Linear differential and recurrence equations viewed as data-structures

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Computer Algebra

Effective mathematics (what can we compute?)
Their complexity (how fast?)

Several million users. 30 years of algorithmic progress.

Thesis in this presentation:
linear differential and recurrence equations are a good data-structure.
Topics

Fast computation with high precision;
automatic proofs of identities;
«computation» of expansions,
of (multiple) integrals, of (multiple) sums.
The objects of study

**Def.** A power series is called **differentially finite (D-finite)** when it is the solution of a linear differential equation with polynomial coefficients.

**Exs:** sin, cos, exp, log, arcsin, arccos, arctan, arcsinh, hypergeometric series, Bessel functions, …

**Def.** A sequence is **polynomially recursive (P-recursive)** when it is the solution of a linear recurrence with polynomial coefficients.

**Prop.** \( f = \sum_{n=0}^{\infty} f_n z^n \) \( \text{D-finite} \Leftrightarrow f_n \text{P-recursive}. \)
Example

Coefficient of $X^{20000}$ in $P(X) = (1+X)^{20000}(1+X+X^2)^{10000}$?

Linear differential equation of order 1
→ linear recurrence of order 2
→ unroll (cleverly).

```markdown
> P := (1+x)^(2*N)*(1+x+x^2)^N:
> deq := gfun:-holexprtodiffeq(P,y(x)):
> rec := gfun:-diffeqtorec(%,y(x),u(k)):
> p := gfun:-rectoprocp(subs(N=10000,rec),u(k)):
> p(20000);
```

23982[...10590 digits...]33952

Total time: 0.5 sec
I. Fast computation at large precision

From large integers to precise numerical values
Fast multiplication

**Fast Fourier Transform** (Gauss, Cooley-Tuckey, Schönhage-Strassen). Two integers of $n$ digits can be multiplied with $O(n \log(n) \log\log(n))$ bit operations.

**Direct consequence** (by Newton iteration):

inverses, square-roots, … : same cost.
Binary Splitting for linear recurrences (70’s and 80’s)

• $n!$ by divide-and-conquer:

$$n! := n \times \cdots \times \lfloor n/2\rfloor \times (\lceil n/2 \rceil + 1) \times \cdots \times 1$$

Cost: $O(n \log^3 n \log \log n)$ using FFT

• linear recurrences of order 1 reduce to

$$p!(n) := (p(n) \times \cdots \times p(\lfloor n/2 \rfloor)) \times (p(\lceil n/2 \rceil + 1) \times \cdots \times p(1))$$

• arbitrary order: same idea, same cost (matrix factorial):

ex:

$$e_n := \sum_{k=0}^{n} \frac{1}{k!}$$

satisfies a 2nd order rec, computed via

$$\begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} n+1 & -1 \\ n & 0 \end{pmatrix} \begin{pmatrix} e_{n-1} \\ e_{n-2} \end{pmatrix} = \frac{1}{n!} A(n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

A(n)
Numerical evaluation of solutions of LDEs

**Principle:**

\[ f(x) = \sum_{n=0}^{N} a_n x^n + \sum_{n=N+1}^{\infty} a_n x^n \]

- fast evaluation
- good bounds

1. linear recurrence in \( N \) for the first sum (easy);
2. tight bounds on the tail (technical);
3. no numerical roundoff errors.

The technique used for fast evaluation of constants like

\[
\frac{1}{\pi} = \frac{12}{C^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! (A + nB)}{(3n)! n!^3 C^{3n}}
\]

with \( A=13591409, \ B=545140134, \ C=640320 \).
Analytic continuation

Compute \( f(x), f'(x), \ldots, f^{(d-1)}(x) \) as new initial conditions and handle error propagation:

**Ex:** \( \text{erf}(\pi) \) with 15 digits:

\[
\begin{align*}
0 & \xrightarrow{200 \text{ terms}} 3.1416 & 3.1415927 & \xrightarrow{6 \text{ terms}} 3.14159265358979 \\
18 & \xrightarrow{18 \text{ terms}} & & \\
& & & \\
& & & 
\end{align*}
\]

Again: computation on integers. No roundoff errors.
II. Proofs of Identities
Confinement

LDE $\iff$ the function and all its derivatives are confined in a finite dimensional vector space

$\Rightarrow$ the sum and product of solutions of LDEs satisfy LDEs

$\Rightarrow$ same property for P-recursive sequences
Proof technique

> series(sin(x)^2+cos(x)^2-1,x,4);

O(x^4)

Proofs of non-linear identities by linear algebra!

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: \( y'' + y = 0 \);
2. their squares and their sum satisfy a 3rd order LDE;
3. the constant -1 satisfies \( y' = 0 \);
4. thus \( \sin^2 + \cos^2 - 1 \) satisfies a LDE of order at most 4;
5. Cauchy’s theorem concludes.
Example: Mehler’s identity for Hermite polynomials

\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \exp \left( \frac{4u(xy-u(x^2+y^2))}{1-4u^2} \right) \frac{1}{\sqrt{1-4u^2}} \]

1. Definition of Hermite polynomials: recurrence of order 2;

2. Product by linear algebra: \( H_{n+k}(x)H_{n+k}(y)/(n+k)! \), \( k \in \mathbb{N} \) generated over \( \mathbb{Q}(x,n) \) by

\[
\begin{align*}
\frac{H_n(x)H_n(y)}{n!}, & \quad \frac{H_{n+1}(x)H_n(y)}{n!}, & \quad \frac{H_n(x)H_{n+1}(y)}{n!}, & \quad \frac{H_{n+1}(x)H_{n+1}(y)}{n!}
\end{align*}
\]

\( \rightarrow \) recurrence of order at most 4;

3. Translate into differential equation.
Dynamic Dictionary of Mathematical Functions

- User need
- Recent algorithmic progress
- Maths on the web

http://ddmf.msr-inria.inria.fr/
Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

This is release 1.9.1 of DDMF
Select a special function from the list

What's new? The main changes in this release 1.9.1, dated May 2013, are:
- Proofs related to Taylor polynomial approximations.

Release history.

More on the project:
- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project
- Article on the project at ICMS'2010
- Source code used to generate these pages
- List of related projects
Guess & prove continued fractions

1. Differential equation produces first terms (easy):

\[
\arctan x = \frac{x}{1 + \frac{1}{3}x^2 \frac{1}{1 + \frac{4}{15}x^2 \frac{1}{1 + \frac{9}{35}x^2 \frac{1}{1 + \cdots}}} \]

2. Guess a formula (easy): \( a_n = \frac{n^2}{4n^2 - 1} \)

3. Prove that the CF with these \( a_n \) satisfies the differential equation.

No human intervention needed.
Automatic Proof of the guessed CF

\[
\text{arctan} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots
\]

- **Aim**: RHS satisfies \((x^2+1)y’-1=0\);
- Convergents \(P_n/Q_n\) where \(P_n\) and \(Q_n\) satisfy the LRE \(u_n=u_{n-1}+a_nu_{n-2}\) (and \(Q_n(0)\neq0\));
- Define \(H_n:=(Q_n)^2((x^2+1)(P_n/Q_n)’-1)\);
- \(H_n\) is a polynomial in \(P_n, Q_n\) and their derivatives;
- therefore, it satisfies a LRE that can be computed;
- from it, \(H_n=O(x^n)\) visible
- from there, \((P_n/Q_n)’-1/(1+x^2)=O(x^n)\) too;
- **conclude** \(P_n/Q_n\to\text{arctan} by integrating.\)
III. Ore Polynomials
From equations to operators

<table>
<thead>
<tr>
<th>$D_x$ ↔ $d/dx$</th>
<th>$S_n$ ↔ $(n\mapsto n+1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ ↔ mult by $x$</td>
<td>$n$ ↔ mult by $n$</td>
</tr>
<tr>
<td>product ↔ composition</td>
<td>product ↔ composition</td>
</tr>
<tr>
<td>$D_xx=xD_x+1$</td>
<td>$S_nn=(n+1)S_n$</td>
</tr>
</tbody>
</table>

**Taylor morphism:** $D_x \mapsto (n+1)S_n$; $x \mapsto S_n^{-1}$

produces linear recurrence from LDE
Framework: Ore polynomials

\[(fg)' = f'g + fg', \quad S_n(fng_n) = f_{n+1}S_n(g_n), \quad \Delta_n(fng_n) = f_{n+1}\Delta_n(g_n) + \Delta_n(f_n)g_n,\]

and many more (e.g., q-analogues) are captured by \(\mathbb{A}\langle \partial \rangle\) (\(\mathbb{A}\) integral domain) with commutation

\[\partial a = \sigma(a)\partial + \delta(a)\]

\(\sigma\) a ring morphism, \(\delta\) a \(\sigma\)-derivation \((\delta(ab) = \sigma(a)\delta(b) + \delta(a)b)\).

**Main property:** \(A, B\) in \(\mathbb{A}\langle \partial \rangle\), then \(\deg AB = \deg A + \deg B\).

**Consequence 1:** (non-commutative) Euclidean division

**Consequence 2:** (non-commutative) Euclidean algorithm
**GCRD & LCLM**

_greatest common right divisor & least common left multiple_

**GCRD**\((A,B)\): maximal operator whose solutions are common to \(A\) and \(B\).  
**LCLM**\((A,B)\): minimal operator having the solutions of \(A\) and \(B\) for solutions.

Example: closure by sum.

Computation: Euclidean algorithm or linear algebra.
Reduction of order

**Input:** a (large) linear recurrence equation + init. cond

**Output:** a factor annihilating *this* solution

- Step 1: use the recurrence and its initial conditions to compute a large number of terms;

- Step 2: *guess* a linear recurrence equation annihilating this sequence (linear algebra);

- Step 3: take the *gcrd* of this operator and the initial one;

- Step 3: *prove* that this factor annihilates the solution by checking sufficiently many initial conditions.
Example from a continued fraction expansion

\[ P_k = a_k x^2 P_{k-2} + P_{k-1}, \quad a_k = \begin{cases} \frac{2k}{(2k+1)(2k+3)}, & k \text{ even,} \\ \frac{-2(k+2)}{(2k+1)(2k+3)}, & k \text{ odd.} \end{cases} \]

Aim: a recurrence for all \( k \).

- Step 1: use both recurrences to find a relation between \( P_k, P_{k+2}, P_{k+4} \) for even \( k \) and one for odd \( k \);

- Step 2: compute their LCLM (order 8);

- Step 3: use the initial conditions to reduce (order 4).
Chebyshev expansions

\[
z = \frac{1}{3} z^3 + \frac{1}{5} z^5 + \cdots
\]

\[
2(\sqrt{2} + 1) \left( \frac{T_1(x)}{(2\sqrt{2} + 3)} - \frac{T_3(x)}{3(2\sqrt{2} + 3)^2} + \frac{T_5(x)}{5(2\sqrt{2} + 3)^3} + \cdots \right)
\]
Ore fractions

Generalize commutative case:

\[ R = Q^{-1}P \] with \( P \) & \( Q \) operators.

\[ B^{-1}A = D^{-1}C \] when \( bA = dC \) with \( bB = dD = \text{LCLM}(B,D) \).

**Algorithms** for sum and product:

\[ B^{-1}A + D^{-1}C = \text{LCLM}(B,D)^{-1}(bA + dC), \] with \( bB = dD = \text{LCLM}(B,D) \)

\[ B^{-1}AD^{-1}C = (aB)^{-1}dC, \] with \( aA = dD = \text{LCLM}(A,D) \).
Application: Chebyshev expansions

Extend Taylor morphism to Chebyshev expansions

Taylor

\[
x^{n+1} = x \cdot x^n \quad \leftrightarrow \quad x \quad \mapsto \quad X := S^{-1}
\]

\[
(x^n)' = nx^{n-1} \quad \leftrightarrow \quad d/dx \quad \mapsto \quad D := (n+1)S
\]

Chebyshev

\[
2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)
\]

\[
\leftrightarrow \quad x \quad \mapsto \quad X := (S + S^{-1})/2
\]

\[
2(1-x^2)T_n'(x) = -nT_{n+1}(x) + nT_{n-1}(x)
\]

\[
\leftrightarrow \quad d/dx \quad \mapsto \quad D := (1-X^2)^{-1}n(S-S^{-1})/2.
\]

Prop. If \( y \) is a solution of \( L(x,d/dx) \), then its Chebyshev coefficients annihilate the numerator of \( L(X,D) \).
IV. Systems of equations
Example: Contiguity of Hypergeometric Series

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1). \]

\[
\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,
\]

\[
\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.
\]

Gauss 1812: contiguity relation.

\[ \text{dim}=2 \Rightarrow S_a^2 F, S_a F, F \text{ linearly dependent} \]

(coordinates in \( \mathbb{Q}(a,b,c,z) \))

\[(a + 1)(z - 1)S_a^2 F + ((b - a - 1)z + 2 - c + 2a)S_a F + (c - a - 1)F = 0 \]
Ore Algebras

\[ \mathcal{O} := \mathbb{K}(x_1, \ldots, x_r)\langle \partial_1, \ldots, \partial_r \rangle := \mathbb{K}(x_1, \ldots, x_r)\langle \partial_1 \rangle \cdots \langle \partial_r \rangle, \]

with commuting \( \partial_i \)'s and for \( i \neq j, \delta_i(\partial_j) = 0 \) and \( \sigma_i(\partial_j) = \partial_j. \)

**Def.** LM (leading monomial) on next slide.

**Main property:** \( A, B \) in \( \mathcal{O} \), then \( \text{LM}(AB) = \text{LM}(A)\text{LM}(B) \).

**Consequence:** (non-commutative) Gröbner bases

Gröbner bases as a data-structure to encode special functions
Gröbner Bases

1. **Monomial ordering**: order on \( \mathbb{N}^k \), compatible with +, 0 minimal.
2. **Leading monomial** of a polynomial: the largest one.
3. **Gröbner basis** of a (left) ideal \( I \): corners of stairs.
4. **Quotient** mod \( I \):
   basis below the stairs (Vect\{\( \partial^\alpha f \}\)).
5. **Reduction** of \( P \):
   Rewrite \( P \) mod \( I \) on this basis.
6. **Dimension**:
   « size » of the quotient.
7. **D-finiteness**: dimension 0.

An access to (finite-dimensional) vector spaces.
Closure Properties

Proposition.
\[
\dim \text{ann}(f + g) \leq \max(\dim \text{ann } f, \dim \text{ann } g),
\]
\[
\dim \text{ann}(fg) \leq \dim \text{ann } f + \dim \text{ann } g,
\]
\[
\dim \text{ann}(\partial f) \leq \dim \text{ann } f.
\]

Algorithms by linear algebra

simple definitions → data-structures for more complicated functions
V. Sums and Integrals
Examples

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}
\]

\[
\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+1} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}
\]

\[
\int_0^{+\infty} x J_1(ax) l_1(ax) Y_0(x) K_0(x) \, dx = -\frac{\ln(1-a^4)}{2\pi a^2}
\]

\[
\frac{1}{2\pi i} \int \frac{(1 + 2xy + 4y^2) \exp \left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} \, dy = \frac{H_n(x)}{[n/2]!}
\]

\[
\sum_{k=0}^{n} \frac{q^{k^2}}{(q; q)_k(q; q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k}(q; q)_{n+k}}
\]

Aims:

1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable
Creative telescoping

\[ I(x) = \int f(x, t) \, dt = ? \quad \text{or} \quad U(n) = \sum_{k} u(n, k) = ? \]

**Input:** equations
(differential for \( f \) or recurrence for \( u \)).

**Output:** equations for the sum or the integral.

**Example:**

\[
\begin{align*}
u(n, k) &= \binom{n}{k} \text{ def. by } \left\{ \begin{array}{l}
\binom{n+1}{k} = \frac{n+1}{n+1-k} \binom{n}{k}, \\
\binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k}
\end{array} \right. \\
S(n+1) &= \sum_{k} \binom{n+1}{k} = \sum_{k} \binom{n+1}{k} - \binom{n+1}{k+1} + \binom{n}{k+1} - \binom{n}{k} + 2 \binom{n}{k} = 2S(n).
\end{align*}
\]

**IF** one knows \( A(n, S_n) \) and \( B(n, k, S_n, S_k) \) such that

\[
(A(n, S_n) + \Delta B(n, k, S_n, S_k)) \cdot u(n, k) = 0,
\]

then the sum telescopes, leading to \( A(n, S_n) \cdot U(n) = 0 \).
Creative Telescoping

\[ I(x) = \int f(x, t) \, dt = ? \]

**IF** one knows \( A(x, \partial_x) \) and \( B(x, t, \partial_x, \partial_t) \) such that

\[
(A(x, \partial_x) + \partial_t B(x, t, \partial_x, \partial_t)) \cdot f(x, t) = 0,
\]

then the integral « telescopes », leading to \( A(x, \partial_x) \cdot I(x) = 0 \).

Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

*Richard P. Feynman 1985*

**Method:** integration (summation) by parts and differentiation (difference) under the integral (sum) sign
Telescoping Ideal

\[ T_t(f) := \left( \text{Ann } f + \partial_t \mathbb{Q}(x, t) \langle \partial_x, \partial_t \rangle \right) \cap \mathbb{Q}(x) \langle \partial_x \rangle. \]

- hypergeometric summation:
  \( \text{dim}=1 + \text{param. Gosper.} \)
  \[ \text{[Zeilberger]} \]

- holonomy: restrict int. by parts to \( \mathbb{Q}(x) \langle \partial_x, \partial_t \rangle \)
  and Gröbner bases.
  \[ \text{[Wilf-Zeilberger, also Sister Celine]} \]

- finite dim, Ore algebras & GB \[ \text{[Chyzak]} \]

- infinite dim & GB

- rational \( f \) and restrict to \( \mathbb{Q}(x)[t, 1/\text{den } f] \langle \partial_x, \partial_t \rangle \) in very good complexity.
Chyzak’s Algorithm

\[ T_t(f) := \left( \text{Ann } f + \partial_t Q(x, t) \langle \partial_x, \partial_t \rangle \right) \cap Q(x) \langle \partial_x \rangle. \]

**Input:** a Gröbner basis \( G \) for \( \text{Ann } f \) in \( \mathbb{A} = \mathbb{Q}(x, t) \langle \partial_x, \partial_t \rangle \)

**Output:** \( P \) in \( \mathbb{Q}(x) \langle \partial_x \rangle \) and \( Q \) in \( \mathbb{A} \), reduced wrt \( G \) and such that \( (P + \partial_t Q)f = 0 \).

For \( r = 1, 2, 3, \ldots \)

1. use indeterminate coefficients to define

\[ Q = \sum_{(i,j) \text{ below stairs}} q_{i,j}(x, t) \partial_x^i \partial_t^j, \quad P = \sum_{|\alpha| \leq r} p_\alpha(x) \partial_x^\alpha. \]

2. reduce \( P + \partial_t Q \) using \( G \), leading to a 1st order system for \( q_{i,j}(x, t) \) and \( p_\alpha(x) \);

3. stop if a rational solution is found.
Examples of applications

- **Hypergeometric**: binomial sums, hypergeometric series;
  \[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{n!^3} \]

- **Higher dimension**: classical orthogonal polynomials, special functions like Bessel, Airy, Struve, Weber, Anger, hypergeometric and generalized hypergeometric,…
  \[ J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1 - t^2}} \, dt \]

- **Infinite dimension**: Bernoulli, Stirling or Eulerian numbers, incomplete Gamma function,…
  \[ \int_0^\infty \exp(-xy) \Gamma(n, x) \, dx = \frac{\Gamma(n)}{y} \left( 1 - \frac{1}{(y + 1)^n} \right) \]
VI. Faster Creative Telescoping
Certificates are big

\[ C_n := \sum_{r,s} (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+s}{s} \binom{n+r}{r} \binom{2n-r-s}{n} f_{n,r,s} \]

\[(n + 2)^3 C_{n+2} - 2(2n + 3)(3n^2 + 9n + 7)C_{n+1} - (4n + 3)(4n + 4)(4n + 5)C_n = 180 \text{ kB} \simeq 2 \text{ pages} \]

\[ l(z) = \int \frac{(1 + t_3)^2 dt_1 dt_2 dt_3}{t_1 t_2 t_3 (1 + t_3 (1 + t_1))(1 + t_3 (1 + t_2)) + z(1 + t_1)(1 + t_2)(1 + t_3)^4} \]

\[ z^2(4z + 1)(16z - 1)l'''(z) + 3z(128z^2 + 18z - 1)l''(z) + (444z^2 + 40z - 1)l'(z) + 2(30z + 1)l(z) = 1080 \text{ kB} \]

\[ \simeq 12 \text{ pages} \]

Next, in \[ T_t(f) := \left( \text{Ann } f + \partial_t \langle Q(x, t) \mid \partial_x, \partial_t \rangle \right) \cap \langle Q(x) \mid \partial_x \rangle . \]

we restrict to rational \( f \) and \( \partial_t \mathcal{Q}(x)[t, 1/\text{den } f] \langle \partial_x, \partial_t \rangle . \]
Bivariante integrals by Hermite reduction

\[ I(t) = \int \frac{P(t, x)}{Q^m(t, x)} \, dx \]

If \( m=1 \), Euclidean division: \( P=aQ+r \), \( \deg_x r < \deg_x Q \)

\[ \frac{P}{Q} = \frac{r}{Q} + \partial_x \int a \]

Def. Reduced form: \[ \left[ \frac{P}{Q} \right] := \frac{r}{Q} \]

If \( m>1 \), Bézout identity and integration by parts

\[ P = uQ + v \partial_x Q \quad \rightarrow \quad \frac{P}{Q^m} = \frac{u + \frac{\partial_x v}{m-1}}{Q^{m-1}} + \partial_x \frac{v/(1-m)}{Q^{m-1}} \]

Algorithm: \( R_0 := [P/Q^m] \)
for \( i=1, 2, \ldots \) do \( R_i := [\partial_t R_{i-1}] \) when there is a relation \( c_0(t)R_0 + \ldots + c_i(t)R_i = 0 \) return \( c_0 + \ldots + c_i \partial_t^i \)
More variables: Griffiths-Dwork reduction

\[ I(t) = \int \frac{P(t, x)}{Q^m(t, x)} \, dx \quad \text{Q square-free} \]
\[ \text{Int. over a cycle where } Q \neq 0. \]

1. Control degrees by homogenizing \((x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_n)\)

2. If \(m=1\), \([P/Q] := P/Q\)

3. If \(m > 1\), reduce modulo Jacobian ideal \(J := \langle \partial_0 Q, \ldots, \partial_n Q \rangle\)

\[
\begin{align*}
P &= r + v_0 \partial_0 Q + \cdots + v_n \partial_n Q \\
\frac{P}{Q^m} &= \frac{r}{Q^m} - \frac{1}{m-1} \left( \partial_0 \frac{v_0}{Q^{m-1}} + \cdots + \partial_n \frac{v_n}{Q^{m-1}} \right) + \frac{1}{m-1} \frac{\partial_0 v_0 + \cdots + \partial_n v_n}{Q^{m-1}} \\
\left[ \frac{P}{Q^m} \right] &= \frac{r}{Q^m} + [A_{m-1}]
\end{align*}
\]

**Thm.** [Griffiths] In the regular case \((Q(t)[x]/J)_{\text{finite dim}},\) if \(R=P/Q^m\) hom of degree \(-n-1\), \([R] = 0 \iff \int R \, dx = 0\).

→ SAME ALGORITHM.
Size and complexity

\[ I(t) = \int \frac{P(t, x)}{Q^m(t, x)} \, dx \]

\[ \in \mathbb{Q}(t, x) \]

\[ N := \deg_x Q, \quad d_t := \max(\deg_t Q, \deg_t P) \]

Thm. A linear differential equation for \( I(t) \) can be computed in \( O(e^{3nN^8d_t}) \) operations in \( \mathbb{Q} \).

It has order \( \leq N^n \) and degree \( O(e^nN^{3n}d_t) \).

Note: generically, the certificate has at least \( N^{n^2/2} \) monomials.

This has consequences for multiple binomial sums.
Conclusion

• Linear differential equations and recurrences are a great data-structure;
• Numerous algorithms have been developed in computer algebra;
• Efficient code is available;
• More is to be found (certificate-free algorithms, diagonals,...)