitz88]

$$a_n(z)y^{(n)}(z) + \ldots + a_0(z)y(z) = 0$$



Thm. If F has degree d in n variables, Diag F satisfies a LDE with order $\approx d^n$, coeffs of degree $d^{O(n)}$.

+ algo in $O(d^{8n})$ ops.

[Bostan-Lairez-S.13,Lairez16]

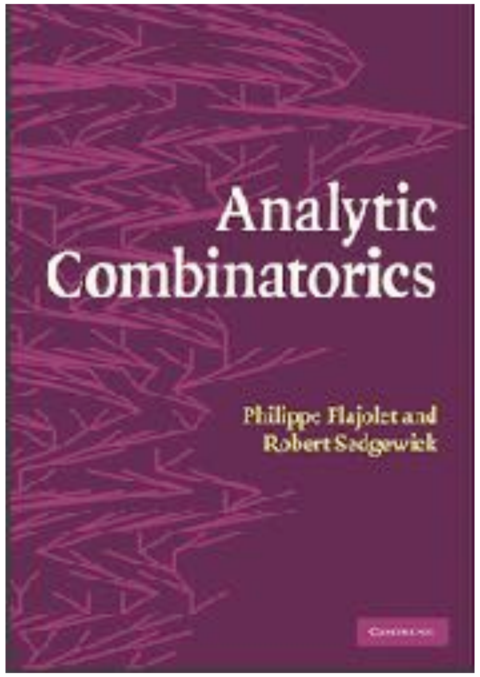
III. Analytic Combinatorics in 1 variable

$$S_{n} = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

$$S(z) = \sum_{n\geq 0} S_n z^n = \operatorname{Diag}\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

$$z^2(4z+1)\left(16z-1\right)S^{(3)}(z)+\cdots+2\,(30z+1)\,S(z)=0$$

$$S_n = 16^n\, n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1-\frac{9}{16n}+O\left(\frac{1}{n^2}\right)\right),\quad n\to\infty$$



Principle

sequence

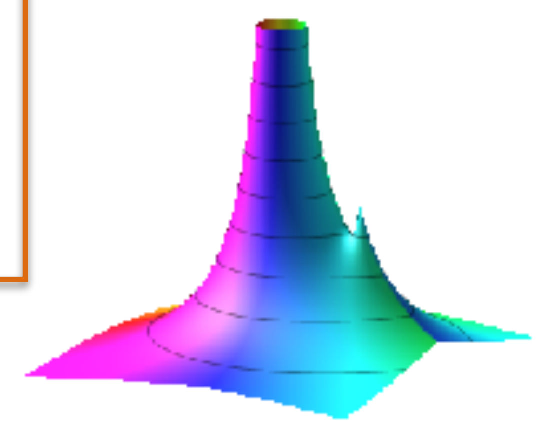
$$(a_n) \mapsto A(z) := \sum_{n\geq 0} a_n z^n$$

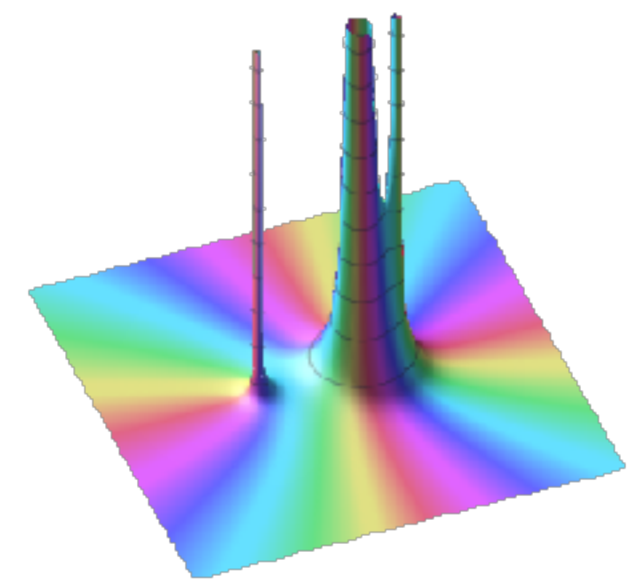
Generating function captures some structure

A 3-Step Method:

1. Locate dominant singularities
2. Compute local behaviour
3. Translate into asymptotics

$$a_n = \frac{1}{2\pi i}\oint \frac{A(z)}{z^{n+1}}\,dz$$





Possible behaviours for diagonals:

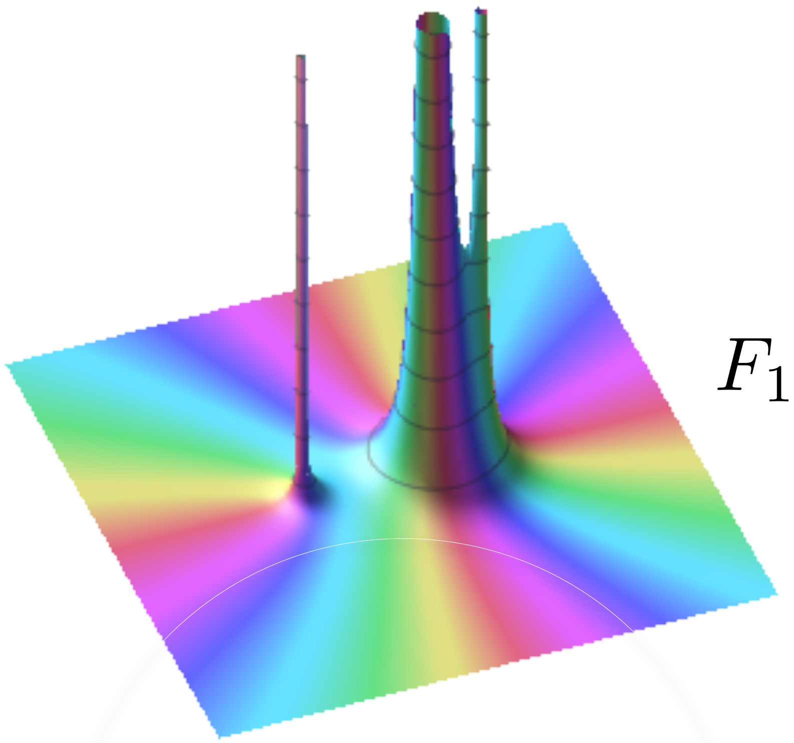
$$A(z) \underset{z\to\rho}{\sim} c\left(1-\frac{z}{\rho}\right)^{\alpha} \log^m \frac{1}{1-\frac{z}{\rho}} \qquad a_n \underset{n\to\infty}{\sim} c\,\rho^{-n}\frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\log^m n$$

full asymptotic expansion available

$(\alpha \notin \mathbb{N})$

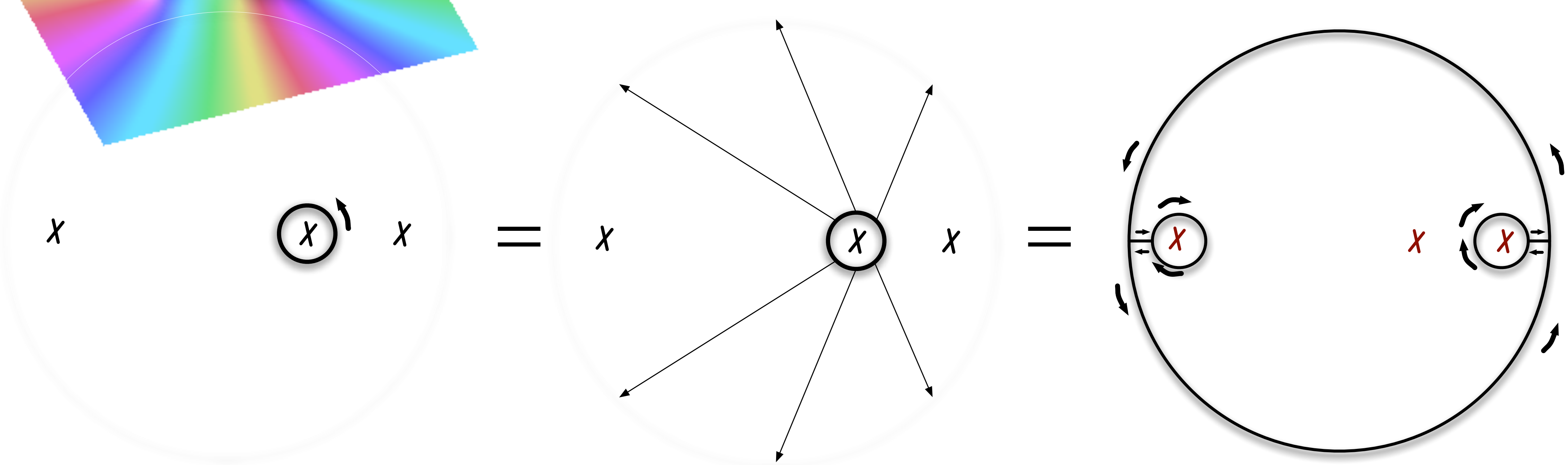
[Flajolet-Odlyzko1990]

Ex: Fibonacci Numbers



$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz$$

$$F_1 = 1 = \frac{1}{2\pi i} \oint \frac{1}{1 - z - z^2} \frac{dz}{z^2}$$

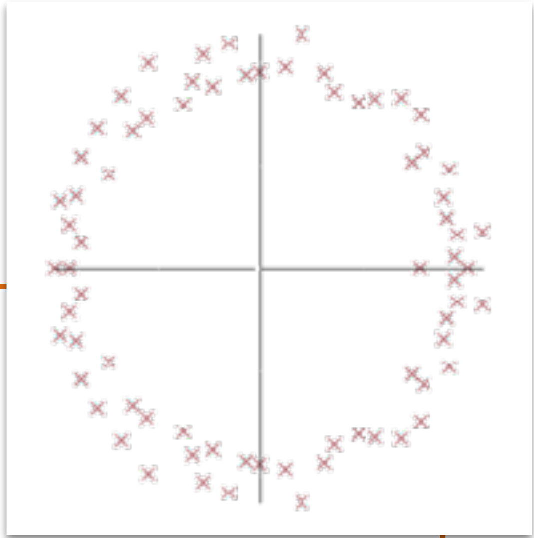


As n increases, the smallest singularities dominate.

$$F_n = \frac{\phi^{-n-1}}{1+2\phi} + \frac{\overline{\phi}^{-n-1}}{1+2\overline{\phi}}$$

Analytic Combinatorics for LDEs

$$q_0(z)A^{(\ell)}(z) + \cdots + q_\ell(z)A(z) = 0$$



Classical properties of LDEs:

1. singularities satisfy $q_0(\rho) = 0$;
2. one can compute a basis of formal solutions at (regular) singular points, of the form

$$\left(1 - \frac{z}{\rho}\right)^{\alpha} \log^{m}\left(\frac{1}{1-\frac{z}{\rho}}\right)(1 + \cdots), \quad \alpha \in \overline{\mathbb{Q}}, m \in \mathbb{N}.$$

convergent

local

(at ρ)

More recently (M. Mezzarobba's `ore_algebra_analytic`):

certified analytic continuation ($\to$ c numerically).

Semi-decision

[Mezzarobba16]

Example: Apéry's Sequences

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad b_n = a_n \sum_{k=1}^{n} \frac{1}{k^3} + \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{(-1)^m \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}$$

and $c_n = b_n - \zeta(3) a_n$ have generating functions that satisfy

vanishes at 0,
$\alpha = 17 - 12\sqrt{2} \simeq 0.03,$
$\beta = 17 + 12\sqrt{2} \simeq 34.$

$$z^2(z^2 - 34z + 1)y''' + \cdots + (z-5)y = 0$$

In the neighborhood of $\alpha,$ all solutions behave like

$$\text{analytic} - \mu\sqrt{\alpha - z}(1 + (\alpha - z)\text{analytic}).$$

Mezzarobba's code gives $\mu_a \simeq 4.55, \quad \mu_b \simeq 5.46, \quad \mu_c \simeq 0.$

Slightly more work gives $\mu_c = 0$, then $c_n \approx \beta^{-n}$
and eventually, a proof that $\zeta(3)$ is irrational.

[Apéry1978]

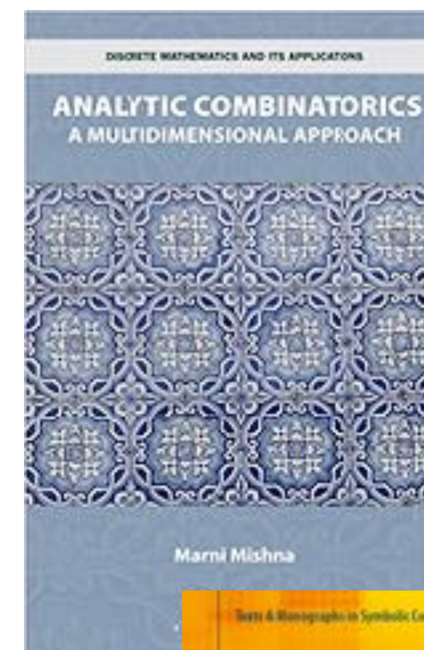
III. Analytic Combinatorics in several variables

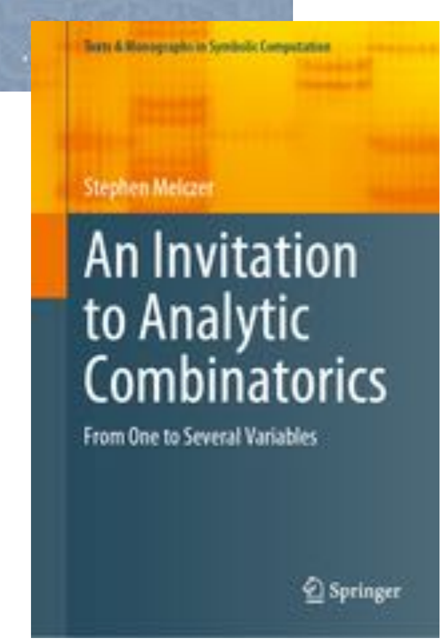
$$S_n = \sum_{r\geq 0}\sum_{s\geq 0}(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}$$

$$S(z) = \sum_{n\geq 0} S_n z^n = \mathrm{Diag}\,\frac{1}{1+t(1+u_1)(1+u_2)(1-u_1u_3)(1-u_2u_3)}$$

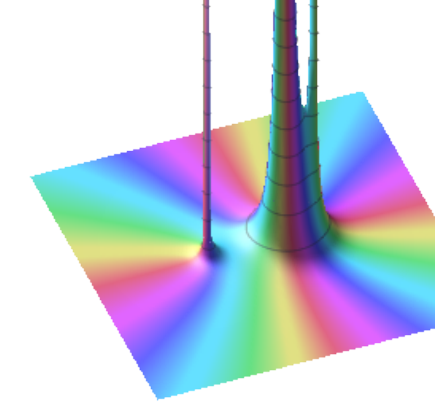
$$z-1)\,S^{(3)}(z) + \cdots + 2\,(30z+1)\,S(z) = 0$$

$$S_n = 16^n\, n^{-3/2}\sqrt{\frac{2}{\pi^3}}\left(1 - \frac{9}{16n} + O\left(\frac{1}{n^2}\right)\right), \quad n \to \infty$$





Coefficients of Diagonals



$$F(\underline{z}) = \frac{G(\underline{z})}{H(\underline{z})} \qquad c_{k,\ldots,k} = \left(\frac{1}{2\pi i}\right)^n \int_T \frac{G(\underline{z})}{H(\underline{z})} \frac{dz_1 \cdots dz_n}{(z_1 \cdots z_n)^{k+1}}$$

Critical points: extrema of $|z_1 \cdots z_n|$ on $\mathscr{V} := \{\underline{z} \mid H(\underline{z}) = 0\}$.

$$\operatorname{rank}\begin{pmatrix} \frac{\partial H}{\partial z_1} & \cdots & \frac{\partial H}{\partial z_n} \\ \frac{\partial (z_1 \cdots z_n)}{\partial z_1} & \cdots & \frac{\partial (z_1 \cdots z_n)}{\partial z_n} \end{pmatrix} \leq 1 \qquad \text{i.e.} \qquad z_1 \frac{\partial H}{\partial z_1} = \cdots = z_n \frac{\partial H}{\partial z_n}$$

J from G-D method

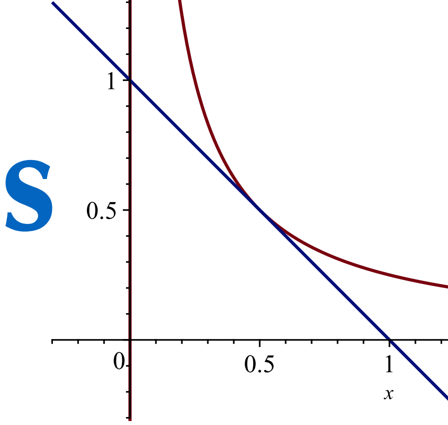
Minimal ones: on the boundary of the domain of convergence.

A 3-step method

1a. locate the critical points (**algebraic** condition);
1b. find the minimal ones (**semi-algebraic** condition);
2. translate (easy in simple cases).

Analytic continuation from the rational function

Ex.: Central Binomial Coefficients

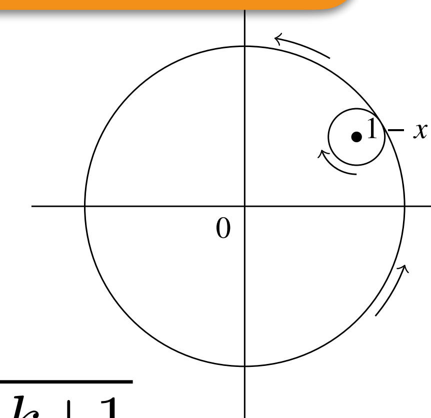


$$\binom{2k}{k}: \qquad \frac{1}{1-x-y} = 1 + x + y + 2xy + x^2 + y^2 + \cdots + 6x^2y^2 + \cdots$$

(1). Critical points: $1 - x - y = 0, x = y \Longrightarrow x = y = 1/2.$

(2). Minimal ones. Easy. In general, this is the difficult step.

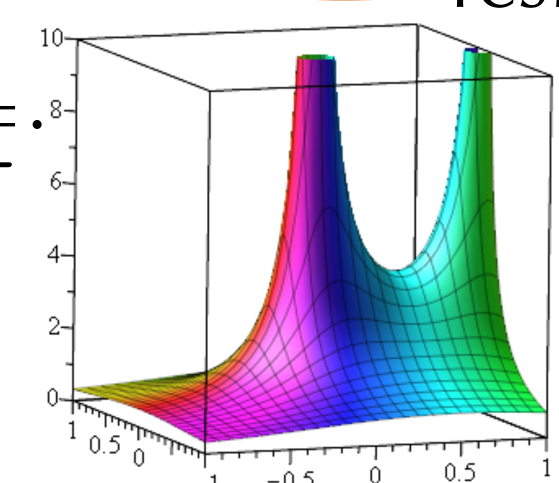
(3). Analysis close to the minimal critical point:



$$a_k = \frac{1}{(2\pi i)^2} \iint \frac{1}{1-x-y} \frac{dx\,dy}{(xy)^{k+1}} \approx \frac{1}{2\pi i} \int \frac{dx}{(x(1-x))^{k+1}}$$

residue

$$\approx \frac{4^{k+1}}{2\pi i} \int \exp(4(k+1)(x-1/2)^2)\,dx \approx \frac{4^k}{\sqrt{k\pi}}.$$



saddle-point approx

Smooth Minimal Critical Point

$\underline{\zeta}$ smooth: $\nabla H(\underline{\zeta}) \neq 0$

$$\text{Wlog } \partial H/\partial z_n(\underline{\zeta}) \neq 0$$

Implicit function theorem

$$g \text{ s.t. } H(\hat{z}, g(\hat{z})) = 0, \quad \hat{z} = (z_1, \ldots, z_{n-1})$$

Step 1. Residue

$$c_{\underline{k}} = \left(\frac{1}{2\pi i}\right)^{n-1} \oint \frac{G(\hat{z}, g(\hat{z}))}{\partial_n H(\hat{z}, g(\hat{z}))} \frac{\partial \hat{z}}{\psi(\hat{z})^{k+1}}, \qquad \psi(\hat{z}) := z_1 \ldots z_{n-1} g(\hat{z})$$

Step 2. Saddle-point analysis

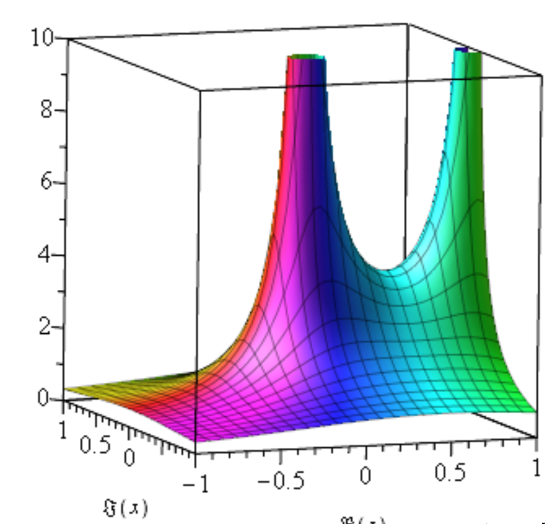
$\underline{\zeta}$ critical

Hessian matrix

$$\psi(\hat{z}) = \zeta_1 \cdots \zeta_n + 0 \cdot (\hat{z} - \hat{\zeta}) + \frac{1}{2}(\hat{z} - \hat{\zeta})^T \mathscr{H}(\zeta)(\hat{z} - \hat{\zeta}) + O(|\hat{z} - \hat{\zeta}|^3)$$

Thm. Under mild conditions,

$$c_{\underline{k}} = \underline{\zeta}^{-k} k^{\frac{1-n}{2}} \left(\frac{(2\pi)^{(1-n)/2}}{\sqrt{(\underline{\zeta}^{3-n}/\zeta_n^2)|\mathscr{H}(\underline{\zeta})|}} \cdot \frac{-G(\underline{\zeta})}{\zeta_n \partial H/\partial z_n(\underline{\zeta})} + O(k^{-1}) \right)$$



[PemantleWilson13;GaoRichmond92]

IV. Computational Aspects

1. Critical Points

Algebraic computation

Kronecker Representation for the Critical Points

Algebraic part: ``compute'' the solutions of the system

$$H(\underline{z}) = 0 \qquad z_1 \frac{\partial H}{\partial z_1} = \cdots = z_n \frac{\partial H}{\partial z_n}$$

$$\deg H = d,\ \max|\mathrm{coeff}(H)| \le 2^h, D := d^n,$$

Prop. Under genericity assumptions, a probabilistic algorithm finds

$$\left.\begin{array}{r} P(u) = 0 \\ P'(u)z_1 - Q_1(u) = 0 \\ \vdots \\ P'(u)z_n - Q_n(u) = 0 \end{array}\right\} \quad \begin{array}{l} \deg \le nD, \\ \text{height} = \tilde{O}(D(d+h)) \end{array}$$

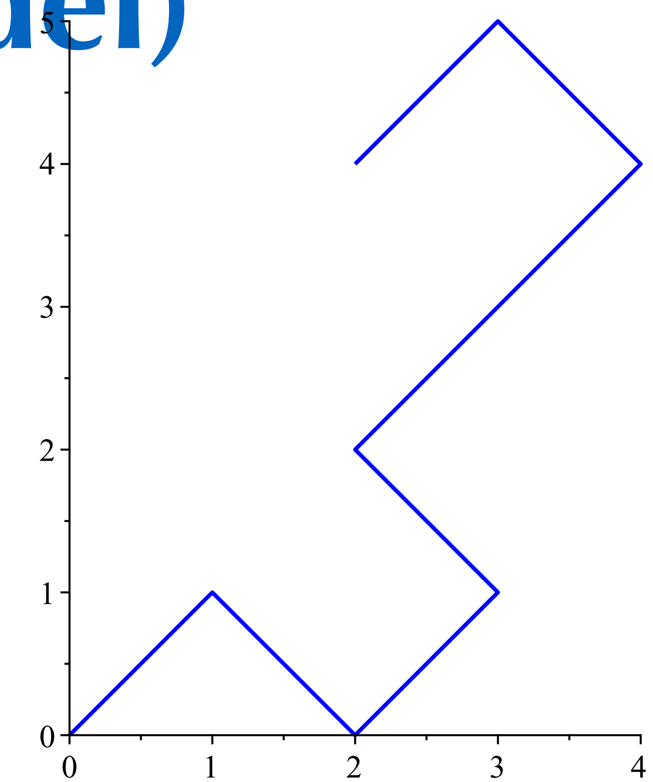
in $\tilde{O}(D^3(d+h))$ bit ops.

System reduced to a univariate polynomial

History and Background: see Castro, Pardo, Hägele, and Morais (2001)

[Giusti-Lecerf-S.01;Schost02;SafeySchost16]

Example (Lattice Path Model)



The number of walks from the origin taking steps *{NW,NE,SE,SW}* and staying in the first quadrant is

$$\operatorname{Diag} \frac{(1+x)(1+y)}{1-t(1+x^2+y^2+x^2+y^2)}$$

Kronecker representation of the critical points:

$$P(u) = 4u^4 + 52u^3 - 4339u^2 + 9338u + 403920$$
$$Q_x(u) = 336u^2 + 344u - 105898$$
$$Q_y(u) = -160u^2 + 2824u - 48982$$
$$Q_t(u) = 4u^3 + 39u^2 - 4339u/2 + 4669/2$$

ie, they are given by:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Which one of these 4 is minimal?

Numerical Kronecker Representation

$$\left.\begin{array}{c} P(u) = 0 \\ P'(u)z_1 - Q_1(u) = 0 \\ \vdots \\ P'(u)z_n - Q_n(u) = 0 \end{array}\right\} \quad + \quad \text{isolating intervals/disks for the real/complex roots of } P$$

degree $\mathscr{D}$, height $\mathscr{H}$ — $\tilde{O}(\mathscr{D}^2(\mathscr{D} + \mathscr{H}))$

all z_i at precision $2^{-\kappa}$ — $\tilde{O}(\mathscr{D}^3 + n(\mathscr{D}^2\mathscr{H} + \mathscr{D}\kappa))$

(Technical) bounds on the complexity to decide whether a polynomial $Q(\underline{z})$
vanishes at some of the solutions,
or is >0 at some of the real solutions;
to group solutions that have the same $|z_i|, i = 1,\ldots,n$.

Also in a multi-degree and/or a straight-line program setting.

[Melczer-S.21]

2. Minimal Critical Points in the Combinatorial Case

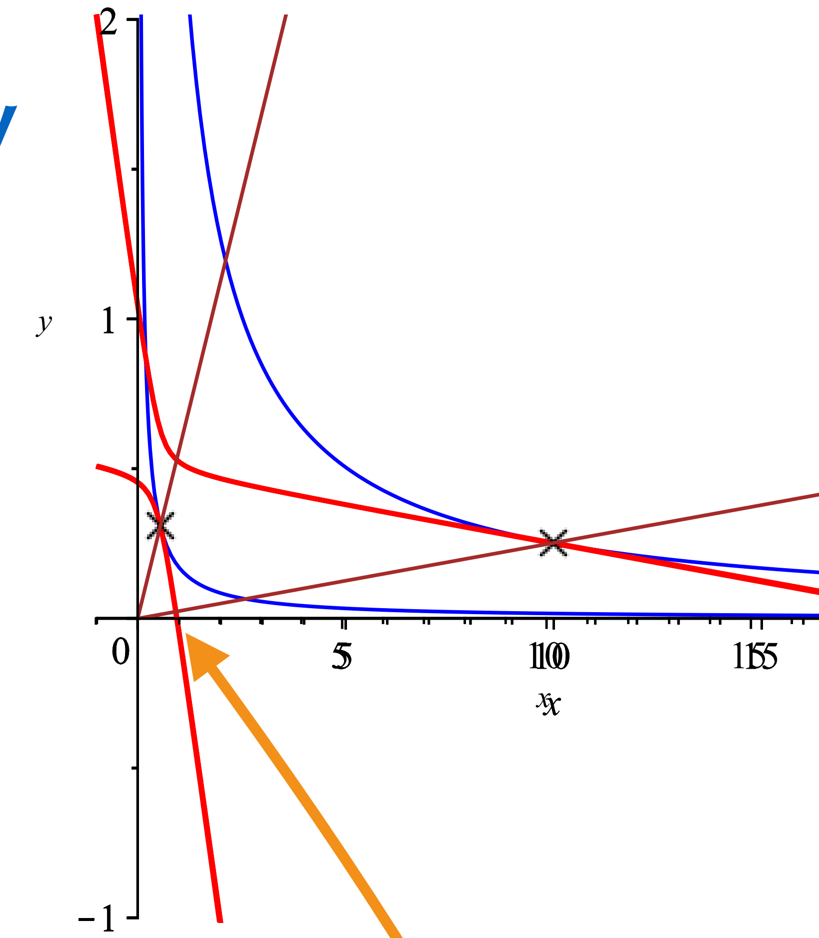
Semi-Algebraic Problem

Combinatorial Generating Functions

Def. $F(z_1, \ldots, z_n)$ is **combinatorial** if every coefficient is ≥ 0.

Prop. [PemantleWilson] In the combinatorial case, one of the minimal critical points has positive real coordinates.

Testing Minimality



$$F = \frac{1}{H} = \frac{1}{(1-x-y)(20-x-40y)-1}$$

Critical point equation $x\frac{\partial H}{\partial x} = y\frac{\partial H}{\partial y}$:

$$x(2x+41y-21) = y(41x+80y-60)$$

→ 4 critical points, 2 of which are real:

$$(x_1, y_1) = (0.2528, 9.9971), \quad (x_2, y_2) = (0.30998, 0.54823)$$

Add $H(tx, ty) = 0$ and compute a Kronecker representation:

$$P(u) = 0, \quad x = \frac{Q_x(u)}{P'(u)}, \quad y = \frac{Q_y(u)}{P'(u)}, \quad t = \frac{Q_t(u)}{P'(u)}$$

Solve numerically and keep the real positive sols:

$$(0.31, 0.55, 0.99), (0.31, 0.55, 1.71), (0.25, 9.99, 0.09), (0.25, 0.99, 0.99)$$

(x_1, y_1) is not minimal, (x_2, y_2) is.

Algorithm and Complexity

Thm. If $F(\underline{z})$ is combinatorial, then under regularity conditions, the points contributing to dominant diagonal asymptotics can be determined in $\tilde{O}(D^4(d+h))$ bit operations. Each contribution has the form

$$A_k = \left(T^{-k} k^{(1-n)/2} (2\pi)^{(1-n)/2}\right) (C + O(1/k))$$

T, C can be found to precision $2^{-\kappa}$ in $\tilde{O}(D^3 d^3 h^3 + D\kappa)$ bit ops.

explicit algebraic numbers

half-integer

This result covers the easiest cases.
All conditions hold generically and can be checked within the same complexity, except combinatoriality.

[Melczer-S. 21]

Example: Apéry's sequence

$$\frac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)} = 1 + \cdots + 5xyzt + \cdots$$

Kronecker representation of the critical points:

$$P(u) = u^2 - 366u - 17711$$

$$x = \frac{2u-1006}{P'(u)}, \quad y = z = -\frac{320}{P'(u)}, \quad t = -\frac{164u+7108}{P'(u)}$$

There are two real critical points, and one is positive. After testing minimality, one has proved asymptotics

```
> A, U := DiagonalAsymptotics(numer(F),denom(F),[t,x,y,z], u, k):
> evala(allvalues(subs(u=U[1],A)));
```

$$\frac{\left(17+12\sqrt{2}\right)^k \sqrt{2}\,\sqrt{24+17\sqrt{2}}}{8\,k^{3/2}\,\pi^{3/2}}$$

Example: Restricted Words in Factors

$$F(x,y) = \frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

words over {0,1} without 10101101 or 1110101

```
> A,U:=DiagonalAsymptotics(numer(F),denom(F),indets(F),u,k,true,u-T,T):
> A;
```

$$\left(\left(\frac{84u^{20}+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}-605u^{10}+2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16}{-12u^{20}+30u^{19}+258u^{18}+500u^{17}+440u^{16}-102u^{15}-378u^{14}-1544u^{13}-2142u^{12}-550u^{11}-2222u^{10}-1644u^{9}+2860u^{8}-1848u^{7}+1230u^{6}+2160u^{5}-2686u^{4}+1494u^{3}-228u^{2}-320u+84}\right)^{k}\right.$$

$$\sqrt{2}\sqrt{\frac{84u^{20}+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}+605u^{10}+2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16}{-162u^{18}-612u^{17}-902u^{16}-616u^{15}+254u^{14}+548u^{13}+2054u^{12}+2156u^{11}+898u^{10}-2268u^{9}+2462u^{8}-2088u^{7}+1312u^{6}-540u^{5}-1410u^{4}+1188u^{3}-290u^{2}+32}}$$

$$\left.\left(12u^{20}+36u^{19}-21u^{18}-170u^{17}-255u^{16}-190u^{15}-19u^{14}+46u^{13}+461u^{12}+628u^{11}+133u^{10}+374u^{9}+161u^{8}-384u^{7}+146u^{6}-138u^{5}-285u^{4}-40u^{3}+91u^{2}-30u+32\right)\right) \Big/ \left(2\sqrt{k}\sqrt{\pi}\,\left(84u^{20}\right.\right.$$

$$\left.\left.+240u^{19}-285u^{18}-1548u^{17}-2125u^{16}-1408u^{15}+255u^{14}+756u^{13}+2509u^{12}+2856u^{11}-605u^{10}+2020u^{9}+1233u^{8}-1760u^{7}+924u^{6}-492u^{5}-675u^{4}+632u^{3}-249u^{2}+24u+16\right)\right)$$

```
> U;
```

$$[\mathit{RootOf}(4\_Z^{21}+12\_Z^{20}-15\_Z^{19}-86\_Z^{18}-125\_Z^{17}-88\_Z^{16}+17\_Z^{15}+54\_Z^{14}+193\_Z^{13}+238\_Z^{12}+55\_Z^{11}+202\_Z^{10}+137\_Z^{9}-220\_Z^{8}+132\_Z^{7}-82\_Z^{6}-135\_Z^{5}+158\_Z^{4}-83\_Z^{3}+12\_Z^{2}$$
$$+16\_Z-4, 0.25574184)]$$

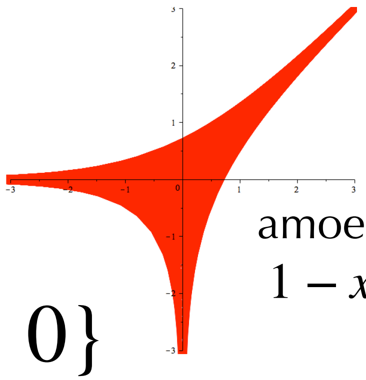
```
> evalf(subs(u=U[1],A));
```

$$\frac{0.60294599\ 3.9101932^{k}}{\sqrt{k}}$$

3. Non-Combinatorial Case

Minimal Critical Points

The connected components of the complement of amoebas are convex



$$\text{amoeba}(H) := \{(\log|z_1|, \ldots, \log|z_n|) \mid \underline{z} \in \mathbb{C}^{*n}, H(\underline{z}) = 0\}$$

$\boxed{\textbf{Consequence:} \text{ With } \mathscr{D} \text{ the domain of convergence of } F, \; \underline{u} \notin \mathscr{D} \Rightarrow \exists t \in (0,1), \underline{z} \in \partial\mathscr{D} \text{ s.t. } |z_j| = t\,|u_j|, j = 1,\ldots,n\,.}$

—> Criterion in the non-combinatorial case

Split into Real & Imaginary Parts

$$f(\underline{z}) \in \mathbb{C}[\underline{z}] \text{ splits into } f(\underline{x} + i\underline{y}) = f^{(R)}(\underline{x}, \underline{y}) + i f^{(I)}(\underline{x}, \underline{y})$$

$$f^{(R)}, f^{(I)} \text{ in } \mathbb{R}[\underline{x}, \underline{y}]$$

—> $2n + 2$ critical point equations in $2n + 2$ real unknowns

Cauchy-Riemann

$$\begin{cases} H^{(R)}(\underline{a}, \underline{b}) = H^{(I)}(\underline{a}, \underline{b}) & = 0 \\ a_j \left(\partial H^{(R)}/\partial x_j\right)(\underline{a}, \underline{b}) + b_j \left(\partial H^{(R)}/\partial y_j\right)(\underline{a}, \underline{b}) - \lambda_R & = 0, \qquad j = 1, \ldots, n \\ a_j \left(\partial H^{(I)}/\partial x_j\right)(\underline{a}, \underline{b}) + b_j \left(\partial H^{(I)}/\partial y_j\right)(\underline{a}, \underline{b}) - \lambda_I & = 0, \qquad j = 1, \ldots, n \end{cases}$$

Minimal Critical Points

Needed: no real zero of $H(\underline{x} + i\underline{y})$ with

$$|x_j + iy_j| = t|a_j + ib_j|, \quad j = 1,\ldots,n$$

with $0 < t < 1$.

Add new equations:

$$H^{(R)}(t\underline{x}, t\underline{y}) = H^{(I)}(t\underline{x}, t\underline{y}) = 0,$$
$$x_j^2 + y_j^2 = t(a_j^2 + b_j^2), \quad j = 1,\ldots,n$$

$n + 2$ eqns in
$2n + 1$ unknowns

And setup a (structured) system for the critical points of

$$\pi_t : (\underline{a}, \underline{b}, \underline{x}, \underline{y}, \lambda_R, \lambda_I, t) \mapsto t\,.$$

$4n + 4$ eqns in
$4n + 4$ unknowns

Bit complexity for min crit pt selection: $\tilde{O}(2^{3n} D^9 d^5 h)$.

Rest as before

Conclusion

Comparison of Approaches

Multiple binomial sum

easy

Diagonal/Integral representation

Griffiths-Dwork reduction

Same ideal

analytic combinatorics in several variables (smooth case)

Kronecker repr. of arith. size $\tilde{O}(d^n)$

arith. size $\tilde{O}(d^{5n})$

Linear differential equation

analytic combinatorics

α arbitrary algebraic num.
C only numerically
full asymptotic expansion

$$S_n = \rho^n n^\alpha C \left(1 + \frac{d_1}{n} + \cdots\right), \quad n \to \infty$$

α half-integer
C explicit
leading term

Next Steps

More ACSV (transverse multiple points; even more degenerate cases; diagonals of meromorphic functions,…);

More general sequences and integrals;

Other ways to get `explicit' constants;

Complete, usable implementations…

The End