Lattice Walks and Reliable Dominant Eigenvalues of the Laplacian on Spherical Triangles

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Joint work with Joel Dahne
Dominant Eigenvalue of the Laplace-Beltrami Operator on the Unit Sphere

Laplace operator in spherical coordinates in $\mathbb{R}^d$

$$\Delta f = r^{1-d} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial f}{\partial r} \right) + r^{-2} \Delta_{\mathbb{S}^{d-1}} f$$

Eigenvalue problem for $\Omega \subset \mathbb{S}^{d-1}$:
$$\Delta_{\mathbb{S}^{d-1}} f + \lambda f = 0 \text{ in } \Omega, \quad f|_{\partial \Omega} = 0.$$

Classical fact: $0 < \lambda_1 < \lambda_2 \leq \ldots$, $\lambda_n \to \infty$

Goal: $(\alpha, \beta, \gamma) \mapsto \lambda_1$ with high precision
(dimension $d=3$)
## Results

<table>
<thead>
<tr>
<th>Angles</th>
<th>BPRT</th>
<th>new</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3\pi/4,\pi/3,\pi/2)$</td>
<td>12.400051</td>
<td>12.40005165284337905… $\pm 10^{-26}$</td>
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<tr>
<td>$(2\pi/3,\pi/3,\pi/2)$</td>
<td>13.74435</td>
<td>13.744355213213231835… $\pm 10^{-94}$</td>
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<tr>
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<td>20.571964</td>
<td>20.571973537984730557… $\pm 10^{-28}$</td>
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<tr>
<td>$(2\pi/3,\pi/3,\pi/3)$</td>
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<td>21.30940763019045206… $\pm 10^{-159}$</td>
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<tr>
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<td>24.456910</td>
<td>24.45691379629911694… $\pm 10^{-40}$</td>
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<tr>
<td>$(2\pi/3,\pi/4,\pi/4)$</td>
<td>49.109942</td>
<td>49.109945263284609920… $\pm 10^{-129}$</td>
</tr>
<tr>
<td>$(2\pi/3,3\pi/4,3\pi/4)$</td>
<td>4.261735</td>
<td>4.3 $\pm 5 \times 10^{-2}$</td>
</tr>
<tr>
<td>$(2\pi/3,2\pi/3,2\pi/3)$</td>
<td>5.159146</td>
<td>5.16 $\pm 5 \times 10^{-3}$</td>
</tr>
<tr>
<td>$(\pi/2,2\pi/3,3\pi/4)$</td>
<td>6.241748</td>
<td>6.2 $\pm 5 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

finite elements & convergence acceleration

I. Why do we care?
II. How do we do it?
III. What is going on?
I. Linear Recurrences with Constant Coefficients
In order to approach this counting problem, one first defines a suitable collection, generically denoted by $C$, of combinatorial classes called configurations, in accordance with the strategy summarized in Figure V.20, p. 356. A configuration relative to an $n \times k$ rectangle is a partial tiling, such that all the first $n-1$ columns are entirely covered by dominoes while between zero and three unit cells of the last column are covered. Here are for instance, configurations corresponding to the example above.

These diagrams suggest the way configurations can be built by successive addition of dominoes. Starting with the empty rectangle $0 \times 3$, one adds at each stage a collection of at most three dominoes in such a way that there is no overlap. This creates a configuration where, like in the example above, the dominoes may not be aligned in a flush-right manner.

Continue to add successively dominoes whose left border is at absciss $a_1, a_2, a_3, \ldots$, in a way that creates no internal "holes".

Depending on the state of filling of their last column, configuration can thus be classified into 8 classes that we may index in binary as $C_{000}, \ldots, C_{111}$. For instance $C_{001}$ represents configurations such that the first two cells (from top to bottom, by convention) are free, while the third one is occupied. Then, a set of rules describes the new type of configuration obtained, when the sweep line is moved one position to the right and dominoes are added. For instance, we have $C_{010} \circ \text{/equal} 1 \Rightarrow C_{101}$.

In this way, one can set up a system of linear equations (resembling a grammar or a deterministic finite automaton) that expresses all the possible constructions of longer rectangles from shorter ones according to the last layer added. The system contains equations like

$$u_{n+k} = a_0 u_n + \cdots + a_{k-1} u_{n+k-1} \quad \text{with initial conditions } u_0, \ldots, u_{k-1}$$

very well understood

$(u_n)$ is a LRS $\iff$ its generating series $U(z) := \sum_{n=0}^{\infty} u_n z^n$ is rational

Ex. Fibonacci: $F_{n+2} = F_{n+1} + F_n$, $F_0 = F_1 = 1$

$$F(z) = \frac{z}{1 - z - z^2} = \frac{(2\phi - 1)/5}{1 - z\phi} - \frac{(2\phi - 1)/5}{1 + z/\phi}$$

$$F_n = \frac{1}{2\pi i} \oint \frac{F(z)}{z^{n+1}} \, dz$$

Numbers divisible by 5 in base 2

Tilings of rectangles of bounded height by dominos and monominos
Classes of Univariate Power Series

\[
U(z) := \sum_{n=0}^{\infty} u_n z^n
\]

- **RATIONAL**
- **ALGEBRAIC**
- **DIAGONAL**
- **D-FINITE**
- **DIFF. ALGEBRAIC**
- **DIFF. TRANSCENDENTAL**

**Christol’s conjecture:** All D-finite power series with integer coefficients and radius of convergence in \((0, \infty)\) are diagonals.

\((u_n)\) satisfies a linear recurrence with polynomial coefficients

\[P(z, U(z)) = 0\] words in context-free languages

\[U(z) \text{ satisfies a linear diff. eqn. with polynomial coefficients}\]

\[P(z, U(z), U'(z), \ldots) = 0\]

Knowing where \(U(z)\) fits helps deduce properties of \((u_n)\).
Lattice Walks: a Mine of Linear Recurrences Waiting for Tools

Num. walks from (0,0) to \((i,j) \in \mathbb{Z}^2\) using \(n\) steps in \(\mathcal{S}\)

**Ex.:** \(\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \nwarrow\}\)

\[ u_{i,j,n} = u_{i-1,j,n-1} + u_{i,j-1,n-1} + u_{i+1,j,n-1} + u_{i,j+1,n-1} + u_{i+1,j+1,n-1} \]

\[ U(x, y, z) := \sum_{i,j,n} u_{i,j,n} x^i y^j z^n, \quad U(0,0,z) = \sum_{n \geq 0} e_n z^n, \quad U(1,1,z) = \sum_{n \geq 0} u_n z^n. \]

**Applications:** queuing theory, statistical physics, combinatorics,..

**Question:** \(\mathcal{S}\), boundary conditions \(\rightarrow\) nature of these series?

**Boundary conditions:**
- no constraint: \(U\) rational;
- \(u_{i,j,n} = 0\) for \(i < 0\): \(U\) algebraic;
- \(u_{i,j,n} = 0\) for \(i < 0\) and \(j < 0\): depends on \(\mathcal{S}\).

[BanderierFlajolet02]
Walks in $\mathbb{N}^2$: Recent Progress

79 quadrant models with small steps

- Creative telescoping algebraic (1965, 1985, 2010)
- D-finite (2010)
- Probabilities & number theory non-D-finite (2014)
- Tutte invariants (2017)
- Differential Galois theory (2018)

Motivation for this work

[Bernardi, Bostan, Bousquet-Mélou, Gessel, Gouyou-Beauchamps, Hardouin, Kauers, Kreweras, Melczer, Mishna, Raschel, Rechnitzer, Roques, Salvy, Singer]
Probabilities & Number Theory for Walks in $\mathbb{N}^d$

$\mathcal{S} = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \land\}$

Idea: Normalize so that the asymptotic behaviour is a Brownian motion

a. fix probabilities for each step that remove drift
b. linear transform to remove correlation $\mathbb{N}^d$ becomes a cone $K$

**Probabilistic** ingredient:

$$e_n \sim K\rho^n n^{-p/2} \text{ with } p = \sqrt{\lambda_1 + (d/2 - 1)^2} - (d/2 - 1) > 0,$$

$\lambda_1$ dominant eigenvalue of $\Delta_{\mathbb{S}^{d-1}}$ on $K \cap \mathbb{S}_{d-1}$.

**Arithmetic** ingredient:

$U(z)$ $D$-finite, convergent, with integer coefficients $\Rightarrow p \in \mathbb{Q}$.

[DenisovWachtel15; Chudnovsky85; André89; Katz70]
Planar Case

\[ \Delta f(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \]

Eigenfunctions: \( \sin(\sqrt{\lambda} \theta + c) \)

Boundary: \[
\begin{align*}
\theta &= 0 \Rightarrow c = 0 \\
\theta &= \alpha \Rightarrow \lambda = \lambda_k := \left( \frac{k\pi}{\alpha} \right)^2, \; k \in \mathbb{N}^*.
\end{align*}
\]

Ex. \( S = \{\uparrow, \downarrow, \rightarrow, \leftarrow, \downarrow\} \)

\[
-\frac{p}{2} = 1 - \frac{\pi}{\arccos(u)}, \; 8u^3 - 8u^2 + 6u - 1 = 0, \; u > 0 \Rightarrow p \notin \mathbb{Q} \Rightarrow U \text{ not D-finite.}
\]

Automatic proof of 51 of the 56 non-Dfinite cases.

[14]
II. 3D Walks: Laplacian on Spherical Triangles

No more closed-forms? Turn to numerical approximation.
Method of Particular Solutions

Eigenvalue problem for $\Omega$:

$$\Delta f + \lambda f = 0 \text{ in } \Omega,$$

$$f|_{\partial \Omega} = 0.$$

A 3-step method:

1. Find a basis of solutions of $\Delta f + \lambda f = 0$ in $\Omega$;
2. Use it to find $(f^*, \lambda^*)$ s.t. $f^*$ is small on $\partial \Omega$;
3. A close-by (eigenvalue,eigenfunction) satisfies

$$\frac{|\lambda - \lambda^*|}{\lambda^*} \leq \frac{\sup_{x \in \partial \Omega} |f^*(x)|}{\|f^*\| \text{ Vol}(\Omega)}.$$
Step 1. EigenFunctions

Separation of variables: \( f = F(\phi)G(\theta) \) gives

\[
F(\phi) = \sin(\mu \phi + c), \quad G(\theta) = P_{\mu \nu}(\cos \theta) \quad (\mu \leq 0)
\]

with \( \lambda = \nu(\nu + 1) \).

First 2 boundaries:

\[
\begin{cases} 
\phi = 0 \rightarrow c = 0 \\
\phi = \phi_{\text{max}} \rightarrow \mu = \mu_k := -\frac{k\pi}{\phi_{\text{max}}}, \; k \in \mathbb{N}.
\end{cases}
\]
Step 2. The Final Boundary

\[ B^N_\nu (\phi) := \sum_{k=0}^{N} c_k \sin(\mu_k \phi) P^\mu_k(\cos \theta(\phi)) \]

**WANTED:** \( \nu \) s.t. \( B^{\infty}_\nu |_{[0,\phi_{\max}]} = 0, \|B_\nu\|_\Omega \approx 1. \)

For each \( \nu \), use [BetckeTrefethen05]

For the "good" case → many digits

For the "not so good" case → fewer digits
Step 3. Rigorous Bound on the Boundary

Use a rigorous polynomial approximation

Taylor coefficients from gfun interval evaluation in Arb.
Summary & Conclusion

Linear recurrences with constant coefficients remain mysterious;
lattice walks provide a simple source of examples;
more and more tools are available;
high-precision is useful in experimental mathematics;
work still in progress (improve speed, work on the bad cases).

Thank you.