MULTIPLE BINOMIAL SUMS

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Multiple Binomial Sums

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$ Dixon

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2$$ Andrews - Paule

$$\sum_{s=0}^{k}\sum_{b\geq 0} (-1)^b \binom{s}{b}\binom{k-s}{2v-b}\binom{k-2v}{s-b} = \binom{k-v}{k-2v} 2^{k-2v}$$ Dent

$$\sum_{r,s} (-1)^{n+r+s} \binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n} = \sum_k \binom{n}{k}^4$$ Petkovšek - Wilf - Zeilberger?

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l}\binom{s-r}{m-n-l}$$ Graham Knuth Patashnik.

Aims: 1. Prove them automatically
2. Find the RHS given the LHS.

APPROACH

$U_n = \sum$

$U(z) = \sum_{n \geq 0} U_n z^n = \sum_n \sum$

$U(z) = \frac{1}{(2\pi i)^k} \oint\!\!\oint \text{rat fcn}$

($|z|$ small enough)

$U(z) = \frac{1}{(2\pi i)^{k'}} \oint\!\!\oint \text{rat fcn} \; (k' < k)$

$\sum a_i(z) U^{(i)}(z) = 0$

$\sum b_i(n) U_{n+i} = 0.$

1. Take the generating series
2. Compute an integral representation
3. Take the contour into account to simplify (optional in theory)
4. Compute a differential equation
5. Translate into a recurrence for U_n

EXAMPLE: APÉRY'S SEQUENCE

$$U_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

1. Take the generating series
2. Compute an integral representation
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1. $U(z) = \sum_{\substack{0 \le k \le n \\ 0 \le n}} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n$

2. $U(z) = \frac{1}{(2\pi i)^6} \oint \cdots$ (|z| small enough)

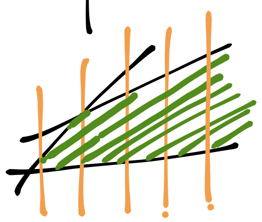
3. $$U(z) = \frac{1}{(2\pi i)^3} \oint \frac{dt_1 \wedge dt_2 \wedge dt_3}{t_1 t_2 t_3 (1 - t_1 t_2 - t_1 t_2 t_3) - (1 + t_1)(1 + t_2)(1 + t_3) z}$$

4. $z(z^2 - 34z + 1) U'''(z) + \cdots + (z-5) U(z) = 0$

5. $(n+2)^3 U_{n+2} - (2n+3)(17n^2 + 51n + 39) U_{n+1} + (n+1)^3 U_n = 0$

MAIN RESULT

Def. $u_{\underline{k},\underline{n}}$ defined over $\mathbb{N}^{m+l}$ has **finite support wrt $\underline{n}$** if for all $\underline{n} \in \mathbb{N}^{l}$,

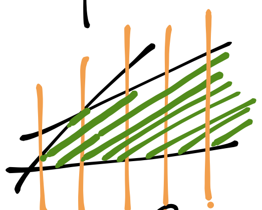


$$\{\underline{k} \in \mathbb{N}^{m} \mid u_{\underline{k},\underline{n}} \neq 0\} \text{ is finite.}$$

MAIN RESULT

Def. $u_{\underline{k},\underline{n}}$ defined over $\mathbb{N}^{m+d}$ has finite support wrt $\underline{n}$
if for all $\underline{n}\in\mathbb{N}^d$,

$$\{\underline{k}\in\mathbb{N}^m \mid u_{\underline{k},\underline{n}}\neq 0\} \text{ is finite.}$$



thm. The sum

$$\sum_{\underline{k} \mid (\underline{k},n)\in K_{\underline{1},n}} \binom{\underline{a_1}\cdot\underline{k}+c_1 n+e_1}{\underline{b_1}\cdot\underline{k}+d_1 n+f_1}\times\cdots\times\binom{\underline{a_r}\cdot\underline{k}+c_r n+e_r}{\underline{b_r}\cdot\underline{k}+d_r n+f_r}\times g_{\underline{k},n}$$

satisfies a linear recurrence of order and degree $\Delta^{O(r+m)}$

in progress: that can be computed in $\Delta^{O(r+m)}$ operations $\left(\Delta \leq \sum |a_i|+|b_i|+|c_i|+\dots+\deg P\right)$

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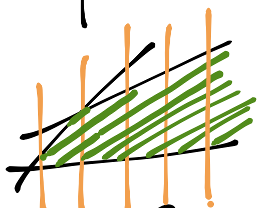

thm. The sum

$$\sum_{\underline{k} \mid (\underline{k},n)\in \boxed{K_{\underline{k},n}}} \binom{\underline{a_1}\cdot\underline{k}+c_1 n+e_1}{\underline{b_1}\cdot\underline{k}+d_1 n+f_1}\times\cdots\times\binom{\underline{a_r}\cdot\underline{k}+c_r n+e_r}{\underline{b_r}\cdot\underline{k}+d_r n+f_r}\times \boxed{g_{\underline{k},\underline{n}}}$$

integer coefficients

coeffs of an algebraic power series ($P(\underline{x},y,G)=0$)

rational polyhedron s.t. $\mathbb{1}_{K_{\underline{k},n}}$ has finite support wrt n

satisfies a linear recurrence of order and degree $\Lambda^{O(r+m)}$

in progress: that can be computed in $\Lambda^{O(r+m)}$ operations $\left(\Lambda \leq \sum |a_i|+|b_i|+|c_i|+\dots+\deg P\right)$

PREVIOUS WORKS

SIMPLE SUMS

- Zeilberger's algorithm (1990)
- Yen (1993) better bounds on order
  (genericity assumption)

MULTIPLE INTEGRALS

- Picard (1906). Simple & double ∫rat.
- Dwork-Griffiths ~ 60's
  general case (rational ∫)
- Zeilberger (1990) Holonomic ∫
- Us (2013) rational ∫:
  bounds on order & degree
  (good) complexity estimate.

MULTIPLE SUMS

- Wilf-Zeilberger (1992)
  + bound on order
- Apagodu-Zeilberger (2006)
  better bounds (genericity assumption)
- Wegschaider (1997)
  improvements & implementation
- Chyzak (2000)
  generalization of Zeilberger 90.

GENERATING SERIES

- Egorychev (1984).
  Encoding into integral
  + add hoc techniques

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GENERATING SERIES

- Egorychev (1984). Encoding into integral + old the techniques

Use creative telescoping

APÉRY'S SEQUENCE BY CREATIVE TELESCOPING

$$u_n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{f_{n,k}}$$

1. Find a telescoping relation (Zeilberger's algorithm)

$$(n+2)^3 f_{n+2,k} + \cdots + (n+1)^3 f_{n,k} = g_{n,k+1} - g_{n,k} \longleftarrow \text{certificate}$$

with $$g_{n,k} = \frac{\mathrm{Pol}(n,k)}{(n+1-k)^2(n+2-k)^2} f_{n,k}$$

2. Sum over k and telescope (with care).

$$(n+2)^3 u_{n+2} + \cdots + (n+1)^3 u_n = 0.$$

AN IDENTITY OF ANDREWS & PAULE

difficult for creative telescoping

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\underbrace{\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i}}_{f_{i,j,n}} = (2n+1)\binom{2n}{n}^2$$

$$\boxed{\text{Needed: } \sum a_k(n) f_{i,j,n+k} = (g_{i+1,j,n} - g_{i,j,n}) + (h_{i,j+1,n} - h_{i,j,n})}$$

- Andrews & Paule (93): two proofs with human insight;
- Wegschaider (97): via a generalization of Sister Celine's technique;
- Koutschan's code (2010).

$$f_{i,j,n} = (g_{i+1,j,n} - g_{i,j,n}) + (h_{i,j+1,n} - h_{i,j,n})$$

g, h are $\dfrac{\mathrm{pol}(i,j,n)}{(1+i)(1+j)(i+j-2n)} f_{i,j,n}$

$\leadsto$ ... more work ... $\to$ result.

An Identity of Andrews & Paule

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2$$

1. $F(z) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} f_{i,j}\, z^n$

2. $= \frac{1}{(2\pi i)^5}\oint\!\!\oint$

3. $= \frac{1}{(2\pi i)^2}\oint \frac{u_2(1+u_2)^2\, du_1 \wedge du_2}{(u_2^2 - z(1+u_2)^4)(u_1(u_2^2+2u_2-u_1) - z(1+u_1)(1+u_2)^4)}$

4. $F^{(6)}(z) + \cdots + (\ \ ) F(z) = 0$

5. $S_{n+4} + \cdots + (\ \ ) S_n = 0$ + initial conditions

$\longrightarrow$ (Petkovšek's algorithm) $\quad S_n = (2n+1)\binom{2n}{n}^2$

1. Take the generating series
2. Compute an integral representation
3. Take the contour into account to simplify (optional in theory)
4. Compute a differential equation
5. Translate into a recurrence for U_n

No human help. Total time < 2 sec. (in step 4).

CONCLUSION: COMPARISON WITH CREATIVE TELESCOPING

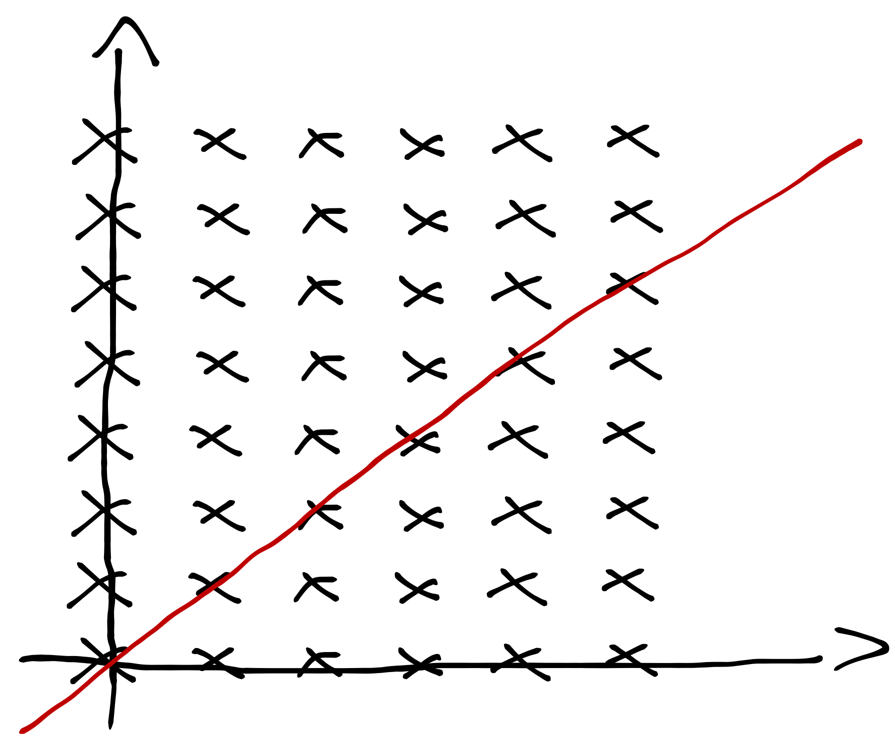
Our approach:

- avoids discussions about summing the certificate;
- deals with multiple sums very easily;
- may lead to smaller order recurrences;
- keeps size & complexity under control;
- is restricted to a specific (but common) class of sums.

I

From Binomial Sums to Diagonals

$$\sum \binom{\ }{\ }\binom{\ }{\ }\cdots\binom{\ }{\ } \Longrightarrow$$



DIAGONALS

Def. $F = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} x^i y^j$ a formal power series with a_{ij} in a ring $\mathbb{A}$, its **diagonal** is

$$\Delta_{x,y} F = \sum_i a_{ii} x^i \in \mathbb{A}[[x]].$$

Note: $\mathbb{A}$ can be a ring of formal power series in other variables.

Application: **Hadamard product** of $F(x) = \sum a_i x^i$ and $G(x) = \sum b_i x^i$:

$$F \odot G(x) := \sum a_i b_i x^i = \Delta_{x,y} F(x) G(y).$$

ENCODING

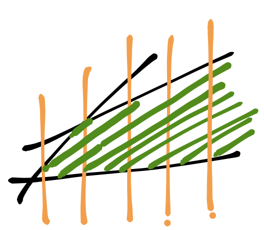
Our sum will be encoded as the diagonal of a rational function:

$$\sum_{\underbrace{\underline{k} \mid (\underline{k},n)\in K_{\underline{1},n}}_{\text{STEP 3}}} \underbrace{\left(\frac{\underline{a_1}\cdot\underline{k}+c_1 n+e_1}{\underline{b_1}\cdot\underline{k}+d_1 n+f_1}\right)\times\cdots\times\left(\frac{\underline{a_r}\cdot\underline{k}+c_r n+e_r}{\underline{b_r}\cdot\underline{k}+d_r n+f_r}\right)}_{\text{STEP 1 (easy)}}\times \underbrace{g_{\underline{k},n}}_{\text{STEP 2}}$$

STEP 3

Brion's formula

+ Hadamard product



STEP 1 (easy)

STEP 2

Denef &

Lipshitz

+ Hadamard prod

GENERATING SERIES FOR BINOMIAL COEFFICIENTS

DEF$^{\underline{N}}$ $$(1+x)^a = \sum_{\ell \geq 0} \binom{a}{\ell} x^\ell$$

GEN FUNC $$\frac{1}{1-t(1+x)^a} = \sum_{\substack{\ell \geq 0 \\ k \geq 0}} \binom{ak}{\ell} x^\ell t^k$$

SELECTION ($\Delta_{x,y}$) $$\frac{1}{1-t(1+x)^a y^b} = \sum_{k,\ell} \binom{ak}{\ell} x^\ell y^{bk} t^k \qquad (b \geq 0)$$

MULTIPLE BINOMIALS $$\frac{1}{1-t(1+x_1)^a(1+x_2)^b} = \sum_{k,\ell_1,\ell_2} \binom{ak}{\ell_1}\binom{bk}{\ell_2} x_1^{\ell_1} x_2^{\ell_2} t^k$$

LINEAR FORM $$\frac{1}{(1-t_1(1+x)^a)(1-t_2(1+x)^b)} = \sum_{k_1,k_2,\ell} \binom{ak_1+bk_2}{\ell} x^\ell t_1^{k_1} t_2^{k}$$

COMBINE ALL $\longrightarrow$

Generating Series for Binomial Coefficients

Prop. With $e_j, f_j, b_{j,i}$ relative integers,

$$\sum_{\substack{0\le k_1,\dots,0\le k_m\\ 0\le \ell_1,\dots,0\le \ell_r}} \left(\prod_{j=1}^r \binom{\sum_i a_{j,i}k_i + e_j}{\ell_j} x_j^{\ell_j+\sum_i b_{j,i}^- k_i + f_j^-} y_j^{\sum_i b_{j,i}^+ k_i + f_j^+} \prod_{i=1}^m t_i^{k_i}\right)$$

$$= \frac{\prod_{j=1}^r (1+x_j)^{e_j} x_j^{f_j^-} y_j^{f_j^+}}{\prod_{i=1}^m \left(1 - t_i \prod_{j=1}^r (1+x_j)^{a_{j,i}} x_j^{b_{j,i}^-} y_j^{b_{j,i}^+}\right)} =: \Phi(\underline{x}, \underline{y}, \underline{t}),$$

where $x^+ := \max(0, x)$, $x^- := \max(0, -x)$. $(x^+ - x^- = x)$

The binomial part of our sum is $\left(\prod_j \Delta_{x_j, y_j}\right) \cdot \Phi \Big|_{\substack{t_1 = \cdots = t_m = 1\\ x_1 = \cdots = x_r = 1}}$.

ALGEBRAIC SERIES

Hyp. $\begin{cases} F(x_1,\dots,x_k) \in \mathbb{K}[[x_1,\dots,x_k]] & \text{char } 0 \\ P(\underline{x}, F(\underline{x})) = 0, & P \in \mathbb{K}[\underline{x}, y] \\ F(0)=0, \ P_y(0,0) \neq 0 & \end{cases}$

Furstenberg ($k=1$): $F(x) = \Delta_{x,y}\, y^2 \dfrac{P_y(xy,y)}{P(xy,y)}$ (by residues).

Denef & Lipshitz: F is the diagonal of a rational series in $2k$ variables.

Lemma. With Hyp., we get

$$F(x_1u, x_2u, \dots, x_ku) = \Delta_{u,y} H(x_1u, \dots, x_ku, y),$$

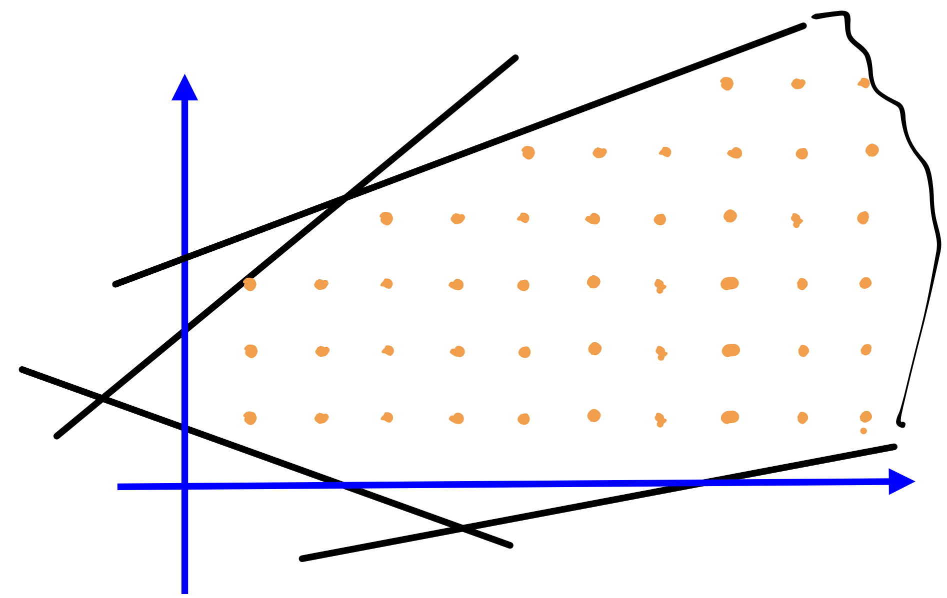
where $H(\underline{x}, y) = y^2 \dfrac{P_y(x_1y, \dots, x_ky, y)}{P(x_1y, \dots, x_ky, y)}$.

(only $k+2$ variables, u can be specialized later).

RATIONAL POLYHEDRA

K a rational polyhedron in $\mathbb{R}^d$

(defined by finitely many $\sum_{j=1}^{d} \underbrace{\alpha_{ij}}_{\in\mathbb{Z}} x_j \leq \underbrace{\beta_j}_{\in\mathbb{Z}}$)



Thm (Brion 1988) $\Psi_{K\cap\mathbb{N}^d}(\underline{x}) := \sum_{m\in K\cap\mathbb{N}^d} x_1^{m_1}\cdots x_d^{m_d}$ is a rational series.

It decomposes nicely as :

$$\sum_{v\in \mathrm{Vert}(K)} \Psi_{\mathbb{Z}^d \cap \text{Cone seen from } v}(\underline{x})$$

Algorithm in small complexity by Barvinok (1994).

ENCODING

Our sum will be encoded as the diagonal of a rational function:

$$\sum_{\underbrace{\underline{k} \mid (\underline{k},n)\in K_{\underline{t},n}}_{\text{STEP 3}}} \underbrace{\left(\frac{\underline{a_1}\cdot\underline{k}+c_1 n+e_1}{\underline{b_1}\cdot\underline{k}+d_1 n+f_1}\right)\times\cdots\times\left(\frac{\underline{a_r}\cdot\underline{k}+c_r n+e_r}{\underline{b_r}\cdot\underline{k}+d_r n+f_r}\right)}_{\text{STEP 1 (easy)}} \times \underbrace{g_{\underline{k},n}}_{\text{STEP 2}}$$

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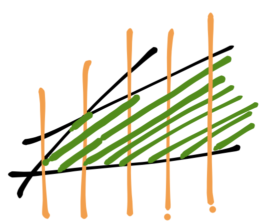


STEP 1 (easy)

STEP 2

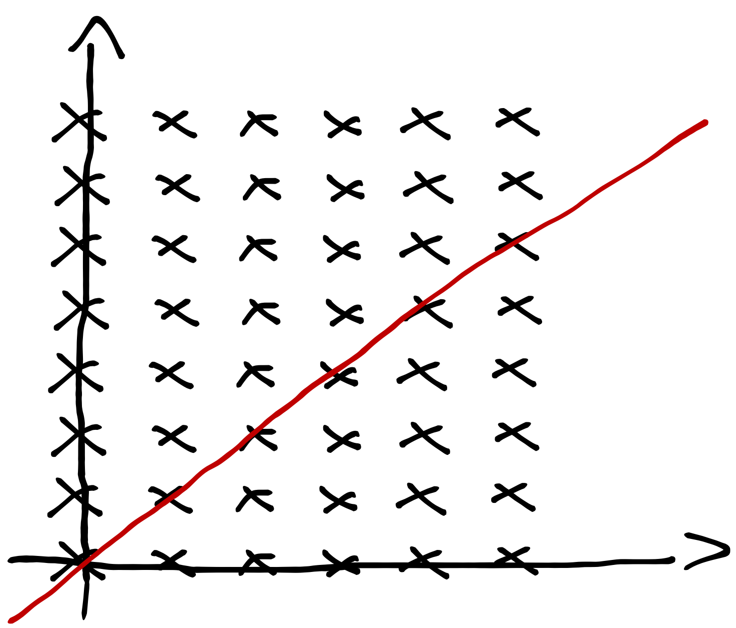
Denef < Lipshitz

+ Hadamard prod

$$\boxed{\Delta_{\underline{x},\underline{y}}\, \Phi(\underline{x},\underline{y},\underline{t},z) \odot_{\underline{t},\underline{u}} G(\underline{u},z) \odot_{\underline{u},\underline{v}} \Psi_{K\cap\mathbb{N}^d}(\underline{v},z) \Big|_{\substack{\underline{t}=1\\ \underline{x}=1,\\ \underline{u}=1}}}$$

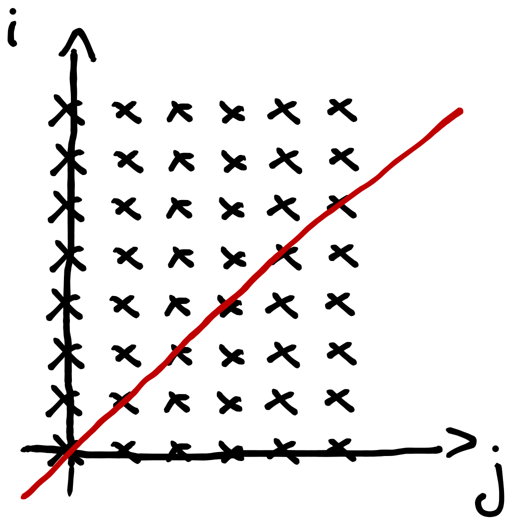
II.

FROM DIAGONALS TO INTEGRALS

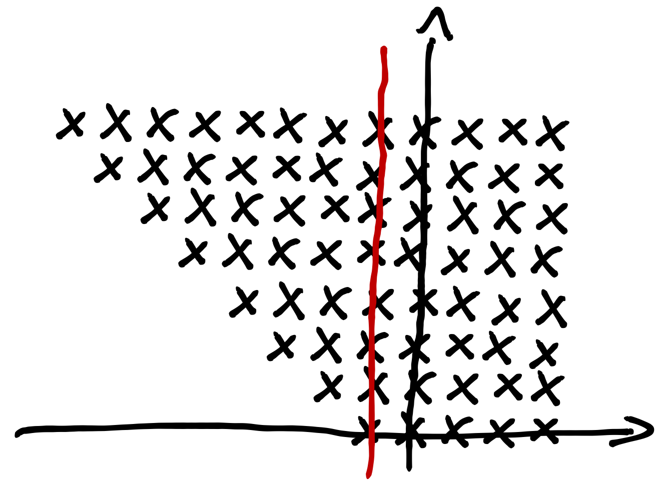


$$\Longrightarrow \oiiint \frac{(\quad)}{(\quad)} .$$

DIAGONALS AS RESIDUES



$F(x,y)=\sum a_{ij}x^iy^j$



$\frac{1}{y}F\left(\frac{x}{y},y\right)$

Lemma: If $F=\sum a_{ij}x^iy^j$ is convergent in a neighborhood of $|x|<r$, $|y|<r'$ of 0, then for any $\varepsilon\in(0,1)$,

$$\Delta_{x,y}F=\sum_i a_{ii}\,x^i=\frac{1}{2\pi i}\oint_{|y|=(1-\varepsilon)r'}F\left(\frac{x}{y},y\right)\frac{dy}{y}\quad\text{for } |x|<rr'(1-\varepsilon).$$

(classical).

DOMAINS OF CONVERGENCE

$$\boxed{\Delta_{\underline{x},\underline{y}}\, \Phi(\underline{x},\underline{y},\underline{t},z) \odot_{\underline{t},\underline{u}} G(\underline{u},z) \odot_{\underline{u},\underline{v}} \Psi(K\cap\mathbb{N}^d)(\underline{v},z)\Big|_{\substack{\underline{t}=1,\\ \underline{x}=1,\\ \underline{u}=1}}}$$

1. Brion's series. Support finite with respect to n $\Rightarrow$

   For arbitrary positive $\underline{r}$, there exists $\rho>0$ s.t.

   $\Psi_{K\cap\mathbb{N}^d}(\underline{v},z)$ converges for $|\underline{v}|<\underline{r}$, $|z|<\rho$.

2. Algebraic Series. $P_y(0,0)\neq 0$ $\Rightarrow$ convergent in a neighborhood of 0.

3. Binomials

$$\frac{\prod_{j=1}^{r}(1+x_j)^{e_j}x_j^{f_j^-}\,y_j^{f_j^+}}{\prod_{i=1}^{m}\left(1-t_i\prod_{j=1}^{r}(1+x_j)^{a_{j,i}}x_j^{b_{j,i}^-}y_j^{b_{j,i}^+}\right)}$$

$\rightarrow$ convergent for $|\underline{y}|<\underline{R}$ arbitrary, $|\underline{x}|<1$, $|\underline{t}|<r(\underline{R})$

ALL TOGETHER $\longrightarrow$

RESULT: INTEGRAL REPRESENTATION

Hypotheses and notation as before

Thm There exist positive $\underline{r}_1, \underline{r}_2, r_3, \rho$ and a rational function R s.t.

$$\sum_{n \geq 0} \sum_{(\underline{k}, n) \in K_{\underline{1}, n}} \binom{\underline{a}_1 \cdot \underline{k} + c_1 n + e_1}{\underline{b}_1 \cdot \underline{k} + d_1 n + f_1} \times \cdots \times \binom{\underline{a}_r \cdot \underline{k} + c_r n + e_r}{\underline{b}_r \cdot \underline{k} + d_r n + f_r} \times g_{\underline{k}, n} \, z^n =$$

$$\frac{1}{(2\pi i)^{m+r+1}} \oint_{\substack{|\underline{y}| = \underline{r}_1 \\ |\underline{v}| = \underline{r}_2 \\ |\tau| = r_3}} R(\underline{y}, \underline{v}, \tau, z) \, \frac{d\underline{y} \, d\underline{v} \, d\tau}{\underline{y} \cdot \underline{v} \cdot \tau}$$

for all $|z| < \rho$.

Works for multiple z's and multivariate generating series.

III.

FROM INTEGRALS TO DIFFERENTIAL EQUATIONS

$$\oiiint \frac{(\quad)}{(\quad)} \longrightarrow y^{(m)}(z) + \cdots + (\quad)\, y(z) = 0$$

TWO VARIABLES: HERMITE NORMAL FORM

$$I(t) = \oint \frac{P(t,x)}{\phi^m(t,x)}\, dx \qquad \phi \text{ square-free}$$

over a cycle where $\phi \neq 0$

If $m=1$, Euclidean division: $P = a\phi + r$ $\quad \deg r < \deg \phi$

$$\frac{P}{\phi} = \frac{r}{\phi} + \partial_x \int a$$

Def: reduced form $\left[\frac{P}{\phi}\right] := \left[\frac{r}{\phi}\right]$.

If $m>1$, Bézout identity and integration by parts

$$P = u\phi + v\,\partial_x\phi \longrightarrow \frac{P}{\phi^m} = \underbrace{\frac{u + \frac{1}{m-1}\partial_x v}{\phi^{m-1}}}_{A_{m-1}} + \partial_x \frac{v/(1-m)}{\phi^{m-1}}$$

$$\left[\frac{P}{\phi^m}\right] := \left[A_{m-1}\right]$$

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$$P = u\Phi + v\,\partial_x\Phi \longrightarrow \frac{P}{\Phi^m} = \underbrace{\frac{u + \frac{1}{m-1}\partial_x v}{\Phi^{m-1}}}_{A_{m-1}} + \partial_x \frac{v/(1-m)}{\Phi^{m-1}}$$

$$\left[\frac{P}{\Phi^m}\right] := \left[A_{m-1}\right]$$

Confined in a vector space of dim $\leq \deg \Phi$

Algorithm: $R_0 := [P/\Phi^m]$

for $i = 1, 2, \dots$

$\quad R_i := [\partial_t R_{i-1}]$;

$\quad$ If there exists a relation $c_0(t)R_0 + \dots + c_i(t)R_i = 0$

$\quad\quad$ return $c_0 + c_1\partial + \dots + c_i\partial^i$.

MORE VARIABLES: GRIFFITHS-DWORK REDUCTION

$$I(t) = \oint \frac{P(t,\underline{x})}{\varphi^m(t,\underline{x})}\, d\underline{x} \qquad \varphi \text{ square-free}$$

over a cycle where $\varphi \neq 0$

1. Control degrees by homogenizing $(x_1, -, x_n) \mapsto (x_0, -, x_n)$
2. If $m=1$, $[P/\varphi] := [P/\varphi]$
3. If $m>1$, reduce modulo Jacobian ideal $\langle \partial_0 \varphi, -, \partial_n \varphi \rangle$:

$$P = r + v_0 \partial_0 \varphi + \cdots + v_n \partial_n \varphi \qquad \text{(via Gröbner bases)}$$

$$\frac{P}{\varphi^m} = \frac{r}{\varphi^m} - \frac{1}{m-1}\left(\partial_0 \frac{v_0}{\varphi^{m-1}} + \cdots + \partial_n \frac{v_n}{\varphi^{m-1}}\right) + \underbrace{\frac{1}{m-1}\,\frac{\partial_0 v_0 + \cdots + \partial_n v_n}{\varphi^{m-1}}}_{A_{m-1}} \qquad \text{(integration by parts)}$$

$$\left[\frac{P}{\varphi^m}\right] := \frac{r}{\varphi^m} + [A_{m-1}]$$

Not too big (no complexity)

thm (Griffiths) In the regular case ($\mathbb{Q}(t)[\underline{x}]/\langle \partial_0 \varphi, -, \partial_n \varphi \rangle$ finite dim),

if $R = P/\varphi^m$ hom. of degree $-n-1$, $[R]=0 \iff \oint_\gamma R\, d\underline{x} = 0$ for all γ where R is continuous.

$\rightarrow$ SAME ALGORITHM.

SIZE AND COMPLEXITY

$$I(t) = \oint \frac{P(t,\underline{x})}{\underbrace{Q(t,\underline{x})}_{\in \mathbb{Q}(t, x_1 - x_n)}}\, d\underline{x}$$

no regularity assumed

$$N := \deg_x Q \ (\geq n+1+\deg_x P), \quad d_t := \max(\deg_t Q, \deg_t P)$$

thm (Bostan, Lairez, S. 2013) A linear differential equation for $I(t)$ can be computed in $O(e^{3n} N^{8n} d_t)$ operations in $\mathbb{Q}$. It has order $\leq N^n$, degree $O(e^n N^{3n} d_t)$.

optimal

might be reducible to N^{2n}.

Note: generically, the certificate has at least $N^{n^2/2}$ monomials.

Non-regular case: by deformation. Better way: Lairez 2014.

IV.

GEOMETRIC REDUCTION

$$\frac{1}{(2\pi i)^k} \oint\!\!\oint \frac{(\quad)}{(\quad)} \longrightarrow \frac{1}{(2\pi i)^{k'}} \oint\!\!\oint \frac{(\quad)}{(\quad)} \qquad \underline{k' < k}.$$

MOTIVATION

What we have:

$$\sum (\ )(\ )(\ )\cdots(\ ) \longrightarrow \oint\oint \frac{(\ )}{(\ )} \longrightarrow \text{linear diff. equation}$$

Does not take the particular cycle into account.

cost suffers from high dimension

Geometric reduction: simplify the integral, paying attention to the cycle of interest.

Technique: detect rational residues.

EXAMPLE

$$D(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{dk}{n}=(-d)^n. \quad (d\in\mathbb{N}) \qquad [\text{GKP}, (5.43)]$$

1. Integral representation

$$\sum_{n\geq 0} D_n z^n = \frac{1}{2\pi i}\oint \frac{dy}{y+z(1+y)^d - z}$$

2. Poles when $z\to 0$: one tends to 0, the other ones to ∞

$\longrightarrow$ one residue $\quad \sum D_n z^n = \dfrac{1}{1+dz}.$

Creative telescoping produces a recurrence of order d.

METHOD

1. Pick an order on the variables of integration compatible with

$$\frac{t_i}{u_i} \to 0, \quad \frac{u_i}{v_i} \to 0, \quad \frac{z}{\text{any var}} \to 0$$

2. For each integration variable w starting from the largest

- $F := \{ \text{factors}(\text{denominator}(R)) \}$;
- Split $F = F_0 \cup F_\infty \cup F_?$) By Newton polygon

  F_0: all roots tend to 0; F_∞: no root tends to 0; $F_?$: some of both

- If $F_? = \emptyset$, set $R = \sum \text{res}(F_0)$ (rational) and remove w from the list of variables

3. Specialize $\underline{y} = \zeta$, $\underline{x} = 1$ and return the remaining $\int R$.

EXAMPLE: DIXON'S IDENTITY

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3}$$

1. Encoding: $$\frac{t_1t_2t_3}{((1+t_1)^2(1+t_2)^2(1+t_3)^2z - t_1^2t_2^2t_3^2)\,(t_1t_2t_3 - t_4)(1+t_4)}.$$

2. Deal with t_4: only one relevant pole $\longrightarrow$

$$\frac{t_1t_2t_3}{((1+t_1)^2(1+t_2)^2(1+t_3)^2z - t_1^2t_2^2t_3^2)\,(1+t_1t_2t_3)}$$

3. t_3: 1st factor only. 2 poles tending to 0 $\longrightarrow$

$$\frac{t_1t_2}{(1+t_1)^2(1+t_2)^2(1-t_1t_2)^2z - t_1^2t_2^2}$$

4. Irreducible, degree 4, 2 poles $\to 0$ & 2 poles $\to \infty$: END of geometric reduction

5. Integral $\longrightarrow$ $z(27z+1)y'' + (54z+1)y' + 6y = 0.$

6. Translation $\longrightarrow$ $(n+1)^2u_{n+1} + 3(3n+1)(3n+2)u_n = 0.$

DEMONSTRATION

CONCLUSION

- An algorithm that is easy to implement (200 l Maple + 500 l Magma)
- Works well in practice
- Complexity under control
- Avoids discussions with certificates
- Restricted to binomial multisums as of now.

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Thank you.