

[Algolib can be downloaded from <http://algo.inria.fr>

[> **libname:="/Users/salvy/lib/maple/Algolib/11", libname:**

▼ Dominant singularity

▼ Rational generating functions

[Fibonacci

```
> infsing(1/(1-z-z^2), z);  
[[ -1/2 + 1/2 sqrt(5) ], polar, false]
```

[asymptotic behaviour:

```
> equivalent(1/(1-z-z^2), z, n);  

$$\frac{(e^{-n})^{-\ln(2) + \ln(-1 + \sqrt{5})}}{-\frac{1}{2} + \frac{1}{2} \sqrt{5} + 2 \left(-\frac{1}{2} + \frac{1}{2} \sqrt{5}\right)^2} + O\left(\frac{(e^{-n})^{-\ln(2) + \ln(-1 + \sqrt{5})}}{n}\right)$$

```

```
> read "conway.mpl";
```

```
GFconway := (-1 + z^2 - z + z^3 + 12 z^78 + 6 z^11 - 20 z^30 - 30 z^29 + z^4 - 20 z^73 + 18 z^76  
- 4 z^69 + 18 z^74 + 31 z^71 - 4 z^68 - z^18 + 3 z^19 - 36 z^24 + 58 z^27 + 13 z^22 + 8 z^12 - 4 z^17  
- 23 z^31 + 15 z^70 - 6 z^23 - 20 z^25 + 8 z^21 - z^13 - z^16 + 6 z^20 - 6 z^9 - 18 z^77 - 5 z^14  
- 18 z^75 - 22 z^72 - 4 z^15 + 45 z^55 - 11 z^63 + 41 z^62 + 54 z^61 - 56 z^60 + 15 z^58 - 44 z^59  
- 27 z^57 + 62 z^66 - 21 z^64 - 19 z^67 - 50 z^65 + 34 z^28 + z^5 - 4 z^8 + 35 z^32 + 7 z^38 + 12 z^36  
- 79 z^39 + 107 z^43 + 8 z^35 - 13 z^40 + 38 z^49 + 16 z^41 - z^26 + z^7 - 64 z^52 - 15 z^56  
+ 89 z^53 - 25 z^50 - 8 z^54 + 126 z^48 - 26 z^34 - 9 z^33 + 42 z^37 - 39 z^47 - 32 z^46 - 66 z^51  
- 33 z^45 + 14 z^42 - 65 z^44) / ((z - 1) (-1 + z^2 + 2 z^3 + z^11 - 8 z^30 - 6 z^29 + z^4 + 6 z^69  
+ 6 z^71 - 12 z^68 + 3 z^18 + 2 z^19 + 3 z^24 + 8 z^27 - z^22 + z^12 + 10 z^17 + 5 z^31 - 3 z^70  
- 9 z^23 + 7 z^25 - 6 z^21 - 2 z^13 + 2 z^16 - 6 z^20 + z^9 - 5 z^14 - 3 z^15 + 7 z^55 - 5 z^62 + 2 z^61  
+ 4 z^60 - 2 z^58 + 12 z^59 - 7 z^57 - 7 z^66 - z^64 + z^10 + 4 z^67 + 7 z^65 - 10 z^28 - 2 z^5 + z^8  
+ 12 z^32 - 10 z^38 - z^36 - z^39 + 3 z^43 - 7 z^35 + 6 z^40 + 2 z^41 + 8 z^26 - z^7 - 3 z^52 - 12 z^56  
+ 4 z^53 + 7 z^50 + 10 z^54 + 8 z^48 + 7 z^34 - 7 z^33 + 3 z^37 - 14 z^47 + 3 z^46 - 9 z^51 - 9 z^45  
+ 10 z^42 - 2 z^44 - 2 z^6))
```

```
> infsing(GFconway, z);
```

```
[[ RootOf(-1 + _Z^2 + 2 _Z^3 + _Z^11 - 8 _Z^30 - 6 _Z^29 + _Z^4 + 6 _Z^69 + 6 _Z^71 - 12 _Z^68  
+ 3 _Z^18 + 2 _Z^19 + 3 _Z^24 + 8 _Z^27 - _Z^22 + _Z^12 + 10 _Z^17 + 5 _Z^31 - 3 _Z^70  
- 9 _Z^23 + 7 _Z^25 - 6 _Z^21 - 2 _Z^13 + 2 _Z^16 - 6 _Z^20 + _Z^9 - 5 _Z^14 - 3 _Z^15  
+ 7 _Z^55 - 5 _Z^62 + 2 _Z^61 + 4 _Z^60 - 2 _Z^58 + 12 _Z^59 - 7 _Z^57 - 7 _Z^66 - _Z^64  
+ _Z^10 + 4 _Z^67 + 7 _Z^65 - 10 _Z^28 - 2 _Z^5 + _Z^8 + 12 _Z^32 - 10 _Z^38 - _Z^36 - _Z^39
```

$$\begin{aligned}
& + 3 z^{43} - 7 z^{35} + 6 z^{40} + 2 z^{41} + 8 z^{26} - z^7 - 3 z^{52} - 12 z^{56} + 4 z^{53} \\
& + 7 z^{50} + 10 z^{54} + 8 z^{48} + 7 z^{34} - 7 z^{33} + 3 z^{37} - 14 z^{47} + 3 z^{46} - 9 z^{51} \\
& - 9 z^{45} + 10 z^{42} - 2 z^{44} - 2 z^6, 0.7671198507)], \text{polar}, \text{false}]
\end{aligned}$$

It's the root of this polynomial that is approximately 0.767. We call it α later.

> **alias(alpha=%[1][1]):**

asymptotic behaviour of the coefficients:

> **equivalent(GFconway, z, n);**

$$\begin{aligned}
& - \left((-1 - 64 \alpha^{52} - 4 \alpha^{15} - 26 \alpha^{34} + 38 \alpha^{49} + 16 \alpha^{41} - \alpha^{26} - 66 \alpha^{51} - 33 \alpha^{45} + 41 \alpha^{62} \right. \\
& + 12 \alpha^{36} + 58 \alpha^{27} + 42 \alpha^{37} - 39 \alpha^{47} - 32 \alpha^{46} + \alpha^7 - \alpha^{16} + 13 \alpha^{22} - \alpha^{13} - 9 \alpha^{33} \\
& + 31 \alpha^{71} - 4 \alpha^{68} - 50 \alpha^{65} - \alpha^{18} + \alpha^5 - 4 \alpha^8 + 35 \alpha^{32} + 54 \alpha^{61} - 56 \alpha^{60} + 6 \alpha^{20} \\
& - 4 \alpha^{17} - 23 \alpha^{31} - 13 \alpha^{40} - 65 \alpha^{44} + 62 \alpha^{66} - 22 \alpha^{72} + 7 \alpha^{38} + 34 \alpha^{28} - 25 \alpha^{50} \\
& - 6 \alpha^{23} - 20 \alpha^{25} + 8 \alpha^{21} + 8 \alpha^{12} - 21 \alpha^{64} - 19 \alpha^{67} + 15 \alpha^{70} + 15 \alpha^{58} + 3 \alpha^{19} - 36 \alpha^{24} \\
& - 15 \alpha^{56} + 89 \alpha^{53} - 44 \alpha^{59} - 27 \alpha^{57} - 8 \alpha^{54} + 126 \alpha^{48} + 45 \alpha^{55} + 14 \alpha^{42} - 79 \alpha^{39} \\
& + 18 \alpha^{74} + \alpha^2 + \alpha^3 + 12 \alpha^{78} - 20 \alpha^{73} - \alpha - 5 \alpha^{14} - 18 \alpha^{75} - 11 \alpha^{63} - 6 \alpha^9 - 18 \alpha^{77} \\
& + 18 \alpha^{76} - 30 \alpha^{29} + \alpha^4 - 20 \alpha^{30} + 6 \alpha^{11} + 107 \alpha^{43} + 8 \alpha^{35} - 4 \alpha^{69}) (4 \alpha^{52} + 2 \alpha^{15} \\
& - 7 \alpha^{34} + 7 \alpha^{49} + 10 \alpha^{41} + 8 \alpha^{26} - 3 \alpha^{51} + 3 \alpha^{45} + 3 \alpha^{36} - 10 \alpha^{27} - 10 \alpha^{37} + 8 \alpha^{47} \\
& - 14 \alpha^{46} + \alpha^7 + 10 \alpha^{16} - 9 \alpha^{22} - 5 \alpha^{13} + 7 \alpha^{33} + 6 \alpha^{68} - 7 \alpha^{65} + 2 \alpha^{18} - 2 \alpha^5 + \alpha^8 \\
& - 7 \alpha^{32} - 5 \alpha^{61} + 2 \alpha^{60} - 6 \alpha^{20} + 3 \alpha^{17} + 12 \alpha^{31} + 2 \alpha^{40} - 9 \alpha^{44} + 4 \alpha^{66} - \alpha^{38} \\
& - 6 \alpha^{28} - 9 \alpha^{50} + 3 \alpha^{23} + 8 \alpha^{25} - \alpha^{21} - 2 \alpha^{12} + 7 \alpha^{64} - 12 \alpha^{67} + 6 \alpha^{70} + 12 \alpha^{58} \\
& - 6 \alpha^{19} + 7 \alpha^{24} - 7 \alpha^{56} + 10 \alpha^{53} + 4 \alpha^{59} - 2 \alpha^{57} + 7 \alpha^{54} - 12 \alpha^{55} + 3 \alpha^{42} + 6 \alpha^{39} \\
& + 2 \alpha^2 + \alpha^3 + \alpha - 3 \alpha^{14} - \alpha^{63} + \alpha^9 - 8 \alpha^{29} - 2 \alpha^4 + 5 \alpha^{30} + \alpha^{11} - 2 \alpha^{43} - \alpha^{35} \\
& \left. - 3 \alpha^{69} - \alpha^6 + \alpha^{10})^{-\ln(e^{-n})} \right) / ((\alpha - 1) (156 \alpha^{52} + 45 \alpha^{15} - 238 \alpha^{34} - 82 \alpha^{41} \\
& - 208 \alpha^{26} + 459 \alpha^{51} + 405 \alpha^{45} + 310 \alpha^{62} + 36 \alpha^{36} - 216 \alpha^{27} - 111 \alpha^{37} + 658 \alpha^{47} \\
& - 138 \alpha^{46} + 7 \alpha^7 - 32 \alpha^{16} + 22 \alpha^{22} + 26 \alpha^{13} + 231 \alpha^{33} - 426 \alpha^{71} + 816 \alpha^{68} - 455 \alpha^{65} \\
& - 54 \alpha^{18} + 10 \alpha^5 - 8 \alpha^8 - 384 \alpha^{32} - 122 \alpha^{61} - 240 \alpha^{60} + 120 \alpha^{20} - 170 \alpha^{17} - 155 \alpha^{31} \\
& - 240 \alpha^{40} + 88 \alpha^{44} + 462 \alpha^{66} + 380 \alpha^{38} + 280 \alpha^{28} - 350 \alpha^{50} + 207 \alpha^{23} - 175 \alpha^{25} \\
& + 126 \alpha^{21} - 12 \alpha^{12} + 64 \alpha^{64} - 268 \alpha^{67} + 210 \alpha^{70} + 116 \alpha^{58} - 38 \alpha^{19} - 72 \alpha^{24}
\end{aligned}$$

$$\begin{aligned}
& + 672 \alpha^{56} - 212 \alpha^{53} - 708 \alpha^{59} + 399 \alpha^{57} - 540 \alpha^{54} - 384 \alpha^{48} - 385 \alpha^{55} - 420 \alpha^{42} \\
& + 39 \alpha^{39} - 2 \alpha^2 - 6 \alpha^3 + 70 \alpha^{14} - 9 \alpha^9 + 174 \alpha^{29} - 4 \alpha^4 + 240 \alpha^{30} - 11 \alpha^{11} - 129 \alpha^{43} \\
& + 245 \alpha^{35} - 414 \alpha^{69} + 12 \alpha^6 - 10 \alpha^{10}) + O\left(\frac{1}{n}(4 \alpha^{52} + 2 \alpha^{15} - 7 \alpha^{34} \right. \\
& + 7 \alpha^{49} + 10 \alpha^{41} + 8 \alpha^{26} - 3 \alpha^{51} + 3 \alpha^{45} + 3 \alpha^{36} - 10 \alpha^{27} - 10 \alpha^{37} + 8 \alpha^{47} - 14 \alpha^{46} \\
& + \alpha^7 + 10 \alpha^{16} - 9 \alpha^{22} - 5 \alpha^{13} + 7 \alpha^{33} + 6 \alpha^{68} - 7 \alpha^{65} + 2 \alpha^{18} - 2 \alpha^5 + \alpha^8 - 7 \alpha^{32} \\
& - 5 \alpha^{61} + 2 \alpha^{60} - 6 \alpha^{20} + 3 \alpha^{17} + 12 \alpha^{31} + 2 \alpha^{40} - 9 \alpha^{44} + 4 \alpha^{66} - \alpha^{38} - 6 \alpha^{28} \\
& - 9 \alpha^{50} + 3 \alpha^{23} + 8 \alpha^{25} - \alpha^{21} - 2 \alpha^{12} + 7 \alpha^{64} - 12 \alpha^{67} + 6 \alpha^{70} + 12 \alpha^{58} - 6 \alpha^{19} \\
& + 7 \alpha^{24} - 7 \alpha^{56} + 10 \alpha^{53} + 4 \alpha^{59} - 2 \alpha^{57} + 7 \alpha^{54} - 12 \alpha^{55} + 3 \alpha^{42} + 6 \alpha^{39} + 2 \alpha^2 + \alpha^3 \\
& + \alpha - 3 \alpha^{14} - \alpha^{63} + \alpha^9 - 8 \alpha^{29} - 2 \alpha^4 + 5 \alpha^{30} + \alpha^{11} - 2 \alpha^{43} - \alpha^{35} - 3 \alpha^{69} - \alpha^6 \\
& \left. + \alpha^{10})^n\right)
\end{aligned}$$

Numerical value:

> **evalf(%)**;

$$2.042160079 \ 1.303577270^{-1 \cdot \ln(e^{-1} \cdot n)} + O\left(\frac{1.303577270^n}{n}\right)$$

> **map(simplify,%) assuming n::posint;**

$$2.042160079 e^{0.2651122315 n} + O\left(\frac{e^{0.2651122315 n}}{n}\right)$$

Meromorphic functions

Derangements

> **derangements := {S=Set(Cycle(Z, card>1))};**

$$\text{derangements} := \{S = \text{Set}(\text{Cycle}(Z, 1 < \text{card}))\}$$

> **combstruct[gfsolve](derangements, labelled, z);**

$$\left\{ Z(z) = z, S(z) = e^{\ln\left(\frac{1}{1-z}\right) - z} \right\}$$

> **der:=simplify(subs(%,S(z)));**

$$\text{der} := -\frac{e^{-z}}{z-1}$$

> **infsing(der, z);**

$$[[1], \text{polar}, \text{false}]$$

asymptotic number:

> **equivalent(der, z, n);**

$$e^{-1} + O\left(\frac{1}{n}\right)$$

Surjections

> **surjections := {S=Sequence(Set(Z, card>0))};**

```

surjections := {S = Sequence(Set(Z, 0 < card)) }
> combstruct[gfsolve](surjections, labelled, z);

$$\left\{ Z(z) = z, S(z) = -\frac{1}{-2 + e^z} \right\}$$

> surj := subs(% , S(z));

$$surj := -\frac{1}{-2 + e^z}$$

> infsing(surj, z);
[[ln(2)], polar, false]
asymptotic number
> equivalent(surj, z, n);

$$\frac{1}{2} \frac{(e^{-n})^{\ln(\ln(2))}}{\ln(2)} + O\left(\frac{(e^{-n})^{\ln(\ln(2))}}{n}\right)$$

> map(simplify, %) assuming n::posint;

$$\frac{1}{2} \ln(2)^{-1-n} + O\left(\frac{\ln(2)^{-n}}{n}\right)$$

Bernoulli numbers
> infsing(z/(exp(z)-1), z);
[[-2 I pi, 2 I pi], polar, false]

```

Iterative constructions

Binary trees of cycles of cycles

```

> btcc := {S = Union(CC, Prod(S, S)), CC = Cycle(Cycle(Z))};
btcc := {S = Union(CC, Prod(S, S)), CC = Cycle(Cycle(Z)) }
> combstruct[gfsolve](btcc, labelled, z);

$$\left\{ Z(z) = z, S(z) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \ln\left(-\frac{1}{-1 + \ln\left(-\frac{1}{z-1}\right)}\right)}, CC(z) = \ln\left(-\frac{1}{-1 + \ln\left(-\frac{1}{z-1}\right)}\right) \right\}$$

> btcc := subs(% , S(z));

$$btcc := \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \ln\left(-\frac{1}{-1 + \ln\left(-\frac{1}{z-1}\right)}\right)}$$

> infsing(btcc, z);

```

$$\left[\left[\frac{\frac{1}{e^4} - 1}{e^{\frac{1}{e^4}} - 1}, \frac{1}{e^{\frac{1}{e^4}}} \right], \text{algebraic, false} \right]$$

Singularity analysis

Binary trees:

> **equivalent((1-sqrt(1-4*z))/2/z, z, n, 5);**

$$\begin{aligned} & \frac{\left(\frac{1}{n}\right)^{3/2} (e^{-n})^{-2\ln(2)}}{\sqrt{\pi}} - \frac{9}{8} \frac{\left(\frac{1}{n}\right)^{5/2} (e^{-n})^{-2\ln(2)}}{\sqrt{\pi}} + \frac{145}{128} \frac{\left(\frac{1}{n}\right)^{7/2} (e^{-n})^{-2\ln(2)}}{\sqrt{\pi}} \\ & - \frac{1155}{1024} \frac{\left(\frac{1}{n}\right)^{9/2} (e^{-n})^{-2\ln(2)}}{\sqrt{\pi}} + \frac{36939}{32768} \frac{\left(\frac{1}{n}\right)^{11/2} (e^{-n})^{-2\ln(2)}}{\sqrt{\pi}} \\ & + O\left(\left(\frac{1}{n}\right)^{13/2} (e^{-n})^{-2\ln(2)}\right) \end{aligned}$$

Cayley trees:

> **Cayley:={T=Prod(Z, Set(T))};**

$$\text{Cayley} := \{T = \text{Prod}(Z, \text{Set}(T))\}$$

> **combstruct[gfsolve](Cayley, labelled, z);**

$$\{Z(z) = z, T(z) = -\text{LambertW}(-z)\}$$

> **equivalent(subs(%, T(z)), z, n);**

$$\frac{1}{2} \frac{\sqrt{2} \sqrt{e} \sqrt{e^{-1}} \left(\frac{1}{n}\right)^{3/2} e^n}{\sqrt{\pi}} + O\left(\frac{e^n}{n^2}\right)$$

> **map(simplify, %) assuming n::posint;**

$$\frac{1}{2} \frac{\sqrt{2} e^n}{\sqrt{\pi} n^{3/2}} + O\left(\frac{e^n}{n^2}\right)$$

Binary trees of cycles of cycles

> **equivalent(btcc, z, n):**

> **map(simplify, %) assuming n::posint;**

$$\frac{1}{2} \frac{\left(e^{\left(\frac{1}{e^4} - 1\right)e^{-\frac{1}{4}}} - 1\right)^{\frac{1}{2} - n} e^{\frac{1}{8} \left(e^{\frac{1}{4}} + 8ne^{\frac{1}{4}} - 8n\right)e^{-\frac{1}{4}}}}{\sqrt{\pi} n^{3/2}}$$

$$+ O\left(\frac{e^{-\left(-e^{\frac{1}{4}} + 1 + \ln\left(e\left(e^{\frac{1}{4}} - 1\right)e^{-\frac{1}{4}} - 1\right)e^{\frac{1}{4}}\right)e^{-\frac{1}{4}} n}}{n^{5/2}}\right)$$

Saddle-point method

Sets

> **equivalent(exp(z), z, n);**

$$\frac{1}{2} \frac{\sqrt{2} \sqrt{\frac{1}{n}} e^n n^{-n}}{\sqrt{\pi}} + O\left(\left(\frac{1}{n}\right)^{3/2} e^n n^{-n}\right)$$

Involutions

> **equivalent(exp(z+z^2/2), z, n);**

$$\frac{1}{2} \frac{e^{-\frac{1}{4}} \sqrt{\frac{1}{n}} e^{\sqrt{\frac{1}{n}}} \sqrt{n^{-n}}}{\sqrt{\pi} \sqrt{e^{-n}}} + O\left(\frac{e^{\frac{\cos\left(\left(\frac{1}{4} - \frac{1}{4 \operatorname{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}} \sqrt{n^{-n}}}{n \sqrt{e^{-n}}}\right)$$

> **map(simplify, %) assuming n::posint;**

$$\frac{1}{2} \frac{e^{-\frac{1}{4} + \sqrt{n} + \frac{1}{2} n} n^{-\frac{1}{2} - \frac{1}{2} n}}{\sqrt{\pi}} + O\left(e^{\sqrt{n} + \frac{1}{2} n} n^{-1 - \frac{1}{2} n}\right)$$

An example with a singularity at finite distance

> **equivalent(exp(z/(1-z)), z, n);**

$$\frac{1}{8} \frac{\sqrt{2} e^{-\frac{1}{2}} 4^{3/4} \left(\frac{1}{n}\right)^{3/4} e^{\sqrt{\frac{1}{n}}}}{\sqrt{\pi}} + O\left(\left(\frac{1}{n}\right)^{5/4} e^{\frac{2 \cos\left(\left(\frac{1}{4} - \frac{1}{4 \operatorname{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}}\right)$$

Set partitions

> **equivalent(exp(exp(z)-1), z, n);**

The saddle point is λ , LambertW(n + 1)

Saddle point's expansion:

$$\ln(n) - \ln(\ln(n)) + \frac{\ln(\ln(n))}{\ln(n)} + O\left(\frac{\ln(\ln(n))^2}{\ln(n)^2}\right)$$

$$\frac{1}{2} \frac{\sqrt{2} e^{-1} \sqrt{e^{-\text{saddlepoint}}} e^{\text{saddlepoint}}}{\text{saddlepoint}^n \sqrt{\pi} \text{saddlepoint}} + O\left(\frac{\sqrt{e^{-\text{saddlepoint}}} e^{\text{saddlepoint}}}{\text{saddlepoint}^2 \text{saddlepoint}^n}\right)$$

>