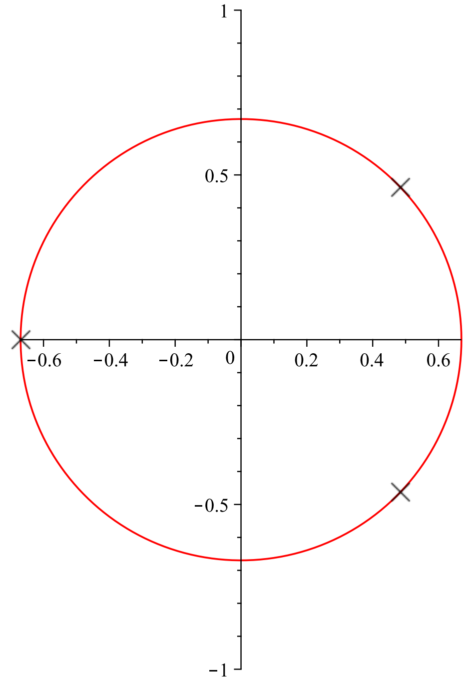
Absolute Root Separation

Bruno Salvy

AriC, Inria at ENS de Lyon



AMM, 2017

arXiv:1606.01131

Exp. Maths., 2022

arXiv:1907.01232

Oberwolfach, Dec. 2022

Joint work with Yann Bugeaud, Andrej Dujella, Wenjie Fang and Tomislav Pejković

Conway's sequence

1,11,21,1211,111221,...

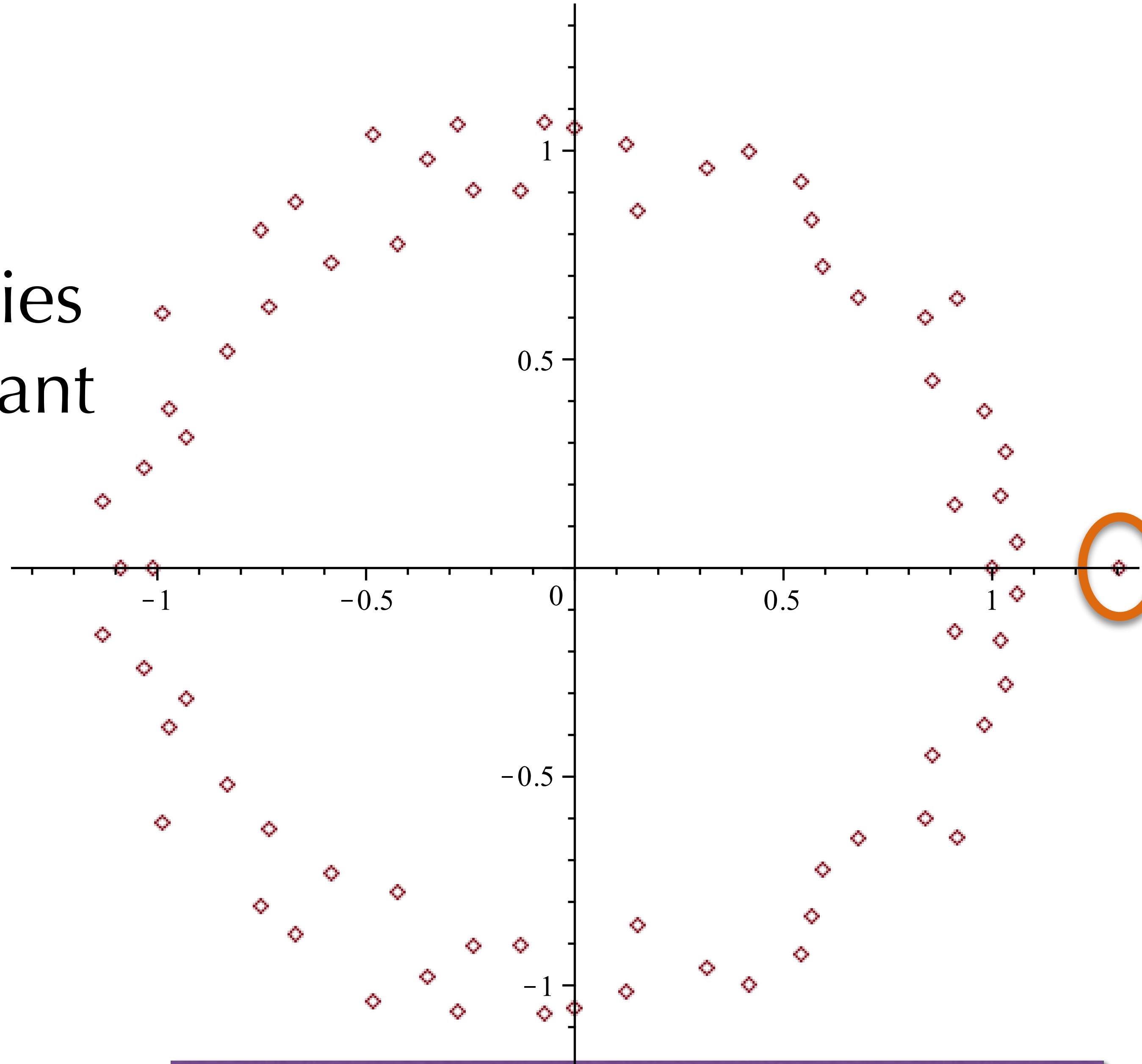
The sequence of *lengths* satisfies a linear recurrence with constant coefficients of order 72.

Largest root of the char. pol.

$\rho \approx 1.30358$

$$\boxed{\ell_n \simeq 2.04216\,\rho^n}$$

remainder exponentially small



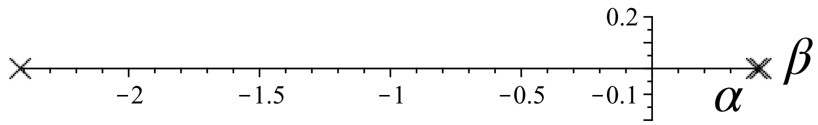
How many digits do we need?

High degree polynomials occur naturally in ACSV

[Conway 1987]

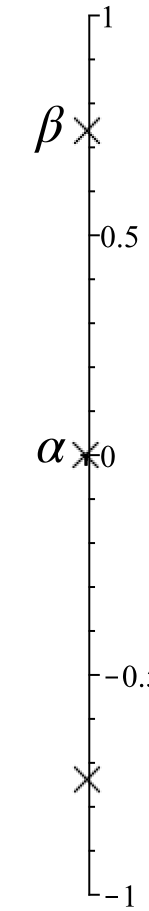
Close Roots of Polynomials

$$P \in \mathbb{Z}[X] \qquad P(\alpha) = P(\beta) = 0$$



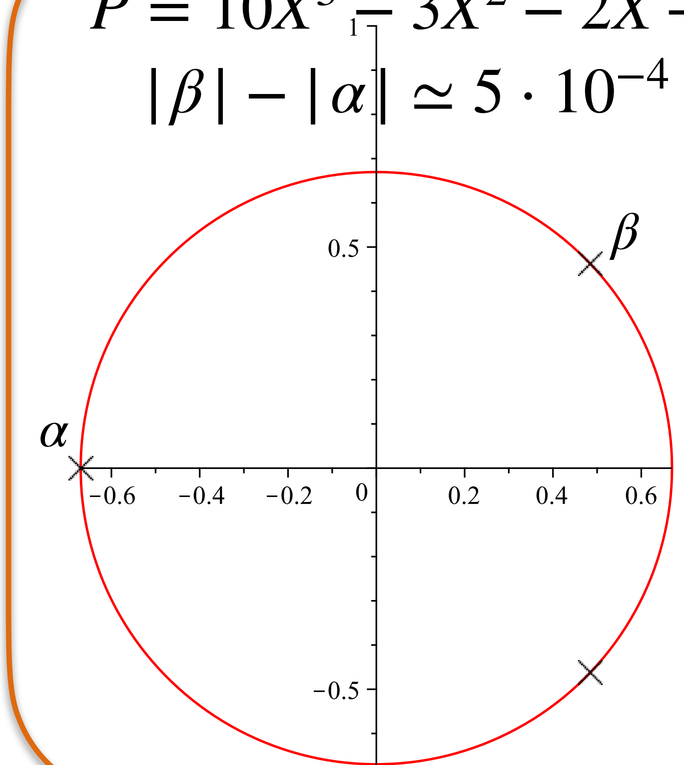
$$P = 5X^3 - 8X^2 - 9X + 2$$
$$\beta - \alpha \simeq 10^{-2}$$

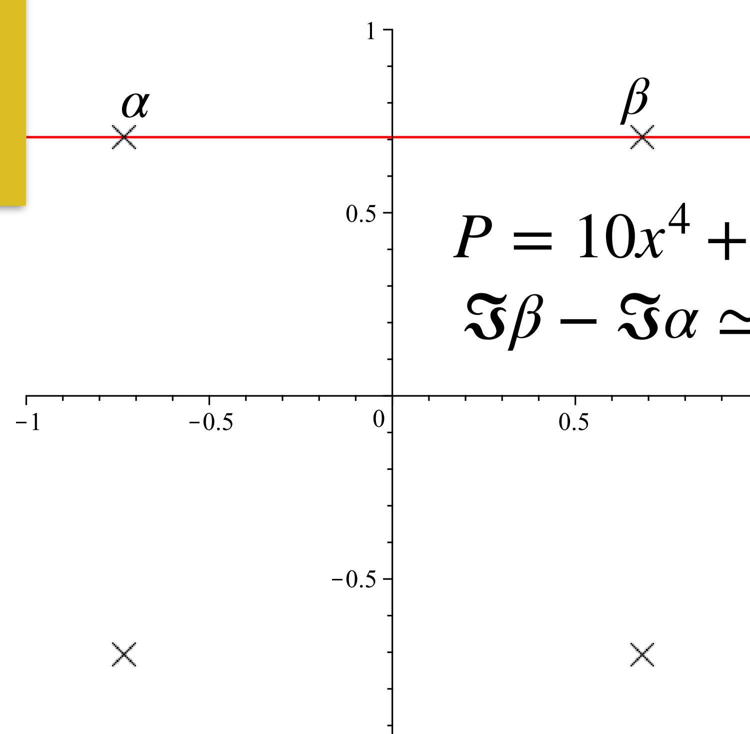
$$P = 7X^3 + 5X^2 + 5X + 1$$
$$\Re\beta - \Re\alpha \simeq 6 \cdot 10^{-4}$$



Aim: Bound precision needed to **decide** that two roots have identical value/real part/imaginary part/absolute value ?

$$P = 10X^3 - 3X^2 - 2X + 3$$
$$|\beta| - |\alpha| \simeq 5 \cdot 10^{-4}$$





$$P = 10x^4 + x^3 + 10$$
$$\Im\beta - \Im\alpha \simeq 6 \cdot 10^{-5}$$

focus for this talk

Mahler's Bound

Def. Separation

$$\operatorname{sep}(P) := \min_{\substack{P(\alpha) = P(\beta) = 0, \\ \alpha \neq \beta}} |\alpha - \beta| \,.$$

Def. Height

$$H\left(\sum_{i=0}^{d} a_i X^i\right) := \max_i |a_i| \,.$$

Thm. If $P \in \mathbb{Z}[X]$ has degree d,

$$\operatorname{sep}(P) > \kappa(d) H(P)^{-d+1} .$$

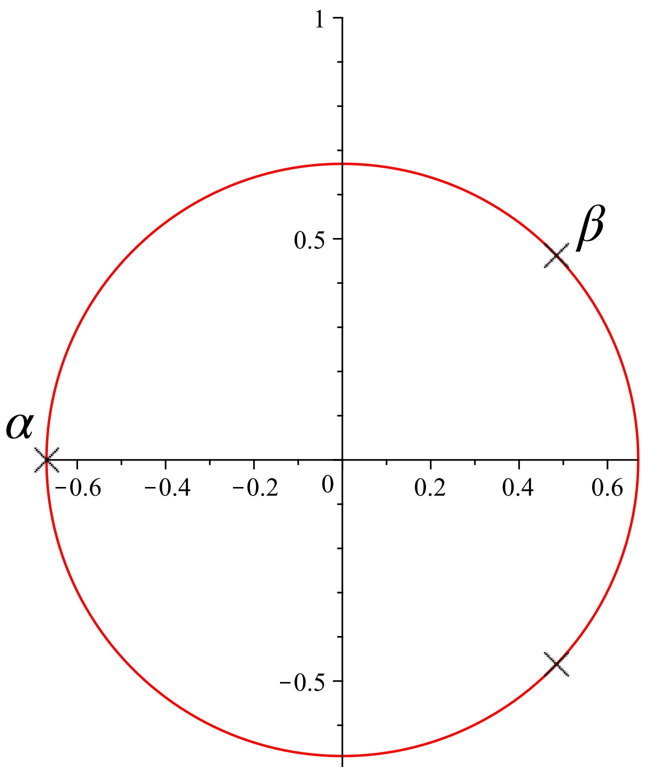
explicit function of d

not known to be tight (except for $d = 3$)

worst known family gives $-(2d-1)/3$.

[Mahler 64; Evertse 04; Schönhage 06; Mignote 82; BugeaudDujella 14]

Absolute Separation



Motivation: asymptotics of linear recurrences & diagonals

Def. Absolute Separation

$$\operatorname{abssep}(P) := \min_{\substack{P(\alpha)=P(\beta)=0,\\ |\alpha|\neq|\beta|}} \Big|\, |\alpha| - |\beta| \,\Big| \,.$$

Aims:

1. $\operatorname{abssep}(P) > \kappa(d)\, H(P)^{-e(d)}$ with $e(d)$ small;

   not the same as before

2. families for small d with

   $\operatorname{abssep}(P_H) \underset{H\to\infty}{\sim} \kappa' H^{-e'}$ and $e'(\leq e(d))$ large.

Note: Isolating disks of radius ε for all roots can be computed in time $\tilde{O}(d^3 + d^2 \log H(P) - d \log \varepsilon)$.

With $\log \varepsilon \sim -e(d) \log H(P)$, last term dominates if $e(d) \gg d$.

Results

Previously	$e(d) \leq d(d^2 + 2d - 1)/2$	1996
	$e(d) \leq d^3/2 - d^2 - d/2 + 2$	2015
	$e(d) \leq d^3/2 - d^2 - d/2 + 1 \quad (d \geq 4)$	2019

New: $e(2) = 1$, $e(3) = 4$, $5 \leq e(4) \leq 12$, $6 \leq e(5) \leq 24$, $7 \leq e(6)$,

$$e(d) \leq (d-1)(d-2)(d-3)/2 = d^3/2 - 3d^2 + \cdots \qquad (d \geq 6).$$

[GourdonSalvy 96; DubickasSha 15; Sha 19]

II. Linear Lower Bounds in Low Degree + a Miracle in Degree 3

Degrees 3 & 4: Exhaustive Search

1. Solve the $(2H+1)^{d+1}$ pols in $\mathbb{Z}[X]_{\leq d}$ with height $\leq H$

and keep the records.

Ex.: $d = 3$, $H = 20 \to$ approx. 300,000 polynomials. (15 min.)
$d = 4$, $H = 20 \to$ approx. $115\ 10^6$ polynomials. (19 h)

2. Refine the search in the neighborhood of those;
look for patterns

	abssep	real root
$17\,x^3 + 9\,x^2 + 7\,x + 8$	$1.9\ 10^{-5}$	-0.7778352845
$102\,x^3 + 97\,x^2 + 71\,x + 40$	$1.5\ 10^{-8}$	-0.7319587393
$153\,x^3 - 97\,x^2 - 71\,x + 60$	$4.5\ 10^{-9}$	-0.7319587525
$181\,x^3 + 153\,x^2 + 112\,x + 71$	$9.0\ 10^{-10}$	-0.7320261422

$\approx 1 - \sqrt{3}$?

Miracle in Degree 3

Key polynomial:

$$P(X,Y) = 6(X^3 - X^2 + 1) + \left(3X^3 - 2X^2 + 4X - 6\right) Y$$

Guessed from numerical coefficients

$$P(X,\sqrt{3}) = (\sqrt{3}+2)\left(X - \sqrt{3} + 1\right)\left(X^2 + aX + (\sqrt{3}-1)^2\right), \quad a < 2(\sqrt{3}-1)$$

discr<0 —> 3 roots with same modulus

Perturbation:

$P(X,\sqrt{3}+\epsilon)$ has a real root at $\sqrt{3} - 1 + (2-\sqrt{3})\epsilon + O(\epsilon^2)$

and a nonreal one with similar modulus, but a different O() term.

If p_n/q_n is the nth convergent of the continued fraction of $\sqrt{3}$,

$$P_n(X) := q_n P(X, p_n/q_n) \in \mathbb{Z}[X], \quad \text{abssep}(P_n) < \kappa\, H(P_n)^{-4}.$$

Proof: $|p_n/q_n - \sqrt{3}| < 1/q_n^2$.

$$\sqrt{3} = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}$$

Perturbative Method (4≤deg≤6)

Principle $$P(X,\epsilon) = R(X) + \epsilon\, Q(X)$$

with roots of identical |.|

with undeterminate coefficients

1. Pick two nonconjugate roots α, β of R

2. Compute expansions $\alpha(\epsilon), \beta(\epsilon)$ of roots of P with $\alpha(0) = \alpha, \beta(0) = \beta$

   in $\mathbb{Q}(\alpha)[q_0, \ldots, q_d][[\epsilon]]$ (or β)

3. Form the expansion of $|\alpha(\epsilon)|^2 - |\beta(\epsilon)|^2$ in $\mathbb{Q}(\alpha,\beta)[q_0, \ldots, q_d][[\epsilon]]$

4. Look for a nondegenerate integer solution of the system formed by its first coefficients

Demo

Results

deg	R	Q	exponent
4	$x^4 - 1$	$x^3 - x^2 + x - 5$	-5
4	$(x^2 - 1)(x^2 + x + 1)$	$x^3 - 3x - 4$	-5
5	$(x^2 + ax + r^2)(x^2 + bx + r^2)$ $\lvert a \rvert < 2r, \lvert b \rvert < 2r$	Too big for the Gröbner basis computation	

Loop over small values of a, b, r gives:

$a = -9,\ b = -11,\ r = 6,$

$$Q = x^5 - 213x^3 + 2404x^2 - 11088x + 20736$$

-6

deg	R	Q	exponent
6	$x^6 - 1$	$9x^5 - 9x^4 - 26x^3 - 9x^2 + 9x - 28$	-7

No luck in degrees 7 & 8

III. Proof Technique for Upper Bounds

Auxiliary Polynomials

From $P(X) = \sum_{i=0}^{d} a_i X^i = a_d \prod_{i=1}^{d} (X - \alpha_i) \in \mathbb{Z}[X]$ of height $H(P)$

construct auxiliary polynomials, bound their nonzero roots.

Prop. 1 [Cauchy] If $\alpha \neq 0$,

$$P(\alpha) = 0 \Rightarrow |\alpha| \geq \frac{1}{1 + H(P)}.$$

Prop. 2 [Symmetric fcns]

$G \in \mathbb{Z}[X_1, \ldots, X_d]$ symmetric with $\deg_{X_i} G \leq k$ for all i

$\Rightarrow a_d^k G(\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}[a_0, \ldots, a_d]$ of total degree $\leq k$.

Example:

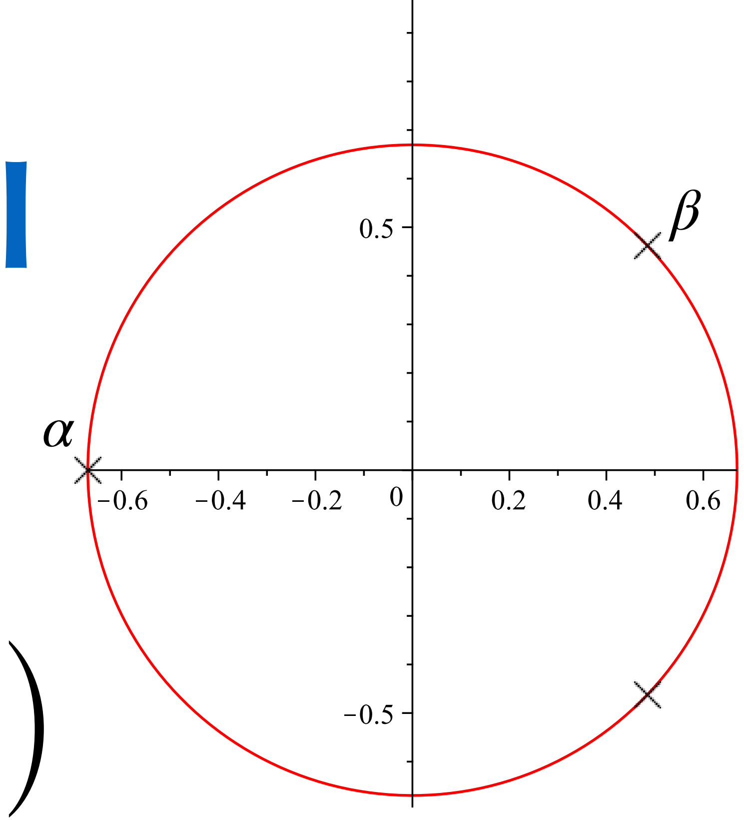
$$M(X) = a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i - \alpha_j)^2) \in \mathbb{Z}[X]$$

coeffs bounded by Prop. 2

$$\rightarrow |\alpha_i - \alpha_j|^2 > \kappa H^{-2(d-1)}.$$

Recovers Mahler's exponent

A Bigger Polynomial



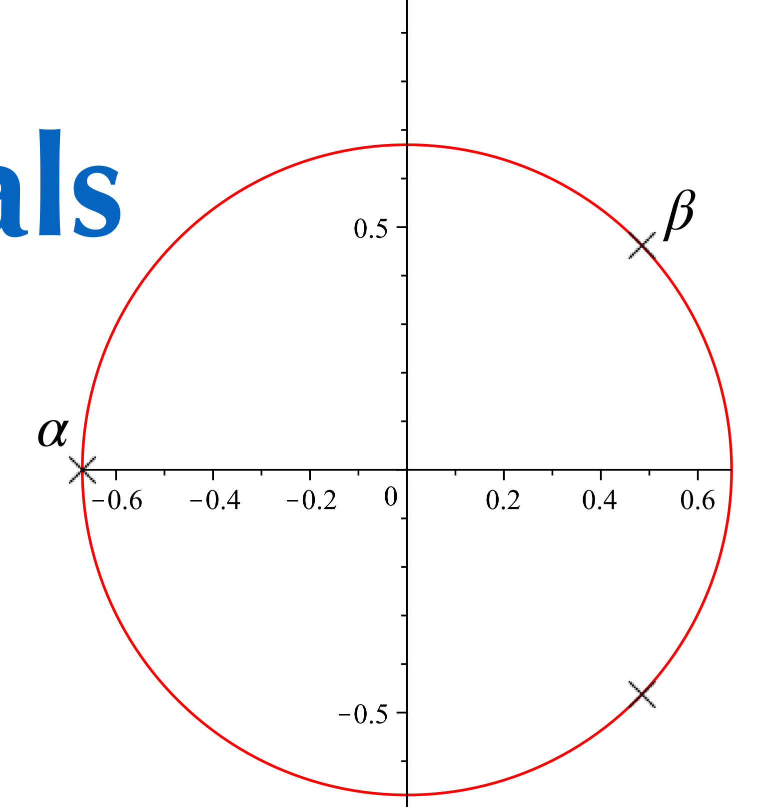
$$a_d^{(d-1)(d-2)(d-3)} \prod_{\substack{i<j,\\ k<\ell,\\ \{i,j\}\cap\{k,\ell\}=\varnothing}} \left(X^{1/2} - (\alpha_i\alpha_j - \alpha_k\alpha_\ell)\right)$$

$$\Rightarrow (|\alpha|^2 - |\beta|^2)^2 > \kappa H^{-(d-1)(d-2)(d-3)}$$

gives exponent $(d-1)(d-2)(d-3)/2$ for the general case

Extra argument when $|\alpha| + |\beta|$ is large, using the reciprocal polynomial.

More Auxiliary Polynomials



$$a_d^{2(d-1)(d-2)} \prod_{i<j, k \notin \{i,j\}} (X - (\alpha_k^2 - \alpha_i \alpha_j))$$

$$\alpha_k \text{ real} \Rightarrow \Big| |\alpha_j| - |\alpha_k| \Big| > \kappa H^{-2(d-1)(d-2)}$$

gives the optimal exponent -4 when $d = 3$

also $e(4) \leq 12$, $e(5) \leq 24$

$$a_d^{2(d-1)} \prod_{i<j} (X - (\alpha_i + \alpha_j)^2)$$

$$\alpha_j, \alpha_k \text{ real} \Rightarrow \Big| |\alpha_j| - |\alpha_k| \Big| > \kappa H^{-(d-1)}$$ optimal

gives $e(2) = 1$.

Other polynomials for the separations between real and imaginary parts, with variants when one of the roots is real.

Conclusion

We have:

. a lower bound $d + 1$ up to degree 6;

. a cubic upper bound.

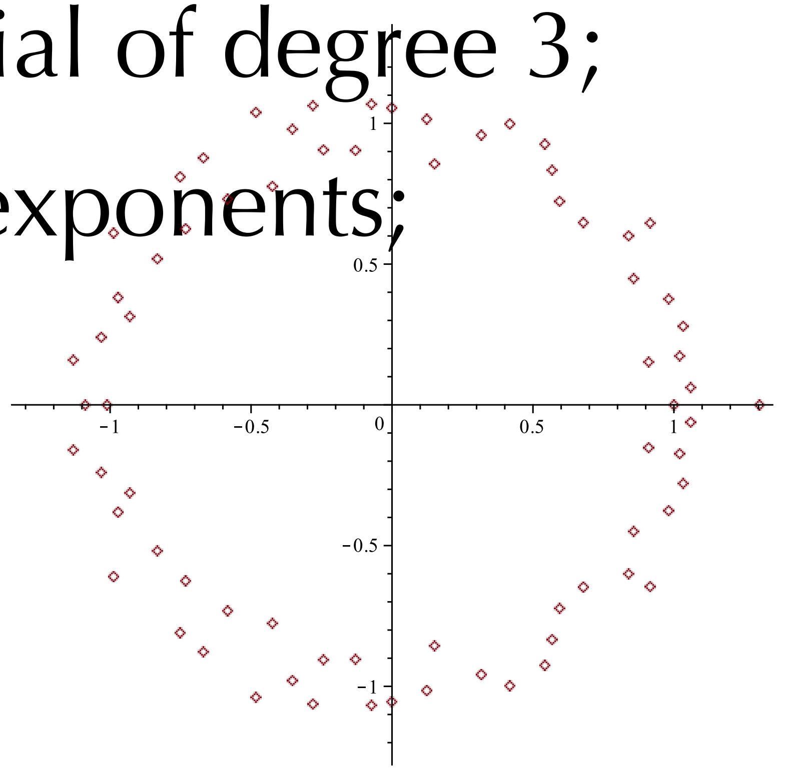
We would like:

. polynomials with small absolute separation in low degree;

. a generalization of the miracle polynomial of degree 3;

. a proof of the (in)existence of subcubic exponents;

. better bounds for the dominant roots?



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